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The nonrelativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of generalized inverse quadratic Yukawa potential as a model

by

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Dedication-and-Acknowledgment

Dedication

To the ones that I am indebted to for eternity: my parents, my sisters Imane and Ahlem ,my brothers Billal and Ali, Ramadan and Ayoub who have encouraged me all the way and whose encouragement, to this day, has made sure that I give it all it takes to finish that which I have started.

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Chapter 1

Introduction

In modern natural science, which includes cosmology, atomic physics, molecular science, materials science, and other fields, quantum mechanics is an essential research tool. The Schrödinger equation that introduced in 1926 by the Austrian physicist Erwin Schrödinger is the foundation of the non-relativistic quantum mechanics at low energies [1]. While Klein-Gordon and Dirac equations are the foundation of the relativistic quantum mechanics at high energies for particles with spin-zero and fermionic particles for spin-1/2. One definition of the Schrödinger equation is a second-order linear differential equation. It has the same general form as the Dirac equation. As both of them relate to first-order time differentials, unlike the Klein-Gordon equation, which includes second-order time differentials. Exact solutions to the Schrödinger equation can be found in limited physical problems, such as the cases of the harmonic oscillator and the hydrogen atom. In relativistic and non-relativistic quantum mechanics, a number of techniques have recently been introduced and used to solve wave equations with a specific provided solvable potential such as factorization method, functional analysis approach, Nikiforov-Uvarov method, exact quantization rule, and asymptotic iteration method are just a few of the numerous methods that are available [2, 3, 4, 5, 6, 7, 8, 9, 10]. It is known to researchers and specialists in physics that quantum mechanics, known in the literature, is based on the following postulates [11, 12]:

$$\begin{cases} [x_i, p_j] = \imath\hbar\delta_{ij}, \\ [x_i, x_j] = [p_i, p_j] = 0. \end{cases}$$

Quantum mechanics on non-commutative space was first proposed by Heisenberg in 1930 [13] and then developed by Snyder in late 1947 [14]. The proposal of extended quantum mechanics came as a possible solution to many physical problems that non-relativistic and relativistic quantum mechanics were unable to find solutions to the divergence problem in the standard model, string theory and quantum gravity [15, 16, 17, 18].

We will reserve this study to obtain a master's degree in theoretical physics from Mohammed Boudiaf University in M'sila to study the generalized inverse quadratic Yukawa potential in the context of nonrelativistic non-commutative quantum mechanics symmetries for the promotion 2023-2024 because non-commutative quantum mechanics includes larger physical symmetries than the quantum mechanics known in the literature.

This master memory is organized as follows. In chapter one, the non-commutative quantum mechanics is represented. In chapter two, the Schrödinger equation is revised under the generalized inverse quadratic Yukawa potential. In chapter three, we study the effect of phase-space non-commutativity deformation on the generalized inverse quadratic Yukawa potential.

Part I

The non-commutative phase-space formalism

1.1 Introduction

This chapter will cover the postulates and hypotheses that define the quantum and physical structures of the non-commutative phase-space and its physical structures. The fundamental ideas that will be covered are as follows:

-A standard quantum structure representation,

-The postulates of the non-commutative phase-space,

-The Star product and its characteristics,

-The generalized inverse quadratic Yukawa potential and its applications using Bopp's Shift method.

1.2 Review of structure of ordinary quantum mechanics

The beginning of the nineteenth century witnessed a radical in the traditional view of light. The prevailing belief was that light had only wave behavior. The wave nature of light was well established among scientists of that period and those before it, and they demonstrated this through well-known interference and diffraction experiments. The fundamental change in knowledge of other behavior of light began convincingly through experiments on the photoelectric effect. In 1900, Planck quantifies the energy of light $E_\gamma = h\nu$, which consider the beginning of quantum physics, here $h \approx 6,6262.10^{-34}$ js. Currently, ordinary quantum mechanics is formulated on the commutative space of the coordinates of variable and the canonical moment of hermetic operators (x_i, p_i), as follows [19, 20]:

$$\left\{ \begin{array}{c} [x_i, p_j] = \imath\hbar\delta_{ij} \\ [x_i, x_j] = 0 \\ [p_i, p_j] = 0 \end{array} \right. \tag{1.1}$$

Here, $\hbar$ and δ_{ij} are the reduced Plank constant $\frac{\hbar}{2\pi}$, and the usual Kronecker symbol, respectively. The above algebra can be generalized to the Dirac and interaction pictures as follows:

$$\left\{ \begin{array}{c} \left[x_i^d\left(t\right), p_j^d\left(t\right)\right] = \left[x_i^I\left(t\right), p_j^I\left(t\right)\right] = \imath\hbar\delta_{ij} \\ \left[x_i^d(t), x_j^d(t)\right] = \left[x_i^I(t), x_j^I(t)\right] = 0 \\ \left[p_i^d\left(t\right), p_j^d\left(t\right)\right] = \left[p_i^I\left(t\right), p_j^I\left(t\right)\right] = 0 \end{array} \right. \tag{1.2}$$

where the usual canonical coordinates (x_i, p_i) and the corresponding time-dependent $x_i\left(t\right)$ and $p_i(t)$ are determined from the projection relations:

$$\left\{ \begin{array}{c} x_i^d\left(t\right) = \exp\left(\frac{\imath}{\hbar}H\left(t-t_0\right)\right)x_i\exp\left(-\frac{\imath}{\hbar}H\left(t-t_0\right)\right) \\ p_i^d\left(t\right) = \exp\left(\frac{\imath}{\hbar}H\left(t-t_0\right)\right)p_i\exp\left(-\frac{\imath}{\hbar}H\left(t-t_0\right)\right) \\ x_i^I\left(t\right) = \exp\left(\frac{\imath}{\hbar}H_0\left(t-t_0\right)\right)x_i\exp\left(-\frac{\imath}{\hbar}H_0\left(t-t_0\right)\right) \\ p_i^I\left(t\right) = \exp\left(\frac{\imath}{\hbar}H_0\left(t-t_0\right)\right)p_i\exp\left(-\frac{\imath}{\hbar}H_0\left(t-t_0\right)\right) \end{array} \right. \tag{1.3}$$

Here, $\{x_i(t)\}$, $\{p_i(t)\}$ and H/H_0 are (total/free) Hermitian operators on a Hilbert space of physical states, which, each, satisfy the Heisenberg equation of motions. We get the following:

$$\begin{cases} \frac{dx_i}{dt} = \frac{\imath}{\hbar}\left[H, x_i(t)\right] \\ \frac{dp_i}{dt} = \frac{\imath}{\hbar}\left[H, p_i(t)\right] \end{cases} \tag{1.4}$$

Both related concepts relating to energy E and impulsion p_i are satisfied by the quantization procedure:

$$\begin{cases} E \rightarrow i\hbar\frac{\partial}{\partial t} \\ p_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x_i} \end{cases} \tag{1.5}$$

It is well known that the classical energy E of a particle of mass m_0 subjected to the external forces produced by a potential $V(\vec{r}, t)$, in a classical mechanic is given by:

$$E = \frac{\overrightarrow{p}^2}{2m_0} + V(\vec{r}, t) \tag{1.6}$$

The quantization process in Equation (1.5) made it possible to derive the Shrödinger equation, which is well-known in the literature's framework for quantum mechanics:

$$\left(-\frac{\hbar^2}{2m}\Delta + V(\vec{r}, t)\right)\psi\left(\overrightarrow{r}, t\right) = i\hbar\frac{\partial\psi\left(\overrightarrow{r}, t\right)}{\partial t} \tag{1.7}$$

Here Δ is the well known Laplacian operator in spherical coordinates $\overrightarrow{r}\,(r, \theta, \varphi)$ as follows:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\varphi^2} \tag{1.8}$$

which can be expressed in Cartesian coordinates $\overrightarrow{r}\,(x, y, z)$ as follows:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.9}$$

while $\psi\left(\overrightarrow{r}, t\right)$ denoting the complex wave function. The probability of finding the particle at time t in an elementary volumes $\left(d^3r, d^3p\right)$, rounding the point r as follows:

$$dp = \begin{cases} \left|\psi\left(\overrightarrow{r}, t\right)\right|^2 d^3r : \text{ In the configuration space} \\ \left|\psi\left(\overrightarrow{p}, t\right)\right|^2 d^3p : \text{ In the momentum space} \end{cases} \tag{1.10}$$

where $\left(d^3r, d^3p\right)$ equals $\left(r^2\sin\theta d\theta d\varphi dr, p^2\sin\theta d\theta d\varphi dp\right)$, respectively.

1.3 Noncommutative phase-space

One of the most essential aspects of quantum physics is dealing with non-commuting operators, specifically the commutation relations between positions x_i and corresponding momenta p_i. Noncommutative quantum mechanics (NCQM) symmetries imply that operators do not commute; for example, consider a situation in which the coordinates and moment operators are non-commutative. In 1930, a hot topic was solving the infinity problem in the newly found quantum field theory (QFT). Heisenberg was the first to propose that non-commutativity be extended to coordinate systems. Then, the concept of NCQM was extended to generalize the usual conception of space-time, in which the non-commutativity of some normally commutative variables is assumed, leading to the formation of different Lie algebras. Connes in 1980 revived the ideas of non-commutative geometry while Woronowicz and Drinfel'd, were generalized the notion of a differential structure to the non-commutative setting [21, 22, 23, 24]. The non-commutativity phase-space concept was characterized as the simplest commutation relation that satisfied the following algebra [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]:

$$\begin{cases} [\hat{x}_i(t) \overset{*}{,} \hat{x}_j(t)] = [\hat{x}_i \overset{*}{,} \hat{x}_j] = \imath\hbar_{eff}\theta_{ij} \\ [\hat{p}_i(t) \overset{*}{,} \hat{p}_j(t)] = [\hat{p}_i \overset{*}{,} \hat{p}_j] = \imath\bar{\theta}_{ij} \end{cases} \tag{1.11}$$

where $\left(\theta_{ij}, \bar{\theta}_{ij}\right) = -\left(\theta_{ji}, \bar{\theta}_{ji}\right) = \epsilon_{ij}\left(\theta, \bar{\theta}\right)$ are constants anti-symmetric tensors of dimensions $[x]^2$ and $[p]^2$, $(\left(\theta, \bar{\theta}\right)$ are the NC parameters and ϵ_{ij} is just an antisymmetric number ($\epsilon_{ij} = -\epsilon_{ji} = 1$ with $i \neq j$ and $\epsilon_{jj} = 0$) and $\hbar_{eff} = \hbar\left(1 + \frac{\theta\bar{\theta}}{4\hbar^2}\right)$ is the effective constant of Planck. The non-commutative coordinates $(\hat{x}_i, \hat{p}_i)$ take the form:

$$\begin{cases} x_i \to \hat{x}_i = f(x_i, p_i) \\ p_i \to \hat{p}_i = f(x_i, p_i) \end{cases} \tag{1.12}$$

In this work, we are interred by the phase-phase has three dimensions $N = 3$, therefore the indices take the values $(i, j = 1, 3)$. In this particular case, the rules of canonical commutations become:

$$\begin{cases} [\hat{x}_1(t), \hat{p}_2(t)] = [\hat{x}_1, \hat{p}_2] = 0 \\ [\hat{x}_1(t), \hat{p}_3(t)] = [\hat{x}_1, \hat{p}_3] = 0 \\ [\hat{x}_2(t), \hat{p}_3(t)] = [\hat{x}_2, \hat{p}_3] = 0 \\ [\hat{x}_1(t), \hat{x}_2(t)] = [\hat{x}_1, \hat{x}_2] = i\theta_{12} \\ [\hat{x}_1(t), \hat{x}_3(t)] = [\hat{x}_1, \hat{x}_3] = i\theta_{13} \\ [\hat{x}_1(t), \hat{x}_3(t)] = [\hat{x}_2, \hat{x}_3] = i\theta_{23} \end{cases} \tag{1.13}$$

and

$$\begin{cases} \left[\hat{x}_1(t), \hat{p}_2(t)\right] = \left[\hat{x}_1, \hat{p}_2\right] = i\hbar_{eff} \\ \left[\hat{x}_1(t), \hat{p}_3(t)\right] = \left[\hat{x}_1, \hat{p}_3\right] = i\hbar_{eff} \\ \left[\hat{x}_2(t), \hat{p}_3(t)\right] = \left[\hat{x}_2, \hat{p}_3\right] = i\hbar_{eff} \\ \left[\hat{p}_1(t), \hat{p}_2(t)\right] = \left[\hat{p}_1, \hat{p}_2\right] = i\bar{\theta}_{12} \\ \left[\hat{p}_1(t), \hat{p}_3(t)\right] = \left[\hat{p}_1, \hat{p}_3\right] = i\bar{\theta}_{13} \\ \left[\hat{p}_2(t), \hat{p}_3(t)\right] = \left[\hat{p}_2, \hat{p}_3\right] = i\bar{\theta}_{23} \end{cases} \tag{1.14}$$

1.4 Weyl's quantization:

The fundamentals of quantum physics inspired many of the broad principles behind non-commutative geometry. Weyl proposed an elegant formulation for mapping quantum operators to classical functions of phase-space variables within the framework of canonical quantification. This method establishes a systematic approach to modeling non-commutative spaces in general and examining ancient ideas based on them [38]. Weyl quantization is a technique for describing quantum physics using classical mechanics' phase space. It is a rule that allows a quantum operator to be associated with a classical function that is dependent on phase space variables. The Weyl quantification also applies to commutative relations in a general form. Consider a $f(x,p)$ and $g(x,p)$ a general two functions, their product in the notion of non-commutative phase-space can be expressed as a new product called the star product or the Weyl-Moyal star product defined on phase space,

$$\begin{gathered} f(x,p) * g(x,p) = \\ f(x,p)\, g(x,p) + \frac{i}{2}\sum_m \theta^{mn} \frac{\partial}{\partial x^m} f(x,p) \frac{\partial}{\partial x^n} + \\ + \frac{i}{2}\sum_m \bar{\theta}^{mn} \frac{\partial}{\partial p^m} f(x,p) \frac{\partial g(x,p)}{\partial p^n} + O\left(\theta^2, \bar{\theta}^2\right) \end{gathered} \tag{1.15}$$

The formalism of the star product initiated by Weyl and Wigner to allow a description of quantum mechanics in terms of phase space, is articulated not around non-commuting operators, as in the operational approach, but around the deformation of the product between the phase space variables. We will see how this formalism can be used in the context of non-commutative phase-space symmetries.

1.5 Properties of the star product

Weyl and Wigner developed the star product formalism, which makes it possible to describe quantum mechanics in terms of phase space. The star product's characteristics are as follows: [39, 40, 41, 42, 43, 44]:

-When $\left(\theta, \bar{\theta}\right) = (0,0)$

$$\underset{\left(\theta,\bar{\theta}\right)\to(0,0)}{Lim} \left(f(x) * g(x)\right) = f(x)g(x) \tag{1.16}$$

-The star product between exponential :

$$e^{\imath kx} * e^{\imath qx} = e^{\imath(k+q)x} e^{-\frac{\imath}{2}(k \curlywedge q)} \qquad \text{with } k \curlywedge q = k^{\imath} q^{j} \theta_{ij} \tag{1.17}$$

-Not commutative property:

$$f(x,p) * g(x,p) \neq g(x,p) * f(x,p) \tag{1.18}$$

-Associative property:

$$(f(x,p) * g(x,p)) * h(x,p) = f(x,p) * (g(x,p) * h(x,p)) \tag{1.19}$$

-The relation of the complex conjugate property:

$$(f(x,p) * g(x,p))^{*} = g(x,p)^{*} * f(x,p) \tag{1.20}$$

-The integral relation property:

$$\left\{ \int d^{D}x\,(f * g) = \int d^{D}x\,(g * f)(x,p) = \int d^{D}x f(x,p)\, g(x,p) \right. \tag{1.21}$$

-Cyclic permutation property:

$$\int d^{D}x\,(f * g * h) = \int d^{D}\,(g * f * h) = \int d^{D}x\,(g * f * h) \tag{1.22}$$

-Satisfies the Leibniz's rule property:

$$\frac{\partial (f * g)}{\partial x^{\alpha}} = \left(\frac{\partial f}{\partial x^{\alpha}}\right) * g + f * \left(\frac{\partial g}{\partial x^{\alpha}}\right) \tag{1.23}$$

1.6 Boop's shift method

It is well known that the physicist Fritz Bopp was the first to examine pseudo-differential operators derived from a symbol using quantization methods:

$$\begin{cases} x \to x + \frac{1}{2}\imath\hbar\frac{\partial}{\partial p} \\ p \to p - \frac{1}{2}\imath\hbar\frac{\partial}{\partial x} \end{cases} \tag{1.24}$$

Instead of the usual correspondence $\left(x \to x,\ p \to -\frac{1}{2}\imath\hbar\frac{\partial}{\partial x}\right)$, the operators $x \to x + \frac{1}{2}\imath\hbar\frac{\partial}{\partial p}$ and $p \to p - \frac{1}{2}\imath\hbar\frac{\partial}{\partial x}$ are known as Bopp's shifts, and this quantization procedure is known as the Bopp quantization procedure. This quantization leads us to obtain the following:

$$\begin{cases} \widehat{x}^{i} = x^{i} - \sum_{j}^{3} \left(\frac{\theta^{ij}}{2} p_{j}\right) \\ \widehat{p}^{i} = p^{i} + \sum_{j}^{3} \left(\frac{\overline{\theta}^{ij}}{2} x_{j}\right) \end{cases} \tag{1.25}$$

To express the Schrödinger equation in the non-commutative phase-space, we will apply the following steps:

1- The ordinary three-dimensional Hamiltonian operators $\hat{H}(p_i, x_i)$ will be replace with new Hamiltonian operator $\hat{H}(\hat{p}_i, \hat{x}_i)$.

2- The ordinary complex wave function $\psi\left(\overrightarrow{r}\right)$ become a new complex wave function $\hat{\psi}(\hat{r})$.

3- The ordinary energie E_{nl} will be replacing with new values E_{nc}^{gqy}.

4- We replace the ordinary product with the star product.

Hence, we get the following Schrödinger equation in the non-commutative phase-space deformation:

$$\begin{cases} H\left(\widehat{x}^i, \widehat{p}^i\right)\hat{\psi}(\vec{r},t) = E_{nc}\hat{\psi}(\vec{r},t) \Rightarrow \\ \hat{H}(x,p) * \hat{\psi}(\vec{r},t) = E_{nc}\hat{\psi}(\vec{r},t) \end{cases} \tag{1.25}$$

The Bopp's shifts method allows to reduce the above deformed Schrödinger equation to the new translated form:

$$H\left(\widehat{x}^i, \widehat{p}^i\right) * \hat{\psi}(\vec{r},t) = E_{nc}\hat{\psi}(\vec{r},t) = E_{nc}\psi(\vec{r}) \tag{1.26}$$

So the Hamiltonian operator takes the three varieties forms as follows [44, 45, 46, 47, 48, 49, 50, **?**]:

$$\begin{cases} H(\hat{p}_i, \hat{x}_i) = H\left(\hat{p}_i = p_i + \sum\limits_{j=1}^{3}\left(\frac{\bar{\theta}_{ij}}{2}x^j\right), \quad \hat{x}_i = x_i - \sum\limits_{j=1}^{3}\left(\frac{\theta_{ij}}{2}p^j\right)\right) \\ \text{For non-commutative phase-space} \\ H(\hat{p}_i, \hat{x}_i) = H\left(\hat{p}_i = p_i, \quad \hat{x}_i = x_i - \sum\limits_{j=1}^{3}\left(\frac{\theta_{ij}}{2}p^j\right)\right) \\ \text{For non-commutative space-space} \\ H(\hat{p}_i, \hat{x}_i) = H\left(\hat{p}_i = p_i + \sum\limits_{j=1}^{3}\left(\frac{\bar{\theta}_{ij}}{2}x^j\right), \ \hat{x}_i = x_i\right) \\ \text{For non-commutative phase-phase} \end{cases} \tag{1.27}$$

The first variety corresponds to non-commutative phase-space (NCPS) symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_i = p_i + \frac{\bar{\theta}^{ij}}{2}x_j, \hat{x}_i = x_i - \frac{\theta^{ij}}{2}p_i\right)$ in Eq. (1.27):

$$\begin{cases} x_i \rightarrow \quad \hat{x}_i = x_i - \sum\limits_{j=1}^{3}\left(\frac{\theta_{ij}}{2}p^j\right) \\ p_i \rightarrow \hat{p}_i = p_i + \sum\limits_{j=1}^{3}\left(\frac{\bar{\theta}_{ij}}{2}x^j\right) \end{cases} \tag{1.28}$$

The second variety corresponds to non-commutative space-space symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_i = p_i, \hat{x}_i = x_i - \frac{\theta^{ij}}{2}p_j\right)$ in Eq. (1.27):

$$\begin{cases} p_i \to \hat{p}_i = p_i \\ x_i \to \hat{x}_i = x_i - \frac{\theta^{ij}}{2} p_j \end{cases} \tag{1.29}$$

The third variety corresponds to non-commutative phase-phase NCPP symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_i = p_i + \frac{\bar{\theta}^{ij}}{2} x_j, \hat{x}_i = x_i\right)$ in Eq. (1.27):

$$\begin{cases} p_i \to \hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2} p_j \\ x_j \to \hat{x}_i = x_i \end{cases} \tag{1.30}$$

In our current master memoir, we are interested in applying the following general procedure to NCPS symmetries which correspond to the first variety of Eq. (1.27). The three-generalized coordinates $(\hat{x} = \hat{x}_1, \hat{y} = \hat{x}_2, \hat{z} = \hat{x}_3)$ in the non-commutative phase-space were depended on corresponding three-usual generalized positions (x, y, z) and three momentum coordinates (p_x, p_y, p_z):

$$\begin{cases} i_1 = 1 \quad \hat{x}_1 = \hat{x} \ \text{ and } \widehat{p}_1 = p_x, \\ i_2 = 2 \quad \hat{x}_2 = \widehat{y} \text{ and } \widehat{p}_2 = p_y, \\ i_3 = 3 \quad \hat{x}_3 = \widehat{z} \ \text{ and } \widehat{p}_3 = p_z. \end{cases} \tag{1.31}$$

It is important to notice that the new operators $\hat{x}_i$ and $\hat{p}_i$ in three-dimensional phase-space non-commutativity was depended on ordinary operator x_i and p_i from the projection relations:

$$\begin{cases} \hat{x}_1 = x_1 - \frac{\theta^{12}}{2} p_2 - \frac{\theta^{13}}{2} p_3, \\ \hat{x}_2 = x_2 - \frac{\theta^{21}}{2} p_1 - \frac{\theta^{23}}{2} p_3, \\ \hat{x}_3 = x_3 - \frac{\theta^{31}}{2} p_1 - \frac{\theta^{32}}{2} p_2, \end{cases} \tag{1.32}$$

and

$$\begin{cases} \hat{p}_1 = p_1 + \frac{\bar{\theta}^{12}}{2} x_2 + \frac{\bar{\theta}^{13}}{2} x_3, \\ \hat{p}_2 = p_2 + \frac{\bar{\theta}^{12}}{2} x_1 + \frac{\bar{\theta}^{23}}{2} x_3, \\ \hat{p}_3 = p_3 + \frac{\bar{\theta}^{31}}{2} x_1 + \frac{\bar{\theta}^{32}}{2} x_2, \end{cases} \tag{1.33}$$

The non-vanish 9-commutators in three-dimensional phase-space non-commutativity can be determined as follows:

$$\begin{cases} \left[\hat{x}(t), \hat{p}_x(t)\right] = \left[\hat{y}(t), \hat{p}_y(t)\right] = \left[\hat{z}(t), \hat{p}_z(t)\right] = \\ = \left[\hat{x}, \hat{p}_x\right] = \left[\hat{y}, \hat{p}_y\right] = \left[\hat{z}, \hat{p}_z\right] = i \\ \left[\hat{x}(t), \hat{y}(t)\right] = \left[\hat{x}, \hat{y}\right] = i\theta_{12}, \\ \left[\hat{x}(t), \hat{z}(t)\right] = \left[\hat{x}, \hat{z}\right] = i\theta_{13}, \\ \left[\hat{y}(t), \hat{z}(t)\right] = \left[\hat{y}, \hat{z}\right] = i\theta_{23}, \\ \left[\hat{p}_y(t), \hat{p}_y(t)\right] = \left[\hat{p}_y, \hat{p}_y\right] = i\bar{\theta}_{12}, \\ \left[\hat{p}_y(t), \hat{p}_z(t)\right] = \left[\hat{p}_y, \hat{p}_z\right] = i\bar{\theta}_{23}, \\ \left[\hat{p}_x(t), \hat{p}_z(t)\right] = \left[\hat{p}_x, \hat{p}_z\right] = i\bar{\theta}_{13}. \end{cases} \tag{1.34}$$

The square of $(\vec{r}, \vec{p})$ are given by :

$$\begin{cases} \overrightarrow{\hat{r}}^{2} = \overrightarrow{\hat{r}}_{x}^{2} + \overrightarrow{\hat{r}}_{y}^{2} + \overrightarrow{\hat{r}}_{z}^{2} \\ \overrightarrow{\hat{p}}^{2} = \overrightarrow{\hat{p}_x}^{2} + \overrightarrow{\hat{p}_y}^{2} + \overrightarrow{\hat{p}_z}^{2} \end{cases} \tag{1.35}$$

To get the solution to the non-commutative Schrödinger equation, we added the star product introduced by Bopp's shift method. That is a consequence of the star product between the potential operator $\hat{V}(\hat{x})$ and the complex wave function $\hat{\Psi}\hat{r}$:

$$\left[\frac{\overrightarrow{\hat{P}}^{2}}{2m} + \hat{V}(\hat{r})\right] * \hat{\Psi}(\hat{r}) = E_{nc}\hat{\psi}(\vec{r}, t)$$

$$\Rightarrow$$

$$\left[\frac{\overrightarrow{p}^{2}_{nc}}{2m} + V(\hat{r})\right]\Psi(\overrightarrow{r}) = E\Psi(\overrightarrow{r}) \tag{1.36}$$

The two operators $\hat{x}$ and $\hat{p}$, when on a non-commutative three-dimensional space-phase, can be written as follows :

$$\begin{cases} \hat{r}^{2} = r^{2} - \overrightarrow{L}\vec{\Theta} \\ \frac{\overrightarrow{p}^{2}_{nc}}{2m_0} = \frac{\overrightarrow{P}^{2}}{2m_0} + \frac{\vec{L}\overrightarrow{\overline{\theta}}}{2m_0} \end{cases} \tag{1.37}$$

Where the two couplings $\overrightarrow{L}\overrightarrow{\Theta}$ and $\vec{L}\overrightarrow{\overline{\theta}}$ are given by the following relations respectively:

$$\begin{cases} \overrightarrow{L}\overrightarrow{\Theta} = L_x\theta_{12} + L_y\theta_{23} + L_z\theta_{13} \\ \overrightarrow{L}\overrightarrow{\overline{\theta}} = L_x\overline{\theta}_{12} + L_y\overline{\theta}_{23} + L_z\overline{\theta}_{13} \end{cases} \tag{1.38}$$

Part II

Reviewed Schrödinger equation with generalized inverse quadratic Yukawa potential in quantum mechanics symmetry

1.7 Introduction

In this part, we want to revise the generalized inverse quadratic Yukawa potential within the framework of ordinary quantum mechanics. Within the context of the three-dimensional Schrödinger equation, we also attempt to revise the associated wave function and energy eigenvalue.

1.8 Schrödinger equation with the generalized inverse quadratic Yukawa potential

The generalized inverse quadratic Yukawa's potential is an exponential potential. It consists of the sum of two potentials, as the first is Yukawa potential $(-\frac{B^{'}}{r}\exp{(-\alpha r)})$ that proposed by Yukawa itself in 1935 [51, 52] while the second is inverse quadratic Yukawa's potential $(-\frac{A^{'}}{r^2}\exp{(-2\alpha r)})$ in addition to the fixed term V_0. Thus, the generalized inverse quadratic Yukawa's potential is as follows [53, 54]:

$$V(r) = -\frac{A^{'}}{r^2}\exp{(-2\alpha r)} - \frac{B^{'}}{r}\exp{(-\alpha r)} - C^{'} \tag{2.1}$$

where $A^{'} = C^{'} = V_0, B^{'} = 2V_0$, and α is the screening parameter. which can be rewritten as follows:

$$V(r) = -V_0\left(1 + \frac{1}{r}\exp{(-\alpha r)}\right)^2 \tag{2.2}$$

Considering that the potential under study (the generalized inverse quadratic Yukawa's potential) is an extension and generalization of Yukawa potential, it is therefore useful to recall the importance of Yukawa potential by recalling some previous studies that dealt with it. The Yukawa potential as a low-energy explanation for nucleon-nucleon interactions induced by the exchange of massive particles known as pions is relatively new to researchers in the field [55]. The Yukawa potential can be found in a variety of physics fields, including plasma physics at low densities and high temperatures, nuclear physics, astrophysics and solid-state physics [56, 57, 58, 59, 60, 61]. Due to the fact that this potential has wide applications, it has received great interest from researchers in the relativistic levels within the framework of the Dirac and Klein Gordon equations, in addition to the non-nonrelativistic framework where the Schrödinger equation is applied using many methods and various approximations [62, 63, 64, 65, 66, 67, 68, 69, 70, 71]. As for the relativistic and non-relativistic level, within the framework of the principles of extended quantum mechanics or non-commutative quantum mechanics symmetries, it has received recent studies, including those published in the references as it is single or in combination with other components [72, 73, 74, 75, 76, 77, 78, 79].

1.9 Reviewing the eigenfunctions and the energy eigenvalues for Yukawa's potential

Schrödinger equation is a fundamental equation of quantum mechanics which describes the evolution of the wave function of a physical system over time. It is a first-order partial differential equation concerning time and a second-order partial differential equation concerning the coordinates of ordinary space. It takes the following form:

$$H\psi\left(\overrightarrow{r},t\right)=E\psi\left(\overrightarrow{r},t\right) \tag{2.3}$$

here $\psi\left(\overrightarrow{r}\right)$ is the complex wave function that satisfies the stationary Schrödinger equation and E is a nonrelativistic eigenvalue of the Hamiltonian H, which is written in the form :

$$H=\frac{\hat{P}^2}{2m_0}-\frac{A^{'}}{r^2}\exp\left(-2\alpha r\right)-\frac{B^{'}}{r}\exp\left(-\alpha r\right)-C^{'} \tag{2.4}$$

where P represents the impulse $\overrightarrow{P}=-i\hbar\overrightarrow{\nabla}$, and $\overrightarrow{\nabla}$ represents the operator of partial derivatives (Nabla). In Cartesian coordinates, it is defined by:

$$\overrightarrow{\nabla}=\overrightarrow{\imath}\frac{\partial}{\partial x}+\overrightarrow{j}\frac{\partial}{\partial y}+\overrightarrow{k}\frac{\partial}{\partial z} \tag{2.5}$$

Hence, Schrödinger's equation becomes:

$$i\hbar\frac{\partial\psi\left(\overrightarrow{r},t\right)}{\partial t}=\left(-\frac{\hbar^2}{2m_0}\Delta-\frac{A^{'}}{r^2}\exp\left(-2\alpha r\right)-\frac{B^{'}}{r}\exp\left(-\alpha r\right)-C^{'}\right)\psi\left(\overrightarrow{r},t\right) \tag{2.6}$$

Since the generalized inverse quadratic generalized inverse quadratic Yukawa potential does not depend on time, solutions can be written separately as a part that is only position-dependent and an only time-dependent part:

$$\Psi\left(\overrightarrow{r},t\right)=\exp(-iE/\hbar t)\Psi\left(\overrightarrow{r}\right) \tag{2.7}$$

And by substituting into Schrödinger equation, we find:

$$\left(\frac{-\hbar^2}{2m}\Delta+V_0\frac{\exp(\alpha r)}{r}\right)\Psi\left(\overrightarrow{r}\right)=E\Psi\left(\overrightarrow{r}\right) \tag{2.8}$$

Using the spherical coordinate system $\overrightarrow{r}\,(r,\theta,\varphi)$, the complex wave function $\Psi\left(\overrightarrow{r}\right)$ can be written as:

$$\Psi\left(r,\theta,\varphi\right)=R_{nl}\left(r\right)Y_{lm}\left(\theta,\varphi\right) \tag{2.9}$$

where $R_{nl}(r)$ is the radial part of the wave function that depends only on radius r, $Y_{l,m}(\theta,\varphi)$ represented the angular part depends on the angles (θ,φ)

and n is the principal quantum number, l the orbital quantum number and m the magnetic quantum number ($-l \leq m \leq +l$). The Schrödinger equation in the spherical coordinate can be expressed as:

$$\left(\frac{-\hbar}{2m_0}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\hat{L}^2}{r^2}-\frac{A^{'}}{r^2}\exp\left(-2\alpha r\right)-\frac{B^{'}}{r}\exp\left(-\alpha r\right)-C^{'}\right)\right)R_{nl}(r)=ER_{nl}\left(r\right) \tag{2.10}$$

In quantum mechanics, the classical momentum obtains the forms $\overrightarrow{L}$ is the orbital angular momentum. The total moment $\overrightarrow{J}$ is given by:

$$\left\{\begin{array}{c} \overrightarrow{J}=\overrightarrow{L}+\overrightarrow{S} \\ \overrightarrow{L}=\overrightarrow{r}\wedge\overrightarrow{p} \end{array}\right. \tag{2.11}$$

here $\overrightarrow{S}$ is the spin. The components L_x , L_y and L_z of $\overrightarrow{L}$ which are expressed in Cartesian coordinates (x, y, z) as:

$$\left\{\begin{array}{c} L_x=\frac{\hbar}{\imath}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right) \\ L_y=\frac{\hbar}{\imath}\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right) \\ L_z=\frac{\hbar}{\imath}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right) \end{array}\right. \tag{2.12}$$

In the spherical coordinate system $\overrightarrow{r}\,(r,\theta,\varphi)$, he components L_x , L_y and L_z of $\overrightarrow{L}$ are expressed as:

$$\left\{\begin{array}{c} L_x=\frac{\hbar}{\imath}\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ L_y=\frac{\hbar}{\imath}\left(\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ L_z=\frac{\hbar}{\imath}\frac{\partial}{\partial\varphi} \end{array}\right. \tag{2.13}$$

Note that the operators H, L^2 and L_z commute with each other and they have formed a common set of eigenfunctions $\psi\left(r,\theta,\varphi\right)$; however, the three components of the angular momentum (L_x, L_y, L_z) do not commute with each other:

$$\left\{\begin{array}{c} \left[H,\mathbf{L}^2\right]=\left[H,L_z\right]=0 \\ \left[L_i,L_j\right]=\imath\hbar\xi_{ijk}L_k \end{array}\right. \tag{2.14}$$

here $\mathbf{L}^2$ is the square of the angular momentum :

$$L^2=-\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)+\hbar^2\frac{\partial^2}{\partial\varphi^2} \tag{2.15}$$

the eigenvalues of $L^2$and L_z are determined from:

$$\left\{\begin{array}{c} \hat{L}^2\psi\left(r,\theta,\varphi\right)=\hbar^2 l\left(l+1\right)\psi\left(r,\theta,\varphi\right) \\ \hat{L}_z\psi\left(r,\theta,\varphi\right)=m\hbar\psi\left(r,\theta,\varphi\right) \end{array}\right. \tag{2.16}$$

We introducing the wave function: we have:

$$R_{n,l}(r) = \frac{u_{n,l}(r)}{r} \tag{2.17}$$

Thus, the new radial part $u_{n,l}(r)$, will be satisfying the following equation:

$$\frac{d^2 u_{n,l}(r)}{dr^2} + \frac{2m_0}{\hbar^2}\left(E_{nl} + \frac{A^{'}}{r^2}\exp(-2\alpha r) + \frac{B^{'}}{r}\exp(-\alpha r) + C^{'} - \frac{l(l+1)}{r^2}\right)u_{n,l}(r) = 0 \tag{2.18}$$

The energy eigenvalues E_{nl} and corresponding eigenfunctions in closed forms were obtained using the parametric Nikiforov-Uvarov approach by the authors of Refs. [54, 55]. They further show that these results are consistent with those obtained previously in other studies using different approaches. They also discovered that when the generalized inverse quadratic generalized inverse quadratic Yukawa potential's screening parameter is set to zero, the energy levels of the familiar pure Coulomb potential energy levels are:

$$E_{n,l} = -V_0 - \frac{2\alpha^2}{\mu}\left[\frac{\left(\frac{B}{2\alpha}+n\right)^2 + \frac{2\mu V_0}{\alpha}(\alpha-1)}{2\left(\frac{B}{2\alpha}+n\right)}\right]^2 \tag{2.19}$$

Where B is determined from:

$$B = \alpha\left[1 + \sqrt{(2l+1)^2 - 8\mu V_0}\right] \tag{2.20}$$

The corresponding radial part $U_{nl}(s)$ of the complex wave function of the generalized inverse quadratic Yukawa potential [54, 55]:

$$U_{nl}(s) = N_{nl} s^{\sqrt{\varepsilon_{nl}}}(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma_l-\eta}}\, P_n^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^2}},2\sqrt{\frac{1}{4}+\gamma_l-\eta}\right)}(1-2s) \tag{2.21}$$

where $P_n^{\left(2\sqrt{\frac{\varepsilon}{4\alpha^2}},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^2}\right)}(1-2s)$ is the Jacobi polynomial (hypergeometric polynomials), ε_{nl}, γ_l , η and s are given by.

$$\begin{cases} \varepsilon_{nl} = \frac{\mu}{2\alpha^2}(E_{n,l}+V_0) \\ \gamma_l = l(l+1) \\ \eta = 2\mu V_0 \\ s = \exp(-2\alpha r) \end{cases} \tag{2.22}$$

Here N is a normalization constant. Thus, the complex wave function $\Psi\left(\overrightarrow{r}\right)$

can be written as:

$$\Psi\left(r,\theta,\varphi,t\right)=N_{nl}\frac{s^{\sqrt{\varepsilon_{nl}}}}{r}\left(1-s\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma_{l}-\eta}}P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(1-2s\right)Y_{lm}\left(\theta,\varphi\right)\exp(-iE_{nl}t) \tag{2.23}$$

In mathematics, Jacobi polynomials (occasionally called hypergeometric polynomials). $P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}$ are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight $\left(1-x\right)^{2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}}}\left(1+x\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}}$ on the interval $[-1,1]$. The global analytical expression of Jacobi polynomials $P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(1-2s\right)$ obtains from the expression:

$$P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(1-2s\right)=\frac{\Gamma\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}}+n+1\right)}{\Gamma\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}}+2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}+n+1\right)}\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)\frac{\Gamma\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}}+2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}+n+m+1\right)}{\Gamma\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}}+m+1\right)}s^{m} \tag{2.24}$$

Here

$$\left(\begin{array}{c}n\\m\end{array}\right)=\left\{\begin{array}{c}\frac{\Gamma(n+1)}{\Gamma(m+n-m+1)}\text{ for }m\geq 0\\0\text{ for }m\leq 0\end{array}\right. \tag{2.25}$$

For n=0

$$P_{0}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(1-2s\right)=1 \tag{2.26}$$

Allow us to obtained, the complex wave function $\Psi\left(\overrightarrow{r}\right)$ can be written for fundamental state as:

$$\Psi\left(r,\theta,\varphi,t\right)=N_{nl}\frac{s^{\frac{\sqrt{\varepsilon_{0l}}}{4\alpha^{2}}}}{r}\left(1-s\right)^{l+1}Y_{lm}\left(\theta,\varphi\right)\exp(-iE_{nl}t) \tag{2.27}$$

Part III

The Effect of non-commutative phase-space on the energy spectrum produced by the generalized inverse quadratic Yukawa potential

1.10 Introduction

The purpose of this chapter is to study the modified Schrödinger equation of generalized inverse quadratic Yukawa potential in non-commutative three-dimensional phase space. Accordingly, we use Bopp's shift method instead of solving the modified Schrödinger equation directly; thus, using the star product and the perturbation theorem to find the corresponding energy correction.

1.11 The Schrödinger equation on a Noncommutative space-time

We simply replace the wave function products (or fields) with the star product or the Moyal product. The Schrödinger equation for a non-commutative space-time has the form:

$$\left\{\begin{array}{c} H(\hat{p}_i,\hat{x}_i)\hat{\Psi}(\overrightarrow{\hat{r}}) = E_{nc}\Psi\left(\overrightarrow{\hat{r}}\right) \\ \Rightarrow \\ \left[\frac{\overrightarrow{\hat{P}^2}}{2m} + V(\hat{r})\right] * \hat{\Psi}\left(\overrightarrow{\hat{r}},\hat{t}\right) = \imath\hbar\frac{\partial}{\partial t}\Psi\left(\overrightarrow{\hat{r}},\hat{t}\right) \end{array}\right. \tag{3.1}$$

According to the method of Bopp's Shift, which we have seen in the first chapter, the above equation can be simplified into the following form:

$$H_{nc-gyp}\left(\hat{p}_i = p_i - \frac{\bar{\theta}_{ij}}{2}x^j, \hat{x}_i = x_i + \frac{\theta_{ij}}{2}p^j\right)\Psi(\overrightarrow{\hat{r}}) = E_{nc}\Psi(\overrightarrow{\hat{r}}) \tag{3.2}$$

where

$$\begin{aligned} H\left(\hat{p}_i,\hat{x}_i\right) &= H\left(\hat{p}_i = p_i - \frac{\overline{\theta}^{ij}}{2}x_j, \quad \hat{x}_i = x_i + \frac{\theta^{ij}}{2}p_i\right) \\ &= \frac{\overrightarrow{\hat{P}}^2}{2m} + V\left(\hat{r}\right) \end{aligned} \tag{3.3}$$

and

$$\left\{\begin{array}{c} V\left(\hat{r}\right) = -\frac{A'}{r^2}\exp\left(-2\alpha r\right) - \frac{B'}{r}\exp\left(-\alpha r\right) - C' \\ \frac{\overrightarrow{P}^2}{2m_0} = \frac{P^2}{2m_0} + \frac{\overrightarrow{L}\vec{\Theta}}{2m_0} \end{array}\right. \tag{3.4}$$

By using Eq.(1.27), we can obtain $-\frac{B'}{\hat{r}}$ and $-\frac{A'}{\hat{r}^2}$ as the sum of corresponding values $-\frac{B'}{r}$ and $-\frac{A'}{r^2}$ in the symmetries of nonrelativistic quantum mechanics plus the induced term $V_0\frac{\overrightarrow{L}\vec{\Theta}}{2r^3}$ with the effect of deformed proprieties of space-space, as follows:

$$\left\{\begin{array}{l} -\frac{B'}{\hat{r}} = -\frac{B'}{r} - B'\frac{\overrightarrow{L}\vec{\Theta}}{2r^3} \\ -\frac{A'}{\hat{r}^2} = -\frac{A'}{r^2} - A'\frac{\overrightarrow{L}\vec{\Theta}}{r^4} \end{array}\right. \tag{3.5}$$

while, the expressions of $\exp(-\alpha\hat{r})$ and $\exp(-2\alpha\hat{r})$ can be written in the symmetries of extended quantum mechanics symmetries as follows:

$$\begin{cases} \exp(-\alpha\hat{r}) = \exp(-\alpha r)\exp\left(\alpha\frac{\overrightarrow{L}\vec{\Theta}}{2r}\right) \backsimeq \exp(-\alpha r) + \exp(-\alpha r)\frac{\alpha\overrightarrow{L}\vec{\Theta}}{2r} \\ \exp(-2\alpha\hat{r}) = \exp(-2\alpha r)\exp\left(\alpha\frac{\overrightarrow{L}\vec{\Theta}}{r}\right) \backsimeq \exp(-2\alpha r) + \exp(-2\alpha r)\frac{\alpha\overrightarrow{L}\vec{\Theta}}{r} \end{cases} \tag{3.6}$$

Allow us to get $\left(-\frac{B^{'}}{\hat{r}}\exp(-\alpha\hat{r})\right)$ and $\left(-\frac{A^{'}}{\hat{r}^2}\exp(-2\alpha\hat{r})\right)r$ as follows:

$$\begin{cases} -\frac{B^{'}}{\hat{r}}\exp(-\alpha\hat{r}) \backsimeq -\frac{B^{'}}{r}\exp(-\alpha r) - \left(\frac{\alpha B^{'}\exp(-\alpha r)}{2r^2} + \frac{B^{'}\exp(-\alpha r)}{2r^3}\right)\overrightarrow{L}\vec{\Theta} \\ -\frac{A^{'}}{\hat{r}^2}\exp(-2\alpha\hat{r}) \backsimeq -\frac{A^{'}}{r^2}\exp(-2\alpha r) - \left(\frac{\alpha A^{'}}{r^3}\exp(-2\alpha r) + \frac{A^{'}\exp(-2\alpha r)}{r^4}\right)\overrightarrow{L}\vec{\Theta} \end{cases} \tag{3.7}$$

Allow us to get the generalized inverse quadratic Yukawa potential in the non-commutative phase-space as follows:

$$\begin{aligned} V(\hat{r}) = & -\frac{A^{'}}{r^2}\exp(-2\alpha r) - \frac{B^{'}}{r}\exp(-\alpha r) - C^{'} \\ & -\left(\frac{\alpha A^{'}\exp(-2\alpha r)}{r^3} + \frac{A^{'}\exp(-2\alpha r)}{r^4} + \frac{\alpha B^{'}\exp(-\alpha r)}{2r^2} + \frac{B^{'}\exp(-\alpha r)}{2r^3}\right)\overrightarrow{L}\vec{\Theta} \end{aligned} \tag{3.8}$$

which can be written as

$$V(\hat{r}) = -\frac{A^{'}}{r^2}\exp(-2\alpha r) - \frac{B^{'}}{r}\exp(-\alpha r) - C^{'} + V(r,\Theta)_{per}^{gyp} \tag{3.9}$$

with

$$V(r,\Theta)_{per}^{gyp} = -\left(\frac{\alpha A^{'}\exp(-2\alpha r)}{r^3} + \frac{A^{'}\exp(-2\alpha r)}{r^4} + \frac{\alpha B^{'}\exp(-\alpha r)}{2r^2} + \frac{B^{'}\exp(-\alpha r)}{2r^3}\right)\overrightarrow{L}\vec{\Theta} \tag{3.10}$$

The global Hamiltonian operator $H_{nc}^{gyp}\left(\hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2}x_j, \hat{x}_i = x_i + \frac{\theta^{ij}}{2}p_i\right)$ in non-commutative three-dimensional phase-space can be written in the following form:

$$\begin{aligned} H_{nc}^{gyp}\left(\hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2}x_j, \hat{x}_i = x_i + \frac{\theta^{ij}}{2}p_i\right) = & \frac{\overrightarrow{p}^2}{2m} + -\frac{A^{'}}{r^2}\exp(-2\alpha r) - \frac{B^{'}}{r}\exp(-\alpha r) - C^{'} \\ & -\left(\frac{\alpha A^{'}\exp(-2\alpha r)}{r^3} + \frac{A^{'}\exp(-2\alpha r)}{r^4} + \frac{\alpha B^{'}\exp(-\alpha r)}{2r^2} + \frac{B^{'}\exp(-\alpha r)}{2r^3}\right)\overrightarrow{L}\vec{\Theta} + \frac{\vec{L}\overrightarrow{\theta}}{2m_0} \end{aligned} \tag{3.11}$$

1-The first two terms in the Hamiltonian operator H_{per}^{gyp}, which corresponds to the generalized inverse quadratic Yukawa potential in Eq. (2.1) and the Kinetic term or dynamic $\frac{\overrightarrow{p}^2}{2m}$ in ordinary commutative space which formed the usual Hamiltonian operator:

$$H(\hat{p}_i = p_i, \hat{x}_i = x_i) = \frac{\overrightarrow{p}^2}{2m} - \frac{A'}{r^2}\exp(-2\alpha r) - \frac{B'}{r}\exp(-\alpha r) - C' \tag{3.12}$$

2-The second and third terms are formed the a new Hamiltonian operator or the additive created term H_{per}^{gyp} which is represent the contributions of the non-commutative space-phase:

$$H_{per}^{gyp} = -\left(\frac{\alpha A'\exp(-2\alpha r)}{r^3} + \frac{A'\exp(-2\alpha r)}{r^4} + \frac{\alpha B'\exp(-\alpha r)}{2r^2} + \frac{B'\exp(-\alpha r)}{2r^3}\right)\vec{L}\vec{\Theta} + \frac{\vec{L}\overrightarrow{\bar{\theta}}}{2m_0} \tag{3.13}$$

where $\vec{L}\vec{\Theta}$ and $\vec{L}\overrightarrow{\bar{\theta}}$ are determined from Eq. (1.39) in the first chapter. According to the mathematical forms of the 2-couplings $\overrightarrow{L}\overrightarrow{\Theta}$ and $\overrightarrow{L}\overrightarrow{\bar{\theta}}$ observed in Eq.(3.13), it is physically possible to replace $\overrightarrow{L}\overrightarrow{\Theta}$ and $\overrightarrow{L}\overrightarrow{\bar{\theta}}$ by $\mu\Theta\overrightarrow{S}\overrightarrow{L}$ and $\mu\bar{\theta}\overrightarrow{S}\overrightarrow{L}$, respectively:

$$\begin{cases} \overrightarrow{L}\overrightarrow{\Theta} \to \mu\Theta\overrightarrow{S}\overrightarrow{L} \\ \overrightarrow{L}\overrightarrow{\bar{\theta}} \to \mu\bar{\theta}\overrightarrow{S}\overrightarrow{L} \end{cases} \tag{3.14}$$

With $\overrightarrow{S}$ denote to the spin of the particle which interacted with generalized inverse quadratic Yukawa potential and μ is a new constant of proportionality. This enables rewriting Eq.(3.13) as follows:

$$H_{per}^{gyp} = \mu\left[\frac{\bar{\theta}}{2m_0} - \left(\frac{\alpha A'\exp(-2\alpha r)}{r^3} + \frac{A'\exp(-2\alpha r)}{r^4} + \frac{\alpha B'\exp(-\alpha r)}{2r^2} + \frac{B'\exp(-\alpha r)}{2r^3}\right)\Theta\right]\overrightarrow{L}\overrightarrow{S} \tag{3.15}$$

The parameters Θ and $\bar{\theta}$ are given by:

$$\begin{cases} \Theta = \left(\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2\right)^{\frac{1}{2}} \\ \bar{\theta} = \left(\bar{\theta}_{12}^2 + \theta_{23}^2 + \bar{\theta}_{13}^2\right)^{\frac{1}{2}} \end{cases} \tag{3.16}$$

In ordinary quantum mechanics, we have the sets of operators $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$**......** which form a complete set of complete observable are commute (ECOC). We apply this rule to the sets of operators $\left(\overrightarrow{\mathbf{J}}^2, \overrightarrow{\mathbf{S}}^2, \overrightarrow{\mathbf{L}}^2 \text{ and } J_z\right)$, i.e.:

$$\begin{cases} \left[\overrightarrow{\mathbf{J}}^2, \overrightarrow{\mathbf{L}}^2\right] = 0 \\ \left[\overrightarrow{\mathbf{J}}^2, \overrightarrow{\mathbf{S}}^2\right] = 0 \\ \left[\overrightarrow{\mathbf{J}}^2, J_z\right] = 0 \end{cases} \tag{3.17}$$

And the corresponding eigenvalues are $j\,(j+1)$ and $l(l+1)$, $s(s+1)$ and $m\,(-l \leq m \leq +l)$ in the system $(c=\hbar=1)$, so:

$$\begin{cases} \overrightarrow{\mathbf{J}}^2 \Psi_{n,l,m_l}(r,\theta,\varphi) = j(j+1)\Psi_{n,l,m_l}(r,\theta,\varphi) \\ \overrightarrow{\mathbf{L}}^2 \Psi_{n,l,m_l}(r,\theta,\varphi) = l(l+1)\Psi_{n,l,m_l}(r,\theta,\varphi) \\ \overrightarrow{\mathbf{S}}^2 \Psi_{n,l,m_l}(r,\theta,\varphi) = s(s+1)\Psi_{n,l,m_l}(r,\theta,\varphi) \\ J_z \Psi_{n,l,m_l}(r,\theta,\varphi) = m\Psi_{n,l,m_l}(r,\theta,\varphi) \end{cases} \tag{3.18}$$

With $\overrightarrow{J}$ being the geometric sum of the moments $\overrightarrow{L}$ and $\overrightarrow{S}$, this allows us to find the spin-orbit coupling $\overrightarrow{L}\overrightarrow{S}$ as follows:

$$\overrightarrow{L}\overrightarrow{S} = \frac{1}{2}\left[\overrightarrow{\mathbf{J}}^2 - \overrightarrow{\mathbf{S}}^2 - \overrightarrow{\mathbf{L}}^2\right] \tag{3.19}$$

An immediate result is:

$$\overrightarrow{L}\overrightarrow{S}\Psi = \frac{1}{2}\left[j(j+1) + l(l+1) + s(s+1)\right]\Psi \tag{3.20}$$

With $j \in [|l-s|, |l+s|]$, this permuted us to obtain j to be equal the values $(|l-s|, |l-s|+1, ..., |l+s|)$. For the two extreme values of the total angular momentum, we can write for $s=\frac{1}{2}$:

$$\overrightarrow{L}\overrightarrow{S}\Psi = \begin{cases} \frac{1}{2}\left\{(l+s)(l+s+1) - l(l+1) - 3/4\right\}\Psi \equiv k_+\Psi & \text{if} \quad j = |l+1/2| \\ \frac{1}{2}\left\{(l-s)(l-s+l) - l(l+1) - 3/4\right\}\Psi \equiv k_-\Psi & \text{if} \quad j = |l-1/2| \end{cases} \tag{3.21}$$

We considered the following approximation type suggested by Greene-Aldrich [54, 55, 80]:

$$\frac{1}{r^2} \approx 4\alpha^2 \frac{\exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2} \Longleftrightarrow \frac{1}{r} \approx 2\alpha \frac{\exp(-\alpha r)}{(1-\exp(-2\alpha r))} \tag{3.22}$$

which allows us to have the following:

$$\frac{1}{r^2} \approx \frac{4\alpha^2 s}{(1-s)^2} \Longleftrightarrow \frac{1}{r} \approx \frac{2\alpha s^{1/2}}{(1-s)} \tag{3.23}$$

This is valid for $ar \ll 1$. Therefore, the perturbative effective generalized inverse quadratic Yukawa Hamiltonian $H_{per}^{gyp}\left(s,\Theta,\bar{\theta}\right)$ in Eq.(3.15) can be written as:

$$H_{per}^{gyp}\left(s,\Theta,\bar{\theta}\right) = \left[\frac{\mu\bar{\theta}}{2m_0} - 4\alpha^3\left(\frac{2\alpha A' s^{5/2}}{(1-s)^3} + \frac{4\alpha A' s^3}{(1-s)^4} + \frac{B' s^{5/2}}{2(1-s)^4} + \frac{B' s^2}{(1-s)^3}\right)\Theta\right]\overrightarrow{L}\overrightarrow{S} \tag{3.24}$$

The generalized inverse quadratic Yukawa Hamiltonian $H(\hat{p}_i = p_i, \hat{x}_i = x_i)$ is extended by including new additive perturbative Hamiltonian $H_{per}^{gyp}\left(s,\Theta,\bar{\theta}\right)$ expressed to the radial terms:

$$\left\{\frac{s^{5/2}}{(1-s)^3}, \frac{s^3}{(1-s)^4}, \frac{s^{5/2}}{2(1-s)^4} \text{and } \frac{s^2}{(1-s)^3}\right\}$$

to become the modified generalized inverse quadratic Yukawa Hamiltonian $H_{nc}^{gyp}\left(\hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2}x_j, \hat{x}_i = x_i + \frac{\theta^{ij}}{2}p_i\right)$ in non-commutative three-dimensional phase-space symmetries. The generated new Hamiltonian $H_{per}^{gyp}\left(s, \Theta, \bar{\theta}\right)$ is also proportional to the infinitesimal parameters Θ and $\overline{\theta}$. This allows us to consider the new additive part of the potential $H_{per}^{gyp}\left(s, \Theta, \bar{\theta}\right)$ as perturbation potential compared with the main Hamiltonian $H(\hat{p}_i = p_i, \hat{x}_i = x_i)$. That is all physical justifications for applying the time-independent perturbation theory become satisfied to calculate the expectation values of previous radial terms. This allows us to give a complete prescription for determining the energy level of the generalized $(n, l, m)^{th}$ excited states. The exact spectrum produced by the spin-orbit effect for the generalized inverse quadratic Yukawa potential in the three dimensional non-commutative phase-space ΔE_{nc}^{gyp} is the sum of the energy corresponding to ordinary space E_{nl} and the corrections E_{per}^{gyp} :

$$E_{nc\_nl}^{gyp} = E_{nl} + \Delta E_{nc}^{gyp}\left(\Theta, \bar{\theta}\right) \tag{3.25}$$

The perturbation theorem allows to obtain the first-order corrections as follows:

$$\Delta E_{nc}^{gyp}\left(\Theta, \bar{\theta}\right) = \langle \Psi^p\left(\overrightarrow{r}\right)| \, H_{per}^{gyp}\left(r, \Theta, \theta\right) |\Psi^p\left(\overrightarrow{r}\right)\rangle \tag{3.26}$$

We can write the equation (3.26) in the form:

$$\Delta E_{nc}^{gyp}\left(\Theta, \bar{\theta}\right) = \int \Psi^{(p)}\left(\overrightarrow{r}\right) H_{per}^{gyp}\left(r, \Theta, \theta\right)\left(r, \Theta, \theta\right) \Psi^{(p)}\left(\overrightarrow{r}\right) d\tau \tag{3.27}$$

where $d\tau$ represent the volume element in spherical coordinates (r, θ, φ), which is given by:

$$d\tau = r^2 dr d\Omega \tag{3.28}$$

With the solid angle

$$d\Omega = \sin\theta d\theta d\varphi$$

and the nonperturbative complex wave function , the wave function is defined by :

$$\Psi^{(p)}\left(\overrightarrow{r}\right) = R_{n,l}\left(r\right) Y_l^m\left(\theta, \phi\right) \tag{3.29}$$

So, we can write the equation (3.27) in the form:

$$\Delta E_{nc}^{gyp}\left(\Theta, \bar{\theta}\right) = \left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle \int_0^{\infty} R_{n,l}^*\left(r\right) H_{per}^{gyp}\left(s, \Theta, \bar{\theta}\right) R_{n,l}\left(r\right) r^2 dr \int_0^{\pi}\int_0^{2\pi} Y_l^{*m_l}\left(\theta, \phi\right) Y_l^m\left(\theta, \phi\right) d\Omega \tag{3.30}$$

The normalized wave function $\Psi\left(\overrightarrow{r}\right)$ allows us to write :

$$\int_0^{\pi}\int_0^{2\pi} Y_l^{*m_l}\left(\theta, \phi\right) Y_l^m\left(\theta, \phi\right) d\Omega = 1 \tag{3.31}$$

This reduces the corrections found by (3.30) to the form:

$$\Delta E_{nc}^{gyp}\left(\Theta,\bar{\theta}\right)=\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle \int_{0}^{\infty} R_{n,l}^{*}\left(r\right) H_{per}^{gyp}\left(s,\Theta,\bar{\theta}\right) R_{n,l}\left(r\right) r^{2}dr \qquad (3.32)$$

We substituted the spin-orbit coupling term $V_{nc}^{py}\left(s\right)$, and we find:

$$\begin{gathered}\Delta E_{nc}^{gyp}\left(\Theta,\bar{\theta}\right)=\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle (\tfrac{\mu\bar{\theta}}{2m_0}\int_{0}^{\infty} R_{n,l}^{*}\left(r\right) R_{n,l}\left(r\right) r^{2}dr \\ -8\alpha^{4}A^{'}\Theta\int_{0}^{\infty} R_{n,l}^{*}\left(r\right)\tfrac{s^{5/2}}{(1-s)^{3}} R_{n,l}\left(r\right) r^{2}dr-16\alpha^{4}A^{'}\Theta\int_{0}^{\infty} R_{n,l}^{*}\left(r\right)\tfrac{s^{3}}{(1-s)^{4}} R_{n,l}\left(r\right) r^{2}dr \\ -2\alpha^{3}B^{'}\Theta\int_{0}^{\infty} R_{n,l}^{*}\left(r\right)\tfrac{s^{5/2}}{(1-s)^{4}} R_{n,l}\left(r\right) r^{2}dr-4\alpha^{3}B^{'}\Theta\int_{0}^{\infty} R_{n,l}^{*}\left(r\right)\tfrac{s^{2}}{(1-s)^{3}} R_{n,l}\left(r\right) r^{2}dr)\end{gathered} \qquad (3.33)$$

We have $s=\exp\left(-2\alpha r\right)$, this allows us to obtain $dr=-\frac{1}{2\alpha}\frac{ds}{s}$. After introducing a new variable $z=1-2s$, we have $s=\frac{1-z}{2}$, $dr=\frac{1}{2\alpha}\frac{dz}{1-z}$ and $1-s=\frac{1+z}{2}$. From the asymptotic behavior of $s=\exp\left(-2\alpha r\right)$ and $z=1-2y$, when $r\rightarrow 0(z\rightarrow -1)$ and $r\rightarrow +\infty(z\rightarrow 1)$, this allows reformulating Eq. (3.31) as follows:

$$\begin{gathered}\Delta E_{nc}^{gyp}\left(\Theta,\bar{\theta}\right)=\frac{\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle}{2\alpha}(\tfrac{\mu\bar{\theta}}{2m_0}-8\alpha^{4}A^{'}\Theta\int_{-1}^{+1} U_{n,l}^{*}\left(r\right)\tfrac{s^{5/2}}{(1-s)^{3}} U_{n,l}\left(r\right)\tfrac{dz}{1-z} \\ -16\alpha^{4}A^{'}\Theta\int_{-1}^{+1} U_{n,l}^{*}\left(r\right)\tfrac{s^{3}}{(1-s)^{4}} U_{n,l}\left(r\right)\tfrac{dz}{1-z}-2\alpha^{3}B^{'}\Theta\int_{-1}^{+1} U_{n,l}^{*}\left(r\right)\tfrac{s^{5/2}}{(1-s)^{4}} U_{n,l}\left(r\right)\tfrac{dz}{1-z} \\ -4\alpha^{3}B^{'}\Theta\int_{-1}^{+1} U_{n,l}^{*}\left(r\right)\tfrac{s^{2}}{(1-s)^{3}} U_{n,l}\left(r\right)\tfrac{dz}{1-z})\end{gathered} \qquad (3.34)$$

If we replace the radial part $U_{n,l}\left(r\right)$ which is expressed as:

$$\begin{aligned}U_{n,l}\left(r\right) &= N_{nl}s^{\sqrt{\varepsilon_{nl}}}\left(1-s\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma_l-\eta}} P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_l-\eta}\right)}\left(1-2s\right)\rightarrow \\ U_{n,l}\left(r\right) &= N_{nl}\left(\frac{1-z}{2}\right)^{\sqrt{\varepsilon_{nl}}}\left(\frac{1+z}{2}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma_l-\eta}} P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_l-\eta}\right)}\end{aligned} \qquad (3.35)$$

We obtain the corrections in Eq. (3.31) as follows :

$$\Delta E_{nc}^{gyp}\left(\Theta,\bar{\theta}\right)=\frac{N_{nl}^{2}\alpha^{-1}\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle }{2^{2+2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}}}$$
$$(\tfrac{\mu\bar{\theta}}{2m_{0}}-8\alpha^{4}A^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+3/2}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-2}\left[P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(z\right)\right]^{2}dz$$
$$-32\alpha^{4}A^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+2}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-3}\left[P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(z\right)\right]^{2}dz$$
$$-2^{-1/2}\alpha^{3}B^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+3/2}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-3}\left[P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(z\right)\right]^{2}dz$$
$$-8\alpha^{3}B^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+1}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-2}\left[P_{n}^{\left(2\sqrt{\frac{\varepsilon_{nl}}{4\alpha^{2}}},2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}\right)}\left(z\right)\right]^{2}dz) \tag{3.36}$$

We have replace $\frac{s^{5/2}}{(1-s)^3}$, $\frac{s^3}{(1-s)^4}$, $\frac{s^{5/2}}{(1-s)^4}$and $\frac{s^2}{(1-s)^3}$ with corresponding new values $\frac{1}{2^{-1/2}}\left(1-z\right)^{5/2}\left(1+z\right)^{-3}$, $\frac{1}{2^{-1}}\left(1-z\right)^{3}\left(1+z\right)^{-4}$, $\frac{1}{2^{-3/2}}\left(1-z\right)^{5/2}\left(1+z\right)^{-4}$ and $\frac{1}{2^{-1}}\left(1-z\right)^{2}\left(1+z\right)^{-3}$, respectively. For the ground state $n=0$, we have

$$P_{0}^{\left(2\frac{\sqrt{\varepsilon}}{4\alpha^{2}},2l+1-\frac{\sqrt{\varepsilon}}{4\alpha^{2}}\right)}\left(z\right)=1, \tag{3.37}$$

Thus the above expectation values in Eqs. (3.33) are reduced to the following simple form:

$$\Delta E_{nc}^{gyp}\left(\Theta,\bar{\theta}\right)=\frac{N_{nl}^{2}\alpha^{-1}\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle }{2^{2+2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}}}$$
$$(\tfrac{\mu\bar{\theta}}{2m_{0}}-8\alpha^{4}A^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+3/2}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-2}dz$$
$$-32\alpha^{4}A^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+2}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-32}dz \tag{3.38}$$
$$-2^{-1/2}\alpha^{3}B^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+3/2}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-3}dz$$
$$-8\alpha^{3}B^{'}\Theta\int_{-1}^{+1}\left(1-z\right)^{2\sqrt{\varepsilon_{nl}}+1}\left(1+z\right)^{2\sqrt{\frac{1}{4}+\gamma_{l}-\eta}-2}dz)$$

Comparing Eq.(3.35) with the integral of the form [81]:

$$\int_{-1}^{+1} (1-x)^{n+\alpha} (1+x)^{n+\beta} dx = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)} \quad \text{for } (n=0,1,....) \tag{3.39}$$

Thus, for $n=0$, the above integral reduce to:

$$\int_{-1}^{+1} (1-x)^{\alpha} (1+x)^{\beta} dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)} \text{for } n=0 \tag{3.40}$$

A direct calculation gives the expression of integrals values in Eq.(3.35) as follows:

$$\int_{-1}^{+1} (1-z)^{2\sqrt{\varepsilon_{nl}}+3/2} (1+z)^{2\sqrt{\frac{1}{4}+\gamma_l-\eta}-2} dz = \frac{2^{2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}+1/2}\Gamma\left(2\sqrt{\varepsilon_{nl}}+5/2\right)\Gamma\left(2\sqrt{\frac{1}{4}+\gamma_l-\eta}-1\right)}{\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}+1/2\right)\Gamma\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}+1/2\right)} \equiv A \tag{3.41}$$

$$\int_{-1}^{+1} (1-z)^{2\sqrt{\varepsilon_{nl}}+2} (1+z)^{2\sqrt{\frac{1}{4}+\gamma_l-\eta}-3} dz = \frac{2^{2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}}\Gamma\left(2\sqrt{\varepsilon_{nl}}+3\right)\Gamma\left(2\sqrt{\frac{1}{4}+\gamma_l-\eta}-2\right)}{\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}\right)\Gamma\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}\right)} \equiv K \tag{3.42}$$

$$\int_{-1}^{+1} (1-z)^{2\sqrt{\varepsilon_{nl}}+3/2} (1+z)^{2\sqrt{\frac{1}{4}+\gamma_l-\eta}-3} dz = \frac{2^{2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}-1/2}\Gamma\left(2\sqrt{\varepsilon_{nl}}+5/2\right)\Gamma\left(2\sqrt{\frac{1}{4}+\gamma_l-\eta}-2\right)}{\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}-1/2\right)\Gamma\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}-1/2\right)} \equiv C \tag{3.43}$$

and

$$\int_{-1}^{+1} (1-z)^{2\sqrt{\varepsilon_{nl}}+1} (1+z)^{2\sqrt{\frac{1}{4}+\gamma_l-\eta}-2} dz =$$

$$\frac{2^{2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}}\Gamma\left(2\sqrt{\varepsilon_{nl}}+2\right)\Gamma\left(2\sqrt{\frac{1}{4}+\gamma_l-\eta}-1\right)}{\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}\right)\Gamma\left(2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}\right)} \equiv D \tag{3.44}$$

Thus, the energy correction for the ground state $n = 0$ reduced to the following simple form:

$$\Delta E_{nc}^{gyp}\left(\Theta, \bar{\theta}\right) = \frac{N_{nl}^2 \alpha^{-1} \left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle}{2^{2+2\sqrt{\varepsilon_{nl}}+2\sqrt{\frac{1}{4}+\gamma_l-\eta}}} \left(\frac{\mu\bar{\theta}}{2m_0} - 8A\alpha^4 A^{'}\Theta - 32K\alpha^4 A^{'}\Theta - 2^{-1/2}C\alpha^3 B^{'}\Theta - 8D\alpha^3 B^{'}\Theta\right) \tag{3.45}$$

The global energy E_{0l}^{py} for the ground state $n = 0$ is the energy spectrums:

$$E_{nc\_0l}^{gyp} = E_{0l}^{gyp} + \Delta E_{nc}^{gyp}\left(\Theta, \bar{\theta}\right) \tag{3.46}$$

Where E_{0l}^{py} is determined from Eq.(2.18) which we have seen in the second chapter:

$$E_{0l}^{py} = -V_0 - \frac{2\alpha^2}{\mu}\left[\frac{B^2}{4\alpha} + \frac{2\mu\alpha V_0}{B}(\alpha - 1)\right]^2 \tag{3.47}$$

here $\left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle$ is determined from:

$$\left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle = \left\{ \begin{array}{c} \frac{1}{2}\left\{(l+s)(l+s+1) - l(l+1) - 3/4\right\} k_+ \quad \text{if } j = |l+1/2| \\ \frac{1}{2}\left\{(l-s)(l-s+l) - l(l+1) - 3/4\right\} \equiv k_- \quad \text{if } \ j = |l-1/2| \end{array} \right. \tag{3.48}$$

It is clear that the following physical limit procedure:

$$\left\{ \begin{array}{c} \underset{\left(\Theta,\bar{\theta}\right)\to(0,0)}{Lim} E_{nc\_0l}^{gyp} = E_{0l}^{py} \\ \underset{\left(\Theta,\bar{\theta}\right)\to(0,0)}{Lim} \Delta E_{nc}^{gyp}\left(\Theta, \bar{\theta}\right) = 0 \end{array} \right. \tag{3.49}$$

Gives us all the results of physical treatments which we have seen in the standard references [54, 55].

Chapter 2

Conclusion

Through this master's memory in physics, theoretical specialty: Promotion 2023-2024

The nonrelativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of generalized inverse quadratic Yukawa potential as a model

This memory aims to study physical systems within the framework of the modified Schrödinger equation with the modified generalized inverse quadratic Yukawa potential, in three-dimensional non-commutative phase-space symmetries.

In the first chapter, we have represented the mathematical and physical formalisms of the non-commutative three-dimensional phase-space.

In the second chapter, we reviewed the Shrodinger equation under the generalized inverse quadratic Yukawa potential Known in the literatures.

In the third chapter, we studied the effect of the non-commutativity phase-space, by applying the generalized Bopp shift method and standard perturbation theory at the first order of parameters $\left(\Theta, \bar{\theta}\right)$, the modifications on the energy corresponding to the ground state are obtained. We can conclude that the application of the phase-space non-commutativity in this work on the modified generalized inverse quadratic Yukawa potential, includes the spin-orbit coupling effect automatically. This is in contrast to what we observe in the framework of quantum mechanics known in the literature, where the spin-orbit interaction appears by external addition and not through spontaneous birth as a result of space deformation.

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Abstract

In our work on this master memory, in theoretical physics (2023/2024): The non-relativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of generalized inverse quadratic Yukawa potential as a model. We have studied the Schrödinger equation with the generalized inverse quadratic Yukawa potential in non-commutative three-dimensional spaces and phases, by applying Boop's Shift method to the first order of the parameters (Θ,θ) , in addition to the standard perturbation the theory, to obtain the spectrum of energy of the system, which is changing radically, and replaced by degenerate new states depending on the discrete atomic quantum numbers (j,n,l,s) and the potential parameterize (A′,C′,B′), and the screening parameter α.

Keywords: Schrödinger equation, generalized inverse quadratic Yukawa potential, non-commutative phase-space, star product, Boop's shift method.

ملخص

في عملنا على الذاكرة الماستر في الفيزياء النظرية (2024/2023):
قمنا بالدراسة غير النسبية للطيف الطاقة الناتج عن جهد مركزي في ميكانيكا الكم التناظري. في حالة نموذج الجهد المعمم العكسي التربيعي ليوكاوا.حيث قمنا بدراسة معادلة شرودينغر مع الجهد المعمم العكسي التربيعي ليوكاوا في الفضاء ثلاثي الأبعاد غير التبادلي و ذلك بالتطبيق طريقة ازاحة بوب بالاضافة الى نظرية الاضطرابات المستقرة للحصول على طيف الطاقة للنظام المراد دراسته و الذي يتغير بشكل جذريا و يتم استبداله بحالات جديدة تعتمد على الاعداد الكم (j,n,l,s) و معلومات الجهد (A′,C′,B′) و معامل الحجبα

الكلمات المفتاحية: معادلة شرودنغر، الجهد المعمم العكسي التربيعي ليوكاوا، فضاء الطور غير التبادلي، نموذج النجمي، طريقة بوب للإزاحة..