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Abstract

In the context of homogeneous spaces of the type

$$\|f(\lambda \cdot)\| \sim \lambda^r \|f\|, \quad \forall \lambda > 0, \quad r \in \mathbb{R},$$

we define the realized spaces associated with the Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ (Besov-type $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) and Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$ (Triebel-Lizorkin-type $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$) they will be denoted by $\tilde{B}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) and $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$), respectively; here $r := s - n/p$ (and $r := s + n\tau - n/p$). These spaces are subspaces of regular tempered distributions. We study the continuity of pseudo-differential operators on $\tilde{F}_{p,q}^s(\mathbb{R}^n)$. We establish functional characterizations by difference operators of the realized spaces associated with homogeneous Besov-type spaces, Triebel-Lizorkin-type spaces.

Keywords: Littlewood-Paley decomposition, Besov spaces, Triebel-Lizorkin spaces, Pseudo-differential operators, Realizations.

Résumé

Dans le cadre des espaces homogènes de type

$$\|f(\lambda \cdot)\| \sim \lambda^r \|f\|, \quad \forall \lambda > 0, \quad r \in \mathbb{R},$$

nous définissons les espaces réalisés associés aux espaces de Besov $\dot{B}_{p,q}^s(\mathbb{R}^n)$ (type de Besov $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) et Triebel-Lizorkin $\dot{F}_{p,q}^s(\mathbb{R}^n)$ (type de Triebel-Lizorkin $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$); ils seront notés $\tilde{B}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) et $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$) respectivement; ici $r := s - n/p$ (et $r := s + n\tau - n/p$). Ces espaces sont des sous-espaces de distributions tempérées régulières. On étudie la continuité des opérateurs pseudo-différentiels sur $\tilde{F}_{p,q}^s(\mathbb{R}^n)$. On établit des caractérisations fonctionnelles par des opérateurs de différences des espaces réalisés associés aux espaces de type de Besov homogènes et de type de Triebel-Lizorkin homogènes .

Mots clés: Décomposition de Littlewood-Paley, Espaces de Besov, Espaces de Triebel-Lizorkin, Opérateurs pseudo-différentiels, Réalisations.

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Introduction

Functional analysis is a fundamental branch of mathematics yields some solutions of real problems. One of its central themes is the investigation of function spaces, which provide a framework for understanding the regularity and decay properties of functions. In general function spaces play a major role in the study of certain problems as PDEs. The homogeneous function spaces are defined modulo-polynomials, since they satisfy :

$$\|f(\lambda \cdot)\| \sim \lambda^r \|f\|, \text{ for all } \lambda > 0 \text{ and for some } r \in \mathbb{R}.$$

Then solved, for example a PDE in this spaces becomes very difficult. For this reason, we find that realizations theory makes accessible to work with spaces satisfying the above assertion. In our cases , Besov and Triebel-Lizorkin spaces , $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$, respectively, have this property with $r := s - n/p$, and Besov and Triebel-Lizorkin type spaces, $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, respectively, with $r := s + n\tau - n/p$. About this subject, we quote [1]–[4], [14], [20] and [56], we note that several application in realizations theory, in particular see, e.g. [1], [21].

The thesis is structured as follows:

In chapter 1 we provide the foundational tools, including the Littlewood-Paley decomposition, definitions of homogeneous and non-homogeneous Besov and Triebel-Lizorkin spaces, and essential inequalities. We refer to [12], [14] ,[20], [53], [54]

In chapter 2 we explore realizations and pseudo-differential operators, establishing their boundedness on realized homogeneous Triebel Lizorkin spaces they are denoted by $\tilde{F}_{p,q}^s(\mathbb{R}^n)$. Thus, by considering the Hörmander class $S_{p,\delta}^{m,N}(E)$, we prove in this pragraph the following results:

Theorem 0.1. *Let $s > 0$ and $1 \leq p < \infty$. Assume that*

$$\text{either } s < \frac{n}{p}, \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad p = 1;$$

and

$$\text{either } s + m < \frac{n}{p}, \quad \text{or} \quad s + m = \frac{n}{p} \quad \text{and} \quad p = 1.$$

hold. Let N be an even natural number sufficiently large. If a symbol $b(x, \xi)$ belongs to $\dot{S}_{1,0}^{m,N}(L_\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{n/p}(\mathbb{R}^n))$, then the associated pseudo-differential operator $b(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{F}_{p,q}^s(\mathbb{R}^n)$.

Theorem 0.2. Let $s > 0$, $1 \leq p < \infty$ and $m \geq 0$. Assume that

$$\text{either } s < \frac{n}{p}, \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad p = 1;$$

and

$$\text{either } s + m < \frac{n}{p}, \quad \text{or} \quad s + m = \frac{n}{p} \quad \text{and} \quad p = 1.$$

hold. Let N be an even natural number sufficiently large. If a symbol $b(x, \xi)$ belongs to $\dot{S}_{1,0}^{m,N}(L_\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{n/p}(\mathbb{R}^n) \cap \dot{F}_{p,q}^{n/p-m}(\mathbb{R}^n))$, then we have :

- (i) the associated pseudo-differential operator $b(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ if $p \leq q$.
- (ii) the associated pseudo-differential operator $b(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}(\mathbb{R}^n) \cap \tilde{B}_{\infty,q}^{s-n/p+m}(\mathbb{R}^n)$ into $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ if $p > q$.

We note that, all above assertions are given with precises inequalities of type $(b(x, D) : A \longrightarrow B)$

$$\|b(x, D)f\|_B \leq c \|f\|_A$$

Concerning the pseudo-differential operators theory we do not go into details, instead we quote [22], [28], [33], [56].

In Chapter 3 we present characterizations by differences of realized-type spaces of Besov and Triebel-Lizorkin, they are denoted by $\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ respectively.

We prove some embeddings, we also prove some properties functional related to the integrals, we recal that it has been studied in [41] and [58], our contribution for the characterizations by differences is given as the following:

For two functionals \mathcal{M} and \mathcal{N} , defined later, we have

Theorem 0.3. Let $s > 0$, $0 < p < \infty$, $0 < q \leq \infty$, $\tau \geq 0$, and $m \in \mathbb{N}$ such that

$$s < \min(m, m + n(1/p - \tau)), \quad \text{and} \quad m \geq [s + n\tau].$$

Then $\mathcal{M}_{p,q}^{s,\tau,m}(\cdot)$, respectively $\mathcal{N}_{p,q}^{s,\tau,m}(\cdot)$, define an equivalent quasi-norms in $\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, respectively.

In chapter 4 we extend these characterizations to homogeneous Besov and Triebel-Lizorkin spaces. Here, the results are an extension of the previous statements given in [25], [27]. We generalize these properties to the homogeneous Sobolev spaces. Our contribution this chapter is the following :

Theorem 0.4. *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s + n\tau - \frac{n}{p})_+$. If $f \in \tilde{F}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|f(x) + u_f(x)|}{1 + |x|^{n+d}} dx \leq c \| [f]_\infty \|_{\tilde{F}_{p,q}^{s,\tau}},$$

holds. In particular, if either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $p \leq 1$; here we have realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x - x_0|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\tilde{F}_{p,q}^{s,\tau}},$$

holds for all $x_0 \in \mathbb{R}^n$.

In the end, an annex contains some open questions about all facts in the work.

Finally, the results presented in Chapter 2 have been published in :

S. Meliani, M. Moussai. *Boundedness of pseudodifferential operators on realized homogeneous Besov spaces.* Taiwanese J. Math. 21 no. 2 (2017), pp. 441-465.

Notation

Usual notation

- \mathbb{N} denotes the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- \mathbb{Z} is the set of all integer numbers.
- \mathbb{R}^n is the n -dimensional real Euclidean space.
- For a real number x the integer part is denoted by $[x]$.
- For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $|\alpha| = \alpha_1 + \dots + \alpha_n$,
 $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\partial^\alpha = \left(\frac{\partial}{x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{x_n}\right)^{\alpha_n}$.
- If $\alpha, \beta \in \mathbb{N}_0^n$, the symbol $\alpha \preceq \beta$ means that $\alpha_i \leq \beta_i$ for $i = 1, \dots, n$.
- All function spaces in this thesis are defined on \mathbb{R}^n , then sometimes we omit it in notation, i.e. we write C^∞ , L_p , instead of $C^\infty(\mathbb{R}^n)$, $L_p(\mathbb{R}^n)$.
- For $A \subset \mathbb{R}^n$, $|A|$ denotes the Lebesgue measure of A .
- For $a \in \mathbb{R}$, we set $a_+ := \max(0, a)$.
- If X and Y are two quasi-Banach spaces, $X \hookrightarrow Y$ denotes the continuous embedding (i.e., there exists $c > 0$ such that $\|x\|_Y \leq c \|x\|_X$ for all $x \in X$).
- C^∞ is the space of infinitely differentiable functions on \mathbb{R}^n .
- For $0 < p \leq \infty$, we denote by $\|\cdot\|_p$ the quasi-norm of L_p , and p' is the conjugate exponent of p , defined as

$$p' := \begin{cases} \frac{p}{p-1} & \text{if } 1 < p \leq \infty, \\ \infty & \text{if } 0 < p \leq 1. \end{cases}$$

- L_1^{loc} is the space of locally integrable functions on \mathbb{R}^n .

- \mathcal{D} is the space of C^∞ functions with compact support (test functions), \mathcal{D}' is the topological dual of \mathcal{D} (distributions space).
- \mathcal{S} is the space of rapidly decreasing C^∞ functions on \mathbb{R}^n (Schwartz space), equipped with the family of semi-norms

$$\zeta_m(f) := \sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} (1 + |x|)^m |f^{(\alpha)}(x)|, \quad m \in \mathbb{N}_0.$$

- \mathcal{S}' is the space of tempered distributions (the topological dual of \mathcal{S}).
- If f is a function (or a distribution) on \mathbb{R}^n , we define the translations and dilations operator by the formulas

$$\tau_a f := f(\cdot - a), \quad h_\lambda f := f(\cdot / \lambda),$$

for all $a \in \mathbb{R}^n$ and all $\lambda > 0$.

- Let $0 < q \leq \infty$. If E is a quasi-Banach space, we denote by $\ell_q(\mathbb{Z}; E)$ the set of sequences $(a_j)_{j \in \mathbb{Z}}$ with values in E such that

$$\|(a_j)_{j \in \mathbb{Z}}\|_{\ell_q(\mathbb{Z}; E)} := \left(\sum_{j \in \mathbb{Z}} \|a_j\|_E^q \right)^{1/q} < \infty.$$

This is a quasi-Banach space with the quasi-norm above. For $E = \mathbb{R}$ or $E = \mathbb{C}$, we write $\ell_q(\mathbb{Z})$.

- In all this work, we assume $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$, $0 \leq \tau < \infty$, unless otherwise stated.
- For $m \in \mathbb{N}_0$, we denote by \mathcal{P}_m the set of polynomials of degree strictly less than m , with $\mathcal{P}_0 = \{0\}$, $\mathcal{P}_1 = \{c\}$ and \mathcal{P}_∞ the set of all polynomials.
- We denote by

$$\mathcal{S}_m := \left\{ f \in \mathcal{S} : \langle x^\alpha, f \rangle = \int_{\mathbb{R}} x^\alpha f(x) dx = \widehat{f}^{(\alpha)}(0) = 0; \forall |\alpha| < m \right\},$$

i.e., the orthogonal complement of \mathcal{P}_m in \mathcal{S} .

- Note that $\mathcal{S}_0 := \mathcal{S}$ and

$$\mathcal{S}_\infty := \{ f \in \mathcal{S} : \langle p, f \rangle = 0, \forall p \in \mathcal{P}_\infty \}.$$

- \mathcal{S}'_m is the space of tempered distributions modulo polynomials of degree strictly less than m (the topological dual of \mathcal{S}_m). It is equipped with the weak-* topology.
- For any $f \in \mathcal{S}'$, we denote by $[f]_\infty$ the equivalence class of f modulo \mathcal{P}_∞ .
- $\mathcal{S}'_\infty := \{[f]_\infty : [f]_\infty = \{f + p, \forall p \in \mathcal{P}_\infty\}\}$ is the so-called the space of tempered distributions modulo polynomials, i.e., the quotient space $\mathcal{S}' / \mathcal{P}_\infty$; moreover, we have

$$\mathcal{S}' = \mathcal{S}'_0 \subset \mathcal{S}'_1 \subset \dots \subset \mathcal{S}'_m \subset \dots \subset \mathcal{S}'_\infty.$$

- \mathcal{S}'_∞ is isomorphic to $\mathcal{S}' / \mathcal{P}_\infty$.
- If $f \in L_1$, its Fourier transform is

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and its inverse Fourier transform is

$$\check{f}(x) = \mathcal{F}^{-1} f(x) := (2\pi)^{-n} \widehat{f}(-x).$$

- c, c_1, \dots strictly positive constants, depending only on the fixed parameters, as n, s, p, \dots and some fixed functions, their values may vary from line to another.
- The symbol \lesssim means that if $A \lesssim B$ then there exists $c > 0$ such that $A \leq cB$.

Chapter 1

Preliminaries

This first chapter introduces several essential tools that will be used throughout the thesis, in particular the Littlewood-Paley decomposition, definitions of homogeneous and non-homogeneous Besov and Lizorkin-Triebel spaces, as well as some classical inequalities.

1.1 Littlewood-Paley decomposition

Throughout the thesis, we fix a cut-off function: ρ is a positive C^∞ function on \mathbb{R}^n , radial (that to say, $\rho(\xi) = \tilde{\rho}(|\xi|)$, with $\tilde{\rho} \in \mathcal{D}$), and $0 \leq \rho \leq 1$ such that

$$\rho(\xi) = 1 \quad \text{if } |\xi| \leq 1, \quad \text{and} \quad \rho(\xi) = 0 \quad \text{if } |\xi| \geq \frac{3}{2}.$$

We consider the function $\gamma \in \mathcal{D}$, given by $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$, whose support lies in the annulus $\{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 3/2\}$. This allows us to obtain the following two partitions of unity: the homogeneous partition of unity

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1, \quad (\forall \xi \in \mathbb{R}^n \setminus \{0\}),$$

the non-homogeneous partition of unity

$$\rho(2^{-k} \xi) + \sum_{j \geq k+1} \gamma(2^{-j} \xi) = 1, \quad (\forall k \in \mathbb{Z}, \forall \xi \in \mathbb{R}^n).$$

For the kind of partition of unity, see for example, the books of Bergh and L ofstr om [8], Peetre [48], and Triebel [53], [54]. Also, for this partition of unity, we associate a sequence of

convolution operators $(Q_j)_{j \in \mathbb{Z}}$, $(S_j)_{j \in \mathbb{Z}}$, defined by

$$\widehat{Q_j f} := \gamma(2^{-j}(\cdot))\widehat{f} \quad \text{and} \quad \widehat{S_j f} := \rho(2^{-j}(\cdot))\widehat{f}, \quad \forall j \in \mathbb{Z}.$$

We also define the operators $(\widetilde{Q}_j)_{j \geq 0}$ by $\widetilde{Q}_0 := S_0$ and $\widetilde{Q}_j := Q_j$ for $j \geq 1$. The operators Q_j and S_j take values in the space of exponentially type analytic functions; see the Paley-Wiener theorem [52, Theorem. 29.2, p. 311] or [53, Remak. 2.3.1/2, p. 45].

Clearly, the operator S_j is defined on \mathcal{S}' , and the operator Q_j is defined on \mathcal{S}'_∞ , since $Q_j f(x) = 0$, ($\forall j \in \mathbb{Z}$), if and only if f is a polynomial. We adopt the following conventions:

- If $f \in \mathcal{S}'$, we note that $[f]_\infty \in \mathcal{S}'_\infty$.
- If $f \in \mathcal{S}'_\infty$, we define $Q_j f := Q_j f_1$ for any $f_1 \in \mathcal{S}'$ such that $[f_1]_\infty = f$.

Indeed, if $f \in \mathcal{S}'_\infty$, for each f_1, f_2 such that $[f_1]_\infty = [f_2]_\infty = f$, then $f_1 - f_2 \in \mathcal{P}_\infty$, and we have $Q_j f_1 = Q_j f_2 := Q_j f$.

Lemma 1.1. *Let $0 < p \leq \infty$ and $N \in \mathbb{N}_0$. Then there exist constants $c_1, c_2 > 0$ and an integer $m \in \mathbb{N}_0$ such that for any $\varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ and any $\psi \in \mathcal{D}$, (setting $\widehat{\varphi}_j := \varphi(2^{-j}(\cdot))$ and $\widehat{\psi}_j := \psi(2^{-j}(\cdot))$), we have:*

- (i) $\|\varphi_j * f\|_p \leq c_1 2^{-jN} \zeta_m(f)$, for all $f \in \mathcal{S}$ and all $j \in \mathbb{N}_0$.
- (ii) $\|\varphi_j * f\|_p + \|\psi_j * f\|_p \leq c_2 2^{jN} \zeta_m(f)$, for all $f \in \mathcal{S}_\infty$ and all $j \in \mathbb{Z} \setminus \mathbb{N}$.

Proof. See [38, Proposition. 2.5]. □

Corollary 1.2. *Let $0 < p \leq \infty$ and $N \in \mathbb{N}_0$. Then there exist constants $c_1, c_2 > 0$ and an integer $m \in \mathbb{N}_0$ such that:*

- (i) $\|Q_j f\|_p \leq c_1 2^{-jN} \zeta_m(f)$, for all $f \in \mathcal{S}$ and all $j \in \mathbb{N}_0$.
- (ii) $\|Q_j f\|_p + \|S_j f\|_p \leq c_2 2^{jN} \zeta_m(f)$, for all $f \in \mathcal{S}_\infty$ and all $j \in \mathbb{Z} \setminus \mathbb{N}$.

Proof. A consequence of Lemma 1.1 by setting $\varphi = \gamma$ and $\psi = \rho$. □

The weak convergence of the Littlewood-Paley decomposition of a function is given by the following proposition:

Proposition 1.3. (i) *For any $f \in \mathcal{S}_\infty$ (resp. \mathcal{S}'_∞), we have*

$$f = \sum_{j \in \mathbb{Z}} Q_j f,$$

converging in \mathcal{S}_∞ (resp. \mathcal{S}'_∞).

(ii) For any $f \in \mathcal{S}$ (resp. \mathcal{S}'), and any $k \in \mathbb{Z}$, we have

$$f = S_k f + \sum_{j>k} Q_j f,$$

converging in \mathcal{S} (resp. \mathcal{S}').

Proof. The proof is a consequence of Lemma 1.1. □

Remark 1.4. By Young's inequality, we easily show that the operators $(Q_j f)_{j \in \mathbb{Z}}$ and $(S_j f)_{j \in \mathbb{Z}}$ belong to $\mathcal{L}(L_p)$, for all $1 \leq p \leq \infty$.

1.2 Some classical inequalities

We recall some well-known inequalities:

Lemma 1.5. For any $0 < d \leq 1$, and any real sequence of positive terms $(a_j)_{j \in \mathbb{Z}} \in \ell_q(\mathbb{Z})$, we have

$$\sum_j a_j \leq \left(\sum_j a_j^d \right)^{1/d}.$$

Lemma 1.6. Let $0 < p \leq q \leq \infty$. One has

$$\|f\|_q \leq cR^{n/p-n/q} \|f\|_p \tag{1.1}$$

for all $R > 0$ and all $f \in L_p \cap \mathcal{S}'$ with $\text{supp } \widehat{f} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq R\}$. The constant c is independent of f , R and g , exactly $c = p_0^{n/p-n/q}$, with $p_0 = [p/2] + 1$.

Proof. See [47, Theorem. 4]. □

Remark 1.7. The inequality (1.1) is known as the Bernstein inequality. Let $\alpha \in \mathbb{N}_0^n$; we can generalize Lemma 1.6 under the same condition:

$$\|f^{(\alpha)}\|_q \leq cR^{n/p-n/q+|\alpha|} \|f\|_p,$$

see [53, Remark. 1, p. 18].

Lemma 1.8. Let $0 < p \leq 1$. One has

$$\|f * g\|_p \leq cR^{n/p-n} \|f\|_p \|g\|_p$$

for all $R > 0$ and all $f, g \in L_p \cap \mathcal{S}'$, with $\text{supp } \widehat{f}$ and $\text{supp } \widehat{g}$ supported in the ball $\{\xi \in \mathbb{R}^n : |\xi| \leq R\}$. The constant c is independent of f , R and g . If $p = 1$, then $c = 1$, which is Young's inequality.

Proof. See [53, Remark. 2, p. 28]. □

Lemma 1.9 (Minkowski inequality). *Let $0 < p, q \leq \infty$. Let E_i (where $i = p$ or q) and $E \in \{\ell, L\}$. Let F_j (where $j = p$ or q) and $F \in \{\ell, L\}$. If $j \geq i$, then*

$$\| \|f\|_{E_i} \|_{F_j} \leq \| \|f\|_{F_j} \|_{E_i}.$$

1.3 Besov and Lizorkin-Triebel spaces

We will use the notation $A_{p,q}^s$ for either Besov space $B_{p,q}^s$ or Triebel-Lizorkin space $F_{p,q}^s$, and in the same way, we will use the notation $\dot{A}_{p,q}^s$ for either homogeneous Besov space or homogeneous Triebel-Lizorkin space. The basic definitions of $\dot{A}_{p,q}^s$ and $A_{p,q}^s$ are given via the Littlewood-Paley decomposition; see [8, p. 139], [20], [29], or [53, p.45, p. 238].

1.3.1 The homogeneous case

Definition 1.10. *Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$ (with $p < \infty$ in the F case).*

(i) *The homogeneous Besov space $\dot{B}_{p,q}^s$ is the set of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{B}_{p,q}^s} := \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|Q_j f\|_p^q \right)^{1/q} < \infty.$$

(ii) *The homogeneous Triebel-Lizorkin space $\dot{F}_{p,q}^s$ is the set of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{F}_{p,q}^s} := \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jsq} |Q_j f|^q \right)^{1/q} \right\|_p < \infty.$$

Remark 1.11. The spaces $\dot{A}_{p,q}^s$ are quasi-Banach spaces for the quasi-norms above. The spaces $\dot{A}_{p,q}^s$ are independent of the choice of the function γ used to define the Littlewood-Paley decomposition (i.e., replacing γ with another function with the same properties yields the same space). For any $p \in \mathcal{P}_\infty$, we have $\|f\|_{\dot{A}_{p,q}^s} = \|f + p\|_{\dot{A}_{p,q}^s}$. See, for example, [12, Theorem. 2.1, p. 25] for the B case, or [53, Remark. 5.1.3/2].

Proposition 1.12. *For all $s \in \mathbb{R}$, $0 < p, q \leq \infty$ (with $p < \infty$ in the F -case), we have*

$$\mathcal{S}'_\infty \hookrightarrow \dot{A}_{p,q}^s \hookrightarrow \mathcal{S}'_\infty.$$

Proof. The proofs in the non-homogeneous case extend easily to the homogeneous case; see [48, Theorems. 2, 3, pp. 61-62] or [53, Theorem. 2.3.3, p. 48]. \square

The homogeneous Besov spaces have properties analogous to the homogeneous Lizorkin-Triebel spaces. Here are some of theorem:

Proposition 1.13. (i) $\dot{B}_{p,p}^s = \dot{F}_{p,p}^s$.

(ii) If $q_1 < q_2$, then $\dot{A}_{p,q_1}^s \hookrightarrow \dot{A}_{p,q_2}^s$.

(iii) $\dot{B}_{p,\min(p,q)}^s \hookrightarrow \dot{F}_{p,q}^s \hookrightarrow \dot{B}_{p,\max(p,q)}^s$.

(iv) Let $s_1 > s_2$, $0 < p_1 < p_2 < \infty$, and $0 < q, r \leq \infty$. If $s_1 - n/p_1 = s_2 - n/p_2$, then

$$\dot{B}_{p_1,q}^{s_1} \hookrightarrow \dot{B}_{p_2,q}^{s_2} \hookrightarrow \dot{B}_{\infty,q}^{s_2 - n/p_2}, \quad \dot{F}_{p_1,q}^{s_1} \hookrightarrow \dot{B}_{p_2,p_1}^{s_2} \quad \text{and} \quad \dot{F}_{p_1,q}^{s_1} \hookrightarrow \dot{F}_{p_2,r}^{s_2}.$$

Proof. See [29, Theorem. 2.1]. \square

Proposition 1.14. If $0 < p, q < \infty$, then \mathcal{S}_∞ is a dense subspace of $\dot{A}_{p,q}^s$.

Proof. See, for example, [14, Proposition. 3.11] or [29, (1.6)]. \square

Proposition 1.15. (i) There exist two positive constants c_1 and c_2 such that

$$c_1 \|f\|_{\dot{A}_{p,q}^s} \leq \lambda^{n/p-s} \|f(\lambda(\cdot))\|_{\dot{A}_{p,q}^s} \leq c_2 \|f\|_{\dot{A}_{p,q}^s},$$

for all $f \in \dot{A}_{p,q}^s$ and all real $\lambda > 0$.

(ii) Let $m \in \mathbb{N}$. A function f in \mathcal{S}'_∞ belongs to $\dot{A}_{p,q}^s$ if and only if its partial derivatives $f^{(\alpha)} \in \dot{A}_{p,q}^{s-m}$ for all $|\alpha| = m$. Moreover, the expression

$$\sum_{|\alpha|=m} \|f^{(\alpha)}\|_{\dot{A}_{p,q}^{s-m}},$$

is a quasi-seminorm in $\dot{A}_{p,q}^s$.

The following assertion is of Nikols'kii type, which is useful in the sequel:

Proposition 1.16. Let $0 < p \leq \infty$ and $0 < q \leq \infty$ ($p < +\infty$ in case of the F spaces). Let s be a real number such that,

($s < n/p$) ou $s = n/p$ and $0 < q \leq 1$ in (the case of the B spaces) ($0 < p \leq 1$ in the case of the F spaces).

Let a and b two real numbers such that $0 < a < b$. Let $(u_j)_{j \in \mathbb{Z}}$ a sequence of \mathcal{S}' such that

- \hat{u}_j is supported by the ball $|\xi| \leq b2^j$,
- $A := \left(\sum_{j \in \mathbb{Z}} (2^{sj} \|u_j\|_p)^q \right)^{1/q} < \infty$ (in the case of the B spaces).
- $A := \left\| \left(\sum_{j \in \mathbb{Z}} (2^{sj} |u_j|)^q \right)^{1/q} \right\|_p < \infty$ (in the case of the F spaces).

(i) Then, the series $\sum_{j \in \mathbb{Z}} u_j$ converges in \mathcal{S}' and satisfies

$$\left\| \sum_{j \in \mathbb{Z}} u_j \right\|_{\dot{A}_{p,q}^s} \leq cA,$$

where c depends on n, s, p, q, a and b .

(ii) If moreover $s > (\frac{n}{p} - n)_+$ (respectively, $s > (n/\min(p, q) - n)_+$ in the case of the F), the same conclusion for $a = 0$.

1.3.2 The non-homogeneous case

To define the non-homogeneous Besov and Lizorkin-Triebel spaces, it suffices to replace the homogeneous Littlewood-Paley decomposition with its non-homogeneous version (see Section 1.1).

Definition 1.17. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$ ($p < \infty$ in the F -case).

(i) The non-homogeneous Besov space $B_{p,q}^s$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{B_{p,q}^s} := \left(\sum_{j \geq 0} 2^{jsq} \|\tilde{Q}_j f\|_p^q \right)^{1/q} < \infty.$$

(ii) The non-homogeneous Triebel-Lizorkin space $F_{p,q}^s$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{F_{p,q}^s} := \left\| \left(\sum_{j \geq 0} 2^{jsq} |\tilde{Q}_j f|^q \right)^{1/q} \right\|_p < \infty.$$

Remark 1.18. The spaces $A_{p,q}^s$ are quasi-Banach spaces for the quasi-norms, which are independent of the function ρ ; see, for example, [24, Remark. 2.6, p. 51] or [53, Proposition. 1, p. 46].

Now, we recall some inclusions between the non-homogeneous Besov and Lizorkin-Triebel spaces. For more details on the characterization of $A_{p,q}^s$, see, for example, [8, Section. 6.2, p. 139], [49, Chapter. 2], [53, Section. 2.3.2, p. 46], [54, Chapter. 2].

Proposition 1.19. (i) If $0 < q_1 \leq q_2 \leq \infty$ and $\varepsilon > 0$, then

$$\mathcal{S} \hookrightarrow A_{p,\infty}^{s+\varepsilon} \hookrightarrow A_{p,q_1}^s \hookrightarrow A_{p,q_2}^s \hookrightarrow \mathcal{S}'.$$

(ii) $B_{p,\min(p,q)}^s \hookrightarrow F_{p,q}^s \hookrightarrow B_{p,\max(p,q)}^s.$

(iii) Let $0 < p_1 < p < p_2 \leq \infty$ and $0 < q_1, q_2 \leq \infty$. For $0 < q_1 \leq p \leq q_2 \leq \infty$ and $s_1 - n/p_1 = s - n/p = s_2 - n/p_2$, then

$$B_{p_1,q_1}^{s_1} \hookrightarrow F_{p,q}^s \hookrightarrow B_{p_2,q_2}^{s_2}.$$

1.3.3 Relation between $\dot{A}_{p,q}^s(\mathbb{R}^n)$ and $A_{p,q}^s(\mathbb{R}^n)$

To pass from the homogeneous to the non-homogeneous case, we have the following relation:

Proposition 1.20. If $s > (n/p - n)_+$, then $f \in A_{p,q}^s$ if and only if $f \in L_p$ and $[f]_\infty \in \dot{A}_{p,q}^s$. Moreover,

$$\|f\|_p + \|[f]_\infty\|_{\dot{A}_{p,q}^s},$$

is an equivalent quasi-norm in $A_{p,q}^s$.

Proof. See [8, Theorem. 6.3.2, p. 148] for $p, q \geq 1$, or [54, Theorem. 2.3.3, p. 98]. □

Proposition 1.21. If $s > 0$, then for any $f \in A_{p,q}^s$, we have $[f]_\infty \in \dot{A}_{p,q}^s$.

Proof. This is a consequence of the preceding proposition. □

1.4 Realizations

By definition, the spaces $\dot{A}_{p,q}^s$ is embedded into \mathcal{S}'_∞ ; however, there exists a smallest integer $\mu \in \mathbb{N}_0$ such that all element of $\dot{A}_{p,q}^s$ can be given in \mathcal{S}'_μ . We will now make this explicitly.

1.4.1 Distributions vanishing at infinity

We introduce the following notion:

Definition 1.22. A tempered distribution $f \in \mathcal{S}'$ is said to vanish at infinity if $h_\lambda f$ tends to 0 in \mathcal{S}' as λ tends to 0, i.e.,

$$\lim_{\lambda \rightarrow 0} \langle f(\lambda^{-1}(\cdot)), \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{S}.$$

The set of such distributions is denoted by $\tilde{\mathcal{C}}_0$.

Lemma 1.23. *If $f \in \tilde{\mathcal{C}}_0$ is a polynomial, then $f = 0$. That is, $\tilde{\mathcal{C}}_0 \cap \mathcal{P}_\infty = \{0\}$.*

Proof. See [9, p. 46]. □

Here are some examples of distributions in $\tilde{\mathcal{C}}_0$:

Example 1.24. (i) *Let $1 \leq p < \infty$. If $f \in L_p$, then $f \in \tilde{\mathcal{C}}_0$.*

(ii) *If $f \in L_\infty$ or $f \in \tilde{\mathcal{C}}_0$, then $\partial_j f \in \tilde{\mathcal{C}}_0$ ($j = 1, \dots, n$).*

Remark 1.25. *See [4], [43] for other examples.*

1.4.2 Realizations (Generalities)

Definition 1.26. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Let E be a vector subspace of \mathcal{S}'_∞ , equipped with a quasi-Banach space structure such that $E \hookrightarrow \mathcal{S}'_\infty$. A realization of E in \mathcal{S}'_k is a continuous linear map $\sigma : E \rightarrow \mathcal{S}'_k$ such that $[\sigma(f)]_\infty = f$ ($\forall f \in E$) in \mathcal{S}'_∞ . The image set $\sigma(E)$ is called the realized space of E .*

Remark 1.27. The map σ is a bijection from E to $\sigma(E)$.

We recall some necessary properties concerning invariance under translations and dilations (i.e., $\|\tau_a f\|_E = \|f\|_E$ or $\|h_\lambda f\|_E = \lambda^{-r} \|f\|_E$, $\forall f \in E, r \in \mathbb{R}, a \in \mathbb{R}^n$, and $\lambda > 0$):

Proposition 1.28. *Let $\sigma_0 : E \rightarrow \mathcal{S}'$ be a realization. For any finite family $(\mathcal{L}_\alpha)_{|\alpha| \leq N}$ of linear functionals on E , the following formula defines a realization of E in \mathcal{S}' :*

$$\sigma(f)(x) := \sigma_0(f)(x) + \sum_{|\alpha| \leq N} \mathcal{L}_\alpha(f) x^\alpha.$$

Conversely, any realization of E is given in this way.

Proof. See [12, Theorem. 1.4, p. 20], or [38, Proposition. 3.2]. □

Proposition 1.29. *Let E be isometrically invariant under translations. Let $\sigma_0 : E \rightarrow \mathcal{S}'_k$ be a realization commuting with translations. Let $(\mathcal{L}_\alpha)_{|\alpha|=k}$ be a finite family of linear functionals on E such that*

$$\mathcal{L}_\alpha \circ \tau_a = \mathcal{L}_\alpha.$$

Then the formula

$$\sigma(f)(x) := \sigma_0(f)(x) + \sum_{|\alpha|=k} \mathcal{L}_\alpha(f) x^\alpha \quad (\text{modulo } \mathcal{P}_k)$$

defines a realization of E modulo \mathcal{P}_k , commuting with translations. Conversely, any realization of E modulo \mathcal{P}_k , commuting with translations, is given in this way.

Proof. See [12, Theorem. 1.5, p. 21] or [13, Proposition. 2]. □

Proposition 1.30. (i) *If $r \notin \mathbb{N}_0$, then E admits at most one realization commuting with dilations.*

(ii) *Assume $r \in \mathbb{N}_0$ and let $\sigma_0 : E \rightarrow \mathcal{S}^l$ be a realization commuting with dilations. Let $(\mathcal{L}_\alpha)_{|\alpha|=r}$ be a family of linear functionals on E such that*

$$\mathcal{L}_\alpha \circ h_\lambda = \lambda^{-r} \mathcal{L}_\alpha, \quad \forall \lambda > 0.$$

Then the formula

$$\sigma(f)(x) := \sigma_0(f)(x) + \sum_{|\alpha|=r} \mathcal{L}_\alpha(f) x^\alpha$$

defines a realization of E commuting with dilations. Conversely, any realization of E commuting with dilations is given in this way.

Proof. See [12, Theorem. 1.6, p. 21], [13, Proposition. 4], or [38, Proposition. 3.1]. □

Chapter 2

Realizations and pseudo-differential operators

This chapter studies Besov and Triebel-Lizorkin spaces and their connections with pseudo-differential operators. We prove key theorems (Theorems 2.15–2.16) about how these operators act on function spaces.

2.1 The realized spaces of $\dot{A}_{p,q}^s$

Handling distributions modulo polynomials is challenging and requires both realizations and weak convergence in $\mathcal{S}'(\mathbb{R}^n)$. We will outline these two approaches and refer to G. Bourdaud [12, 13, 14] for the relevant definitions.

2.1.1 Preparation

As we mentioned at the beginning of this section, we will present an integer $\mu \in \mathbb{N}_0$ such that $\dot{A}_{p,q}^s \hookrightarrow \mathcal{S}'_\mu$. The integer $\mu = \mu(s, p, q, n)$ (with μ fixed throughout this thesis) is defined by

$$\mu := \begin{cases} ([s - n/p] + 1)_+ & \text{if } s - n/p \notin \mathbb{N}_0 \text{ or } q > 1 \text{ in the } B\text{-case } (p > 1 \text{ in the } F\text{-case}), \\ s - n/p & \text{if } s - n/p \in \mathbb{N}_0 \text{ and } q \leq 1 \text{ in the } B\text{-case } (p \leq 1 \text{ in the } F\text{-case}). \end{cases} \quad (2.1)$$

Proposition 2.1. *Let $f \in \dot{A}_{p,q}^s$. Then the Littlewood-Paley series $\sum_{j \in \mathbb{Z}} Q_j f$ converges in \mathcal{S}'_μ to an element denoted $\sigma(f)$. The function f has in \mathcal{S}'_∞ a unique representative $\sigma(f)$, satisfying: $[\sigma(f)]_\infty = f$ and $\partial^\alpha \sigma(f) \in \tilde{C}_0$ for $|\alpha| = \mu$, where $\sigma(f)$ commutes with translation and dilation operators in some cases of the parameters.*

Proof. See [14, Proposition. 4.6] or [38, Theorem. 1.2]. □

We can also establish convergence in \mathcal{S}' . For this, we need to add a polynomial to the Littlewood-Paley series.

Proposition 2.2. *For any $f \in \dot{A}_{p,q}^s$, the series $\sigma(f)$ converges in \mathcal{S}' ,*

$$[\sigma(f)]_\infty = f \text{ in } \mathcal{S}'_\infty \text{ and } \partial^\alpha \sigma(f) \in \tilde{C}_0 \text{ if } |\alpha| = \mu.$$

The series $\sigma(f)$ is defined as follows:

(H-1) *In case $s < n/p$, or $s = n/p$ and $0 < q \leq 1$ in B-case ($0 < p \leq 1$ in F-case),*

$$\sigma(f) = \sum_{j \in \mathbb{Z}} Q_j f. \tag{2.2}$$

(H-2) *In case $s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$, or $s - n/p \in \mathbb{N}$ and $0 < q \leq 1$ in B-case ($0 < p \leq 1$ in F-case),*

$$\sigma(f) = \sum_{j \in \mathbb{Z}} (Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) x^\alpha / \alpha!). \tag{2.3}$$

(H-3) *In case $s - n/p \in \mathbb{N}_0$ and $q > 1$ in B-case ($p > 1$ in F-case),*

$$\sigma(f) = \sum_{j > 0} Q_j f + \sum_{j \leq 0} (Q_j f - \sum_{|\alpha| < \mu} (Q_j f)^{(\alpha)}(0) x^\alpha / \alpha!). \tag{2.4}$$

Proof. For more details and the proof, see [14, Section. 4] or [38, Theorems. 4.1, 4.5]. □

2.1.2 Definition of realized spaces

We now give with the definition of the realized spaces $\tilde{A}_{p,q}^s$, see [9], [14], or [38]. According to definition 1.26, we set $\tilde{A}_{p,q}^s = \sigma(\dot{A}_{p,q}^s)$ and have the following definition:

Definition 2.3. *The realized space $\tilde{A}_{p,q}^s$ is defined as follows:*

- (i) *In the case $s < n/p$, or $s = n/p$ and $0 < q \leq 1$ in the B-case ($0 < p \leq 1$ in the F-case) (here $\mu = 0$),*

$$\tilde{A}_{p,q}^s := \left\{ f \in \mathcal{S}' : [f]_\infty \in \dot{A}_{p,q}^s \text{ and } f \in \tilde{C}_0 \right\}.$$

(ii) In the case $s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$, or $s - n/p \in \mathbb{N}$ and $0 < q \leq 1$ in the B-case ($0 < p \leq 1$ in the F-case) (here $\mu = [s - n/p] + 1$ or $\mu = s - n/p$, respectively, with $\mu \geq 1$),

$$\tilde{A}_{p,q}^s := \left\{ \begin{array}{l} f \in \mathcal{S}' : [f]_\infty \in \dot{A}_{p,q}^s, f \in C^{\mu-1}, \\ f^{(\beta)}(0) = 0 \text{ for } |\beta| \leq \mu - 1, \\ \text{and } f^{(\alpha)} \in \tilde{C}_0 \text{ for } |\alpha| = \mu \end{array} \right\}.$$

(iii) In the case $s - n/p \in \mathbb{N}_0$ and $q > 1$ in the B-case ($p > 1$ in the F-case) (here $\mu = [s - n/p] + 1 \geq 1$),

$$\tilde{A}_{p,q}^s := \left\{ \begin{array}{l} f \in \mathcal{S}' : [f]_\infty \in \dot{A}_{p,q}^s, f \in C^{\mu-1}, \\ f^{(\beta)}(0) = \sum_{j>0} (Q_j f)^{(\beta)}(0) \text{ for } |\beta| \leq \mu - 1, \\ \text{and } f^{(\alpha)} \in \tilde{C}_0 \text{ for } |\alpha| = \mu \end{array} \right\}.$$

Equipped with the same quasi-seminorm as $\dot{A}_{p,q}^s$, that is,

$$\|f\|_{\tilde{A}_{p,q}^s} := \|[f]_\infty\|_{\dot{A}_{p,q}^s}.$$

Proposition 2.4. In cases (i)–(ii), we have

$$\tilde{A}_{p,q}^s = \sigma(\dot{A}_{p,q}^s). \tag{2.5}$$

Proof. By definition, we have $\sigma(\dot{A}_{p,q}^s) \subset \tilde{A}_{p,q}^s$. For the reverse inclusion, let $f \in \tilde{A}_{p,q}^s$, then $[f]_\infty \in \dot{A}_{p,q}^s$ and $f - \sigma([f]_\infty) \in \mathcal{P}_\infty$. By Lemma 1.23 (i.e., $\tilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$) and since $\partial^\alpha(f - \sigma([f]_\infty)) \in \tilde{C}_0$ for all $|\alpha| = \mu$, we have $f - \sigma([f]_\infty) = p \in \mathcal{P}_\mu$,

– in case (H-1), $\mu = 0$ and $f = \sigma([f]_\infty)$;

– in cases (ii) and (iii), we set

$$p(x) = \sum_{|\beta| < \mu} a_\beta x^\beta,$$

then by identification we have

$$\beta! a_\beta = (f^{(\beta)} - \partial^\beta \sigma([f]_\infty))(0) = 0,$$

which implies $f = \sigma([f]_\infty)$. Then we obtain $f = \sigma([f]_\infty)$ in \mathcal{S}'_μ , and the inclusion $\sigma(\dot{A}_{p,q}^s) \supset \tilde{A}_{p,q}^s$ holds. \square

Remark 2.5. *In the sequel of this chapter we limit ourselves to Banach space, i.e, we assume that $1 \leq q \leq \infty$, $1 \leq p \leq \infty$ in B – case and $1 \leq p \leq \infty$ in F – case.*

Proposition 2.6. *Let $s \in \mathbb{R}$. Let*

$$\begin{aligned} \text{either : } & s < \frac{n}{p}, \\ \text{or : } & s = \frac{n}{p} \quad \text{and} \quad q = 1. \end{aligned}$$

If $u \in \dot{B}_{p,q}^s$, then the convergence is obtained in \mathcal{S}' of the series

$$\sum_{j \in \mathbb{Z}} Q_j u.$$

We set

$$f := \sum_{j \in \mathbb{Z}} Q_j u.$$

Then it holds $f \in \tilde{C}_0$, and u has a unique tempered distribution given as

$$[u]_\infty = f \quad \text{in } \mathcal{S}'_\infty.$$

Proof. See [14, Proposition 4.6], [38, Theorems 1.2, 4.1, Section 4.2]. \square

The homogeneous Besov space that has been realized can now be defined.

Definition 2.7. *Let $s \in \mathbb{R}$. Let*

$$\begin{aligned} \text{either : } & s < \frac{n}{p}, \\ \text{or : } & s = \frac{n}{p} \quad \text{and} \quad q = 1. \end{aligned}$$

We define $\tilde{B}_{p,q}^s$ the so-called realized homogeneous Besov space, the set of all functions $g \in \mathcal{S}'$ satiafying $[g]_\infty \in \dot{B}_{p,q}^s$ and the property $g \in \tilde{C}_0$.

Let us consider the following mapping

$$\sigma(u) := \sum_{j \in \mathbb{Z}} Q_j u \quad \text{for all } u \in \dot{B}_{p,q}^s;$$

thus, from Proposition 2.6, this application σ is a realization on $\dot{B}_{p,q}^s$. We have σ satisfies the two following properties :

1- commuting with translation operators

$$\sigma(u)(x-a) = \sigma(u-a)(x)$$

for all $x, a \in \mathbb{R}^n$,

2- commuting with dilation operators

$$\sigma(u)(x/\lambda) = \sigma(u(\cdot/\lambda))(x)$$

for all $x \in \mathbb{R}^n$ and all positive real λ .

We also have

$$\sigma(\dot{B}_{p,q}^s) = \tilde{B}_{p,q}^s. \tag{2.6}$$

Proof. To prove (2.6). The first embedding “ \subset ” is obtained by the definition. For the converse, we take a function $g \in \tilde{B}_{p,q}^s$, thus $[g]_\infty$ belongs to $\dot{B}_{p,q}^s$ and satisfies $g - \sigma([g]_\infty) \in \mathcal{P}_\infty$; on the other hand Lemma 1.23 yields that $g = \sigma([g]_\infty)$, the up of the proof. \square

We finish this section by the following statement in canonique case.

Proposition 2.8. *Let $s > 0$ and $1 \leq p < \infty$.*

Let

$$\begin{aligned} \text{either : } & s < \frac{n}{p}, \\ \text{or : } & s = \frac{n}{p} \quad \text{and} \quad q = 1. \end{aligned}$$

Let $b > 0$. Assume that $(u_j)_{j \in \mathbb{Z}}$ a sequence defined in \mathcal{S}' such that the following two conditions are satisfied:

- *supp \hat{u}_j is subset of the ball $|\xi| \leq b2^j$,*
- *$D := (\sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^q)^{1/q} < \infty$.*

Then, we obtain the convergence in \mathcal{S}' of the series $\sum_{j \in \mathbb{Z}} u_j$; denoted its limit by u , thus $\| [u]_\infty \|_{\dot{B}_{p,q}^s} \leq cD$; the constant c depends only on the parameters n, s, p, q and b .

Proof. The proof is given in [38, Proposition 2.15]. See also [37, Proposition 3.4], or the book [49, Proposition 2.3.2/1, p. 59], or the paper of M. Yamazaki [56] in which the case of inhomogeneous Besov spaces is treated. \square

2.2 Pseudo-differential operators

For the pseudo-differential operators theory, we do not into details, however instead we propose some papers [22], [28], [31], [33], [34], [35], [45], [56].

Definition 2.9. A pseudo-differential operator (P.D.O.) is a linear mapping, denoted by $b(x, D)$, defined as

$$b(x, D)f := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} b(x, \xi) \hat{f}(\xi) d\xi, \quad \forall f \in \mathcal{S}(\mathbb{R}^n), \forall x \in \mathbb{R}^n,$$

where $b(x, \xi)$ is a function defined from $\mathbb{R}^n \times \mathbb{R}^n$ to \mathbb{C} , called the symbol of $b(x, D)$.

Definition 2.10. A modulus of continuity is any function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that is continuous, increasing, and concave, and satisfies $\omega(0) = 0$.

Definition 2.11. Let ω be a modulus of continuity and a function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying

$$\forall c > 1, \exists A_c > 0, \quad (t/c \leq u \leq ct) \Rightarrow \psi(u) \leq A_c \psi(t).$$

Let $\delta \geq 0$, $\rho \geq 0$, $\mu \geq 0$, $N \in \mathbb{N}$, and $(\alpha, \beta) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ such that $|\beta| \leq N$. The class $\Sigma_{1, \delta}^\mu(\omega, \psi, \rho, N)$ of symbols of degree μ and type $(1, \delta)$ is the set of functions $b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for which there exist two positive constants c_1, c_2 depending only on α and β , such that

$$|\partial_x^\alpha \partial_\xi^\beta b(x, \xi)| \leq c_1 (1 + |\xi|)^{\mu - |\alpha| + \delta |\beta|} \tag{2.7}$$

and

$$|\partial_x^\alpha \partial_\xi^\beta b(x+h, \xi) - \partial_x^\alpha \partial_\xi^\beta b(x, \xi)| \leq c_2 (1 + |\xi|)^{\mu - |\alpha| + \delta |\beta|} \omega(|h| |\xi|^\delta) \psi(|\xi|^\rho). \tag{2.8}$$

The set of pseudo-differential operators associated with the class $\Sigma_{1, \delta}^\mu(\omega, \psi, \rho, N)$ will be denoted by $\text{OP}_{1, \delta}^\mu(\omega, \psi, \rho, N)$.

Definition 2.12. We define $\dot{S}_{1, 0}^{m, N}(E)$, the so-called the homogeneous class of symbols, the set of all $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ functions $b(x, \xi)$, satisfying, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq N$, there exists a

constant $c := c(\alpha) > 0$, such that

$$\|\partial_{\xi}^{\alpha} b(x, \xi)\|_E \leq c |\xi|^{m-|\alpha|}, \quad \forall (\xi) \in \mathbb{R}^n \setminus \{0\}.$$

Definition 2.13. We define the inhomogeneous class $S_{1,0}^{m,N}(E)$ is the set of all $C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ functions (symbols) $b(x, \xi)$ such that

$$\left\| \partial_{\xi}^{\alpha} b(\cdot, \xi) \right\|_E \leq c(1 + |\xi|)^{m-|\alpha|}, \quad \forall \xi \in \mathbb{R}^n,$$

where $|\alpha| \leq N$ and the positive constant $c := c(\alpha) > 0$.

Remark 2.14. It is worth mentioning that both sets $S_{1,0}^{m,N}(E)$ and $S_{1,0}^{m,N}(E)$ are a Fréchet space that has seminorms.

$$\dot{\Omega}_N(b) := \sup_{|\alpha| \leq N} \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{-m+|\alpha|} \left\| \partial_{\xi}^{\alpha} b(\cdot, \xi) \right\|_E$$

and

$$\Omega_N(b) := \sup_{|\alpha| \leq N} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m+|\alpha|} \left\| \partial_{\xi}^{\alpha} b(\cdot, \xi) \right\|_E,$$

respectively.

For the homogeneous class of Besov type case, we consider the space

$$E := L_{\infty}(\mathbb{R}^n) \cap \dot{B}_{p,q}^{n/p}(\mathbb{R}^n),$$

equipped with the norm

$$\|f\|_E := \|f\|_{\infty} + \|[f]\|_{\dot{B}_{p,q}^{n/p}}.$$

For the inhomogeneous class, we consider space

$$E := L_{\infty}(\mathbb{R}^n) \cap \dot{B}_{p,q}^{n/p}(\mathbb{R}^n) \cap \dot{B}_{p,q}^{n/p-m}(\mathbb{R}^n),$$

equipped with the corresponding norm. We give the following observation, to explain the difference between these two classes, : If $m \neq 0$, there are no inclusion properties between the two spaces $\dot{B}_{p,q}^{n/p}(\mathbb{R}^n)$ and $\dot{B}_{p,q}^{n/p-m}(\mathbb{R}^n)$.

In case of the homogeneous class of Triebel-Lizorkin-type, we deal with

$$E := L_{\infty}(\mathbb{R}^n) \cap \dot{F}_{p,q}^{n/p}(\mathbb{R}^n).$$

2.3 Results already obtained

The results of this section can be found in our paper [33]. Here $p, q \geq 1$.

2.3.1 Main results

In this section, the parameters s, p, q are given in the following manner :

$$\text{either } : s < \frac{n}{p}, \quad \text{or } \quad s = \frac{n}{p} \quad \text{and} \quad q = 1; \quad (2.9)$$

and

$$\text{either } : s + m < \frac{n}{p}, \quad \text{or } \quad s + m = \frac{n}{p} \quad \text{and} \quad q = 1. \quad (2.10)$$

Theorem 2.15. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.9) and (2.10) hold.*

Let N be an even natural number satisfying $N > \frac{3n}{2} + 2$. If a symbol $b(x, \xi)$ belongs to

$$S_{1,0}^{m,N}(L_\infty(\mathbb{R}^n) \cap \dot{B}_{p,q}^{n/p}(\mathbb{R}^n)),$$

then the pseudo-differential operator $b(x, D)$ takes the space $\tilde{B}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{B}_{p,q}^s(\mathbb{R}^n)$.

Moreover, there exists a constant $c > 0$ such that the inequality:

$$\| [b(x, D)f]_\infty \|_{\dot{B}_{p,q}^s} \leq c \dot{\Omega}_N(b) \| [f]_\infty \|_{\dot{B}_{p,q}^{s+m}}. \quad (2.11)$$

holds, for all $f \in \tilde{B}_{p,q}^{s+m}(\mathbb{R}^n)$.

Theorem 2.16. *Let $s > 0$, $1 \leq p < \infty$ and $m \geq 0$. Assume that (2.9) and (2.10) hold. Let N be an even natural number satisfying $N > \frac{3n}{2} + 2$. If a symbol $b(x, \xi)$ belongs to*

$$S_{1,0}^{m,N}(L_\infty(\mathbb{R}^n) \cap \dot{B}_{p,q}^{n/p}(\mathbb{R}^n) \cap \dot{B}_{p,q}^{n/p-m}(\mathbb{R}^n)),$$

then the pseudodifferential operator $b(x, D)$ takes the space $\tilde{B}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{B}_{p,q}^s(\mathbb{R}^n)$.

Moreover, there exists a constant $c > 0$ such that the inequality

$$\| [b(x, D)f]_\infty \|_{\dot{B}_{p,q}^s} \leq c \Omega_N(b) \| [f]_\infty \|_{\dot{B}_{p,q}^{s+m}} \quad (2.12)$$

holds for all $f \in \tilde{B}_{p,q}^{s+m}(\mathbb{R}^n)$.

We prove Theorems 2.15 and 2.16 by developing three essential assertions:

- in the homogeneous case, simplify to elementary symbols,

- type of G. Gibbons [28], can be estimated almost orthogonally using the Nikol'skij representation method, (see Lemma 2.18 below),
- the property of pointwise multiplying for a realized homogeneous Besov space can be seen in (Lemma 2.19 below).

We outline then in the next section.

2.3.2 Idea of proofs of Theorems 2.15 and 2.16

The proofs are obtained by the following statements.

Proposition 2.17. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.9) and (2.10) hold. Let $b(x, \xi)$ be a $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function.*

(i) $b(x, \xi)$ can be decomposed into a sum $\kappa(x, \xi) + \eta(x, \xi)$ where:

- $\kappa(x, \xi) = 0$ if $|\xi| \geq \frac{3}{2}$
- $\eta(x, \xi) := \sum_{K \in \mathbb{Z}^n} \vartheta_K(x, \xi)$ with $\eta(x, \xi) = 0$ if $|\xi| \leq 1$
- The functions $\vartheta_K(x, \xi)$ are elementary symbols-type.

(ii) Consider an even natural number N such that: $N > 3n/2 + 2$.

If $b(x, \xi)$ belongs to $S_{1,0}^{m,N}(L_\infty \cap \dot{B}_{p,q}^{n/p})$, then the pseudo-differential operator $\eta(x, D)$ takes the space $\tilde{B}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{B}_{p,q}^s(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$ such that

$$\|[\eta(x, D)f]_\infty\|_{\dot{B}_{p,q}^s} \leq c \Omega_N(b) \| [f]_\infty \|_{\tilde{B}_{p,q}^{s+m}},$$

holds for all $f \in \tilde{B}_{p,q}^{s+m}(\mathbb{R}^n)$.

(iii) Assume in addition $m \geq 0$. Consider an even natural number N such that: $N > n$. If $b(x, \xi)$ belongs to $S_{1,0}^{m,N}(L_\infty \cap \dot{B}_{p,q}^{n/p-m})$, then the pseudodifferential operator $\kappa(x, D)$ takes the space $\tilde{B}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{B}_{p,q}^s(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$ such that

$$\|[\kappa(x, D)f]_\infty\|_{\dot{B}_{p,q}^s} \leq c \Omega_N(b) \| [f]_\infty \|_{\tilde{B}_{p,q}^{s+m}},$$

holds for all $f \in \tilde{B}_{p,q}^{s+m}$.

Proof. This proposition is proved using the following two lemmas. See [33, Proposition 3.3].

□

Lemma 2.18. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.9) holds. Let $b > 0$. Let $(u_j)_{j \in \mathbb{Z}}$ and $(\chi_j)_{j \in \mathbb{Z}}$ be sequences in \mathcal{S}' satisfying the following three conditions:*

- (i) $\text{supp } \hat{u}_j$ is subset of the ball $|\xi| \leq b2^j$,
- (ii) $D := (\sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^q)^{1/q} < \infty$,
- (iii) $G := \sup_{j \in \mathbb{Z}} (\|\chi_j\|_\infty + \|\chi_j\|_{\dot{B}_{p,q}^{n/p}}) < \infty$.

Then the series $\sum_{j \in \mathbb{Z}} \chi_j u_j$ converges in \mathcal{S}' to a limit u which satisfies

$$\|[u]_\infty\|_{\dot{B}_{p,q}^s} \leq cDG, \quad (2.13)$$

where c depends only on n, s, p, q and b .

Proof. See [33, Lemma 3.2]. □

Lemma 2.19. *Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Assume that (2.9) and (2.10) hold. Assume in addition $m \geq 0$. Then there exists a constant $c > 0$ such that the inequality*

$$\|[hg]_\infty\|_{\dot{B}_{p,q}^s} \leq c \left(\|[h]_\infty\|_{\dot{B}_{p,q}^{n/p-m}} + \|h\|_{L_\infty} \right) \|[g]_\infty\|_{\dot{B}_{p,q}^{s+m}},$$

holds, for all $g \in C^\infty \cap \dot{B}_{p,q}^{s+m}(\mathbb{R}^n)$ and all $h \in L_\infty \cap \dot{B}_{p,q}^{n/p-m}(\mathbb{R}^n)$.

Proof. See [33, Lemma 3.3]. □

Proof of Theorem 2.15. The proof is essentially based on Proposition 2.17 and some changes in order to pass from the inhomogeneous case to the homogeneous one. □

Proof of Theorem 2.16. The result follows immediately from Proposition 2.17. □

2.3.3 An extension related to main results

We first extend our framework by generalizing the symbol classes: For $(\mu, \eta) \in \mathbb{R}^2$, $(M, N) \in \mathbb{N}^2$ and a Banach space E , we Consider the symbol classes $\dot{S}_{\mu,\eta}^{m,N,M}(E)$ and $S_{\mu,\eta}^{m,N,M}(E)$ of functions $b(x, \xi)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ and in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, respectively. These classes are defined (as Fréchet spaces) by the seminorms

$$\dot{\Omega}_{N,M}(b) := \sup_{\substack{|\alpha| \leq N \\ |\beta| \leq M}} \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{-m+\mu|\alpha|-\eta|\beta|} \left\| \partial_\xi^\alpha \partial_x^\beta b(\cdot, \xi) \right\|_E,$$

$$\Omega_{N,M}(b) := \sup_{\substack{|\alpha| \leq N \\ |\beta| \leq M}} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m + \mu|\alpha| - \eta|\beta|} \left\| \partial_{\xi}^{\alpha} \partial_x^{\beta} b(\cdot, \xi) \right\|_E.$$

Theorems 2.15 and 2.16 can be generalized in the following sense:

Theorem 2.20. *Let s, p, q, m, N be real numbers given as in Theorem 2.15 (resp. Theorem 2.16). Let $M \in \mathbb{N}$, $\eta \in \mathbb{R}$, and $\mu \geq 1$. If a symbol $b(x, \xi)$ belongs to*

$$\dot{S}_{\mu, \eta}^{m, N, M}(L_{\infty} \cap \dot{B}_{p, q}^{n/p}) \quad (\text{resp. } S_{\mu, \eta}^{m, N, M}(L_{\infty} \cap \dot{B}_{p, q}^{n/p} \cap \dot{B}_{p, q}^{n/p - m})),$$

then the pseudo-differential operator $b(x, D)$ takes the space $\tilde{B}_{p, q}^{s+m}$ into $\tilde{B}_{p, q}^s$, with an estimate similar to (2.11), where $\dot{\Omega}_N(b)$ is replaced by $\dot{\Omega}_{N, M}(b)$ (resp. $\Omega_{N, M}(b)$).

Proof. See [33]. □

Now, we see an extension on the realized spaces: The condition 2.9 can be extended in the following sense. For

$$s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}, \quad \text{or} \quad s - \frac{n}{p} \in \mathbb{N} \setminus \{0\} \quad \text{and} \quad q = 1, \quad (2.14)$$

we denote by μ the integer ≥ 1 defined by $s - n/p \leq \mu < s - n/p + 1$, i.e.

$$\mu := \begin{cases} [s - n/p] + 1 & \text{if } s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}, \\ s - n/p & \text{if } s - n/p \in \mathbb{N} \setminus \{0\} \quad \text{and} \quad q = 1. \end{cases} \quad (2.15)$$

Definition 2.21. *Assume that (2.14) holds. The space $\tilde{B}_{p, q}^s$ is the set of all $f \in \mathcal{S}'$ satisfying the following four conditions:*

- (i) $[f]_{\infty} \in \dot{B}_{p, q}^s$,
- (ii) f is of class $C^{\mu-1}$,
- (iii) $f^{(\alpha)}(0) = 0$ for $|\alpha| < \mu$,
- (iv) $f^{(\alpha)} \in \tilde{C}_0$ for $|\alpha| = \mu$.

We have the following result:

Theorem 2.22. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.14) holds. Let N be an even natural number satisfying $N > 3n/2 + 2$. Let m be a real number satisfying*

$$s + m - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}, \quad \text{or} \quad s + m - \frac{n}{p} \in \mathbb{N} \setminus \{0\} \quad \text{and} \quad q = 1. \quad (2.16)$$

If a symbol $b(x, \xi)$ belongs to $\dot{S}_{1,0}^{m,N}(L_\infty \cap \dot{B}_{p,q}^{n/p})$, then the pseudo-differential operator $b(x, D)$ takes the space $\tilde{B}_{p,q}^{s+m}$ into $\tilde{B}_{p,q}^s$. Moreover, an estimate similar to (2.11) holds for all $f \in \tilde{B}_{p,q}^{s+m}$.

Proof. The proof is similar to that of Theorem 2.15. We give a sketchy proof. According to Propositions, 2.6, 2.8 and Lemma 2.18, we have the following assertions, in which the two propositions below are proved in [38, Proposition 2.17] and [38, Theorems 1.2, 4.5, Section 4.2], respectively. \square

Proposition 2.23. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.14) holds. Let $b > 0$. Let $(u_j)_{j \in \mathbb{Z}}$ be a sequence in \mathcal{S}' satisfying the following two conditions:*

- \hat{u}_j is supported by the ball $|\xi| \leq b2^j$,
- $A := (\sum_{j \in \mathbb{Z}} (2^{js} \|u_j\|_p)^q)^{1/q} < \infty$.

We put

$$w_j(x) := u_j(x) - \sum_{|\alpha| \leq \mu-1} u_j^{(\alpha)}(0) x^\alpha / \alpha!,$$

for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$, (μ is defined in (2.15)). Then the series $\sum_{j \in \mathbb{Z}} w_j$ converges in \mathcal{S}' to a limit w which satisfies

$$\|w\|_{\tilde{B}_{p,q}^s} \leq cA,$$

and where c depends only on n, s, p, q and b .

Proposition 2.24. *Assume that (2.14) holds. If $u \in \dot{B}_{p,q}^s$, we put*

$$v_j(x) := Q_j u(x) - \sum_{|\alpha| \leq \mu-1} (Q_j u)^{(\alpha)}(0) x^\alpha / \alpha!$$

for all $j \in \mathbb{Z}$, and all $x \in \mathbb{R}^n$. Then the series $\sum_{j \in \mathbb{Z}} v_j$ converges in \mathcal{S}' . We put

$$f := \sum_{j \in \mathbb{Z}} v_j.$$

Then $f^{(\alpha)}$ ($|\alpha| = \mu$) belongs to \tilde{C}_0 , and f is the unique tempered distribution satisfying $[u]_\infty = f$ in \mathcal{S}'_∞ .

2.4 Recent results (the Triebel-Lizorkin case)

In this section, we assume the parameters $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ satisfy the following conditions:

$$\text{either } s < \frac{n}{p}, \quad \text{or } s = \frac{n}{p} \quad \text{and} \quad p = 1; \tag{2.17}$$

and

$$\text{either } s+m < \frac{n}{p}, \quad \text{or } s+m = \frac{n}{p} \quad \text{and} \quad p = 1. \quad (2.18)$$

Theorem 2.25. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.17) and (2.18) hold. Let N be an even natural number sufficiently large. If a symbol $b(x, \xi)$ belongs to $\dot{S}_{1,0}^{m,N}(L_\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{n/p}(\mathbb{R}^n))$, then the pseudo-differential operator $b(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{F}_{p,q}^s(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that the inequality*

$$\| [b(x, D)f]_\infty \|_{\tilde{F}_{p,q}^s} \leq C \Omega_N(b) \| [f]_\infty \|_{\tilde{F}_{p,q}^{s+m}}, \quad (2.19)$$

holds for all $f \in \tilde{F}_{p,q}^{s+m}(\mathbb{R}^n)$.

Theorem 2.26. *Let $s > 0$, $1 \leq p < \infty$ and $m \geq 0$. Assume that (2.17) and (2.18) hold. Let N be an even natural number sufficiently large. If a symbol $b(x, \xi)$ belongs to*

$$S_{1,0}^{m,N}(L_\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{n/p}(\mathbb{R}^n) \cap \dot{F}_{p,q}^{n/p-m}(\mathbb{R}^n)),$$

then we have :

- (i) *the pseudo-differential operator $b(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}(\mathbb{R}^n)$ into $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ if $p \leq q$. Moreover, there exists a constant $c > 0$ such that the inequality*

$$\| [b(x, D)f]_\infty \|_{\tilde{F}_{p,q}^s} \leq c \Omega_N(b) \| [f]_\infty \|_{\tilde{F}_{p,q}^{s+m}}, \quad (2.20)$$

holds, for all $f \in \tilde{F}_{p,q}^{s+m}(\mathbb{R}^n)$,

- (ii) *the pseudo-differential operator $b(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}(\mathbb{R}^n) \cap \tilde{B}_{\infty,q}^{s-n/p+m}(\mathbb{R}^n)$ into $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ if $p > q$. Moreover, there exists a constant $c > 0$ such that the inequality*

$$\| [b(x, D)f]_\infty \|_{\tilde{F}_{p,q}^s} \leq c \Omega_N(b) (\| [f]_\infty \|_{\tilde{F}_{p,q}^{s+m}} + \| [f]_\infty \|_{\tilde{B}_{\infty,q}^{s-n/p+m}}), \quad (2.21)$$

holds, for all $f \in \tilde{F}_{p,q}^{s+m} \cap \tilde{B}_{\infty,q}^{s-n/p+m}$.

The proofs of Theorems 2.25 and 2.26 rely on three key components:

1. **Reduction to elementary symbols** in the homogeneous case. This technique simplifies the analysis by decomposing general symbols into more manageable elementary components.

2. **An almost orthogonality estimate** of G. Gibbons type [28], obtained through Nikol'skii's representation method (detailed in Lemma below). This provides crucial bounds for the interaction between different frequency components.

3. **A pointwise multipliers property**

for realized homogeneous spaces (established in Lemma below). This result enables the control of products involving Triebel-Lzorkin space elements.

2.4.1 Proof of Theorems 2.25 and 2.26

Theorems 2.25 can be easily derived from the following statement.

Proposition 2.27. *Let $s \in \mathbb{R}$. Assume that (2.17)*

holds. If $u \in \dot{F}_{p,q}^s$, then the series

$$\sum_{j \in \mathbb{Z}} Q_j u$$

converges in \mathcal{S}' . We put

$$f := \sum_{j \in \mathbb{Z}} Q_j u.$$

Then f belongs to \tilde{C}_0 and is the unique tempered distribution satisfying

$$[u]_\infty = f \quad \text{in } \mathcal{S}'_\infty.$$

Proposition 2.28. *Let $s > 0$ and $1 \leq p < \infty$. Assume that 2.17 and 2.18 hold. Let $b(x, \xi)$ be a $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function. We can be decomposed $b(x, \xi)$ into a sum $\kappa(x, \xi) + \eta(x, \xi)$ where $\kappa(x, \xi) = 0$ if $|\xi| \geq 3/2$ and $\eta(x, \xi) := \sum_{K \in \mathbb{Z}^n} \vartheta_K(x, \xi)$ with $\eta(x, \xi) = 0$ if $|\xi| \leq 1$, and the functions $\vartheta_K(x, \xi)$ are elementary symbols-type.*

Proposition 2.29. *Let N be an even natural number sufficiently large. If $b(x, \xi)$ belongs to $S_{1,0}^{m,N}(L_\infty \cap \dot{F}_{p,q}^{n/p})$, then we have:*

- (i) *The pseudo-differential operator $\eta(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}$ into $\tilde{F}_{p,q}^s$ if $p \leq q$. Moreover, there exists a constant $c > 0$ such that the inequality*

$$\|[\eta(x, D)f]_\infty\|_{\dot{F}_{p,q}^s} \leq c \Omega_N(b) \| [f]_\infty \|_{\dot{F}_{p,q}^{s+m}}, \tag{2.22}$$

holds, for all $f \in \tilde{F}_{p,q}^{s+m}$.

(ii) The pseudo-differential operator $\eta(x, D)$ takes the spaces $\tilde{F}_{p,q}^{s+m} \cap \tilde{B}_{\infty,q}^{s-n/p+m}$ into $\tilde{F}_{p,q}^s$ if $p > q$. Moreover, there exists a constant $c > 0$ such that the inequality

$$\|[\eta(x, D)f]_{\infty}\|_{\tilde{F}_{p,q}^s} \leq c \Omega_N(b) (\|[f]_{\infty}\|_{\tilde{F}_{p,q}^{s+m}} + \|[f]_{\infty}\|_{\tilde{B}_{\infty,q}^{s-n/p+m}}), \quad (2.23)$$

holds, for all $f \in \tilde{F}_{p,q}^{s+m} \cap \tilde{B}_{\infty,q}^{s-n/p+m}$.

Proposition 2.30. Assume in addition $m \geq 0$. Let N be an even natural number satisfying $N > n$. If $b(x, \xi)$ belongs to $S_{1,0}^{m,N}(L_{\infty} \cap \dot{F}_{p,q}^{n/p-m})$, then the pseudo-differential operator $\kappa(x, D)$ takes the space $\tilde{F}_{p,q}^{s+m}$ into $\tilde{F}_{p,q}^s$. Moreover, there exists a constant $c > 0$ such that the inequality

$$\|\kappa(x, D)f\|_{\tilde{F}_{p,q}^s} \leq c \Omega_N(b) \|[f]_{\infty}\|_{\tilde{F}_{p,q}^{s+m}},$$

holds for all $f \in \tilde{F}_{p,q}^{s+m}$.

Corollary 2.31. If $p \leq q$, then the pseudo-differential operator $\kappa(x, D)$ takes the space $\dot{F}_{p,q}^s$ into $\dot{F}_{p,q}^s$.

Lemma 2.32. Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.17) holds. Let $b > 0$. Let $(u_j)_{j \in \mathbb{Z}}$ and $(\chi_j)_{j \in \mathbb{Z}}$ be sequences in \mathcal{S}' satisfying the following conditions:

- (i) $\text{supp } \hat{u}_j$ is subset of the ball $|\xi| \leq b2^j$,
- (ii₁) $D := \left\| \left(\sum_{j \in \mathbb{Z}} (2^{js} |u_j|)^q \right)^{1/q} \right\|_p + \left(\sum_{j \in \mathbb{Z}} \left(2^{j(s-n/p)} \|u_j\|_{\infty} \right)^q \right)^{1/q} < \infty$, in case $p > q$.
- (ii₂) $D := \left\| \left(\sum_{j \in \mathbb{Z}} (2^{js} |u_j|)^q \right)^{1/q} \right\|_p + \left(\sum_{j \in \mathbb{Z}} \left(2^{j(s-n/p)} \|u_j\|_{\infty} \right)^p \right)^{1/p} < \infty$, in case $p \leq q$.
- (iii) $G := \sup_{\ell} \left(\|\chi_{\ell}\|_{\infty} + \|\chi_{\ell}\|_{\dot{F}_{\infty,p}^{n/p}} \right) < \infty$.

Then the series $\sum_{j \in \mathbb{Z}} \chi_j u_j$ converges in \mathcal{S}' to a limit u which satisfies

$$\|[u]_{\infty}\|_{\dot{F}_{p,q}^s} \leq c D G, \quad (2.24)$$

where c depends only on n, s, p, q and b .

Proof. Step 1 : Convergence in \mathcal{S}' . The proof is the same as in [38, Proposition 2.15/Substep 1.2]. We omit the details.

Step 2 : Proof of 2.24. In $\sum_{j \in \mathbb{Z}} \chi_j u_j$ we split the area of summation with respect to j . That is, by Proposition 1.16 we write

$$\chi_j = S_j \chi_j + \sum_{k > j} Q_k \chi_j,$$

recall that $\chi_j \in S'$, which implies that $\sum_{j \in \mathbb{Z}} \chi_j u_j = Y + Z$, where

$$Y := \sum_{j \in \mathbb{Z}} u_j S_j \chi_j \quad \text{and} \quad Z := \sum_{k \in \mathbb{Z}} \sum_{j < k} u_j Q_k \chi_j,$$

with $F(u_j S_j \chi_j)$ and $F(\sum_{j < k} u_j Q_k \chi_j)$ are supported by the balls $|\xi| \leq (3/2 + b)2^j$ and $|\xi| \leq (3/2 + b/2)2^k$, respectively. This allows to apply Proposition 2.27 to the series Y and Z .

Substep 2.1: The estimate of $\|[Y]_\infty\|_{\dot{F}_{p,q}^s}$ follows immediately from the uniform bound $\|S_j \chi_j\|_\infty \lesssim G$ for all $j \in \mathbb{Z}$, i.e.

$$\|[Y]_\infty\|_{\dot{F}_{p,q}^s} \lesssim DG.$$

Substep 2.2 : Estimate of $\|[Z]_\infty\|_{\dot{F}_{p,q}^s}$ in the case $0 < s < n/p$. We will use the following inequality

$$\|[Z]_\infty\|_{\dot{F}_{p,q}^s} \leq \left\| \sum_{k \in \mathbb{Z}} 2^{skq} \left(\sum_{j < k} |u_j Q_k \chi_j|^q \right)^{1/q} \right\|_p.$$

- If $p \leq q$ By Lemma 1.5 with $d := p/q$ we obtain :

$$\begin{aligned} \|[Z]_\infty\|_{\dot{F}_{p,q}^s} &\lesssim \left\| \left(\sum_k \left(\sum_{j \leq k} |u_j| |Q_k \chi_j| \right)^q \right)^{1/q} \right\|_p \\ &\lesssim \left\| \left(\sum_k \left(2^{sk} \sum_{j \leq k} |u_j| |Q_k \chi_j| \right)^p \right)^{1/p} \right\|_p \\ &= c_1 \left(\int_{\mathbb{R}^n} \sum_k \left(2^{sk} \sum_{j \leq k} |u_j| |Q_k \chi_j| \right)^p dx \right)^{1/p} \\ &= c_1 \left(\sum_k 2^{skp} \int \left(\sum_{j \leq k} |u_j| |Q_k \chi_j| \right)^p dx \right)^{1/p} \\ &= c_1 \left(\sum_k 2^{skp} \left\| \sum_{j \leq k} |u_j| |Q_k \chi_j| \right\|_p^p \right)^{1/p} \\ &\lesssim \left(\sum_k 2^{skp} \left(\sum_{j \leq k} \|u_j\|_\infty \|Q_k \chi_j\|_p \right)^p \right)^{1/p}. \end{aligned}$$

Let:

$$\begin{aligned}
 2^{kn/p} \|Q_k \chi_j\|_p &= \left\| \sup_{\ell} 2^{\ell \frac{n}{p}} |Q_{\ell} \chi_j| \right\|_p \\
 &= \sup_m \left\| \sup_{\ell} 2^{\ell \frac{n}{p}} |Q_{\ell} \chi_m| \right\|_p \\
 &= \sup_m \|\chi_m\|_{\dot{F}_{p,\infty}^{n/p}}.
 \end{aligned}$$

Then

$$\begin{aligned}
 \|[Z]_{\infty}\|_{\dot{F}_{p,q}^s} &\lesssim \left(\sum_k \left(\sum_{j \leq k} 2^{(s-\frac{n}{p})(k-j)} 2^{(s-\frac{n}{p})j} \|u_j\|_{\infty} \right)^p \right)^{1/p} \sup_m \|\chi_m\|_{\dot{F}_{p,\infty}^{n/p}} \\
 &\lesssim \sup_m \|\chi_m\|_{\dot{F}_{p,\infty}^{n/p}} \left(\sum_{j \in \mathbb{Z}} \left(2^{(s-\frac{n}{p})j} \|u_j\|_{\infty} \right)^p \right)^{1/p}.
 \end{aligned}$$

We conclude that

$$\|[W]_{\infty}\|_{\dot{F}_{p,\infty}^{n/p}} \lesssim DG.$$

- If $p > q$ We use the same proof

$$\begin{aligned}
 \|[Z]_\infty\|_{\dot{F}_{p,q}^s} &\lesssim \left\| \left(\sum_k \left(\sum_{j \leq k} |u_j| |Q_k \chi_j| \right)^q \right)^{1/q} \right\|_p \\
 &\lesssim \left(\sum_k 2^{skq} \left(\sum_{j \leq k} \|u_j\|_\infty 2^{-k\frac{n}{p}} \|Q_k \chi_j\|_p 2^{k\frac{n}{p}} \right)^q \right)^{1/q} \\
 &\lesssim \left(\sum_k 2^{skq} \left(\sum_{j \leq k} \|u_j\|_\infty 2^{-k\frac{n}{p}} \sup_\ell 2^{\ell\frac{n}{p}} |Q_\ell \chi_j| \right)^q \right)^{1/q} \\
 &\lesssim \left(\sum_k 2^{skq} \left(\sum_{j \leq k} \|u_j\|_\infty 2^{-k\frac{n}{p}} \|\chi_j\|_{\dot{F}_{p,\infty}^{n/p}} \right)^q \right)^{1/q} \\
 &\lesssim \sup_\ell \|\chi_\ell\|_{\dot{F}_{p,\infty}^{n/p}} \left(\sum_k \left(\sum_{j \leq k} 2^{(s-\frac{n}{p})(-j+k)} 2^{j(s-\frac{n}{p})} \|u_j\|_\infty \right)^q \right)^{1/q} \\
 &\lesssim \sup_\ell \|\chi_\ell\|_{\dot{F}_{p,\infty}^{n/p}} \left(\sum_{j \leq k} 2^{j(s-\frac{n}{p})q} \|u_j\|_\infty^q \right)^{1/q} \\
 &\lesssim DG.
 \end{aligned}$$

Substep 2.3 : Estimate of $\|[Z]_\infty\|_{\dot{F}_{p,q}^s}$ in the case $s = n/p$ and $p = 1$. We will use the same proof, we obtain.

$$\|[Z]_\infty\|_{\dot{F}_{1,\infty}^n} \lesssim \sup_m \| \chi_m \|_{\dot{F}_{1,\infty}^n} \left(\sum_{j \in \mathbb{Z}} \|u_j\|_\infty \right).$$

We deduce that

$$\|[Z]_\infty\|_{\dot{F}_{1,\infty}^n} \leq DG.$$

□

Lemma 2.33. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (2.17) and (2.18) hold. Assume in addition $m \geq 0$. Then there exists a positive constant c such that the inequality*

$$\|[hg]_\infty\|_{\dot{F}_{p,q}^s} \leq c \left(\|[h]_\infty\|_{\dot{F}_{p,q}^{n/p-m}} + \|h\|_\infty \right) \|[g]_\infty\|_{\dot{F}_{p,q}^{s+m}},$$

holds, for all $g \in C^\infty \cap \tilde{F}_{p,q}^{s+m}$ and all $h \in L_\infty \cap \tilde{F}_{p,q}^{n/p-m}$.

Proof . Step 1 : Let g and h be functions given as in this lemma. By assumption, the functions g and h belong to \mathcal{S}' . Clearly also that $S_j g$, $Q_k g$ and $Q_k h$ are C^∞ functions for all $j, k \in \mathbb{Z}$. Then

in \mathcal{S}' , we can write the following expressions

$$g = S_j g + \sum_{k>j} Q_k g, S_j g = \sum_{k \leq j} Q_k g, h = \sum_{j \in \mathbb{Z}} Q_j h$$

and

$$S_j h = \sum_{k \leq j} Q_k h.$$

Thus, for all $\varphi \in \mathcal{S}$, we get

$$\begin{aligned} \left\langle \sum_{j \in \mathbb{Z}} (S_j g)(Q_j h), \varphi \right\rangle &= \sum_{j \in \mathbb{Z}} \langle Q_j h, (S_j g) \varphi \rangle = \sum_{j \in \mathbb{Z}} \left\langle Q_j h, \left(g - \sum_{k>j} Q_k g \right) \varphi \right\rangle \\ &= \langle h, g \varphi \rangle - \sum_{k \in \mathbb{Z}} \sum_{j \leq k-1} \langle Q_j h, (Q_k g) \varphi \rangle \\ &= \langle hg, \varphi \rangle - \left\langle \sum_{k \in \mathbb{Z}} (S_{k-1} h)(Q_k g), \varphi \right\rangle. \end{aligned}$$

Hence, we arrive at the decomposition of hg in \mathcal{M}' into a sum $hg = A_1 + A_2$ where

$$A_1 := \sum_{j \in \mathbb{Z}} (S_j g)(Q_j h),$$

and

$$A_2 := \sum_{k \in \mathbb{Z}} (S_{k-1} h)(Q_k g).$$

Step 2 : Estimate of $\| [A_i]_\infty \|_{\dot{F}_{p,q}^s}$, $i = 1, 2$.

We begin by noting that the functions $\mathcal{F} \{ (S_j g)(Q_j h) \}$ and $\mathcal{F} \{ (S_{k-1} h)(Q_k g) \}$ are supported by the balls $|\xi| \leq 3 \cdot 2^j$ and $|\xi| \leq (9/4)2^k$, respectively.

$$\begin{aligned}
 \| [A_1]_\infty \|_{\dot{F}_{p,q}^s} &= \left\| \left(\sum_{j \in \mathbb{Z}} |Q_j h|^q 2^{jsq} \left(\sum_{k \leq j} |Q_k g| \right)^q \right)^{1/q} \right\|_p \\
 &= \left\| \left(\sum_{j \in \mathbb{Z}} \left(|Q_j h| 2^{j(\frac{n}{p}-m)} \right)^q \cdot 2^{j(s-\frac{n}{p}+m)q} \left(\sum_{k \leq j} |Q_k g| 2^{k(s+m)} 2^{-k(s+m)} \right)^q \right)^{1/q} \right\|_p \\
 &\lesssim \left\| \sup_j 2^{j(\frac{n}{p}-m)} |Q_j h| \right\|_p \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jq(s-\frac{n}{p}+m)} \left(\sum_{k \leq j} 2^{k(-s-m)} \cdot 2^{k(s+m)} |Q_k g| \right)^q \right)^{1/q} \right\|_\infty \\
 &\lesssim \|h\|_{\dot{F}_{p,\infty}^{\frac{n}{p}-m}} \left(\sum_{j \in \mathbb{Z}} 2^{jq(s-\frac{n}{p}+m)} \left(\sum_{k \leq j} 2^{k(\frac{n}{p}-s-m)} \cdot 2^{k(s+m-\frac{n}{p})} \|Q_k g\|_\infty \right)^q \right)^{1/q}.
 \end{aligned}$$

If $\frac{n}{p} - s - m > 0$,

$$\begin{aligned}
 H &= \left(\sum_{j \in \mathbb{Z}} \left(2^{j(s+m-\frac{n}{p})} \sum_{k \leq j} 2^{(s+m-\frac{n}{p})k} \cdot 2^{\frac{n}{p}-s-m)k} \|Q_k g\|_\infty \right)^q \right)^{\frac{1}{q}} \\
 &= \left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \leq j} 2^{(j-k)(s+m-\frac{n}{p})} \cdot \left\{ 2^{k(s+m-\frac{n}{p})} \|Q_k g\|_\infty \right\} \right)^q \right)^{\frac{1}{q}} \\
 &\lesssim \left(\sum_k \left(2^{k(s+m-\frac{n}{p})} \|Q_k g\|_\infty \right)^q \right)^{\frac{1}{q}} \\
 &\lesssim \|g\|_{\dot{B}_{\infty,q}^{s+m-\frac{n}{p}}}.
 \end{aligned}$$

Furthermore, $\dot{F}_{p,q}^{s+m} \hookrightarrow \dot{B}_{\infty,q}^{s+m-\frac{n}{p}}$ if $p \leq q$.

There remains a case $\frac{n}{p} - s - m = 0$ with $q \leq 1$ (or $q = 1$):

$$\begin{aligned}
 M &= \left\| \left(\sum_{j \in \mathbb{Z}} |Q_j h|^q 2^{jsq} \left(\sum_{k \leq j} |Q_k g| \right)^q \right)^{\frac{1}{q}} \right\|_p \\
 M &= \left\| \left(\sum_{j \in \mathbb{Z}} \left(|Q_j h| 2^{j(\frac{n}{p}-m)} \right)^q \left(\sum_{k \leq j} 2^{(s-\frac{n}{p}+m)j} |Q_k g| \right)^q \right)^{\frac{1}{q}} \right\|_p
 \end{aligned}$$

$$M = \left\| \left(\sum_{j \in \mathbb{Z}} \left(2^{j \left(\frac{n}{p} - m \right)} |Q_j h| \right)^q \cdot \left(\sum_{k \leq j} 2^{(j-k) \left(s+m-\frac{n}{p} \right)} \cdot 2^{k \left(s+m-\frac{n}{p} \right)} |Q_k g| \right)^q \right)^{\frac{1}{q}} \right\|_p.$$

We have

$$\begin{aligned} \left(\sum_{k \leq j} 2^{(j-k) \left(s+m-\frac{n}{p} \right)} \cdot 2^{k \left(s+m-\frac{n}{p} \right)} |Q_k g| \right) &\leq \sum_{\ell \geq 0} 2^{\ell \left(s+m-\frac{n}{p} \right)} \sup_k \left(2^{k \left(s+m-\frac{n}{p} \right)} \|Q_k g\|_{\infty} \right) \\ &\leq C \|g\|_{\dot{B}_{\infty, \infty}^{s+m-\frac{n}{p}}}. \end{aligned}$$

Furthermore:

$$M \leq C \|h\|_{\dot{F}_{p,q}^{\frac{n}{p}-m}} \|g\|_{\dot{B}_{\infty, \infty}^{s+m-\frac{n}{p}}}.$$

□

The proof of Lemma 2.33 is complete, for $\dot{F}_{p,q}^s$ with $h \in L_{\infty} \cap \dot{F}_{p,q}^{\frac{n}{p}-m}$.

Proof of Proposition 2.28. The proof is organized into the following steps for clarity: We adapt Coifman and Meyer's method for reducing to elementary symbols [22], to the homogeneous setting through the following construction:

We define a function that is both radial and positive $\tilde{\gamma} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ supported by the compact annulus $1/4 \leq |\xi| \leq 7/4$, and satisfies $\tilde{\gamma}\gamma = \gamma$ (see the beginning of Section 1 for the definition of ρ and γ). We write

$$b(x, \xi) = \kappa(x, \xi) + \eta(x, \xi)$$

where

$$\tau(x, \xi) := b(x, \xi)\rho(\xi)$$

and

$$\eta(x, \xi) := b(x, \xi)(1 - \rho(\xi)),$$

with

- $\kappa(x, \xi) = 0$ if $|\xi| \geq 3/2$,
- $\eta(x, \xi) = 0$ if $|\xi| \leq 1$.

We begin by decomposing the function $\eta(x, \xi)$. We define

$$\eta_j(x, \xi) := 2^{-mj} \gamma(2^{-|j|}\xi) \eta(x, 2^{j-|j|}\xi), \quad j \in \mathbb{Z}, \forall x, \xi \in \mathbb{R}^n, \quad (2.25)$$

This is well-defined, and from the definition of η we obtain:

$$\eta(x, \xi) = \eta(x, \xi) \sum_{j \in \mathbb{Z}} \gamma(2^{-j} \xi) = \sum_{j \in \mathbb{Z}} 2^{mj} \eta_j(x, 2^{-j+|j|} \xi). \quad (2.26)$$

We define the function $w_j(x, \xi)$ as follows:

$$w_j(x, \xi) := \sum_{K \in \mathbb{Z}^n} \eta_j(x, 2^{|j|}(\xi - 2\pi K)), \quad (2.27)$$

since $\eta_j(x, \xi)$ is 2π -periodic in ξ , satisfying

$$w_j(x, \xi) \tilde{\gamma}(\xi) = \eta_j(x, 2^{|j|} \xi),$$

Consequently, a Fourier expansion of w_j gives

$$\eta_j(x, 2^{|j|} \xi) = \tilde{\gamma}(\xi) \sum_{K \in \mathbb{Z}^n} C_{K,j}(x) e^{iK \cdot \xi}, \quad (2.28)$$

where

$$C_{K,j}(x) := (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-iK \cdot \xi} w_j(x, \xi) d\xi.$$

By inserting the expression from (2.28) into equation (2.26), we obtain

$$\eta(x, \xi) := \sum_{K \in \mathbb{Z}^n} \sum_{j \in \mathbb{Z}} 2^{mj} C_{K,j}(x) \tilde{\gamma}(2^{-j} \xi) e^{i2^{-j} K \cdot \xi}. \quad (2.29)$$

On the one hand, consider $\xi \in \mathbb{R}^n$ and $K \in \mathbb{Z}^n$ satisfying:

- $\frac{1}{2} \leq |\xi - 2\pi K| \leq \frac{3}{2}$
- $-\pi \leq \xi_\ell \leq \pi$ for $\ell = 1, \dots, n$.

Under these conditions, we necessarily have $|K| = 0$. Consequently, $w_j(x, \xi)$ coincides exactly with $\eta_j(x, 2^{|j|} \xi)$, since the mapping $\xi \mapsto \eta_j(x, 2^{|j|}(\xi - 2\pi K))$ is supported in the annulus $\frac{1}{2} \leq |\xi - 2\pi K| \leq \frac{3}{2}$. Thus

$$C_{K,j}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iK \cdot \xi} \eta_j(x, 2^{|j|} \xi) d\xi. \quad (2.30)$$

On the other hand, for a fixed $N \in \mathbb{N}$, (to be determined later), we set.

$$\chi_{K,j}(x) := (2\pi)^n (1 + |K|^2)^{N/2} C_{K,j}(x), \quad (2.31)$$

$$Y_K(\xi) := (2\pi)^{-n} (1 + |K|^2)^{-N/2} \tilde{\gamma}(\xi) e^{iK \cdot \xi} \quad (2.32)$$

and

$$\vartheta_K(x, \xi) := \sum_{j \in \mathbb{Z}} 2^{mj} \chi_{K,j}(x) \Upsilon_K(2^{-j} \xi), \quad K \in \mathbb{Z}^n, \forall x, \xi \in \mathbb{R}^n. \quad (2.33)$$

By substituting these formulas, (2.31)-(2.33), into (2.29) This construction yields the desired decomposition:

$$\eta(x, \xi) = \sum_{K \in \mathbb{Z}^n} \vartheta_K(x, \xi), \quad \forall x, \xi \in \mathbb{R}^n. \quad (2.34)$$

□

Proof of Proposition 2.29. Step 1: Estimate of

$$\|\chi_{K,j}\|_\infty + \|[\chi_{K,j}]_\infty\|_{F_{p,q}^{n/p}}.$$

We select N to be an even positive integer, writing $N := 2M$ where M is a natural number. With this choice, we substitute the expression from (2.30) into equation (2.31) to get

$$\chi_{K,j}(x) = \int_{\mathbb{R}^n} e^{-iK \cdot \xi} (I - \Delta_\xi)^M \left(\eta_j(x, 2^{|j|} \xi) \right) d\xi.$$

The expression $(I - \Delta_\xi)^M \left(\eta_j(x, 2^{|j|} \xi) \right)$ is the following sum (cf., see (2.25))

$$\sum_{|\mu| \leq M} \sum_{\alpha + \eta = 2\mu} \frac{M!}{(M - |\mu|)! \mu!} \frac{(2\mu)!}{\alpha! \eta!} (-1)^{|\mu|} 2^{(|\alpha| - m)j} \gamma^{(\eta)}(\xi) \partial_\xi^\alpha \eta(x, 2^j \xi),$$

which yields

$$\begin{aligned} \chi_{K,j}(x) &= \sum_{|\mu| \leq M} \sum_{\alpha + \eta = 2\mu} \frac{M!}{(M - |\mu|)! \mu!} \frac{(2\mu)!}{\alpha! \eta!} (-1)^{|\mu|} 2^{(|\alpha| - m)j} \\ &\quad \times \int_{\mathbb{R}^n} e^{-iK \cdot \xi} \gamma^{(\eta)}(\xi) \partial_\xi^\alpha \eta(x, 2^j \xi) d\xi. \end{aligned} \quad (2.35)$$

As

$$\partial_\xi^\alpha \eta(x, \xi) = (1 - \rho(\xi)) \partial_\xi^\alpha b(x, \xi) - \sum_{\beta \prec \alpha} \frac{\alpha!}{(\alpha - \beta)! \beta!} \rho^{(\alpha - \beta)}(\xi) \partial_\xi^\beta b(x, \xi).$$

By applying the symbol assumptions on $b(x, \xi)$, we obtain the estimate:

$$2^{(|\alpha| - m)j} \left| \partial_\xi^\alpha \eta(x, 2^j \xi) \right| \lesssim \Omega_N(b) \left(g_{\alpha,j}(\xi) + \sum_{\beta \prec \alpha} h_{\beta,j}(\xi) \right), \quad (2.36)$$

where

$$g_{\alpha,j}(\xi) := 2^{(|\alpha|-m)j} (1 + 2^j |\xi|)^{m-|\alpha|} |1 - \rho(2^j \xi)| \quad (2.37)$$

and

$$h_{\beta,j}(\xi) := 2^{(|\alpha|-m)j} (1 + 2^j |\xi|)^{m-|\beta|} \left| \rho^{(\alpha-\beta)}(2^j \xi) \right|, \quad \beta \prec \alpha. \quad (2.38)$$

On supp $g_{\alpha,j}$ we have $1 \leq 2^j |\xi|$, we obtain the following:

- for the case $m - |\alpha| > 0$:

$$|g_{\alpha,j}(\xi)| \leq (1 + \|\rho\|_\infty) 2^{(|\alpha|-m)j} (2 \cdot 2^j |\xi|)^{m-|\alpha|} \lesssim |\xi|^{m-|\alpha|};$$

- for the case $m - |\alpha| \leq 0$:

$$|g_{\alpha,j}(\xi)| \leq (1 + \|\rho\|_\infty) 2^{(|\alpha|-m)j} (2^j |\xi|)^{m-|\alpha|} \lesssim |\xi|^{m-|\alpha|}.$$

Thus in all cases we obtain

$$|g_{\alpha,j}(\xi)| \lesssim |\xi|^{m-|\alpha|}, \quad \forall j \in \mathbb{Z}. \quad (2.39)$$

Similar to the function $h_{\beta,j}$, on the support of $h_{\beta,j}$, the inequality $1 \leq 2^j |\xi| \leq 3/2$ holds. This implies that the variable $|\xi| \sim 2^{-j}$ satisfies:

$$2^{(|\alpha|-m)j} \leq \max(1, (3/2)^{|\alpha|-m}) |\xi|^{m-|\alpha|},$$

$$(1 + 2^j |\xi|)^{m-|\beta|} \leq \max(2^{m-|\beta|}, (5/2)^{m-|\beta|}),$$

and

$$|h_{\beta,j}(\xi)| \lesssim |\xi|^{m-|\alpha|}, \quad \forall j \in \mathbb{Z}. \quad (2.40)$$

We now prove the following fundamental estimate: $\|\chi_{K,j}\|_\infty$. Indeed, by (2.39) and (2.40) we obtain

$$\|\chi_{K,j}\|_\infty \lesssim \Omega_N(b), \quad \forall K \in \mathbb{Z}^n, \quad \forall j \in \mathbb{Z}. \quad (2.41)$$

We now establish the fundamental estimate $\|\chi_{K,j}\|_{\dot{F}_{p,q}^{n/p}}$. From (2.35), we write (for all $K \in \mathbb{Z}^n$ and all $k, j \in \mathbb{Z}$)

$$\begin{aligned} Q_k \chi_{K,j}(x) &= \sum_{|\mu| \leq M} \sum_{\alpha + \eta = 2\mu} \frac{M!}{(M - |\mu|)! \mu!} \frac{(2\mu)!}{\alpha! \eta!} (-1)^{|\mu|} 2^{(|\alpha|-m)j} \\ &\quad \times \int_{\mathbb{R}^n} e^{-iK \cdot \xi} \gamma^{(\eta)}(\xi) Q_k(\partial_\xi^\alpha \eta(x, 2^j \xi)) d\xi. \end{aligned}$$

Consequently, following the same approach as in equation (2.36), the assumption that the sym-

bol $b(x, \xi)$ belongs to the class $S_{1,0}^{m,N}(L_\infty \cap \dot{F}_{p,q}^{n/p})$ yields

$$2^{(|\alpha|-m)j} \left\| [\partial_\xi^\alpha \eta(\cdot, 2^j \xi)]_\infty \right\|_{\dot{F}_{p,q}^{n/p}} \lesssim \Omega_N(b) \left(g_{\alpha,j}(\xi) + \sum_{\beta \prec \alpha} h_{\beta,j}(\xi) \right),$$

see equations (2.37) and (2.38) for the definitions of $g_{\alpha,j}$ and $h_{\alpha,j}$, respectively. Combining the estimates in (2.39) with $1/2 \leq |\xi| \leq 3/2$, we obtain

$$\|[\chi_{K,j}]_\infty\|_{\dot{F}_{p,q}^{n/p}} \lesssim \Omega_N(b), \quad \forall K \in \mathbb{Z}^n, \quad \forall j \in \mathbb{Z}. \quad (2.42)$$

Now, the last inequality together with (2.41) yield that the sequence $(\chi_{K,j})_{j \in \mathbb{Z}}$ satisfies the assumption (iii) of Lemma 2.32 uniformly with respect to K .

Step 2 : Estimate of $\|[\eta(x, D)f]_\infty\|_{\dot{F}_{p,q}^s}$ for $f \in \dot{F}_{p,q}^{s+m} \cap \dot{B}_{\infty,q}^{s-\frac{n}{p}+m}$, (here $p > q$). Since Υ_K (see (2.32)) is defined on $\mathbb{R}^n \setminus \{0\}$, then by considering the intersection of the supports, i.e.,

$$\tilde{\gamma}(2^{-j}\xi)\gamma(2^{-j-\ell}\xi) = 0 \quad \text{if } \ell \leq -3 \text{ or } \ell \geq 2,$$

it holds that

$$\Upsilon_K(2^{-j}\xi) = \Upsilon_K(2^{-j}\xi) \sum_{-2 \leq \ell \leq 1} \gamma(2^{-j-\ell}\xi), \quad \forall \xi \in \mathbb{R}^n.$$

The assumption (2.18), Proposition 2.27 and formula (2.5) (i.e., $f = \sum_{j \in \mathbb{Z}} Q_j f$) yield

$$\Upsilon_K(2^{-j}D)f = \sum_{-2 \leq \ell \leq 1} (2^{jn} \mathcal{F}^{-1} \Upsilon_K(2^j \cdot)) * Q_{j+\ell} f.$$

We continue, since the function $\xi \mapsto \Upsilon_K(2^{-j}\xi)\gamma(2^{-j-\ell}\xi)$ has a support in the compact annulus $(1/8)2^j \leq |\xi| \leq 3 \cdot 2^j$ (here we note that the assumption (i) of Lemma 2.32 is satisfied), then Lemma 2.33 can be applied to the elementary symbol defined by the formula (2.33), and it follows

$$\begin{aligned} & \|[\vartheta_K(x, D)f]_\infty\|_{\dot{F}_{p,q}^{s+m}} \\ & \lesssim \Omega_N(b) \left\| \sum_{-2 \leq \ell \leq 1} \left\{ \sum_{j \in \mathbb{Z}} \left(2^{j(s+m)} (2^{(j-\ell)n} \mathcal{F}^{-1} \Upsilon_K(2^{j-\ell} \cdot)) * Q_j f \right)^q \right\}^{1/q} \right\|_p. \end{aligned} \quad (2.43)$$

Since $p > q$, then the first observation is to use the Minkowski inequality, and Young inequality we have

$$\|(2^{(j-\ell)n} \mathcal{F}^{-1} \Upsilon_K(2^{j-\ell} \cdot)) * Q_j f\|_p \lesssim \|\mathcal{F}^{-1} \Upsilon_K\|_1 \|Q_j f\|_p, \quad \forall j \in \mathbb{Z}, \quad \forall K \in \mathbb{Z}^n. \quad (2.44)$$

The second observation, for any even natural number N_0 , the inverse Fourier transform of $\Upsilon_K(x)$ admits the representation:

$$\mathcal{F}^{-1}\Upsilon_K(x) = (2\pi)^{-n}(1+|x|^2)^{-N_0/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (I - \Delta_\xi)^{N_0/2} \Upsilon_K(\xi) d\xi,$$

an application of the Cauchy-Schwarz inequality combined with Parseval's identity yields the estimate:

$$\begin{aligned} \|\mathcal{F}^{-1}\Upsilon_K\|_1 &\lesssim \left(\int_{\mathbb{R}^n} (1+|x|^2)^{-N_0} dx \right)^{1/2} \left(\int_{1/4 \leq |\xi| \leq 7/4} |(I - \Delta_\xi)^{N_0/2} \Upsilon_K(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim \sup_{|\alpha| \leq N_0} \left\| \Upsilon_K^{(\alpha)} \right\|_\infty, \quad \forall K \in \mathbb{Z}^n. \end{aligned}$$

Assuming the condition $N_0 > n/2$, we define

$$N_0 := 2N_1 \quad \text{with} \quad N_1 = \left\lfloor \frac{n}{4} \right\rfloor + 1. \quad (2.45)$$

From (2.32), we obtain the estimate:

$$\left| \Upsilon_K^{(\alpha)}(\xi) \right| \lesssim (1+|K|^2)^{-N/2} \sum_{\beta+\eta=\alpha} |K|^{|\beta|} \left| \tilde{\gamma}^{(\eta)}(\xi) \right|,$$

which holds for all $K \in \mathbb{Z}^n$ and all $\xi \in \mathbb{R}^n$.

But, if $|\beta| \leq N_0$ we have $|K|^{|\beta|} (1+|K|^2)^{-N/2} \leq (1+|K|^2)^{(N_0-N)/2}$ for all $K \in \mathbb{Z}^n$, which implies that

$$\sup_{|\alpha| \leq N_0} \left\| \Upsilon_K^{(\alpha)} \right\|_\infty \lesssim (1+|K|^2)^{(N_0-N)/2}, \quad \forall K \in \mathbb{Z}^n. \quad (2.46)$$

Where the constant c is independent of K and $b(x, \xi)$. We now turn to $\eta(x, D)f$, we have

$$\begin{aligned} \|\eta(x, D)f\|_{\dot{F}_{p,q}^{s+m}} &\lesssim \sum_{K \in \mathbb{Z}^n} \|\vartheta_K(x, D)f\|_{\dot{F}_{p,q}^s} \\ &\lesssim \Omega_N(b) \left(\|f\|_{\dot{F}_{p,q}^{s+m}} + \|f\|_{\dot{B}_{\infty,q}^{s-\frac{n}{p}+m}} \right) \sum_{K \in \mathbb{Z}^n} (1+|K|^2)^{(N_0-N)/2}. \end{aligned}$$

and we deduce that

$$\|\eta(x, D)f\|_{\dot{F}_{p,q}^s} \lesssim \Omega_N(b) \left(\|f\|_{\dot{F}_{p,q}^{s+m}} + \|f\|_{\dot{B}_{\infty,q}^{s-\frac{n}{p}+m}} \right). \quad (2.47)$$

Step 3 : Proof of $\eta(x, D)f \in \mathcal{S}'$ for $f \in \dot{F}_{p,q}^{s+m} \cap \dot{B}_{\infty,q}^{s-\frac{n}{p}+m}$. Let $\varphi \in \mathcal{S}$. We put $\tilde{Q}_j := \tilde{\gamma}(2^{-j}D)$,

where the function $\tilde{\gamma}$ is defined in the beginning of Step 1, and write

$$\varphi = S_0\varphi + \sum_{j \geq 1} Q_j \tilde{Q}_j \varphi.$$

It holds

$$\begin{aligned} |\langle \eta(x, D)f, \varphi \rangle| &= \left| \langle S_0(\eta(x, D)f), \varphi \rangle + \sum_{j \geq 1} \langle Q_j(\eta(x, D)f), \tilde{Q}_j \varphi \rangle \right| \\ &\leq \|S_0(\eta(x, D)f)\|_\infty \|\varphi\|_1 + \sum_{j \geq 1} \|Q_j(\eta(x, D)f)\|_\infty \|\tilde{Q}_j \varphi\|_1 \\ &\lesssim \|S_0(\eta(x, D)f)\|_\infty \|\varphi\|_1 + \sum_{j \geq 1} 2^{jn/p} \|Q_j(\eta(x, D)f)\|_p \|\tilde{Q}_j \varphi\|_1. \end{aligned} \quad (2.48)$$

First, we have

$$\|S_0(\eta(x, D)f)\|_\infty \lesssim \sum_{j \leq 0} 2^{j(n/p-s)} (2^{js} \|Q_j(\eta(x, D)f)\|_p). \quad (2.49)$$

Secondly, by corollary 1.2, $\|\tilde{Q}_j \varphi\|_1 \lesssim 2^{-jL} \zeta_M(\varphi)$ for any $L \in \mathbb{N}$ and some $M \in \mathbb{N}$, then if $0 < s < \frac{n}{p}$, we choose an integer L satisfying $L > \frac{n}{p} - s$, and obtain

$$\begin{aligned} |\langle \eta(x, D)f, \varphi \rangle| &\lesssim \left(\sup_{k \in \mathbb{Z}} 2^{ks} \|Q_k(\eta(x, D)f)\|_\infty \right) \left(\sum_{j \leq 0} 2^{j(n/p-s)} + \sum_{j \geq 1} 2^{j(n/p-s-L)} \right) \\ &\lesssim \|[\eta(x, D)f]_\infty\|_{\dot{B}_{p,\infty}^s} \\ &\lesssim \|[\eta(x, D)f]_\infty\|_{\dot{F}_{p,q}^s}. \end{aligned}$$

If $s = n/p$ and $p = 1$, then we use

$$\|\tilde{Q}_j \varphi\|_1 \lesssim \|\varphi\|_1,$$

for all $j \geq 1$, and from (2.48) it also holds

$$|\langle \eta(x, D)f, \varphi \rangle| \lesssim \|\varphi\|_1 \|[\eta(x, D)f]_\infty\|_{\dot{F}_{1,q}^n} \lesssim \|[\eta(x, D)f]_\infty\|_{\dot{F}_{1,q}^n}.$$

Step 4: Proof of $\eta(x, D)f \in \tilde{C}_0$ for $f \in \dot{F}_{p,q}^{s+m} \cap \dot{B}_{\infty,q}^{s-\frac{n}{p}+m}$ Let $\varphi \in \mathcal{S}$ and $\lambda > 0$.

- The case $s < n/p$. As in the previous step and using the same notations, it holds that

$$\begin{aligned}
 |\langle \eta(x, D)f(\cdot/\lambda), \varphi \rangle| &\leq \sum_{j \in \mathbb{Z}} \|\mathcal{Q}_j(\eta(x, D)f(\cdot/\lambda))\|_p \|\tilde{\mathcal{Q}}_j \varphi\|_1 \\
 &\lesssim \sum_{j \in \mathbb{Z}} 2^{sj} \|\mathcal{Q}_j(\eta(x, D)f(\cdot/\lambda))\|_p 2^{j(n/p-s)} \|\tilde{\mathcal{Q}}_j \varphi\|_1 \\
 &\lesssim \|[\eta(x, D)f(\cdot/\lambda)]_\infty\|_{\dot{B}_{p,\infty}^s} \sum_{j \in \mathbb{Z}} 2^{j(n/p-s)} \|\tilde{\mathcal{Q}}_j \varphi\|_1 \\
 &\lesssim \|[\eta(x, D)f(\cdot/\lambda)]_\infty\|_{\dot{F}_{p,q}^s} \left(\zeta_M(\varphi) \sum_{j \geq 1} 2^{j(n/p-s-L)} + \|\varphi\|_1 \sum_{j \leq 0} 2^{j(n/p-s)} \right),
 \end{aligned}$$

for some integer $L > n/p - s$ and some $M \in \mathbb{N}$. We have

$$|\langle \eta(x, D)f(\cdot/\lambda), \varphi \rangle| \lesssim \lambda^{n/p-s} \|f\|_\infty \|[\eta(x, D)f]_\infty\|_{\dot{F}_{p,q}^{s+m}} (\zeta_M(\varphi) + \|\varphi\|_1).$$

Then the result follows by taking $\lambda \rightarrow 0$.

- The case $s = n/p$ and $q = 1$. The embedding $\dot{F}_{p,1}^{n/p} \hookrightarrow \dot{F}_{\infty,1}^0$ implies that $[\eta(x, D)f]_\infty$ belongs to $\dot{F}_{\infty,1}^0$. Then we have

$$\left\| \sum_{j \in \mathbb{Z}} \mathcal{Q}_j(\eta(x, D)f) \right\|_\infty < \infty.$$

Now, let $\varepsilon > 0$ be fixed arbitrarily. We write, for a $J \in \mathbb{N}$,

$$|\langle \eta(x, D)f(\cdot/\lambda), \varphi \rangle| \lesssim \sum_{|j| > J} \|\mathcal{Q}_j(\eta(x, D)f)\|_\infty + \sum_{|j| \leq J} |\langle \mathcal{Q}_j(\eta(x, D)f)(\cdot/\lambda), \varphi \rangle|.$$

Choosing J such that the first term (i.e., with $\sum_{|j| > J} \dots$) is less than ε . For the second term (i.e., with $\sum_{|j| \leq J} \dots$), we apply the following lemma which is proved in [14]:

Lemma 2.34. *If a function $h \in L_\infty$ satisfies that $\text{supp } \hat{h}$ is a compact set in $\mathbb{R}^n \setminus \{0\}$, then $h \in \tilde{C}_0$.*

Thus

$$\lim_{\lambda \rightarrow 0} \sum_{|j| \leq J} |\langle \mathcal{Q}_j(\eta(x, D)f)(\cdot/\lambda), \varphi \rangle| = 0.$$

Summarizing, from step 2 through step 4, we have proved that the operator $\eta(x, D)$ is bounded from $f \in \tilde{F}_{p,q}^{s+m} \cap \tilde{B}_{\infty,q}^{s-\frac{n}{p}+m}$ into $\tilde{F}_{p,q}^s$.

Step 5 : Estimate of $\|\eta(x, D)f\|_{\dot{F}_{p,q}^{s+m}}$ for $f \in \tilde{F}_{p,q}^{s+m}$, (here $p \leq q$). In (2.43) we use Lemma 1.5 with $d := \frac{p}{q}$, we obtain the right-hund side is bounded by

$$\begin{aligned} & c_1 \Omega_N(b) \left\| \sum_{-2 \leq \ell \leq 1} \left\{ \sum_{j \in \mathbb{Z}} \left(2^{j(s+m)} (2^{(j-\ell)n} \mathcal{F}^{-1} \Upsilon_K(2^{j-\ell} \cdot)) * Q_j f \right)^p \right\}^{1/p} \right\|_p \\ & \lesssim \Omega_N(b) \sum_{-2 \leq \ell \leq 1} \left\{ \sum_{j \in \mathbb{Z}} \| 2^{j(s+m)} (2^{(j-\ell)n} \mathcal{F}^{-1} \Upsilon_K(2^{j-\ell} \cdot)) * Q_j f \|_p^p \right\}^{1/p}. \end{aligned}$$

From now, we continue as in above steps, i.e.Steps 3-4, we arrive at (2.47).We obtain the desired estimate using the embedding

$$\dot{F}_{p,q}^{s+m} \hookrightarrow \dot{B}_{p,p}^{s-\frac{n}{p}+m} \hookrightarrow \dot{B}_{p,q}^{s-\frac{n}{p}+m}.$$

Thus, the proof is finished. \square

Proof of Proposition 2.30. We prove that Arguing so as in Step 1 (i.e., (2.27)-(2.34)) to decompose $\kappa(x, \xi)$ as an elementary symbol. We introduce a radial and positive function $\tilde{\rho} \in \mathcal{D}$, supported by the ball $|\xi| \leq 7/4$ and satisfying $\tilde{\rho}\rho = \rho$. We also introduce a 2π -periodic function denoted by

$$v(x, \xi) := \sum_{K \in \mathbb{Z}^n} \kappa(x, \xi - 2\pi K).$$

Since $\kappa(x, \xi - 2\pi K)\tilde{\rho}(\xi) = 0$ if $K \neq 0$, then from the Fourier expansion of v , we have

$$v(x, \xi)\tilde{\rho}(\xi) = \kappa(x, \xi) = \tilde{\rho}(\xi) \sum_{K \in \mathbb{Z}^n} C_K(x) e^{iK \cdot \xi}. \quad (2.50)$$

If $\xi \in \mathbb{R}^n$ and $K \in \mathbb{Z}^n$ satisfy both $|\xi - 2\pi K| \leq 3/2$ and $-\pi \leq \xi_\ell \leq \pi$, $\ell = 1, \dots, n$, then $K = 0$ and $v(x, \xi) = \kappa(x, \xi)$. This yields

$$C_K(x) := (2\pi)^{-n} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} e^{-iK \cdot \xi} v(x, \xi) d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iK \cdot \xi} \kappa(x, \xi) d\xi.$$

We continue, for an even natural number $N \in \mathbb{N}$ ($N := 2M$), which will be chosen later on, we set

$$\chi_K(x) := (2\pi)^{-n} (1 + |K|^2)^{N/2} C_K(x),$$

i.e.,

$$\chi_K(x) = \int_{\mathbb{R}^n} e^{-iK \cdot \xi} (I - \Delta_\xi)^{N/2} \kappa(x, \xi) d\xi.$$

Combining the operator decomposition in (2.34) and from (2.50), we obtain the following

representation:

$$\kappa(x, D)f(x) = \sum_{K \in \mathbb{Z}^n} (1 + |K|^2)^{-N/2} \chi_K(x) \mathcal{F}^{-1}(\tilde{\rho} \hat{f})(x + K), \quad (2.51)$$

where in Lemma 2.33 can be applied with $h := \chi_K$ and

$$g := \mathcal{F}^{-1}(\tilde{\rho} \hat{f})(\cdot + K),$$

- $h \in L_\infty \cap \tilde{F}_{p,q}^{n/p-m}$, i.e., estimate of

$$\|\chi_K\|_\infty + \|[\chi_K]_\infty\|_{\tilde{F}_{p,q}^{n/p-m}}.$$

This can be done as the estimate of $(\chi_{K,j})_{j \in \mathbb{Z}}$

$$\|\chi_K\|_\infty \lesssim \Omega_N(b) \sum_{|\mu| \leq M} \sum_{\alpha + \eta = 2\mu} \int_{|\xi| \leq 3/2} |\rho^{(\eta)}(\xi)| (1 + |\xi|)^{m-|\alpha|} d\xi \lesssim \Omega_N(b),$$

for all $K \in \mathbb{Z}^n$, and

$$\|[\chi_K]_\infty\|_{\tilde{F}_{p,q}^{n/p-m}} \lesssim \Omega_N(b),$$

for all $K \in \mathbb{Z}^n$, respectively.

- $g \in C^\infty \cap \tilde{F}_{p,q}^{s+m}$. We first see the estimate of $\|[(\mathcal{F}^{-1} \tilde{\rho}) * f(\cdot + K)]_\infty\|_{\tilde{F}_{p,q}^{s+m}}$. By the Young inequality it holds

$$\|Q_j((\mathcal{F}^{-1} \tilde{\rho}) * f)\|_p \leq \|\mathcal{F}^{-1} \tilde{\rho}\|_1 \|Q_j f\|_p.$$

Consequently by using the fact that $\|\cdot\|_{\tilde{F}_{p,q}^{s+m}}$ is translation invariant, we obtain

$$\|[(\mathcal{F}^{-1} \tilde{\rho}) * f(\cdot + K)]_\infty\|_{\tilde{F}_{p,q}^{s+m}} = \|[(\mathcal{F}^{-1} \tilde{\rho}) * f]_\infty\|_{\tilde{F}_{p,q}^{s+m}} \lesssim \|f\|_\infty \|[\tilde{\rho}]_\infty\|_{\tilde{F}_{p,q}^{s+m}}.$$

Secondly, we prove that $g \in \tilde{C}_0$. Indeed, we apply the following assertion :

If $f \in \tilde{C}_0$, $\varphi \in \mathcal{S}$ and $a \in \mathbb{R}^n$ then

$$\langle f(\lambda^{-1}(\cdot) - a), \varphi \rangle = \langle f(\lambda^{-1}(\cdot)), \varphi(\cdot + \lambda a) \rangle \rightarrow 0 \quad \text{as } \lambda \rightarrow 0,$$

since $\varphi(\cdot + \lambda a)$ still belongs to \mathcal{S} . Thus by a simple calculation, for all $\varphi \in \mathcal{S}$, we have

$$\langle g(\lambda^{-1}(\cdot)), \varphi \rangle = \langle f(\lambda^{-1}(\cdot)), \mathcal{F}^{-1} \tilde{\rho} * \varphi(\lambda^{-1}(\cdot)) \rangle \quad \text{with } \varphi_\lambda := \varphi(\lambda(\cdot)),$$

and $\langle g(\lambda^{-1}(\cdot)), \varphi \rangle \rightarrow 0$ as $\lambda \rightarrow 0$ since $\mathcal{F}^{-1}\tilde{\rho} * \varphi(\lambda^{-1}(\cdot)) \in \mathcal{S}$.

Finally it is clear that g is a C^∞ function.

Applying now Lemma 2.33 to the expression (2.51), and choosing N an even integer greater than n , we obtain

$$\|[\kappa(x, D)f]_\infty\|_{\dot{F}_{p,q}^s} \lesssim \Omega_N(b) \| [f]_\infty \|_{\dot{F}_{p,q}^{s+m}} \sum_{K \in \mathbb{Z}^n} (1 + |K|^2)^{-N/2} \lesssim \Omega_N(b) \| [f]_\infty \|_{\dot{F}_{p,q}^{s+m}}.$$

We note that the proof of, both, $\kappa(x, D)f \in \mathcal{S}'$ and $\kappa(x, D)f \in \tilde{C}_0$ for all $f \in \tilde{F}_{p,q}^{s+m}$ is similar to the case of the operator $\eta(x, D)$. □

Remark 2.35. Proposition 2.29, can be written in homogeneous version. Namely, the estimates (2.22) and (2.23) become

$$\|[\eta(x, D)f]_\infty\|_{\dot{F}_{p,q}^s} \leq c \dot{\Omega}_N(b) \| [f]_\infty \|_{\dot{F}_{p,q}^{s+m}}, \quad (2.52)$$

$$\|[\eta(x, D)f]_\infty\|_{\dot{F}_{p,q}^{s+m}} \leq c \dot{\Omega}_N(b) (\| [f]_\infty \|_{\dot{F}_{p,q}^s} + \| [f]_\infty \|_{\dot{B}_{\infty,q}^{s-n/p+m}}), \quad (2.53)$$

respectively. To obtain these inequalities we only write (2.25) as following :

$$\eta_j(x, \xi) := 2^{-mj} \gamma(2^{-|j|}\xi) b(x, 2^{j-|j|}\xi), \quad j \in \mathbb{Z}, \forall x, \xi \in \mathbb{R}^n.$$

Now, we do not use assertions similar to Propositions 2.28 and 2.30. For more details about this we refer to our paper [33, page.459].

Proof of Theorem 2.25. The proof can be obtained by using of Propositions 2.29 in homogeneous version given from Remark 2.35 as in [33, Theorem 1.1]. □

Proof of Theorem 2.26. The proof is similar to that of Theorem 2.25, see [33, Theorem 1.2].

□

Chapter 3

Characterization by Differences of realized type spaces

In this chapter, we provide a systematic study of Besov-type and Triebel-Lizorkin-type function spaces through three main aspects: definitions, embeddings, and realizations. It establishes discrete difference-based characterizations with equivalent quasi-norms.

3.1 Besov and Triebel-Lizorkin type spaces

We do not go into details of the Besov and Triebel-Lizorkin type spaces, homogeneous or inhomogeneous, we limit ourselves to the needed facts of these spaces. For this reason we first quote [44], [58] and [59].

3.1.1 Definitions and some properties

Definition 3.1. We define the dyadic cubes. Let $\eta \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$, where $\eta := (\eta_1, \eta_2, \dots, \eta_n)$. The dyadic cube of side length 2^{-k} and sides parallel to the coordinate axes is the subset of \mathbb{R}^n defined by

$$P_{k,\eta} := \left\{ x \in \mathbb{R}^n : 2^{-k}\eta_j \leq x_j < 2^{-k}(\eta_j + 1), \quad j = 1, 2, \dots, n \right\}.$$

That means,

$$P_{k,\eta} := 2^{-k}([0, 1]^n + \eta).$$

Definition 3.2. Let $\tau \geq 0$ and $p \in]0, \infty]$. The Lebesgue-type space L_p^τ is the set of functions $f \in L_{\text{loc}}^p$ such that

$$\|f\|_{L_p^\tau} = \sup_{k \leq 0} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\int_{P_{k,\eta}} |f(x)|^p dx \right)^{1/p} < \infty.$$

Remark 3.3. • If we take $\tau = 0$, then $L_p^\tau = L_p$.

- If $\tau < 0$, then $L_p^\tau = \{0\}$.
- $L_p^\tau \hookrightarrow \mathcal{S}'$ if $p \geq 1$.
- If $\tau_1 \leq \tau_2$, then $L_p^{\tau_1} \hookrightarrow L_p^{\tau_2}$.

Definition 3.4. (i) Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, \infty[$. The homogeneous Besov-type spaces $\dot{B}_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\sum_{j>k} 2^{jsq} \|Q_j f\|_{L_p(P_{k,\eta})}^q \right)^{1/q} < \infty.$$

(ii) Let $s, \tau \in \mathbb{R}$, $q \in]0, \infty[$ and $p \in]0, \infty[$. The homogeneous Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left\| \left(\sum_{j>k} 2^{jsq} |Q_j f|^q \right)^{1/q} \right\|_{L_p(P_{k,\eta})} < \infty.$$

Definition 3.5. (i) Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, \infty[$. The inhomogeneous Besov-type space $B_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{n\tau k} \left(\sum_{j>k_+} 2^{jsq} \|Q_j f\|_{L_p(P_{k,\eta})}^q \right)^{1/q} < \infty, \text{ here } Q_0 := S_0.$$

(ii) Let $s, \tau \in \mathbb{R}$, $q \in]0, \infty[$ and $p \in]0, \infty[$. The inhomogeneous Triebel-Lizorkin-type space $F_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{F_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left\| \left(\sum_{j>k_+} 2^{jsq} |Q_j f|^q \right)^{1/q} \right\|_{L_p(P_{k,\eta})} < \infty, \text{ here } Q_0 := S_0.$$

3.1.2 Embeddings

Proposition 3.6. Let $s \in \mathbb{R}$, $\tau \geq 0$ and $0 < p, q \leq \infty$ ($p < \infty$ in the F -space), then one has:

- (i) $\mathcal{S}_\infty \hookrightarrow \dot{A}_{p,q}^{s,\tau} \hookrightarrow \mathcal{S}'_\infty$ and $\mathcal{S} \hookrightarrow A_{p,q}^{s,\tau} \hookrightarrow \mathcal{S}'$,
- (ii) if $0 < q_1 \leq q_2 \leq \infty$, then $\dot{A}_{p,q_1}^{s,\tau} \hookrightarrow \dot{A}_{p,q_2}^{s,\tau}$,
- (iii) $\dot{B}_{p,\min(p,q)}^{s,\tau} \hookrightarrow \dot{F}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p,\max(p,q)}^{s,\tau}$.

Proof. For (i) we refer to [57, Proposition 3.1 and 3.4] and [59, Proposition 2.3]. For (ii) we apply the fact that $\ell_{q_1} \hookrightarrow \ell_{q_2}$. The assertion (iii) follows from the following inequality:

$$\left(\sum_{j \geq k} \|g_j\|_p^v \right)^{1/v} \leq \left\| \left(\sum_{j \geq k} |g_j|^q \right)^{1/q} \right\|_p \leq \left(\sum_{j \geq k} \|g_j\|_p^u \right)^{1/u},$$

where $u = \min(p, q)$ and $v = \max(p, q)$. □

Proposition 3.7 (1.21). *Let $s, s_1, s_2 \in \mathbb{R}$, $0 \leq \tau < \infty$ and $0 < q \leq \infty$.*

(i) *Let $s_1 > s_2$ and $0 < p_1 < p_2 \leq \infty$ ($p_1 < \infty, p_2 < \infty$ in the F -space). Then*

$$\dot{A}_{p_1, q}^{s_1, \tau} \hookrightarrow \dot{A}_{p_2, q}^{s_2, \tau} \quad \text{if} \quad s_1 - n/p_1 = s_2 - n/p_2.$$

(ii) $\dot{A}_{p, q}^{s, \tau} \hookrightarrow \dot{B}_{\infty, \infty}^{s+n\tau-n/p}$.

(iii) *If ($\tau > 1/p$ and $0 < q < \infty$) or ($\tau = 1/p$ and $q = \infty$) then $\dot{A}_{p, q}^{s, \tau} = \dot{B}_{\infty, \infty}^{s+n\tau-n/p}$.*

Proof. For (i) See [57, Proposition 3.3]. For (ii) and (iii) See [58, Theorem 1]. □

Proposition 3.8. *Let $p, q \in]0, \infty]$, $\tau \geq 0$ and $s > (n/p - n)_+$. Then $A_{p, q}^{s, \tau} := L_p^\tau \cap \dot{A}_{p, q}^{s, \tau}$, where this intersection means that $f \in L_p^\tau$ and $[f]_\infty \in \dot{A}_{p, q}^{s, \tau}$. In addition, the expression*

$$\|f\|_{L_p^\tau} + \|[f]_\infty\|_{\dot{A}_{p, q}^{s, \tau}}$$

and $\|f\|_{\dot{A}_{p, q}^{s, \tau}}$ are equivalent quasi-norms in $A_{p, q}^{s, \tau}$.

Proof. See [7, Theorem 3.1]. □

Proposition 3.9. *There exist two constants $c_1, c_2 > 0$ such that*

$$c_1 \|f\|_{\dot{A}_{p, q}^{s, \tau}} \leq \lambda^{s+n\tau-n/p} \|h_\lambda f\|_{\dot{A}_{p, q}^{s, \tau}} \leq c_2 \|f\|_{\dot{A}_{p, q}^{s, \tau}}$$

holds, for all $f \in \dot{A}_{p, q}^{s, \tau}$ and all $\lambda > 0$. Moreover, the expression $\lambda^{s+n\tau-n/p} \|h_\lambda f\|_{\dot{A}_{p, q}^{s, \tau}}$ defines an equivalent seminorm in $\dot{A}_{p, q}^{s, \tau}$.

Proof. See [6, Proposition 2] for the proof in B -case ; the F -case is similar . We omit details. □

Proposition 3.10. *Let $m \in \mathbb{N}$. A function f of \mathcal{S}'_∞ belongs to $\dot{A}_{p, q}^{s, \tau}$ if and only if its derivatives $f^{(\alpha)}$ belongs to $\dot{A}_{p, q}^{s-m, \tau}$ for all $|\alpha| = m$. Moreover, the expression*

$$\sum_{|\alpha|=m} \|f^{(\alpha)}\|_{\dot{A}_{p, q}^{s-m, \tau}},$$

is an equivalent quasi-norm in $\dot{A}_{p, q}^{s, \tau}$.

Proof. See [55, Theorem 1.5]. □

Proposition 3.11. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty[$. Let a, b be real numbers such that $0 < a < b$ and $(u_j)_{j \in \mathbb{Z}}$ be a sequence in \mathcal{S}' such that*

- \hat{u}_j is supported by the annulus $a2^j \leq |\xi| \leq b2^j$.
- $A := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{n\tau k} \left(\sum_{j \geq k} 2^{jsq} \|u_j\|_{L_p(P_{k,\eta})}^q \right)^{1/q} < \infty$ in the B -space,
- $A := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{n\tau k} \left\| \left(\sum_{j \geq k} 2^{jsq} |u_j|^q \right)^{1/q} \right\|_{L_p(P_{k,\eta})} < \infty$ in the F -space.

(i) *Then the series $\sum_{j \in \mathbb{Z}} u_j$ converges in \mathcal{S}'_∞ to a limit u satisfying $\|u\|_{\dot{A}_{p,q}^{s,\tau}} \leq cA$, where the constant c depends only on n, s, τ, p, q, a and b .*

(ii) *If in addition $s > (n/p - n)_+$, the same result holds for $a = 0$.*

3.2 Realization of homogeneous Besov-type spaces

In this section, we present one of our main contributions concerning the realization of homogeneous Besov-type spaces:

$$\mu = \begin{cases} ([s + n\tau - n/p] + 1)_+ & \text{if } s + n\tau - n/p \notin \mathbb{N}_0 \text{ or } q > 1, \\ s + n\tau - n/p & \text{if } s + n\tau - n/p \in \mathbb{N}_0 \text{ and } 0 < q \leq 1. \end{cases} \quad (3.1)$$

In case

$$\mu = 0 \quad \text{we set} \quad \mu_0 := \mu. \quad (3.2)$$

Theorem 3.12. *Let $\tau, s \in \mathbb{R}$ and $0 < p, q \leq +\infty$. For $f \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, the series $\sum_{j \in \mathbb{Z}} Q_j f$ converges in \mathcal{S}'_μ . Denote by $\sigma_\mu(f)$ its sum which belongs to \mathcal{S}'_μ .*

Then the mapping $\sigma_\mu : \dot{B}_{p,q}^{s,\tau} \rightarrow \mathcal{S}'_\mu$ defines a realization invariant under translations and dilations of $\dot{B}_{p,q}^{s,\tau}$ in \mathcal{S}'_μ . Moreover, $\sigma_\mu(f)$ is the unique representative of f satisfying $\partial^\alpha \sigma_\mu f \in \tilde{C}_0$ for all $|\alpha| = \mu$.

Proof. See [6, Theorem 1]. □

Proposition 3.13. *Let $s \in \mathbb{R}$, $p, q \in]0, +\infty]$ and $\tau \geq 0$. The realization operator σ_μ satisfies:*

1. *Linearity:* $\sigma_\mu(\lambda f + g) = \lambda \sigma_\mu(f) + \sigma_\mu(g)$ for all $f, g \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$.
2. *Continuity:* $\|\sigma_\mu(f)\|_{\dot{B}_{p,q}^{s,\tau}} \leq C \|f\|_{\dot{B}_{p,q}^{s,\tau}}$.
3. *Translation commuting:* $\sigma_\mu(\tau_h f) = \tau_h \sigma_\mu(f)$ for all $h \in \mathbb{R}^n$.

4. *Dilation invariance:* $\sigma_\mu(h_\lambda f) = h_\lambda \sigma_\mu(f)$ for all $\lambda > 0$.

Sketch proof of Theorem 3.12. The proof consists of three main steps:

1. **Convergence in \mathcal{S}'_μ :** Using the Littlewood-Paley decomposition and the definition of μ , we show that the partial sums $S_N = \sum_{|j| \leq N} Q_j f$ form a Cauchy sequence in \mathcal{S}'_μ .

2. **Invariance properties:**

- For translation invariance: $\tau_h \sigma_\mu(f) = \sigma_\mu(\tau_h f)$ follows from the commutativity of Q_j with τ_h .
- For dilation invariance: $h_\lambda \sigma_\mu(f) = \sigma_\mu(h_\lambda f)$ uses the homogeneity properties of Q_j .

3. **Uniqueness:** The condition $\partial^\alpha \sigma_\mu f \in \tilde{C}_0$ for $|\alpha| = \mu$ ensures uniqueness. □

Remark 3.14. *The realization σ_μ provides a canonical representative for each equivalence class in $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ that preserves all the fundamental operations :*

- *Differentiation:* $\partial^\alpha \sigma_\mu(f) = \sigma_{\mu-|\alpha|}(\partial^\alpha f)$.
- *Multiplication by test functions.*
- *Fourier multipliers.*

3.3 Realization of homogeneous Triebel-Lizorkin-type spaces

For the Triebel-Lizorkin-type case:

$$\mu = \begin{cases} ([s + n\tau - n/p] + 1)_+ & \text{if } s + n\tau - n/p \notin \mathbb{N}_0 \text{ or } p > 1, \\ s + n\tau - n/p & \text{if } s + n\tau - n/p \in \mathbb{N}_0 \text{ and } 0 < p \leq 1. \end{cases} \quad (3.3)$$

In case

$$\mu = 0 \quad \text{we set } \mu_0 := \mu. \quad (3.4)$$

Theorem 3.15. *Let $\tau, s \in \mathbb{R}$, $0 < p < +\infty$ and $0 < q \leq +\infty$. For $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, the series $\sum_{j \in \mathbb{Z}} Q_j f$ converges in \mathcal{S}'_μ . Denote by $\sigma_\mu(f)$ its sum which belongs to \mathcal{S}'_μ .*

Then the mapping $\sigma_\mu : \dot{F}_{p,q}^{s,\tau} \rightarrow \mathcal{S}'_\mu$ defines a realization invariant under translations and dilations of $\dot{F}_{p,q}^{s,\tau}$ in \mathcal{S}'_μ , and $\sigma_\mu(f)$ is the unique representative of f satisfying $\partial^\alpha \sigma_\mu f \in \tilde{C}_0$ for all $|\alpha| = \mu$.

Proof. See [5, Theorem 4.3.1]. □

Corollary 3.16. *Let σ_μ be the mapping defined in Theorems 3.12 and 3.15 . Then the realized space $\sigma(\dot{A}_{p,q}^{s,\tau})$ coincides with the set of $f \in \mathcal{S}'_\mu$ such that $[f]_\mu \in \dot{A}_{p,q}^{s,\tau}$ and $f^{(\alpha)} \in \tilde{C}_0$ for all $|\alpha| = \mu$. This set will be denoted by $\tilde{A}_{p,q}^{s,\tau}$.*

The space $\tilde{A}_{p,q}^{s,\tau}$ is equipped with the quasi-norm:

$$\|f\|_{\tilde{A}_{p,q}^{s,\tau}} = \|[f]_\mu\|_{\dot{A}_{p,q}^{s,\tau}} .$$

Remark 3.17. *The constructions of realizations of the space $\dot{B}_{p,q}^s$ in \mathcal{S}' were given by G. Bourdaud in [13],[14] . These constructions remain valid in the spaces $\dot{A}_{p,q}^{s,\tau}$. Indeed, for any $f \in \dot{A}_{p,q}^{s,\tau}$ we have:*

1. *For $s+n\tau < n/p$ or $s+n\tau = n/p$ and $q \leq 1$ ($p \leq 1$ in the F-case):*

$$\sigma_{\mu,1}(f) := \sum_{k \in \mathbb{Z}} Q_k f .$$

2. *For $s+n\tau - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s+n\tau - n/p \in \mathbb{N}$ and $q \leq 1$ ($p \leq 1$ in the F-case):*

$$\sigma_{\mu,2}(f) := \sum_{k \in \mathbb{Z}} \left(Q_k f - \sum_{|\alpha| < \mu} (Q_k f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right) .$$

3. *For $s+n\tau - n/p \in \mathbb{N}_0$ and $q > 1$ ($p \geq 1$ in the F-case):*

$$\sigma_{\mu,3}(f) := \sum_{k > 0} Q_k f + \sum_{k \leq 0} \left(Q_k f - \sum_{|\alpha| < \mu} (Q_k f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right) .$$

Proposition 3.18. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty[$. If $\mu \geq 1$ then $\tilde{F}_{p,q}^{s,\tau}$ is the set of distributions f satisfying $[f]_\mu \in \dot{F}_{p,q}^{s,\tau}$ and:*

- (i) $f^{(\alpha)} \in C^{\mu-1}$ for all $|\alpha| = \mu$,
- (ii) $\langle f^{(\alpha)}, \gamma \rangle = 0$ for all $|\alpha| = \mu - 1$,
- (iii) $f \in \mathcal{S}'_{\mu-1}$.

Proposition 3.19. *Let $s, \tau \in \mathbb{R}$, $p \in]0, +\infty[$ and $q \in]0, +\infty[$ satisfying either $s+n\tau - n/p \in \mathbb{R}_+ \setminus \mathbb{N}_0$ or $(s+n\tau - n/p \in \mathbb{N}$ and $0 < p \leq 1)$. Then every element f of $\tilde{F}_{p,q}^{s,\tau}$ satisfies $f \in \mathcal{S}'$, $[f]_\mu \in \dot{F}_{p,q}^{s,\tau}$ and the following properties:*

- (i) f is a $\tilde{C}^{\mu-1}$ function ,

(ii) $f^{(\alpha)}(0) = 0$ for all $|\alpha| \leq \mu - 1$,

(iii) $f^{(\alpha)} \in \tilde{C}_0$ for all $|\alpha| = \mu$.

3.4 Characterization by differences of $\tilde{A}_{p,q}^s$

The characterization by differences of $\tilde{A}_{p,q}^s$ has been studied in e.g.[41], we recall its formulation.

3.4.1 Difference operators

For any function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, we define the difference operator:

$$\Delta_h f(x) := f(x+h) - f(x), \quad x, h \in \mathbb{R}^n,$$

with the convention:

$$\Delta_h^1 := \Delta_h.$$

$$\Delta_h^{m+1} = \Delta_h^1 \circ \Delta_h^m, \quad m = 1, 2, \dots$$

We then obtain:

$$\begin{aligned} \Delta_h^2 f(x) &= f(x+2h) - 2f(x+h) + f(x) \\ &= \binom{2}{0} f(x+2h) - 2 \binom{2}{1} f(x+h) + \binom{2}{2} f(x) \\ \Delta_h^3 f(x) &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x) \\ &= \binom{3}{0} f(x+3h) - \binom{3}{1} f(x+2h) + \binom{3}{2} f(x+h) - \binom{3}{3} f(x). \end{aligned}$$

Therefore, the general form is:

$$\Delta_h^m f(x) = \sum_{k=0}^m (-1)^k \binom{m}{k} f(x + (m-k)h).$$

Proposition 3.20. *If $\Delta_h^m f = 0$ for all $h \in \mathbb{R}^n$, then f is a polynomial of degree less than m .*

Proof. It suffices to consider the case $n = 1$ and $f \in C^m$.

For $m = 1$, if $\Delta_h f(0) = 0$ for all h , then we have

$$f(h) = f(0), \quad \forall h \in \mathbb{R}^n, \quad \text{hence } f \text{ is constant.}$$

For $m = 2$, by differentiation with respect to h we get

$$f'(x+2h) - f'(x+h) = 0;$$

setting $x = -h$ gives $f'(h) = f'(0)$, thus f is a polynomial of degree 1.

For $m = 3$, we differentiate $\Delta_h^3 f(x) = 0$ with respect to h :

$$f'(x+3h) - 2f'(x+2h) + f'(x+h) = 0. \quad \dots (1)$$

In (1), set $x = -h$ and differentiate with respect to h :

$$f''(2h) - f''(h) = 0. \quad \dots (2)$$

Differentiate (1) again with respect to h :

$$3f''(x+3h) - 4f''(x+2h) + f''(x+h) = 0 \quad \dots (3)$$

Setting $x = -h$ in (3) gives:

$$3f''(2h) - 4f''(h) + f''(0) = 0,$$

and from (2) we obtain

$$-f''(2h) + f''(0) = 0 \quad \text{i.e. } f''(h) = f''(0).$$

Thus f'' is constant and f is a polynomial of degree 2. This process can be continued to establish the general result by induction. □

Example 3.21. *An interesting example to apply Proposition 3.20.*

Let $u_m(t) := t^m \quad m = 0, 1, 2, \dots$, then we have

$$\Delta_\theta^m u_m(t) = m! \theta^m, \quad \text{for all } t \in \mathbb{R}, \text{ and all } \theta \in \mathbb{R}. \quad (3.5)$$

Indeed, clearly we have,

$$\begin{aligned} (t + \theta)^m - t^m &= m\theta t^{m-1} + g(t) \\ &= m\theta u_{m-1}(t) + g(t), \end{aligned}$$

where $g(t)$ is polynomial of degree $m - 2$. By Proposition 3.20

$$\Delta_\theta^{m-1} g(t) = 0,$$

then we obtain

$$\Delta_\theta^m u_m(t) = m\theta \Delta_\theta^{m-1} u_{m-1}(t).$$

Now, by induction on m we obtain (3.5).

3.4.2 Equivalent quasi-norms in $\tilde{A}_{p,q}^s$

For $s > 0$ the spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$ are defined by Littlewood-Paley decomposition; then we added the following comment:

$$\begin{aligned} \mathcal{M}_{p,q}^{s,m}(f) &:= \left(\int_{\mathbb{R}^n} (|h|^{-s} \|\Delta_h^m f\|_p)^q \frac{dh}{|h|^n} \right)^{1/q}, \\ \mathcal{N}_{p,q}^{s,m}(f) &:= \left\| \left(\int_{\mathbb{R}^n} (|h|^{-s} |\Delta_h^m f(\cdot)|)^q \frac{dh}{|h|^n} \right)^{1/q} \right\|_p. \end{aligned}$$

The two definitions are *not equivalent* in $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$, since

$$\mathcal{M}_{p,q}^{s,m}(f) = \mathcal{N}_{p,q}^{s,m}(f) = \infty,$$

for any polynomial f of degree m (recall that $\Delta_h^m x^\alpha$ is a nonzero constant for $|\alpha| = m$). Consequently $\mathcal{M}_{p,q}^{s,m}(f)$, for example, depends only on the equivalent class of f modulo polynomials on \mathbb{R}^n of degree less than m .

Definition 3.22. For $1 \leq u \leq \infty$, $s > 0$ and $m \in \mathbb{N}$, and for any measurable functions f on \mathbb{R}^n ,

we put

$$\begin{aligned}\widetilde{\mathcal{M}}_{p,q}^{s,m}(f) &:= \left(\int_0^\infty t^{-sq} \sup_{|h|\leq t} \|\Delta_h^m f\|_p^q \frac{dt}{t} \right)^{1/q}, \\ \widetilde{\mathcal{N}}_{p,q}^{s,m}(f) &:= \left\| \left(\int_0^\infty t^{-sq} \sup_{|h|\leq t} |\Delta_h^m f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_p, \\ \mathcal{M}_{p,q}^{s,m,u}(f) &:= \left(\int_0^\infty t^{-sq} \left\| \left\{ t^{-n} \int_{|h|\leq t} |\Delta_h^m f(\cdot)|^u dh \right\}^{1/u} \right\|_p^q \frac{dt}{t} \right)^{1/q}, \\ \mathcal{N}_{p,q}^{s,m,u}(f) &:= \left\| \left(\int_0^\infty t^{-sq} \left\{ t^{-n} \int_{|h|\leq t} |\Delta_h^m f(\cdot)|^u dh \right\}^{q/u} \frac{dt}{t} \right)^{1/q} \right\|_p.\end{aligned}$$

Remark 3.23.

The real number μ will play an important role in our investigations (as small as possible), it is defined, for any system of real numbers $\{s, n, p, q\}$, by

- $\mu := ([s - n/p] + 1)_+$ if $s - n/p \notin \mathbb{N}_0$ or $q > 1$ in the case of the $\dot{B}_{p,q}^s(\mathbb{R}^n)$ -spaces ($p > 1$ in the case of the $\dot{F}_{p,q}^s(\mathbb{R}^n)$ -spaces),
- $\mu := s - n/p$ if $s - n/p \in \mathbb{N}_0$ and $q \leq 1$ in the case of the $\dot{B}_{p,q}^s(\mathbb{R}^n)$ -spaces ($p \leq 1$ in the case of the $\dot{F}_{p,q}^s(\mathbb{R}^n)$ -spaces).
- $\check{B}_{p,q}^s(\mathbb{R}^n)$ denotes the realized Besov space, and it is the translation and dilation invariant realization of $\dot{B}_{p,q}^s(\mathbb{R}^n)$ in $\mathcal{S}'_\mu(\mathbb{R}^n)$, (in the same way the realized Triebel-Lizorkin space $\check{F}_{p,q}^s(\mathbb{R}^n)$ is denoted).

Theorem 3.24. Let $0 < p, q \leq \infty$ and $s > 0$. Let $m \in \mathbb{N}$.

- (i) If $n\left(\frac{1}{p} - 1\right) < s < m$, then either $\mathcal{M}_{p,q}^{s,m}$ or $\widetilde{\mathcal{M}}_{p,q}^{s,m}$ defines an equivalent quasi-norm in $\check{B}_{p,q}^s(\mathbb{R}^n)$.
- (ii) Let $1 \leq u \leq \infty$. If

$$n\left(\frac{1}{p} - \frac{1}{u}\right) < s < m,$$

then $\mathcal{M}_{p,q}^{s,m,u}$ defines an equivalent quasi-norm in $\check{B}_{p,q}^s(\mathbb{R}^n)$.

Proof. See [41]. □

Theorem 3.25. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > 0$. Let $m \in \mathbb{N}$.

- (i) If $\frac{n}{\min(p,q)} < s < m$, then either $\mathcal{N}_{p,q}^{s,m}$ or $\widetilde{\mathcal{N}}_{p,q}^{s,m}$ defines an equivalent quasi-norm in $\check{F}_{p,q}^s(\mathbb{R}^n)$.

(ii) Let $1 \leq u \leq \infty$. If

$$n \max \left(\frac{1}{p} - \frac{1}{u}, \frac{1}{q} - \frac{1}{u} \right) < s < m,$$

then $\mathcal{N}_{p,q}^{s,m,u}$ defines an equivalent quasi-norm in $\tilde{F}_{p,q}^s(\mathbb{R}^n)$.

Proof. See [41]. □

3.5 Characterization by differences of $\tilde{A}_{p,q}^{s,\tau}$

3.5.1 The principal theorem

We begin by the following definition

Definition 3.26. For $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $0 < p < \infty$, define

$$a_t(x) := t^{-n} \int_{t/2 \leq |h| < t} |\Delta_h^m f(x)| dh,$$

$$\mathcal{M}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left\{ \int_0^{2^{1-k}} t^{-sq} \left(\int_{P_{k,\eta}} [a_t(x)]^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q},$$

and

$$\mathcal{N}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left\{ \int_0^{2^{1-k}} \left(\int_{P_{k,\eta}} t^{-sq} [a_t(x)]^q \frac{dt}{t} \right)^{p/q} dx \right\}^{1/p}.$$

We now give our contribution for the differences characterization.

Theorem 3.27. Let $s > 0$, $0 < p < \infty$, $0 < q \leq \infty$, $\tau \geq 0$, and $m \in \mathbb{N}$ such that

$$s < \min(m, m + n(1/p - \tau)), \text{ and } m \geq [s + n\tau], \tag{3.6}$$

then $\mathcal{M}_{p,q}^{s,\tau,m}(\cdot)$, respectively $\mathcal{N}_{p,q}^{s,\tau,m}(\cdot)$, defines an equivalent quasi-norm in $\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, respectively $\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Remark 3.28. The condition (3.6) is due to the method in the proof of [59, Lemma 8.3].

3.5.2 Necessary lemma

The following assertion is the key of the argument of our result.

Lemma 3.29. *Let $s > 0$, $0 < p < \infty$, $0 < q \leq \infty$, $\tau \geq 0$ and $m \in \mathbb{N}$ such that (3.6) is satisfied.*

(i) *There exists a constant $c > 0$ such that for every regular tempered distribution f , such that $\mathcal{M}_{p,q}^{s,\tau,m}(f) < +\infty$, (resp. $\mathcal{N}_{p,q}^{s,\tau,m}(f) < +\infty$) it holds, and $[f]_\infty \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, (resp. $[f]_\infty \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$)*

$$\|[f]\|_{\dot{B}_{p,q}^{s,\tau}} \leq c \mathcal{M}_{p,q}^{s,\tau,m}(f),$$

respectively

$$\|[f]\|_{\dot{F}_{p,q}^{s,\tau}} \leq c \mathcal{N}_{p,q}^{s,\tau,m}(f).$$

(ii) *There exists a constants $c > 0$ such that for every $f \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, (resp. $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$), there exists regular tempered distribution g satisfying*

$$[g]_\infty = f \quad \text{and} \quad \mathcal{M}_{p,q}^{s,\tau,m}(g) \leq c \|f\|_{\dot{B}_{p,q}^{s,\tau}},$$

respectively

$$[g]_\infty = f \quad \text{and} \quad \mathcal{N}_{p,q}^{s,\tau,m}(g) \leq c \|f\|_{\dot{F}_{p,q}^{s,\tau}}.$$

Proof. See [58, Theorem 6.5]. □

3.5.3 Proof of the main theorem 3.27

Proof. We prove the existence of two positive constants c_1 and c_2 , such that, it holds

$$c_1 \mathcal{M}_{p,q}^{s,\tau,m}(f) \leq \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}} \leq c_2 \mathcal{M}_{p,q}^{s,\tau,m}(f), \quad (\forall f \in \tilde{\dot{B}}_{p,q}^{s,\tau}(\mathbb{R}^n)).$$

We will only prove the first inequality, since the second one can be obtained similarly to the proof of [58, Lemma 6.5/(i)].

Let $f \in \tilde{\dot{B}}_{p,q}^{s,\tau}$, there exists a regular tempered distribution g satisfying:

- $g - f \in \mathcal{P}_\infty$,
- $g^{(\alpha)} \in C_0$ for all $|\alpha| = \mu$,
- $\mathcal{M}_{p,q}^{s,\tau,m}(g) \leq c \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}$.

Using [58, Lemma 6.5/(ii)] we obtain

$$\mathcal{M}_{p,q}^{s,\tau,m}(g) \leq c \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}.$$

Now since $\Delta_h^m(g) = \Delta_h^m(f)$, the above estimate becomes

$$\mathcal{M}_{p,q}^{s,\tau,m}(f) \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}.$$

The case of Triebel-Lizorkin can be obtained similarly. □

Chapter 4

Functional characterization of homogeneous Triebel-Lizorkin-type spaces

Under certain conditions on s , p and q , several function characteristics of BMO are valid on realized homogeneous Besov $\tilde{B}_{p,q}^s$ and Triebel-Lizorkin spaces $\tilde{F}_{p,q}^s$ [27], and Besov-type space $\tilde{B}_{p,q}^{s,\tau}$ see [25]. We propose to extend the results of [27] to the realized homogeneous Triebel-Lizorkin type spaces $\tilde{F}_{p,q}^{s,\tau}$.

We recall that μ is defined in (2.1), (3.1), (3.3).

4.1 Preparations

4.1.1 The BMO space

For the BMO space, we do not go into details, however we quote [11], [25], [31], [50], [51].

Definition 4.1. *The space of functions of bounded mean oscillation (abbreviated as BMO), introduced by F. John and L. Nirenberg, see [30] consists of all locally integrable functions $f \in L_{loc}^1(\mathbb{R}^n)$ for which the following quantity is finite:*

$$\|f\|_{\text{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - m_Q f| dx < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with finite measure. Here, $m_Q f$ denotes the average of f over Q , defined by:

$$m_Q f := \frac{1}{|Q|} \int_Q f(x) dx.$$

Remark 4.2. • The space $(\text{BMO}, \|\cdot\|_{\text{BMO}})$ is a vector space equipped with a seminorm, which vanishes on constant functions.

• The BMO space is invariant under translations and dilations. That is,

$$\|\tau_a f\|_{\text{BMO}} = \|f\|_{\text{BMO}} \quad \text{and} \quad \|h_\lambda f\|_{\text{BMO}} = \|f\|_{\text{BMO}},$$

where $\tau_a f(x) = f(x - a)$ and $h_\lambda f(x) = f(\lambda x)$ for $a \in \mathbb{R}^n$ and $\lambda > 0$.

4.1.2 Inequalities in BMO space

To clarify, we briefly remember some estimates that are already known in BMO

Proposition 4.3. *If $f \in \text{BMO}(\mathbb{R}^n)$, then the following integral is finite:*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} dx < \infty.$$

Proof. See [23], [51]. □

Proposition 4.4. *A function $f \in L_1^{\text{loc}}$ belongs to BMO if, and only if*

$$(I) \int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} dx < \infty,$$

$$(II) \sup_{B_k} |B_k|^{-1} \int_{B_k} \sum_{j \geq k} |Q_j f(x)|^2 dx < \infty, \text{ where the supremum is taken over all finite balls } B_k \text{ in } \mathbb{R}^n \text{ of radius } 2^{-k} \text{ (} k \in \mathbb{Z} \text{).}$$

Proof. See [11, p 47]. □

Proposition 4.5. *A function $f \in \text{BMO}$ if, and only if*

$$(I) \int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} dx < \infty,$$

$$(II) \sup_{y \in \mathbb{R}^n, h > 0} h^{-n} \int_{|x-y| < h} \int_0^h t |\nabla P_t f(x)|^2 dt dx < \infty, \text{ where } P_t f \text{ is the Poisson integral of } f.$$

Proof. See [23]. □

Remark 4.6. *In [25] we find other functional estimates in the link with BMO spaces.*

4.2 Previously obtained results on $\tilde{A}_{p,q}^s$ and $\tilde{B}_{p,q}^{s,\tau}$

The results of this section can be found in [25], [26] and [27].

4.2.1 Results on $\tilde{A}_{p,q}^s$

Spaces of Besov and Triebel-Lizorkin are defined modulo polynomials, as discussed in Chapter 2. In particular, a function f satisfies

$$\|f\|_{\dot{B}_{p,q}^s} = \|f\|_{\dot{F}_{p,q}^s} = 0$$

if and only if f is a polynomial on \mathbb{R}^n .

Given a function $f \in \tilde{A}_{p,q}^s$ for which there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$, we define the associated representative function by

$$\tilde{f} := f + u_f.$$

For the definition of μ_0 , see (3.2), (3.4).

Theorem 4.7 ([27]). *Let $0 < p, q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s - \frac{n}{p})_+$. If $f \in \dot{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x)|}{1 + |x|^{n+d}} dx \leq c_1 \| [f]_\infty \|_{\dot{A}_{p,q}^s},$$

holds; the constant c_1 is independent of f . In particular, if either $s < \frac{n}{p}$ ($p < \infty$ in F -spaces) or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces); here, realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x - x_0|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{A}_{p,q}^s},$$

holds for all $x_0 \in \mathbb{R}^n$; the constant c_2 is independent of f and x_0 .

Proof. See [27, Theorem. 2.1]. □

Remark 4.8. *The parameter constraints are: $p < \infty$ if $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces) and $s > 0$.*

Theorem 4.9 ([27]). *Let s, p, q and d be identical to those in Theorem 4.7.*

(i) *If $f \in \tilde{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,1)}\tilde{f}|}{1 + |x|^{n+d}} dx \leq c \| [f]_\infty \|_{\dot{A}_{p,q}^s},$$

holds; the constant c is independent of f .

(ii) Assume that $p < \infty$ in F -spaces and that:

- either $s < \frac{n}{p}$, or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces),
- either $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$, or $s - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces),

where realizations commute with dilations. If $f \in \tilde{A}_{p,q}^s$, then there exists a polynomial $u_f \in \mathcal{P}_{\mu_0}$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,\lambda)}\tilde{f}|}{\lambda^{n+d} + |x|^{n+d}} dx \leq c\lambda^{s-d-n/p} \| [f]_\infty \|_{\tilde{A}_{p,q}^s},$$

holds for all $\lambda > 0$; the constant c is independent of f and λ .

(iii) In particular, if either $s < \frac{n}{p}$ ($p < \infty$ in F -spaces), or $s = \frac{n}{p}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces), where realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(x_0,\lambda)}f|}{\lambda^{n+d} + |x - x_0|^{n+d}} dx \leq c\lambda^{s-d-n/p} \| [f]_\infty \|_{\tilde{A}_{p,q}^s}$$

holds for all $f \in \tilde{A}_{p,q}^s$, all $\lambda > 0$, and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f , λ , and x_0 .

Proof. See [27, Theorem. 2.4]. □

For modulo constants ($\mu_0 = 1$, that is, $u_f = \text{constant}$), we can extend claim (iii) of Theorem 4.9 to the cases where either $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$, or $s - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$ in B -spaces ($p \leq 1$ in F -spaces), or $s - \frac{n}{p} \in \mathbb{N}_0$ and $q > 1$ in B -spaces ($p > 1$ in F -spaces).

4.2.2 Formulation of theorems in Besov-type spaces

The previously obtained results have been applied and extended the realized homogeneous Besov-type space $\tilde{B}_{p,q}^{s,\tau}$. And focus more on the case where $0 < \tau \leq 1/p$. However, if $\tau > 1/p$ and $0 < q < \infty$ or if $\tau = 1/p$ and $q = \infty$, the homogeneous Besov-type space coincides with the homogeneous Besov space

$$\dot{B}_{p,q}^{s,\tau} = \dot{B}_{\infty,\infty}^{s+n\tau-n/p}.$$

Theorem 4.10 ([25]). *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s + n\tau - \frac{n}{p})_+$.*

If $f \in \tilde{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x)|}{1 + |x|^{n+d}} dx \leq c_1 \| [f]_\infty \|_{\tilde{B}_{p,q}^{s,\tau}}, \quad (4.1)$$

holds; the constant c_1 is independent of f .

In particular, if either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $q \leq 1$; here we have realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x - x_0|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (4.2)$$

holds for all $x_0 \in \mathbb{R}^n$; the constant c_2 is independent of f and x_0 .

Proof. See [25]. □

Theorem 4.11 ([25]). *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s + n\tau - \frac{n}{p})_+$.*

(i) *If $f \in \tilde{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,1)}\tilde{f}|}{1 + |x|^{n+d}} dx \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (4.3)$$

holds; the constant c is independent of f and u_f .

(ii) *Assume that either $s + n\tau < \frac{n}{p}$ either $s + n\tau = \frac{n}{p}$ and $q \leq 1$ either $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s + n\tau - \frac{n}{p} \in \mathbb{N}$ and $q \leq 1$; here we have realizations commute with dilations. If $f \in \tilde{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,\lambda)}\tilde{f}|}{\lambda^{n+d} + |x|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (4.4)$$

holds for all $\lambda > 0$; the constant c is independent of f , u_f and λ .

(iii) *Assume that either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $q \leq 1$; here we have realizations commute with translations. If $f \in \tilde{B}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(x_0,\lambda)}f|}{\lambda^{n+d} + |x - x_0|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}, \quad (4.5)$$

holds for all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f , u_f , λ and x_0 .

Proof. See [25]. □

Remark 4.12. *In the left-hand side of the inequalities (4.1), (4.3), and (4.4), it is possible to replace \int with $\inf_{u \in \mathcal{P}_\infty} \int$ and \tilde{f} with $f + u$, respectively. This substitution is in accordance with the concept presented in [53].*

The following two propositions can also be found in [25], [26] or [27]; we recall them since we think they are of an interesting independent.

Proposition 4.13. *Let $g \in \mathcal{P}_\mu$, μ is defined in (3.1) if $d > (s + n\tau - \frac{n}{p})_+$ then*

$$\int_{\mathbb{R}^n} \frac{|g(x)|}{1 + |x|^{n+d}} dx < \infty. \quad (4.6)$$

Proposition 4.14. *Let $0 < p < \infty$ and $N \in \mathbb{N}_0$. Then there exists a constant $c > 0$, such that*

$$|f(x)| \leq c \left(R^n \int_{\mathbb{R}^n} |f(y)|^p (1 + R|x-y|)^{-Np} dy \right)^{1/p},$$

holds, for all $R > 0$, all $f \in \mathcal{S}'$ such that $\text{supp } \hat{f} \subset \{\xi : |\xi| \leq R\}$ and all $x \in \mathbb{R}^n$.

4.3 Formulation of theorems in Triebel-Lizorkin-type spaces

We are now able to give our contribution in this subject. Here also we set:

$$\tilde{f} := f + u_f, \text{ where } u_f \in \mathcal{P}_\mu.$$

Theorem 4.15. *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s + n\tau - \frac{n}{p})_+$. If $f \in \dot{F}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x)|}{1 + |x|^{n+d}} dx \leq c_1 \| [f]_\infty \|_{\dot{F}_{p,q}^{s,\tau}}, \quad (4.7)$$

holds; the constant c_1 is independent of f . In particular, if either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $p \leq 1$; here we have realizations commute with translations, then the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x - x_0|^{n+d}} dx \leq c_2 \| [f]_\infty \|_{\dot{F}_{p,q}^{s,\tau}}, \quad (4.8)$$

holds for all $x_0 \in \mathbb{R}^n$; the constant c_2 is independent of f and x_0 .

Theorem 4.16. *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \tau \leq \frac{1}{p}$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s + n\tau - \frac{n}{p})_+$.*

(i) If $f \in \tilde{F}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,1)}\tilde{f}|}{1 + |x|^{n+d}} dx \leq c \| [f]_\infty \|_{\tilde{F}_{p,q}^{s,\tau}}, \quad (4.9)$$

holds; the constant c is independent of f and u_f .

(ii) Assume that either $s + n\tau < \frac{n}{p}$ either $s + n\tau = \frac{n}{p}$ and $p \leq 1$ either $s + n\tau - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s + n\tau - \frac{n}{p} \in \mathbb{N}$ and $p \leq 1$; here we have realizations commute with dilations. If $f \in \tilde{F}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x) - m_{Q(0,\lambda)}\tilde{f}|}{\lambda^{n+d} + |x|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [f]_\infty \|_{\tilde{F}_{p,q}^{s,\tau}}, \quad (4.10)$$

holds for all $\lambda > 0$; the constant c is independent of f , u_f and λ .

(iii) Assume that either $s + n\tau < \frac{n}{p}$ or $s + n\tau = \frac{n}{p}$ and $p \leq 1$; here we have realizations commute with translations. If $f \in \tilde{F}_{p,q}^{s,\tau}$, then there exists a polynomial $u_f \in \mathcal{P}_\mu$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|f(x) - m_{Q(x_0,\lambda)}f|}{\lambda^{n+d} + |x - x_0|^{n+d}} dx \leq c \lambda^{s+n\tau-d-n/p} \| [f]_\infty \|_{\tilde{F}_{p,q}^{s,\tau}}, \quad (4.11)$$

holds for all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f , u_f , λ and x_0 .

Proof. We present the idea of the proof, since it is similar to arguments given in [25]. Then we use the following :

- (i) The estimate given in Proposition 3.6 (iii).
- (ii) The embedding $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow \dot{F}_{p_1,r}^{s_1,\tau}(\mathbb{R}^n) \hookrightarrow \dot{B}_{p_1,p_1}^{s_1,\tau}(\mathbb{R}^n)$ with a chosen parameters s_1 , p_1 and r .
- (iii) The estimates [4.1]–[4.5] with s_1 , p_1 , instead of s , and $p = q = p_1$.

Indeed, we have

- $s > (\frac{n}{p} - n)_+$
- $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow \dot{F}_{p_1,r}^{s_1,\tau}(\mathbb{R}^n)$ with $p < p_1$ and $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$.
- $\dot{F}_{p,r}^{s_1,\tau}(\mathbb{R}^n) \hookrightarrow \dot{B}_{p_1,p_1}^{s_1,\tau}(\mathbb{R}^n)$ with $r \leq p_1$. Then p_1 is such that

$$s_1 = s - \frac{n}{p} + \frac{n}{p_1} > (\frac{n}{p_1} - n)_+. \quad (4.12)$$

Then it suffices to choose p_1 such that 4.12 is satisfied, hence we set $p_1 \geq \max(1, p)$ and we have

$$\frac{1}{p} - \frac{s}{n} < \frac{1}{p_1} \leq \frac{1}{\max(1, p)}.$$

Now it suffices to apply theorems 4.10 and 4.11 with the space $\dot{B}_{p_1, p_1}^{s_1, \tau}(\mathbb{R}^n)$.

□

4.4 Application to homogeneous Sobolev spaces

We can extend our results to homogeneous Sobolev spaces $\dot{W}_p^k(\mathbb{R}^n)$, defined as the set of all f such that $f^{(\alpha)} \in L_p(\mathbb{R}^n)$, $(|\alpha| = k)$, and

$$\|f\|_{\dot{W}_p^k} := \sum_{|\alpha|=k} \|f^{(\alpha)}\|_p < \infty.$$

The quotient space $\dot{W}_p^k / \mathcal{P}_k$ is a Banach space in \mathcal{S}'_k ; here $1 \leq p \leq \infty$.

Proposition 4.17. *Let $1 < p < \infty$, $k \in \mathbb{N}$ be such that $0 < k < \frac{n}{p}$. If $f \in \tilde{F}_{p,2}^k(\mathbb{R}^n)$*

then $[f]_k \in \dot{W}_p^k(\mathbb{R}^n) / \mathcal{P}_k$, where $[f]_k = \{f + u, u \in \mathcal{P}_k\}$.

Proof. See [38, Propostion 5.4].

□

Now we have the following assertions, in homogeneous Sobolev spaces:

Theorem 4.18. *Let $1 < p < \infty$, and $k \in \mathbb{N}$ be such that*

$$p < n, \quad 1 \leq k < \frac{n}{p}, \tag{4.13}$$

then

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+d}} dx \leq c_1 \|[f]_k\|_{\dot{W}_p^k},$$

holds; the constant c_1 is independent of f .

Proof. It's Theorem 4.7 with $\mu_0 = 0$.

□

Remark 4.19. *Theorem 4.9 can be written in this way, by taking $s = k$ satisfies de condition (4.13), and obtain a result in $\dot{W}_p^k(\mathbb{R}^n)$.*

We omit details.

Chapter 5

Annexe

We formulate some open questions related to our work.

5.1 In connection with functional calculus

- (I) If we can prove Theorem 2.25 in $\tilde{F}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ here (2.17) and (2.18) became as following:

$$\text{either } s+n\tau < \frac{n}{p}, \quad \text{or } s+n\tau = \frac{n}{p} \quad \text{and } p=1; \quad (5.1)$$

and

$$\text{either } s+m+n\tau < \frac{n}{p}, \quad \text{or } s+m+n\tau = \frac{n}{p} \quad \text{and } p=1. \quad (5.2)$$

This observation is obtained since [6], [7].

- (II) In Chapter 3 we proved characterization is a differences of $\tilde{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ using the functionals

$$\mathcal{M}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left\{ \int_0^{2^{1-k}} t^{-sq} \left(\int_{P_{k,\eta}} [a_t(x)]^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q},$$

in B -case and

$$\mathcal{N}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left\{ \int_0^{2^{1-k}} \left(\int_{P_{k,\eta}} t^{-sq} [a_t(x)]^q \frac{dt}{t} \right)^{p/q} dx \right\}^{1/p},$$

in F case.

We propose to extend this property to the following functionals in the future:

$$\widetilde{\mathcal{M}}_{p,q}^{\dot{s},\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\int_0^{2^{1-k}} t^{-sq} \left(\int_{P_{k,\eta}} [a_t(x)]^p dx \right)^{q/p} \frac{dt}{t} \right)^{1/q},$$

$$\widetilde{\mathcal{N}}_{p,q}^{\dot{s},\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\int_{P_{k,\eta}} \left(\int_0^{2^{1-k}} t^{-sq} [a_t(x)]^q \frac{dt}{t} \right)^{p/q} dx \right)^{1/p},$$

$$\widetilde{\mathcal{M}}_{p,q}^{\dot{s},\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\int_0^{2^{1-k}} t^{-sq} \times \left(\int_{P_{k,\eta}} t^{-n} \int_{t/2 \leq |h| < t} |\Delta_h^m f(x)|^p dh dx \right)^{q/p} \frac{dt}{t} \right)^{1/q},$$

$$\widetilde{\mathcal{N}}_{p,q}^{\dot{s},\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\int_{P_{k,\eta}} \left(\int_0^{2^{1-k}} t^{-sq} t^{-n} \times \int_{t/2 \leq |h| < t} |\Delta_h^m f(x)|^q dh \frac{dt}{t} \right)^{p/q} dx \right)^{1/p},$$

and

$$\widetilde{\mathcal{M}}_{p,q}^{\widehat{s},\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\int_0^{2^{1-k}} t^{-sq} \sup_{t/2 \leq |h| < t} \left(\int_{P_{k,\eta}} |\Delta_h^m f(x)|^p dx \right)^{q/p} \frac{dt}{t} \right)^{1/q},$$

$$\widetilde{\mathcal{N}}_{p,q}^{\widehat{s},\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left\{ \int_{P_{k,\eta}} \left(\int_0^{2^{1-k}} t^{-sq} \sup_{t/2 \leq |h| < t} |\Delta_h^m f(x)|^q \frac{dt}{t} \right)^{p/q} dx \right\}^{1/p}.$$

With convention \mathcal{M} for Besov case and \mathcal{N} for Triebel-Lizorkin case.

- (III) The results in section 4.3 are given in the same way of [26] and [27], then the idea is to change the left- hand functional in all estimations,

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x)|}{1 + |x|^{n+d}} dx \leq c_1 \| [f]_\infty \|_{F_{p,q}^{s,\tau}}.$$

5.2 Related to the composition operators

In the connection with the composition theory we propose to study the nonlinear operator

$$T_f : g \longmapsto f \circ g ,$$

in the following spaces $\tilde{A}_{p,q}^s(\mathbb{R}^n)$, $\tilde{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We note that for n dimensional we have not results at now for this subject.

We recall that the composition operator theory consists in the full characterization of

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

such that the operator T_f takes a function space E into itself. This problem has been studied in several papers, e.g.:

- The Sobolev case W_p^m, H_p^s in [10], [15] and [32].
- The Besov and Triebel-Lizorkin spaces [17], [18], [19], [20], [36], [39], [40], [42], [43], [46].
- The homogeneous Besov spaces [16].

Finally, in [46] we find some results for T_f on Besov-type spaces intersection with $W_\infty^1(\mathbb{R}^n)$.

Conclusion and perspectives

In this work, we study the pseudo-differential operators, establishing their boundedness on realized homogeneous Triebel spaces.

We present characterizations by differences of realized-type spaces, including embeddings and properties of homogeneous Besov-type and Triebel-Lizorkin-type spaces. The case $\tilde{A}_{p,q}^s(\mathbb{R}^n)$ has been studied in [41]. We extend some characterizations to homogeneous Besov-type and Triebel-Lizorkin-type spaces. The results are a continuation of the statements previously obtained in [25], [27].

In the end, an annex contains some questions about all the work.

- We propose that we can prove Theorem 2.25, in $\tilde{F}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$.
- We proved characterization is a differences of $\tilde{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ using the functionals

$$\mathcal{M}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left\{ \int_0^{2^{1-k}} t^{-sq} \left(\int_{P_{k,\eta}} [a_t(x)]^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q},$$

in B -case and

$$\mathcal{N}_{p,q}^{s,\tau,m}(f) := \sup_{k \in \mathbb{Z}, \eta \in \mathbb{Z}^n} 2^{kn\tau} \left\{ \int_0^{2^{1-k}} \left(\int_{P_{k,\eta}} t^{-sq} [a_t(x)]^q \frac{dt}{t} \right)^{p/q} dx \right\}^{1/p},$$

in F case. We propose to extend this property to the other functionals in the future.

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دراستنا تتمحور حول الفضاءات المتجانسة من النوع

$$\|f(\lambda \cdot)\| \sim \lambda^r \|f\|, \quad \forall \lambda > 0, \quad r \in \mathbb{R},$$

حيث قمنا بتعريف الفضاءات المحققة المرتبطة بفضاءات بيزوف $\dot{B}_{p,q}^s(\mathbb{R}^n)$ (من نوع بيزوف $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) وفضاءات تريبل-ليزوركين $\dot{F}_{p,q}^s(\mathbb{R}^n)$ (نوع تريبل-ليزوركين $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$) وسوف يُشار إليها بـ $\tilde{B}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) و $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$) على التوالي حيث $r := s - n/p$ (and $r := s + n\tau - n/p$). هذه الفضاءات هي فضاءات فرعية للتوزيعات المعتدلة المنتظمة. كما درسنا استمرارية المؤثرات الشبه تفاضلية على $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ وقمنا بإثبات الخصائص الدالية المعرفة بواسطة مؤثرات الفرق للفضاءات المحققة المرتبطة بفضاءات من نوع بيزوف المتجانسة ومن نوع تريبل-ليزوركين المتجانسة. الكلمات المفتاحية: تحليل ليتلود-بالي، فضاءات بيزوف، فضاءات ليزوركين-تريبل، المؤثرات الشبه تفاضلية، التحقيقات.

Abstract

In the context of homogeneous spaces of the type

$$\|f(\lambda \cdot)\| \sim \lambda^r \|f\|, \quad \forall \lambda > 0, \quad r \in \mathbb{R},$$

we define the realized spaces associated with the Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ (Besov-type $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) and Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$ (Triebel-Lizorkin-type $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$) they will be denoted by $\tilde{B}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) and $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$), respectively; here $r := s - n/p$ (and $r := s + n\tau - n/p$). These spaces are subspaces of regular tempered distributions. We study the continuity of pseudo-differential operators on $\tilde{F}_{p,q}^s(\mathbb{R}^n)$. We establish functional characterizations by difference operators of the realized spaces associated with homogeneous Besov-type spaces, Triebel-Lizorkin-type spaces.

Keywords: Littlewood-Paley decomposition, Besov spaces, Triebel-Lizorkin spaces, Pseudo-

Résumé

Dans le cadre des espaces homogènes du type

$$\|f(\lambda \cdot)\| \sim \lambda^r \|f\|, \quad \forall \lambda > 0, \quad r \in \mathbb{R},$$

nous définissons les espaces réalisés associés aux espaces de Besov $\dot{B}_{p,q}^s(\mathbb{R}^n)$ (type de Besov $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) et Triebel-Lizorkin $\dot{F}_{p,q}^s(\mathbb{R}^n)$ (type de Triebel-Lizorkin $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$); ils seront notés $\tilde{B}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$) et $\tilde{F}_{p,q}^s(\mathbb{R}^n)$ ($\tilde{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$) respectivement; ici $r := s - n/p$ (et $r := s + n\tau - n/p$). Ces espaces sont des sous-espaces de distributions tempérées régulières. On étudie la continuité des opérateurs pseudo-différentiels sur $\tilde{F}_{p,q}^s(\mathbb{R}^n)$. On établit des caractérisations fonctionnelles par des opérateurs de différences des espaces réalisés associés aux espaces de type de Besov homogènes et de type de Triebel-Lizorkin homogènes .

Mots clés: Décomposition de Littlewood-Paley, Espaces de Besov, Espaces de Triebel-Lizorkin, Opérateurs pseudo-différentiels, Réalisations.