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CONTENTS

DEDICATION

To the ones that I am indebted to for eternity: my dear father and mother who have encouraged me all the way and whose encouragement, to this day, has made sure that I give it all it takes to finish that which I have started. my dear sisters: Dr. Ikhlass, Dr. Toka, Dr. Maroua and Ghaya, thank you. My love for you all can never be quantified.

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INTRODUCTION

The last century witnessed the birth of two new revolutions in the field of science, which led to a great development that we observe today. Where the first axis of the contemporary scientific revolution was focused on the macroscopic scale. This resulted in the birth of both special relativity and general relativity. It was Einstein who reaped the fruit of the two theories by relying on the research of many scientists in his time and centuries before his era, where classical mechanics was the dominant force for centuries by relying on Galilean relativity. Classical mechanics was used to predicting the dynamics of material bodies, and Maxwell's electromagnetism provided the proper framework to study radiation; matter and radiation were described in terms of particles and waves, respectively [1].

The second part of the scientific revolution was concerned with the microscopic field, where distances are of the order of Planck's constant. This is at the atomic and subatomic levels. At this level of infinitesimal dimensions, the energies range from ev to Mev, where strong and weak nuclear interactions prevail.

By considering the solutions of the basic equations known in the literature,it is possible to study various physical information about the systems, such as their energy, the wave function, and others. The Schrödinger equation describes the state of atomic particles with energies in the ev range, which corresponds to low energy. The Schrödinger equation must be replaced by the Klein- Gordon equation or the Dirac equation at an energy of the order of Mev, which corresponds to high energy. The Schrodinger equation is a fundamental equation in nonrelativistic quantum mechanics that describes the time evolution of a quantum object's state by studying wave function and energy spectrum of various conceivable states. Among the most important principles in quantum mechanics is the Heisenberg uncertainty principle. It states that the position and speed of a particle cannot be measured at the same time with very high precision because the uncertainty Δx in the measurement of its position and the uncertainty Δp on its momentum must satisfy the relation:

$$\Delta x.\Delta p = \hbar$$

This is how position x and momentum p become operators $\hat{x}$ and $\hat{p}$ act on the state space, which is a Hilbert space; their commutator is given by [1, 2]:

$$\begin{aligned} [x_i, p_j] &= \imath\hbar\delta_{ij} \\ [x_i, x_j] &= [p_i, p_j] = 0 \end{aligned}$$

Quantum mechanics on noncommutative space was first proposed by Heisenberg in 1930 [3] and then developed by Snyder in late 1947 [4]. The proposal of extended quantum mechanics came as a possible solution to many physical problems that non-relativistic and relativistic quantum mechanics were unable to find solutions to the divergence problem in the standard model, string theory and quantum gravity [5, 6, 7, 8].

We will reserve this study to obtain a master's degree in theoretical physics from Mohammed Boudiaf University in M'sila to study the Yukawa potential in the context of nonrelativistic noncommutative quantum mechanics symmetries for the promotion 2021-2022 because noncommutative quantum mechanics includes larger physical symmetries than the quantum mechanics known in the literature.

This master memory is organized as follows. In chapter one, the noncommutative quantum mechanics is represented. In chapter two, the Schrödinger equation is revised under the Yukawa potential. In chapter three, we study the effect of noncommutativity properties on the Yukawa potential.

Part I

The noncommutative phase-space formalism

1.1 Introduction

The postulates and hypotheses that identify the quantum and physical structures of the noncommutative phase-space and its physical structures will be discussed in this chapter. The following are the foundational principles that we shall discuss:

-An example of a common quantum structure

-The noncommutative space-new phase's postulates

-The Moyal-Weyl recipe, the Five-Star product and its properties

-The method of Bopp's Shift and its applications to Yukawa's potential

1.2 Review of structure of ordinary quantum mechanics

In 1900, Planck quantifies the energy of light $E_\gamma = hv$, which consider the beginning of quantum physics, here $h \approx 6,6262.10^{34}\ js$. Currently, ordinary quantum mechanics is formulated on the commutative space of the coordinates of variable and the canonical moment of hermetic operators $(x_i;\ p_i)$, as follows [9, 10]:

$$\begin{cases} [x_i, p_j] = \imath\hbar\delta_{ij} \\ [x_i, x_j] = 0 \\ [p_i, p_j] = 0 \end{cases} \tag{1.1}$$

Here, $\hbar = \frac{\hbar}{2\pi}$ is the reduced Plank constant, and δ_{ij} is the Kronecker ordinary symbol. The above algebra can be generalized to the Dirac picture as follows:

$$\begin{cases} [x_i(t), p_j(t)] = \imath\hbar\delta_{ij} \\ [x_i(t), x_j(t)] = 0 \\ [p_i(t), p_j(t)] = 0 \end{cases} \tag{1.2}$$

where the usual canonical coordinates and new momentum $x_i(t)$ and $p_i(t)$ are determined from the projection relations:

$$\begin{aligned} x_i(t) &= \exp(\imath H(t-t_0))\, x_i \exp(-\imath H(t-t_0)) \\ p_i(t) &= \exp(\imath H(t-t_0))\, p_i \exp(-\imath H(t-t_0)) \end{aligned} \tag{1.3}$$

Here, $\{x_i(t)\}$, $\{p_i(t)\}$ and H are Hermitian operators on a Hilbert space of physical states, which, each, satisfy the Heisenberg equation of motions. We get the following:

$$\begin{cases} \frac{dx_i}{dt} = \imath\,[H, x_i(t)] \\ \frac{dp_i}{dt} = \imath\,[H, p_i(t)] \end{cases} \tag{1.4}$$

Both related concepts relating to energy E and impulsion p_i are satisfied by the quantization procedure:

$$\begin{cases} E \to i\hbar\frac{\partial}{\partial t} \\ p_i \to i\hbar\frac{\partial}{\partial t} \end{cases} \tag{1.5}$$

It is well known that the classical energy E of a particle of massm_0 subjected to the external forces produced by a potential $V(\vec{r}, t)$, in a classical mechanic is given by:

$$E = \frac{\overrightarrow{P^2}}{2m} + V(\vec{r}, t) \tag{1.6}$$

The quantization procedure in $Eq.(1.5)$ permitted to obtain the Shrödinger equation known in the framework of quantum mechanics known in the literature:

$$\left(\frac{-\hbar^2}{2m}\Delta - V(r)\right)\psi\left(\overrightarrow{r},t\right) = i\hbar\frac{\partial\psi\left(\overrightarrow{r},t\right)}{\partial t} \tag{1.7}$$

Here Δ is the Laplacian operator in spherical coordinates $\overrightarrow{r}(r,\theta,\varphi)$:

$$\Delta = \bigtriangledown^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\Phi^2} \tag{1.8}$$

which can be expressed in Cartesian coordinates $\overrightarrow{r}(x,y,z)$ as follows:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.9}$$

while $\psi\left(\overrightarrow{r},t\right)$ denoting the complex wave function. The probability of finding the particle at time t in an elementary volumes (d^3r, d^3p), rounding the point r as follows:

$$dp = \begin{cases} \left|\psi\left(\overrightarrow{r},t\right)\right|^2 d^3r \\ In\ the\ configuration\ space \\ \left|\psi\left(\overrightarrow{p},t\right)\right|^2 d^3p \\ In\ the\ momentum\ space \end{cases} \tag{1.10}$$

Where (d^3r, d^3p) equals $(r^2\sin\theta d\theta d\varphi dr, p^2\sin\theta d\theta d\varphi dr)$, respectively.

1.3 Noncommutative phase-space

One of the most essential aspects of quantum physics is dealing with non-commuting operators, specifically the commutation relations between positions

x_i and corresponding momenta p_i. Noncommutative quantum mechanics NCQM symmetries is a concept that implies that operators do not commute; for example, consider a situation in which the coordinates and moment operators are noncommutative. During 1930, a hot topic was how to solve the infinity problem in the newly found quantum field theory QFT. Heisenberg was the first to propose that noncommutativity be extended to coordinate systems. Then, the concept of NCQM was extended to generalize the usual conception of space-time, in which the noncommutativity of some normally commutative variables is assumed, leading to the formation of different Lie algebras. Connes in 1980 revived the ideas of noncommutative geometry while Woronowicz and Drinfel'd, were generalized the notion of a differential structure to the noncommutative setting [11, 12, 13, 14]. The simplest commutation relation that described the noncommutativity idea satisfying the following algebra [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]:

$$\begin{cases} [\hat{x}_i, \hat{x}_j] = \imath \hbar_{eff} \theta_{ij} \\ [p_i, p_j] = \imath \theta_{ij} \end{cases} \tag{1.11}$$

Where $\left(\theta_{ij}, \bar{\theta}^{ij}\right) = -\left(\theta_{ij}, \bar{\theta}^{ij}\right) = \epsilon_{ij}\left(\theta, \bar{\theta}\right)$ are constants anti-symmetric tensors of dimensions $[x]^2$ and $[p]^2$, $\left(\theta, \bar{\theta}\right)$ are the NC parameters and ϵ_{ij} is just an antisymmetric number ($\epsilon_{ij} = -\epsilon_{ij} = 1$ with $i \neq j$ and ϵ_{ij}) and $\hbar_{eff} = \hbar\left(1 + \frac{\theta\bar{\theta}}{4\hbar^2}\right)$ is the effective constant of Planck. The noncommutative coordinates $(\hat{x}_i, \hat{p}_i)$ take the form:

$$\begin{cases} x_i \rightarrow \hat{x}_i = f\left(x_i, p_i\right) \\ p_i \rightarrow \hat{p}_i = f\left(x_i, p_i\right) \end{cases} \tag{1.12}$$

In this work, we interred by the phase-phase has three dimensions $N = 3$, therefore the indices take the values $(i, j = 1; 3)$. In this particular case, the rules of canonical commutations become:

$$\begin{cases} [\hat{x}_1, \hat{p}_2] = 0 \\ [\hat{x}_1, \hat{p}_3] = 0 \\ [\hat{x}_2, \hat{p}_3] = 0 \\ [\hat{x}_1, \hat{x}_2] = i\theta_{12} \\ [\hat{x}_1, \hat{x}_3] = i\theta_{13} \\ [\hat{x}_2, \hat{x}_3] = i\theta_{23} \end{cases} \tag{1.13}$$

and

$$\begin{cases} [\hat{x}_1, \hat{p}_2] = i\hbar_{eff} \\ [\hat{x}_1, \hat{p}_3] = i\hbar_{eff} \\ [\hat{x}_2, \hat{p}_3] = i\hbar_{eff} \\ [\hat{p}_1, \hat{p}_2] = i\bar{\theta}_{12} \\ [\hat{p}_1, \hat{p}_3] = i\bar{\theta}_{13} \\ [\hat{p}_2, \hat{p}_3] = i\bar{\theta}_{32} \end{cases} \tag{1.14}$$

1.4 Weyl's quantization

The fundamentals of quantum physics inspired many of the broad principles behind noncommutative geometry. Weyl proposed an elegant formulation for mapping quantum operators to classical functions of phase-space variables within the framework of canonical quantification. This method establishes a systematic approach to modeling noncommutative spaces in general and examining ancient ideas based on them [25]. Weyl quantization is a technique for describing quantum physics using classical mechanics' phase space. It is a rule that allows a quantum operator to be associated with a classical function that is dependent on phase space variables. The Weyl quantification also applies to commutative relations in a general form. Consider a $f(x,p)$ and $g(x,p)$ a general two functions, their product in the notion of noncommutative phase-space can be expressed as a new product called the star product or the Weyl-Moyal star product defined on phase space,

$$\begin{aligned}(f * g)(x,p) &= f(x,p)\, g(x,p) + \frac{i}{2}\sum \theta^{mn} \frac{\partial}{\partial x^m} f(x,p) \frac{\partial}{\partial x^n} + O\left(\theta^2\right) \quad (1.15) \\ &+ \frac{\imath}{2}\sum \bar{\theta}^{mn} \frac{\partial}{\partial p^m} f(x,p) \frac{\partial}{\partial p^n} g(x,p) + O\left(\theta^2, \bar{\theta}^2\right)\end{aligned}$$

The formalism of the star product initiated by Weyl and Wigner to allow a description of quantum mechanics in terms of phase space, is articulated not around non-commuting operators, as in the operational approach, but around the deformation of the product between the phase space variables. We will see how this formalism can be used in the context of noncommutative quantum mechanics (NCQM) symmetries.

1.5 Properties of the star product

The formalism of the star product was initiated by Weyl and Wigner to allow a description of quantum mechanics in terms of phase space, the properties of the star product are presented as follows [27, 28, 29, 30, 31, 32]:

-When $\left(\theta,\bar{\theta}\right)=(0,0)$

$$f\left(x\right)*g\left(x\right)=f\left(x\right)g\left(x\right) \tag{1.16}$$

Thus we find the commutative case.

-The star product between exponential:

$$e^{\imath kx}*e^{\imath qx}=e^{\imath(k+q)x}e^{-\frac{\imath}{2}(k\curlywedge q)} \text{ with } k\curlywedge q=k^{\imath}q^{\jmath}\theta_{ij} \tag{1.17}$$

-Not commutative:

$$f\left(x,p\right)*g\left(x,p\right)\neq g\left(x,p\right)*f\left(x,p\right) \tag{1.18}$$

-Associative:

$$\left(f\left(x,p\right)*g\left(x,p\right)\right)*h\left(x,p\right)=f\left(x,p\right)*\left(g\left(x,p\right)*h\left(x,p\right)\right) \tag{1.19}$$

-The relation of the complex conjugate:

$$\left(f\left(x,p\right)*g\left(x,p\right)\right)^{*}=g\left(x,p\right)^{*}*f\left(x,p\right) \tag{1.20}$$

The integral relation:

$$\left\{\begin{array}{l}\int d^{D}x\left(f*g\right)=\int d^{D}x\left(g*f\right)\left(x,p\right)\\ \qquad=\int d^{D}xf\left(x,p\right)g\left(x,p\right)\end{array}\right. \tag{1.21}$$

-Cyclic permutation:

$$\int d^D x \left(f * g * h\right) = \int d^D \left(g * f * h\right) = \int d^D x \left(g * f * h\right) \tag{1.22}$$

-Satisfies the Leibniz's rule:

$$\partial_\mu \left(f * g\right) = \partial_\mu f * g + f * \partial_\mu g \tag{1.23}$$

1.6 Boop's shift method

In his study, physicist Fritz Bopp was the first to examine pseudo-differential operators derived from a symbol using quantization methods:

$$\begin{cases} x \rightarrow x + \frac{1}{2}\imath\hbar\partial_p \\ p \rightarrow p - \frac{1}{2}\imath\hbar\partial_x \end{cases} \tag{1.24}$$

Instead of the usual correspondence $\left(x \rightarrow x,\ p \rightarrow -\frac{1}{2}\imath\hbar\partial_x\right)$,the operators $x \rightarrow x + \frac{1}{2}\imath\hbar\partial_p$ and $p \rightarrow p - \frac{1}{2}\imath\hbar\partial_x$ are known as Bopp's shifts, and this quantization procedure is known as the Bopp quantization procedure. This quantization leads us to obtain the following:

$$\begin{cases} \hat{x}^i = \hat{x}^i + \sum_j \frac{\theta_{ij}}{2} p_j \\ \hat{p}^i = \hat{p}^i + \sum_j \frac{\theta_{ij}}{2} x_j \end{cases} \tag{1.25}$$

To write the Schrodinger equation in the noncommutative phase-space, we follow these steps:

1- The Ordinary three dimensional Hamiltonian operators $\hat{H}\left(p_i, x_i\right)$ are replace with new Hamiltonian operators $\hat{H}\left(\hat{p}_i, \hat{x}_i\right)$.

2- The ordinary complex wave function $\psi\left(\overrightarrow{r}\right)$ become a new complex wave function $\hat{\psi}\left(\hat{r}\right)$.

3- The ordinary two energies E are replace with new values $E_{nc-iqym}$.

4- We replace the ordinary product with the star product.

Hence, we get the following Schrodinger equation in the noncommutative space:

$$\begin{cases} H\left(\hat{x}^{i},\hat{p}^{i}\right)\hat{\psi}\left(\vec{r},t\right)=E_{nc}\hat{\psi}\left(\vec{r},t\right) \\ \hat{H}\left(x^{i},p^{i}\right)\hat{\psi}\left(\vec{r},t\right)=E_{nc}\hat{\psi}\left(\vec{r},t\right) \end{cases} \tag{1.26}$$

The Bopp's shifts method allows to reduce the above deformed Shrodinger equation to the new translated form:

$$H\left(\hat{x}^{i},\hat{p}^{i}\right)*\hat{\psi}\left(\vec{r},t\right)=E_{nc}\hat{\psi}\left(\vec{r},t\right)=E_{nc}\hat{\psi}\left(\vec{r}\right) \tag{1.27}$$

So the Hamiltonian operator takes the three varieties forms as follows [33, 34, 35, 36, 37, 38, 39, 40]:

$$\begin{cases} H\left(\hat{p}_{i},\hat{x}_{i}\right)=H\left(\hat{p}_{i}=p_{i}-\frac{\bar{\theta}^{ij}}{2}x_{j}, \quad \hat{x}_{i}=x_{i}-\frac{\theta^{ij}}{2}p_{i}\right) \\ \text{For Non commutative phase-space} \\ H\left(\hat{p}_{i},\hat{x}_{i}\right)=H\left(\hat{p}_{i}=p_{i}, \quad \hat{x}_{i}=x_{i}-\frac{\theta^{ij}}{2}p_{j}\right) \\ \text{For Non commutative space-space} \\ H\left(\hat{p}_{i},\hat{x}_{i}\right)=H\left(\hat{p}_{i}=p_{i}-\frac{\bar{\theta}^{ij}}{2}x_{j}, \quad \hat{x}_{i}=x_{i}\right) \\ \text{For Non commutative phase-phase} \end{cases} \tag{1.28}$$

The first variety corresponds to noncommutative phase-space NCSP symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_{i}=p_{i}-\frac{\bar{\theta}^{ij}}{2}x_{j}, \quad \hat{x}_{i}=x_{i}-\frac{\theta^{ij}}{2}p_{j}\right)$

in Eq. (1.28):

$$\begin{cases} p_i \to \hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2} x_j \\ x_i \to \hat{x}_i = x_i - \frac{\bar{\theta}^{ij}}{2} p_j \end{cases} \tag{1.29}$$

The second variety corresponds to noncommutative space-space symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_i = p_i, \quad \hat{x}_i = x_i - \frac{\theta^{ij}}{2} p_j\right)$ in Eq. (1.28):

$$\begin{cases} p_i \to \hat{p}_i = p_i \\ x_i \to \hat{x}_i = x_i - \frac{\bar{\theta}^{ij}}{2} p_j \end{cases} \tag{1.30}$$

The third variety corresponds to noncommutative phase-phase NCPP symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_i = p_i + \frac{\theta^{ij}}{2} x_j, \quad \hat{x}_i = x_i\right)$ in Eq. (1.28):

$$\begin{cases} p_i \to \hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2} x_j \\ x_j \to \hat{x}_i = x_i \end{cases} \tag{1.31}$$

In our current master memoir, we are interested in applying the following general procedure to NCSP symmetries which correspond to the first variety of Eq. (1.28). The three-generalized coordinates $(\hat{x} = \hat{x}_1,\ \hat{y} = \hat{x}_2,\ z = \hat{x}_3)$ in the noncommutative phase-space were depended on corresponding three-usual generalized positions (x, y, z) and three momentum coordinates (p_x, p_y, p_z):

$$\begin{cases} i_1 = 1 \quad \hat{x}_1 = \hat{x}_{p_1} = p_x \\ i_2 = 2 \quad \hat{x}_2 = \hat{x}_{p_2} = p_y \\ i_3 = 3 \quad \hat{x}_3 = \hat{x}_{p3} = p_z \end{cases} \tag{1.32}$$

It is important to notice that the new operators $\hat{x}_i$ and $\hat{p}_i$ in (NC-3D) was depended on ordinary operator x_i and p_i from the projection relations [28]:

$$\begin{cases} \hat{x}_1 = x_1 - \frac{\theta^{12}}{2}p_2 - \frac{\theta^{13}}{2}p_3 \\ \hat{x}_2 = x_2 - \frac{\theta^{21}}{2}p_1 - \frac{\theta^{23}}{2}p_3 \\ \hat{x}_3 = x_3 - \frac{\theta^{31}}{2}p_1 - \frac{\theta^{32}}{2}p_2 \end{cases} \tag{1.33}$$

and

$$\begin{cases} \hat{p}_1 = p_1 - \frac{\bar{\theta}^{12}}{2}x_2 - \frac{\bar{\theta}^{13}}{2}x_3 \\ \hat{p}_2 = p_2 - \frac{\bar{\theta}^{12}}{2}x_1 - \frac{\bar{\theta}^{23}}{2}x_3 \\ \hat{p}_3 = p_3 - \frac{\bar{\theta}^{31}}{2}x_1 - \frac{\bar{\theta}^{32}}{2}x_2 \end{cases} \tag{1.34}$$

The non-vanish 9-commutators in (NC-3D) can be determined as follows [29]:

$$\begin{cases} [\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i \\ [\hat{x}, \hat{y}] = i\theta_{12}, \\ [\hat{x}, \hat{z}] = i\theta_{13}, \\ [\hat{y}, \hat{z}] = i\theta_{23}, \\ [\hat{p}_y, \hat{p}_y] = i\bar{\theta}_{12}, \\ [\hat{p}_y, \hat{p}_z] = i\bar{\theta}_{23}, \\ [\hat{p}_x, \hat{p}_z] = i\bar{\theta}_{13}. \end{cases} \tag{1.35}$$

The square of$(\overrightarrow{r}, \overrightarrow{p})$ are given by :

$$\begin{cases} \overrightarrow{\hat{r}}^2 = \overrightarrow{\hat{r}}^2_x + \overrightarrow{\hat{r}}^2_y + \overrightarrow{\hat{r}}^2_z \\ \overrightarrow{\hat{p}}^2 = \overrightarrow{\hat{p}_x}^2 + \overrightarrow{\hat{p}_y}^2 + \overrightarrow{\hat{p}_z}^2 \end{cases} \tag{1.36}$$

To get the solution to the noncommutative Schrodinger equation, we added the star product introduced by Bopp's shift method. That is a consequence of the star product between the potential operator $\hat{V}(\hat{x})$ and the complex wave function $\hat{\psi}\hat{r}$:

$$\left[\frac{\overrightarrow{\hat{p}}^{2}}{2m}+\hat{V}\left(\hat{r}\right)\right]*\hat{\psi}\left(\hat{r}\right)=E_{nc}\hat{\psi}\left(\vec{r},t\right)$$

$$\Longrightarrow$$

$$\left[\frac{\vec{p}_{nc}^{2}}{2m}+V\left(\hat{r}\right)\right]\psi\left(\vec{r}\right)=E\psi\left(\vec{r}\right) \tag{1.37}$$

The two operators $\hat{x}$ and $\hat{p}$, when on a noncommutative three-dimensional space-phase, can be written as follows :

$$\begin{cases} \hat{r}^{2}=r^{2}-\overrightarrow{L}\vec{\Theta} \\ \frac{\overrightarrow{p_{nc}}^{2}}{2m_{0}}=\frac{P^{2}}{2m_{0}}+\frac{\vec{L}\overrightarrow{\Theta}}{2m_{0}} \end{cases} \tag{1.38}$$

Where the two couplings $\overrightarrow{L}\overrightarrow{\Theta}$ and$L\overrightarrow{\overline{\theta}}$ are given by the following relations respectively:

$$\begin{cases} \overrightarrow{L}\overrightarrow{\Theta}=L_{x}\theta_{23}+L_{y}\theta_{13}+L_{z}\theta_{12} \\ L\overrightarrow{\overline{\theta}}=L_{x}\overline{\theta}_{23}+L_{y}\overline{\theta}_{13}+L_{z}\overline{\theta}_{12} \end{cases} \tag{1.39}$$

Part II

Reviewed SE with Yukawa potential in QM

1.7 Introduction

We revised the Yukawa potential within the framework of ordinary quantum mechanics in this chapter. Within the context of the three-dimensional Schrodinger equation, we also attempt to revise the associated wave function and energy eigenvalue.

1.8 Schrodinger equation with the Yukawa potential

Yukawa's potential (also known as static screened Coulomb, Debye-Hückel potential, or Thomas-Fermi potential) is an exponential potential proposed by Yukawa in 1935[41] which has the form:

$$V(r) = V_0 \frac{\exp(\alpha r)}{r} \tag{2.1}$$

where $V_0 = aZ$, $a = (137.037)^{-1}$ is the fine-structure constant, Z is the atomic number, and is the screening parameter [42]. This potential as a low-energy explanation for nucleon-nucleon interactions induced by the exchange of massive particles known as pions is relatively new to researchers in the field [43]. The Yukawa potential can be found in a variety of physics fields, including plasma physics at low densities and high temperatures, nuclear physics, astro-physics and solid-state physics [44, 45, 46, 47, 48, 49, 50, 51, 52, 53]. Due to the fact that this potential has wide applications, it has received great interest from researchers in the relativistic levels within the framework of the Dirac and Klein Gordon equations, in addition to the non-nonrelativistic framework where the Schrödinger equation is applied using many methods and various approximations [52, 53, 54, 55, 56, 57, 58, 59, 60]. As for the relativistic and non-relativistic level, within the framework of the principles of extended quantum mechanics or NCQM symmetries, it has received recent studies, including those published in the references as it is single or in combination with other components [61, 62, 63, 64, 65, 66, 67, 68].

1.9 Reviewing the eigenfunctions and the energy eigenvalues for Yukawa's potential

Schrodinger equation is a fundamental equation of quantum mechanics which describes the evolution of the wave function of a physical system over time. It is a first-order partial differential equation concerning time and a second-order partial differential equation concerning the coordinates of ordinary space. It takes the following form:

$$H\psi\left(\overrightarrow{r},t\right) = E\psi\left(\overrightarrow{r},t\right) \tag{2.2}$$

here $\psi\left(\overrightarrow{r}\right)$ is the complex wave function that satisfies the stationary Schrodinger equation and E is a nonrelativistic eigenvalue of the Hamiltonian H, which is written in the form :

$$H = \frac{\hat{P}}{2m} + V_0\frac{\exp(\alpha r)}{r} \tag{2.3}$$

where P represents the impulse $\overrightarrow{P} = -i\hbar\overrightarrow{\nabla}$, and $\overrightarrow{\nabla}$ represents the operator of partial derivatives (nabla). In Cartesian coordinates, it is defined by:

$$\overrightarrow{\nabla} = \overrightarrow{\imath}\frac{\partial}{\partial x} + \overrightarrow{j}\frac{\partial}{\partial y} + \overrightarrow{k}\frac{\partial}{\partial z} \tag{2.4}$$

Hence, Schrodinger's equation becomes:

$$i\hbar\frac{\partial\psi\left(\overrightarrow{r},t\right)}{\partial t}=\left(-\frac{\hbar^2}{2m}\Delta+V_0\frac{\exp(\alpha r)}{r}\right)\psi\left(\overrightarrow{r},t\right)\tag{2.5}$$

Since the Yukawa potential does not depend on time, solutions can be written separately as a part that is only position-dependent and an only time-dependent part:

$$\Psi\left(\overrightarrow{r},t\right)=\exp(-iE/\hbar)\Psi\left(\overrightarrow{r}\right)\tag{2.6}$$

And by substituting into the Shrodinger equation, we find:

$$\left(-\frac{\hbar^2}{2m}\Delta+V_0\frac{\exp(\alpha r)}{r}\right)\psi\left(\overrightarrow{r},t\right)=E\psi\left(\overrightarrow{r}\right)\tag{2.7}$$

Using the spherical coordinate system $\overrightarrow{r}\,(r,\theta,\varphi)$,the complex wave function $\psi\left(\overrightarrow{r}\right)$ can be written as:

$$\psi\left(r,\theta,\varphi\right)=R_{nl}\left(r\right)Y_{lm}\left(\theta,\varphi\right)\tag{2.8}$$

where$R_{nl}(r)$ is the radial part of the wave function that depends only on radius$r$, $Y_{l,m}\left(\theta,\varphi\right)$represented the angular part depends on the angles (θ,φ) and n is the principal quantum number, l the orbital quantum number and m the magnetic quantum number$(-l\preceq m\preceq +l)$. The Schrodinger equation in the spherical coordinate can be expressed as:

$$\left(\frac{-\hbar}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{r^2}\right)+V_0\frac{\exp(\alpha r)}{r}\right)R_{nl}(r)=E_{n,l}R_{nl}\left(r\right)\tag{2.9}$$

In quantum mechanics, the classical momentum obtains the forms $\overrightarrow{L}$ is the orbital angular momentum. The total moment $\overrightarrow{J}$ is given by:

$$\begin{cases} \overrightarrow{J} = \overrightarrow{L} + \overrightarrow{S} \\ \overrightarrow{L} = \overrightarrow{r} \wedge \overrightarrow{p} \end{cases} \tag{2.10}$$

here $\overrightarrow{S}$ is the spin. The components L_x, L_y and L_z of $\overrightarrow{L}$ which are expressed in Cartesian coordinates (x, y, z) as:

$$\begin{cases} L_x = \frac{\hbar}{\imath}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y = \frac{\hbar}{\imath}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z = \frac{\hbar}{\imath}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{cases} \tag{2.11}$$

In the spherical coordinate system $\overrightarrow{r}(r, \theta, \varphi)$, he components L_x, L_y and $L_z$of L are expressed as:

$$\begin{cases} L_x = \frac{\hbar}{\imath}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ L_y = \frac{\hbar}{\imath}\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ L_z = \frac{\hbar}{\imath}\frac{\partial}{\partial\varphi} \end{cases} \tag{2.12}$$

Note that the operators H, L^2 and L_z commute with each other and they have formed a common set of eigenfunctions $\psi(r, \theta, \varphi)$; however, the three components of the angular momentum (L_x, L_y, L_z) do not commute with each other:

$$\begin{cases} [H, L^2] = [H, L_z] = 0 \\ [L_i, L_j] = \imath\hbar\xi_{ijk}L_k \end{cases} \tag{2.13}$$

here L^2 is the square of the angular momentum :

$$L^2 = -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right) + \hbar^2\frac{\partial^2}{\partial\varphi^2} \tag{2.14}$$

the eigenvalues of L^2 and L_z are determined from:

$$\begin{cases} \hat{L}^2\psi\left(r,\theta,\varphi\right) = \hbar^2 l\left(l+1\right)\psi\left(r,\theta,\varphi\right) \\ \hat{L}_z\psi\left(r,\theta,\varphi\right) = m\hbar\psi\left(r,\theta,\varphi\right) \end{cases} \tag{2.15}$$

We introducing the wave function: we have:

$$R(r) = \frac{u\left(r\right)}{r} \tag{2.16}$$

Thus, the new radial part $u_{n,l}(r)$, will be satisfying the following equation:

$$\frac{d^2u_{n,l}(r)}{dr^2} + \frac{2m}{\hbar^2}\left(E_{nl} - V_0\frac{\exp\left(-\alpha r\right)}{r} - \frac{l\left(l+1\right)}{r^2}\right)u_{n,l}(r) = 0 \tag{2.17}$$

The energy eigenvalues E_{nl} and corresponding eigenfunctions in closed forms were obtained using the parametric Nikiforov-Uvarov approach by the authors of Ref. [42]. They further show that these results are consistent with those obtained previously in other studies using different approaches. They also discovered that when the Yukawa potential's screening parameter is set to zero, the energy levels of the familiar pure Coulomb potential energy levels are:

$$E_{nl} = -\frac{\alpha^2}{2m}\frac{\left[\frac{mV_0}{a} - (n+l+l)^2\right]^2}{(n+l+l)} \tag{2.18}$$

and the wave function of the Yukawa potential [42]:

$$R_{nl}\left(s\right) = s^{\frac{\sqrt{\varepsilon}}{4\alpha^2}}\left(1-s\right)^{l+1}P_n^{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^2}\right)}\left(1-2s\right) \tag{2.19}$$

by substituting

$$s = \exp\left(-2\alpha r\right) \tag{2.20}$$

we have:

$$R_{nl}\left(s\right) = N\exp(-\sqrt{\varepsilon r})\left(1-\exp\left(-2\alpha r\right)\right)^{l+1} P_n^{\left(2\frac{\sqrt{\varepsilon}}{4\alpha^2},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^2}\right)}\left(1-2\exp\left(-2\alpha r\right)\right) \tag{2.21}$$

Where N is a normalization constant.

Part III

The Effect of NC on the energy spectrum produced by the Yukawa potential in 3D-NCQM

1.11 Introduction

The purpose of this chapter is to study the modified Schrodinger equation of Yukawa potential in noncommutative three-dimensional phase space. Accordingly, we use Bopp's shift method instead of solving the modified Schrodinger equation directly; thus, using the star product and the perturbation theorem to find the corresponding energy correction.

1.12 The Schrodinger equation on a Noncommutative space-time

We simply replace the wave function products (or fields) with the star product or the Moyal product. The Schrodinger equation for a non-commutative space-time has the form:

$$\begin{cases} H(\hat{p}_i, \hat{x}_i)\hat{\psi}(\overrightarrow{\hat{r}}) = E_{nc}\psi\left(\overrightarrow{\hat{r}}\right) \\ \left[\frac{\overrightarrow{\hat{p}}^2}{2m} + V(\hat{r})\right] * \hat{\psi}\left(\overrightarrow{\hat{r}}, \hat{t}\right) = \imath\hbar\frac{\partial}{\partial t}\psi\left(\overrightarrow{\hat{r}}, \hat{t}\right) \end{cases} \tag{3.1}$$

According to the method of Bopp's Shift, which we have seen in the first chapter, the above equation can be simplified into the following form:

$$H_{nc-yp}\left(\hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2}x_j, \quad \hat{x}_i = x_i\frac{\theta^{ij}}{2}p_i\right)\psi(\overrightarrow{\hat{r}}) = E_{nc-yp}\psi(\overrightarrow{\hat{r}}) \tag{3.2}$$

Where

$$\begin{aligned} H(\hat{p}_i, \hat{x}_i) &= H\left(\hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2}x_j, \quad \hat{x}_i = x_i\frac{\theta^{ij}}{2}p_i\right) \\ &= \frac{\overrightarrow{\hat{P}}^2}{2m} + \frac{V_0}{r}\exp(-\alpha r) \end{aligned} \tag{3.3}$$

and

$$\begin{cases} V(\hat{r}) = V_0 \frac{\exp(-\alpha r)}{\hat{r}} \\ \frac{\overrightarrow{P}^2}{2m_0} = \frac{P^2}{2m_0} + \frac{\overrightarrow{L}\overrightarrow{\Theta}}{2m_0} \end{cases} \tag{3.4}$$

By using Eq.(1.27), we can obtain $\frac{V_0}{\hat{r}}$ as the sum of corresponding values $\frac{V_0}{r}$ in the symmetries of nonrelativistic quantum mechanics plus the induced term $V_0 \frac{\overrightarrow{L}\overrightarrow{\Theta}}{2r^3}$ with the effect of deformed proprieties of space-space, as follows:

$$\frac{V_0}{\hat{r}} = \frac{V_0}{r} + V_0 \frac{\overrightarrow{L}\overrightarrow{\Theta}}{2r^3} \tag{3.5}$$

while, the expression of $\exp(-\alpha r)$ can be written in the symmetries of extended quantum mechanics symmetries as follows [61]:

$$\begin{aligned} \exp(-\alpha r) &= \exp(-\alpha r) \exp\left(\alpha \frac{\overrightarrow{L}\overrightarrow{\theta}}{2r}\right) \\ &\backsimeq \exp\left(1 + \frac{\alpha \overrightarrow{L}\,\overrightarrow{\theta}}{2r}\right) \end{aligned} \tag{3.6}$$

Allow us to get the Yukawa potential in the noncommutative phase-space as follows:

$$V(\hat{r}) = \frac{V_0}{r} \exp(-\alpha r) + V_0 \left(\frac{\exp(-\alpha r)}{2r^3} + \frac{\alpha \exp(-\alpha r)}{2r} \right) \overrightarrow{L}\,\overrightarrow{\Theta} \tag{3.7}$$

The global Hamiltonian operator $H_{nc-yp}\left(\hat{p}_i = p_i - \frac{\overline{\theta}^{ij}}{2} x_j, \quad \hat{x}_i = x_i \frac{\theta^{ij}}{2} p_i\right)$ in noncommutative three-dimensional phase-space can be written in the following form:$L\,\overrightarrow{\overline{\theta}}$

$$H_{nc-yp}\left(\hat{p}_i = p_i - \frac{\overline{\theta}^{ij}}{2} x_j, \quad \hat{x}_i = x_i \frac{\theta^{ij}}{2} p_i\right) =$$

$$\frac{\hat{\overrightarrow{P}}^2}{2m} + \frac{V_0}{r} \exp(-\alpha r) +$$

$$+V_0\left(\frac{\exp\left(-\alpha r\right)}{2r^3}+\frac{\alpha\exp\left(-\alpha r\right)}{2r}\right)\overrightarrow{L}\overrightarrow{\Theta}+\frac{\overrightarrow{L}\overrightarrow{\theta}}{2m_0} \tag{3.8}$$

1-The first two terms in the Hamiltonian operator H_{nc-yp}, which corresponds to the Yukawa potential in Eq. (2.1) and the Kinetic term or dynamic $\frac{\overrightarrow{p}^2}{2m}$ in ordinary commutative space which formed the usual Hamiltonian operator:

$$H\left(\hat{p}_i=p_i,\ \hat{x}_i=x_i\right)=\frac{\overrightarrow{p}^2}{2m}++\frac{V_0}{r}\exp\left(-\alpha r\right) \tag{3.9}$$

2-The second and third terms are formed the a new Hamiltonian operator or the additive created term H_{nc-yp} which is represent the contributions of the noncommutative space-phase:

$$H_{nc-yp}=V_0\left(\frac{\exp\left(-\alpha r\right)}{2r^3}+\frac{\alpha\exp\left(-\alpha r\right)}{2r}\right)\vec{L}\vec{\Theta}+\frac{L\overrightarrow{\theta}}{2m_0} \tag{3.10}$$

where $\overrightarrow{L}\overrightarrow{\Theta}$ and $L\overrightarrow{\theta}$ are determined from Eq. (1.39) in the first chapter. According to the mathematical forms of the 2-couplings $\overrightarrow{L}\overrightarrow{\Theta}$ and $L\overrightarrow{\theta}$ observed in Eq.(3.8), it is physically possible to replace $\overrightarrow{L}\overrightarrow{\Theta}$ and $L\overrightarrow{\theta}$ by $\mu\Theta\overrightarrow{S}\overrightarrow{L}$ and $\mu\bar{\theta}\overrightarrow{S}\overrightarrow{L}$:

$$\begin{cases}\overrightarrow{L}\overrightarrow{\Theta}\rightarrow\mu\Theta\overrightarrow{S}\overrightarrow{L}\\ \overrightarrow{L}\overrightarrow{\theta}\rightarrow\mu\bar{\theta}\overrightarrow{S}\overrightarrow{L}\end{cases} \tag{3.11}$$

With $\overrightarrow{S}$ denote to the spin of the particle which interacted with Yukawa potential and μ is a new constant of proportionality. This enables rewriting Eq.(3.11) as follows:

$$H_{per-yp}=\mu\left[\frac{\bar{\theta}}{2m}-V_0\left(\frac{e^{(-\alpha r)}}{2r^3}+\frac{\alpha e^{(-\alpha r)}}{2r}\right)\Theta\right]\overrightarrow{L}\overrightarrow{S} \tag{3.12}$$

The parameters Θ and $\bar{\theta}$ are given by:

$$\begin{cases} \Theta = (\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2)^{\frac{1}{2}} \\ \theta = \left(\bar{\theta}_{12}^2 + \theta_{23}^2 + \bar{\theta}_{13}^2\right)^{\frac{1}{2}} \end{cases} \tag{3.13}$$

In ordinary quantum mechanics, we have the sets of operators $\hat{\mathbf{A}}$**,** $\hat{\mathbf{B}}$**,** $\hat{\mathbf{C}}$**......** which form a complete set of complete observables are commute (ECOC). We apply this rule to the sets of operators $\left(\overrightarrow{J^2}, \overrightarrow{S^2}, \overrightarrow{L^2} \text{ and } J_z\right)$, i.e.:

$$\begin{cases} \left[\overrightarrow{J^2}, \overrightarrow{L^2}\right] = 0 \\ \left[\overrightarrow{J^2}, \overrightarrow{S^2}\right] = 0 \\ \left[\overrightarrow{J^2}, J_z\right] = 0 \end{cases} \tag{3.14}$$

And the corresponding eigenvalues are $j\,(j+1)$ and $l(l+1)$, $s(s+1)$ and $m\,(-l \leq m \leq +l)$ in the system $(c = \hbar = 1)$, so:

$$\begin{cases} \overrightarrow{J^2}\Psi_{n,l,m_l}(r,\theta,\varphi) = j(j+1)\Psi_{n,l,m_l}(r,\theta,\varphi) \\ \overrightarrow{L^2}\Psi_{n,l,m_l}(r,\theta,\varphi) = l(l+1)\Psi_{n,l,m_l}(r,\theta,\varphi) \\ \overrightarrow{S^2}\Psi_{n,l,m_l}(r,\theta,\varphi) = s(s+1)\Psi_{n,l,m_l}(r,\theta,\varphi) \\ J_z\Psi_{n,l,m_l}(r,\theta,\varphi) = m\Psi_{n,l,m_l}(r,\theta,\varphi) \end{cases} \tag{3.15}$$

With $\overrightarrow{J}$ being the geometric sum of the moments $\overrightarrow{L}$ and $\overrightarrow{S}$, this allows us to find the spin-orbit coupling $\overrightarrow{L}\overrightarrow{S}$ as follows:

$$\overrightarrow{L}\overrightarrow{S} = \frac{1}{2}\left[\overrightarrow{J^2} - \overrightarrow{S^2} - \overrightarrow{L^2}\right] \tag{3.16}$$

An immediate result is:

$$\overrightarrow{L}\overrightarrow{S}\psi = \frac{1}{2}\left[j(j+1) + l(l+1) + s(s+1)\right]\psi \tag{3.17}$$

With

$$|l-s| \leq j \leq |l+s|$$

So:

$$j=|l-s|\,,|l-s|+1,......,|l+s| \tag{3.18}$$

For the two extreme values of the total angular momentum, we can write for $s=\frac{1}{2}:$

$$\overrightarrow{L}\overrightarrow{S}\Psi=\begin{cases}\frac{1}{2}\left\{(l+s)\,(l+s+1)-l(l+1)-3/4\right\}\psi \\ \qquad \equiv k_{+}\psi \quad \text{if} \qquad j{=}\,|l+1/2| \\ \frac{1}{2}\left\{(l+s)\,(l-s+1)-l(l+1)-3/4\right\}\psi \\ \qquad \equiv k_{-}\psi \quad \text{if} \qquad j{=}\,|l+1/2|\end{cases} \tag{3.19}$$

We considered the following approximation type suggested by Greene-Aldrich [42, 69]:

$$\frac{1}{r^2}\approx 4\alpha^2\frac{\exp\left(-2\alpha r\right)}{(1-\exp\left(-2\alpha r\right))^2}\Longleftrightarrow\frac{1}{r}\approx 2\alpha\frac{\exp\left(-\alpha r\right)}{(1-\exp\left(-2\alpha r\right))} \tag{3.20}$$

Which allows us to have the following:

$$\frac{1}{r^2}\approx\frac{4\alpha^2 s}{(1-s)^2}\Longleftrightarrow\frac{1}{r}\approx\frac{2\alpha s^{1/2}}{(1-s)} \tag{3.21}$$

This is valid for $\alpha r \ll 1$. Therefore, the perturbative effective Yukawa potential in Eq. (3.12) can be written as:

$$V_{nc}^{py}\left(r\right)=\mu\left[\frac{\bar{\theta}}{2m}-\alpha^2 V_0\left(\frac{4\alpha s^{3/2}}{\left(1-s\right)^2}+\frac{s}{1-s}\right)\Theta\right]\overrightarrow{L}\overrightarrow{S} \tag{3.22}$$

The Yukawa potential is extended by including new additive potential $V_{nc}^{py}(r)$ expressed to the radial terms:

$$\left\{\frac{s^{3/2}}{(1-s)^2} \text{ and } \frac{s}{1-s}\right\}$$

to become the modified Yukawa potential in noncommutative three-dimensional phase-space symmetries. The generated new potential $V_{nc}^{py}(r)$ is also proportional to the infinitesimal parameters Θ and θ. This allows us to consider the new additive part of the potential $V_{nc}^{py}(r)$ as perturbation potential compared with the main potentials $V(r)$. That is all physical justifications for applying the time-independent perturbation theory become satisfied to calculate the expectation values of previous radial terms. This allows us to give a complete prescription for determining the energy level of the generalized $(n, l, m)^{th}$ excited states. The exact spectrum produced by the spin-orbit effect for the Yukawa potential in the three dimensional noncommutative space-phase E_{nc-yp} is the sum of the energy corresponding to ordinary space E_{nl} and the corrections E_{yp-per} :

$$E_{nc-0l}^{yp}\left(\Theta, \bar{\theta}\right) = E + \Delta E_{nc-yp} \tag{3.23}$$

The perturbation theorem allows to obtain the first-order corrections as follows:

$$\Delta E_{nc-yp}\left(\Theta, \bar{\theta}\right) = \left\langle \psi^p\left(\overrightarrow{r}\right) \left| H_{per-yp}\left(r, \Theta, \theta\right) \right| \psi^p\left(\overrightarrow{r}\right) \right\rangle \tag{3.24}$$

We can write the equation (3.24) in the form:

$$\Delta E_{nc-yp}\left(\Theta, \bar{\theta}\right) = \int \psi^p\left(\overrightarrow{r}\right) H_{per-yp}\left(r, \Theta, \theta\right) \psi^p\left(\overrightarrow{r}\right) d\tau \tag{3.25}$$

where $d\tau$ represent the volume element in spherical coordinates r, which is given by:

$$d\tau = r^2 dr d\Omega \tag{3.26}$$

With the solid angle

$$d\Omega = \sin\theta d\theta d\varphi$$

and the non perturbative complex wave function , the wave function is defined by :

$$\psi^{(p)}\left(\overrightarrow{r}\right) = R_{n,l}\left(r\right) Y_l^m\left(\theta, \phi\right) \tag{3.27}$$

So, we can write the equation (3.27) in the form:

$$E_{nc-yp}\left(\Theta, \theta\right) = \left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle$$

$$\int_0^\infty R_{n,l}^*\left(r\right) V_{nc}^{py}\left(s\right) R_{n,l}\left(r\right) r^2 dr \int_0^\pi \int_0^{2\pi} Y_l^{*m_l}\left(\theta, \phi\right) Y_l^m\left(\theta, \phi\right) d\Omega \tag{3.28}$$

The normalized wave function $\psi\left(\overrightarrow{r}\right)$ allows us to write:

$$\int_0^\pi \int_0^{2\pi} Y_l^{*m_l}\left(\theta, \phi\right) Y_l^m\left(\theta, \phi\right) d\Omega = 1 \tag{3.29}$$

This reduces the corrections found by Eq. (3.27) to the form:

$$\Delta E_{nc-yp}\left(\Theta, \bar{\theta}\right) = \left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle \int_0^\infty R_{n,l}^*\left(r\right) V_{nc}^{py}\left(s\right) R_{n,l}\left(r\right) r^2 dr \tag{3.30}$$

We substituted the spin-orbit coupling term $V_{nc}^{py}\left(s\right)$, and we find:

$$\Delta E_{nc-yp}\left(\Theta, \bar{\theta}\right) = \left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle \tag{3.31}$$

$$(\frac{\mu\bar{\theta}}{2m_0}\int_0^\infty R_{n,l}^*(r)\, R_{n,l}(r)\, r^2 dr$$

$$-8\mu V_0\alpha^3\int_0^\infty R_{n,l}^*(r)\,\frac{s^{3/2}}{(1-s)^2}R_{n,l}(r)\, r^2 dr$$

$$-\mu V_0\alpha^2\int_0^\infty R_{n,l}^*(r)\,\frac{s}{1-s}R_{n,l}(r)\, r^2 dr)$$

We have $s=\exp(-2\alpha r)$, this allows us to obtain $dr=-\frac{1}{2\alpha}\frac{ds}{s}$ After introducing a new variable $z=1-2s$, we have $s=\frac{1-z}{2}$, $dr=\frac{1}{2\alpha}\frac{dz}{1-z}$ and $1-s=\frac{1+z}{2}$. From the asymptotic behavior of $s=\exp(-2\alpha r)$ and $z=1-2y$, when $r\to 0\,(z\to -1)$ and $r\to +\infty\,(z\to 1)$, this allows reformulating Eq. (3.31) as follows:

$$\Delta E_{nc-yp}\left(\Theta,\bar{\theta}\right)=\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle(\frac{\mu\bar{\theta}}{2m_0}-\frac{\mu V_0\alpha}{2}$$

$$-4V_0\alpha^2\int_{-1}^{+1} R_{n,l}^*(r)\,\frac{s^{3/2}}{(1-s)^2}R_{n,l}(r)\, r^2\frac{dz}{1-z} \tag{3.32}$$

$$-\int_0^\infty R_{n,l}^*(r)\,\frac{s}{1-s}R_{n,l}(r)\, r^2\frac{dz}{1-z})$$

Now, we replace r^2 by its expression in Eq. (3.21), we found:

$$E_{nc-yp}\left(\Theta,\bar{\theta}\right)=\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle(\frac{\mu\bar{\theta}}{2m_0}$$

$$-\mu V_0\int_{-1}^{+1} R_{n,l}^*(r)\, s^{1/2}R_{n,l}(r)\,\frac{dz}{1-z} \tag{3.33}$$

$$-\frac{\mu V_0}{8\alpha}\int\limits_0^{\infty} R_{n,l}^{*}\left(r\right)\left(1-s\right)R_{n,l}\left(r\right)\frac{dz}{1-z})$$

If we replace the radial part $R_{n,l}(r)$ which is expressed as:

$$R_{n,l}(s)=R_{nl}\left(s\right)=Ns^{\frac{\sqrt{\varepsilon}}{4\alpha^2}}\left(1-s\right)^{l+1}P_n^{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^2}\right)}\left(1-2s\right) \tag{3.34}$$

The corrections in Eq. (3.32) will be simplified to the form :

$$\Delta E_{nc-yp}\left(\Theta,\bar{\theta}\right)=\left\langle\overrightarrow{L}\overrightarrow{S}\right\rangle N^2(\frac{\mu\bar{\theta}}{2m_0}$$

$$-\frac{\mu V_0}{2^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+5/2}}\int\limits_{-1}^{+1}\left(1-z\right)^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1/2}$$

$$\left(1+z\right)^{2l+2}\left[P_n^{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^2}\right)}\left(z\right)\right]^2dz \tag{3.35}$$

$$-\frac{\mu V_0}{\alpha^{(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+6)}}\left(1-z\right)^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1}\left(1-s\right)^{2l+3}$$

$$\left[P_n^{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^2}\right)}\left(z\right)\right]^2dz)$$

For the ground state$n=0$, we have

$$P_n^{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^2}\right)}\left(z\right)=1,$$

thus the above expectation values in Eqs. (3.35) are reduced to the following simple form:

$$E_{nc-yp}\left(\Theta,\bar{\theta}\right)=\left\langle\overrightarrow{L}\overrightarrow{S}\right\rangle N^2(\frac{\mu\bar{\theta}}{2m_0}-\frac{\mu V_0}{2^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+5/2}}$$

$$\int\limits_{-1}^{+1}\left(1-z\right)^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1/2}\left(1+z\right)^{2l+2}dz \tag{3.36}$$

$$-\frac{\mu V_0}{\alpha 2^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+6}}\int_{-1}^{+1}(1-z)^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1}(1-s)^{2l+3}\,dz)$$

Comparing Eq. (3.35) with the integral of the form [70]:

$$\int_{-1}^{+1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}P_m^{(\alpha,\beta)}(x)\,dx=2^{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)} \tag{3.37}$$

for n=0,1,...

A direct calculation gives the expression of integrals values in Eq.(3.36) as follows:

$$\int_{-1}^{+1}(1-z)^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1/2}(1+z)^{2l+2}\,dz=$$

$$2^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+5/2}\frac{\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+1/2\right)\Gamma(2l+3)}{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+5/2\right)\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1/2+2l+3\right)} \tag{3.38}$$

and

$$\int_{-1}^{+1}(1-z)^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1/2}(1-s)^{2l+3}\,dz=$$

$$2^{2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+3}\frac{\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}\right)\Gamma(2l+4)}{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+3\right)\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+3\right)} \tag{3.39}$$

Thus, the energy correction for the ground state $n=0$ reduced to the following simple form:

$$\Delta E_{nc-yp}\left(\Theta,\bar{\theta}\right)=\left\langle\overrightarrow{L}\overrightarrow{S}\right\rangle N^2\Big(\frac{\mu\bar{\theta}}{2m_0}$$

$$-\mu V_0\frac{\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+1/2\right)\Gamma(2l+3)}{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+5/2\right)\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}-1/2+2l+3\right)} \tag{3.40}$$

$$-\frac{\mu V_0}{\alpha}\frac{\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}\right)\Gamma(2l+4)}{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+3\right)\Gamma\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}}+2l+3\right)}$$

The global energy E_{0l}^{py} for the ground state $n = 0$ is the energy spectrums:

$$E_{nc\_0l}^{yp} = E_{0l}^{yp} + \Delta E_{nc-yp}\left(\Theta, \bar{\theta}\right) \tag{3.41}$$

Where E_{0l}^{py} is determined from Eq. (2.18) which we have seen in the second chapter:

$$E_{0l}^{py} = -\frac{\sqrt{\frac{\varepsilon_{0l}}{4\alpha^2}}}{2m} \frac{\left[\frac{mV_0}{a} - (l+l)^2\right]^2}{(l+1)^2} \tag{3.42}$$

here $\left\langle \overrightarrow{L}\,\overrightarrow{S} \right\rangle$ is determined from:

$$\left\langle \overrightarrow{L}\,\overrightarrow{S} \right\rangle = \begin{cases} \frac{1}{2}\left\{(l+s)(l+s+1) - l(l+1) - 3/4\right\} \\ \qquad \equiv k_+ \quad \text{if} \quad j{=}\left|l+1/2\right| \\ \frac{1}{2}\left\{(l+s)(l-s+1) - l(l+1) - 3/4\right\} \\ \qquad \equiv k_- \quad \text{if} \quad j{=}\left|l+1/2\right| \end{cases} \tag{3.43}$$

It is clear that the following physical limit procedure:

$$\begin{cases} \underset{(\Theta,\theta)\to(0,0)}{Lim} E_{nc\_0l}^{yp} = E_{nc\_0l}^{py} \\ \underset{(\Theta,\theta)\to(0,0)}{Lim} \Delta E_{nc-yp}\left(\Theta, \bar{\theta}\right) = 0 \end{cases} \tag{3.44}$$

Gives us all the results of physical treatments which we have seen in the standard reference [42].

CONCLUSION

Through this master's memory in physics, theoretical specialty:

Promotion 2021-2022

The nonrelativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of Yukawa potential as a model

This memory aims to study physical systems within the framework of the modified Schrödinger equation with the modified Yukawa potential, in three-dimensional non-commutative quantum mechanics.

In the first chapter, we have represented the mathematical and physical formalisms of the noncommutative three-dimensional space-phase. and apply these principles to the atoms of modified Yukawa potential.

In the second chapter, we reviewed the Shrodinger equation under the Yukawa potential based on many works.

In the third chapter, we studied the effect of the noncommutativity of the three-dimensional phase-space, by applying the generalized Bopp shift method and standard perturbation theory at the first order of parameters (Θ, θ) the modifications on the energy corresponding to the ground state are obtained. We can conclude that the application of the noncommutativity in this work on the modified Yukawa potential, includes the spin-orbit coupling effect automatically.

This is in contrast to what we observe in the framework of quantum mechanics known in the literature, where the spin-orbit interaction appears by external addition and not through spontaneous birth as a result of space deformation.

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Résumé

Dans notre travail sur ce mémoire de master, en physique théorique (2021/2022): Une étude d'un spectre d'énergie non relativiste produit avec un potentiel isotrope dans le cadre des symétries étendues de la mécanique quantique : le cas du potentiel de Yukawa. Nous avons étudié l'équation de Schrödinger avec le potentiel de Yukawa dans des espaces et des phases tridimensionnelles non commutatives, en appliquant la méthode de Boopp's Shift au premier ordre des paramètres (θ, Θ), en plus de la théorie standard des perturbations, pour obtenir le spectre d'énergie du système, qui change radicalement, et remplacé par de nouveaux états dégénérés dépendant des nombres quantiques atomiques discrets (j, n, l, s). Ce résultat a été produit par l'interaction spin-orbite.

Mots-clés: équation de Schrödinger, potentiel de Yukawa, mécanique quantique noncommutative, produit star, la méthode de Boopp's Shift

Abstract

In our work on this these, in theoretical physics (2021/2022): A study of a nonrelativistic energy spectrum produced with an isotropic potential in the framework of extended quantum mechanics symmetries: the case of Yukawa potential. We have studied the Schrödinger equation with the Yukawa potential in noncommutative three-dimensional spaces and phases, by applying the Boop's Shift method to the first order of the parameters (θ, Θ), in addition to the standard perturbation theory, to obtain the spectrum of energy of the system, which is changing radically, and replaced by degenerate new states depending on the discrete atomic quantum numbers (j, n, l, s). This result was produced by the spin-orbit interaction.

Keywords: Schrödinger equation, Yukawa potential, noncommutative quantum mechanics, star product, Boop's shift method

ملخص

تناولنا في هذه المذكرة الدراسة اللانسبية لطيف الطاقة الناتج عن كمون متماثل المناحي في إطار تناظرات ميكانيك الكم الواسع: حالة كمون يوكاوا. قمنا بدراسة هذا الكمون في الفضاء الاتبادلي الثلاثي البعد والطور , وذلك بتطبيق مبدأ بوب الذي يوافق الحد (θ, Θ), بالإضافة إلى تطبيق نظرية االضطراب المعيارية للحصول على طيف الطاقة للجملة والذي يتغير بشكل جذري بحيث يستبدل بحالات منحلة جديدة والتي تتعلق بالأعداد الكمية المتقطعة (j, n, l, s). هذه النتائج تأتي من تأثير مفعول السبين-مدار المستحدث تلقائيا.

الكلمات المفتاحية : معادلة شرودينغر ، كمون يوكاوا ، ميكانيك الكم الالتبادلي، الجداء النجمي، طريقة بوب.

LIST OF ABBREVIATIONS

NCQM: NonCommutative Quantum Mechanics
QFT: Quntum Field Theory
NCSP: NonCommutative Space-Phase
NCPP: NonCommutative Phase-Phase
SE: Shrodinger Equation
QM: Quantum Mechanics
ECOC: Complete set of Commuting Observable