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Abdelhamid TALLAB

Intitulée

Généralisation des propriétés des opérateurs p -sommants aux opérateurs Lipschitz p -sommants

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M. MOUSSAI
L. MEZRAG
L. ABBAOUI
D. ACHOUR
A. AIBECHE
F. MOKHTARI

Prof. Université Mohamed Boudiaf - M'sila
Prof. Université Mohamed Boudiaf - M'sila
Prof. Université Sétif 1
Prof. Université Mohamed Boudiaf - M'sila
Prof. Université Sétif 1
MCA Université Alger 1

Président
Encadreur
Rapporteur
Rapporteur
Rapporteur
Rapporteur

Didications

To my parents; my mother Aicha and the soul of my father Brahim.

To my brothers and sisters; the soul of Benalia, Chaabane, Salah, Rabeh, Hadda, and Messouda.

To my dear wife Hadjer.

To my angel girle Rahik.

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Résumé.

L'idée de cette thèse nous a été inspirée de l'article de J.-D. Farmer et W.-B. Johnson intitulé, "*les opérateurs Lipschitz p -sommants*". Ils ont posé dans le problème six la question suivante: peut-on généraliser certaines propriétés des opérateurs linéaires p -sommants aux opérateurs Lipschitz p -sommants? Certains auteurs ont répondu à cette question par la généralisation de quelques notions dans le cas linéaire au cas lipschitzien, par exemple les notions de p -nucléaires, p -intégrals, opérateurs fortement lipschitziens p -sommants, les opérateurs (p, σ) -absolument lipschitziens, les opérateurs compacts et faiblement compacts parmi tant d'autres. Dans cette thèse, nous allons généraliser ce concept aux opérateurs $\tau(p)$ -sommants qui sont les opérateurs Lipschitz $\tau(p)$ -sommants et leur version "*multiple $\tau(p)$ -sommant*" suivi de quelques applications sur les espaces L_p . En utilisant l'idée de Defant et Floret (i.e., T est un opérateur linéaire p -sommant si et seulement si $\widehat{T} : \ell_0^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_{\Delta_p} F$ est continu), nous avons généralisé aussi cette idée au cas des opérateurs multilinéaires. Deux extensions naturelles des opérateurs p -sommants ont été étudiées dans le cas multilinéaire qui sont les opérateurs multiples p -sommants et les opérateurs multilinéaires (γ, α, p) -sommants. Ces deux classes ont été intensivement étudiées et sont bien connues.

Mots-clés: opérateurs Lipschitz Cohen p -nucléaires, opérateurs Lipschitz p -sommants, opérateurs Lipschitz $\tau(p)$ -sommants, opérateurs Lipschitz fortement p -sommants, espace de Arens-Eels, opérateurs multiples p -sommants, opérateurs multilinéaires (γ, α, p) -sommants, norme tensorielle.

Abstract.

The idea of this thesis was inspired by the article of J.-D. Farmer and W.-B. Johnson entitled, "*the Lipschitz p -summing operators*". They posed in problem six the following question: what results about p -summing operators have analogues for Lipschitz p -summing operators? Some authors have answered this question by generalizing some concepts in the linear case to the Lipschitz case, for example p -nuclears, p -integrals, Lipschitz strongly p -summing operators, Lipschitz (p, σ) -absolutely operators, compact operators and weakly compact operators among many others. In this thesis, we will generalize this concept to

$\tau(p)$ -summing operators which Lipschitz $\tau(p)$ -summing operators and their version "*multiple $\tau(p)$ -summing*" with some applications on L_p -spaces. Using the idea of Defant and Floret (i.e., T is absolutely p -summing if and only if $\widehat{T} : \ell_0^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_{\Delta_p} F$ is continuous). We have also generalized this idea to the case of multilinear operators. Two natural extensions of p -summing operators have been studied in the multilinear case which are the multiple p -summing operators and (γ, α, p) -summing multilinear operators. These two classes have been intensively studied and are well known

Keywords :Cohen Lipschitz p -nuclear operators, Lipschitz p -summing operators, Lipschitz $\tau(p)$ -summing operators, Lipschitz strongly p -summing operators, Arens–Eells space, multiple summing operators, (γ, α, p) -summing multilinear operators, tensor norm.

Notations

\mathbb{K}	The field of real or complex numbers
\mathbb{R}_+	The field of non negative real numbers
p^*	The conjugate of the number p ($1 \leq p \leq \infty$), that is $\frac{1}{p} + \frac{1}{p^*} = 1$
E, F	Banach spaces
X, Y	Metric spaces
E^*	The topological dual of E
B_E	The closed unit ball of E
$L(E, F)$	The set of all linear operators
$\mathcal{L}(E, F)$	The set of all bonded linear operators
w	The weak topology
w^*	The weak * topology
\mathcal{U}	The ideal operator
\mathcal{U}^{inj}	The injective hull
\mathcal{U}^{surj}	The surjectiv hull
$\mathcal{L}_{p,q}$	(p, q) -factorable operators
$\mathcal{L}_{p,q}^{inj}$	The injective hull of (p, q) -factorable operators
$\mathcal{L}_{p,q}^{sur}$	The surjectiv hull of (p, q) -factorable operators
$\text{Lip}_0(X, Y)$	The set of all Lipschitz operators between X and Y that vanish at 0
$X^\# = \text{Lip}_0(X, \mathbb{R})$	The Lipschitz dual of the pointed metric space X
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X
$m_{xx'}$	The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$, where χ_A is the characteristic function of the set A
$\mathcal{A}(X)$	The Arens-Eells space of X
$\mathcal{F}(X)$	The Lipschitz free space of X
T^*	The adjoint linear operator of T
$T^\#$	The Lipschitz adjoint operator of T
T_L	The linearization of the Lipschitz operator T
$T^t = T^\# \setminus E^*$	The Lipschitz transpose of Lipschitz map $T : X \rightarrow E$
$J_p(1 \leq p < \infty)$	The canonical inclusion map defined between $C(K)$ and $L_p(\mu)$

$I_{\infty,p}$	The formal inclusion map defined between $L_{\infty}(\mu)$ and $L_p(\mu)$ for $(1 \leq p < \infty)$
i_E	The isometric embedding, $i_E : E \longrightarrow C(E^*)$ given by $i_E(x) := \langle x, \cdot \rangle$
Π_p	The set of all linear p -summing operators $(1 \leq p < \infty)$
Π_p^L	The set of all Lipschitz p -summing operator $(1 \leq p < \infty)$
$\Pi_{r,p,q}^L$	The set of all Lipschitz (r, p, q) -summing operators $(1 \leq p, r, q \leq \infty)$
\mathcal{D}_p	The set of all strongly linear p -summing operators $(1 < p \leq \infty)$
$\mathcal{D}_{st,p}^L$	The set of all strongly Lipschitz p -summing operators $(1 < p \leq \infty)$
\mathcal{N}_p	The set of all p -nuclear linear operators $(1 \leq p < \infty)$
\mathcal{N}_p^L	The set of all Lipschitz Cohen p -nuclear operators $(1 \leq p < \infty)$
$\Pi_{\tau(p,q)}$	The vector space of all $\tau(p, q)$ -summing linear operators
$\Pi_{\tau(p,q)}^L$	The vector space of all Lipschitz $\tau(p, q)$ -summing operators
$\Pi_p(E_1 \times \cdots \times E_n, F)$	The class of multiple p -summing n -linear operators
$E \otimes F$	The tensor product between two Banach spaces
$E \otimes_{\alpha} F$	The tensor product $E \otimes F$ endowed with the tensor norm α
$E \widehat{\otimes}_{\alpha} F$	The completion of $E \otimes_{\alpha} F$
α^t	The transpose of the tensor norm α
α' and α^*	The dual and the conjugate of the tensor norm α
$\backslash \alpha$ and $/ \alpha$	The left injective and left projective associate tensor norms
$\alpha \backslash$ and $\alpha /$	The right projective and right injective associate tensor norms
d_p and g_p	Chavet-Saphar tensor norms
ε and π	The injective and projective tensor norms
$X \boxtimes Y$	Lipschitz tensor product between X and Y
$X \widehat{\boxtimes}_{\varepsilon} E$	The injective Lipschitz tensor product of X and E
$X \widehat{\boxtimes}_{\pi} E$	The projectiv tensor product of X and E .

List of publications

1. L.MEZRAG AND A.TALLAB, On Lipschitz $\tau(p)$ –summing operators, *Colloq. Math.* **147** (1) (2017), 95-114.
2. P. RUEDA, E.A. SÁNCHEZ-PÉREZ AND A.TALLAB, *Traced tensor norms and multiple summing multilinear operators*. *Linear and multilinear Algebra*, **65**(4) (2017), 768-786.

Introduction

The notion of p -summing linear operators goes back to Grothendieck in the 1950s, but just in 1967 and 1968, the classical works of Pietsch [Pie67] and Lindenstrauss-Pełczyński [LP68] clarified Grothendieck's precious ideas and contributed clearly to the vigorous development of this notion. A linear operator u between two Banach spaces E and F is called p -summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all x_1, \dots, x_n in E , we have

$$\left(\sum_{i=1}^n \|u(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}}.$$

The infimum of such constants is denoted by $\pi_p(T)$ and the space of all p -summing operators denoted by $\Pi_p(E, F)$ which a Banach space under the norm $\pi_p(T)$.

In fact, the concept of Lipschitz 1-summing operators the first time appeared in a paper of Bourgain [Bou85] proving that every n point metric space can be embedded into a Hilbert space with distortion at most $C \log n$, where C is an absolute constant. Farmer and Johnson [FJ09] proved a Pietsch domination theorem for this class of Lipschitz operators, used it to show that for linear operators in Banach spaces this new notion coincides with the classical notion of p -summing operators, and finished their work with a couple of interesting problem. There are very concrete problems like (decide whether or not there is a Grothendieck theorem for Lipschitz summing operators, and less concrete ones as: what results about p -summing operators have analogues for Lipschitz p - summing operators?).

The aim of this thesis is the generalization and extension of some results and properties about linear p -summing operators to nonlinear operators. Especially (the Lipschitz and the multilinear cases). Let X be pointed metric space, i.e., a metric space with a distinguished

element e_X and let E be a Banach space. According to Framer and Johnson [FJ09], a Lipschitz map $T : X \rightarrow E$ is called Lipschitz p -summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all $x_1, \dots, x_n, x'_1, \dots, x'_n$ in X and all non negative reals $\lambda_1, \dots, \lambda_n$ we have

$$\left(\sum_{i=1}^n \lambda_i \|T(x_i) - T(x'_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x'_i))|^p \right)^{\frac{1}{p}}.$$

The infimum of such constants is denoted by $\pi_p^L(T)$ and the space of all p -summing operators denoted by $\Pi_p^L(X, E)$ endowed with the norm $\pi_p^L(T)$ is a Banach space. We denote by $\mathcal{B}_{X^\#}$ the unit ball of the Lipschitz dual $X^\#$ of X which is a compact Hausdorff space in the topology of pointwise convergence on X .

In the First part of this thesis, we based on the paper [RST17] introduced by Rueda, Sánchez-Pérez and Tallab, which is entitled by "*Traced tensor norms and multiple summing multilinear operators*". Using a general tensor norm approach, our aim is to show that some distinguished classes of summing operators can be characterized by means of an "*order reduction*" procedure for multiple summing multilinear operators, which becomes the keystone of our arguments and can be considered our main result (see [RST17]). We work in a tensor product framework involving traced tensor norms (for this notion see [Def86] and [DF93]) and the representation theorem for maximal operator ideals. Several applications are given not only to multi-ideals, but also to linear operator ideals. In particular, we get applications to multiple p -summing bilinear operators, (p, q) -factorable linear operators, $\tau(p)$ -summing linear operators which as a special case of the definition introduced [Muj08] in the multilinear operators and absolutely p -summing linear operators, providing a characterization of this later class whenever the absolutely p -summing linear operators take values in an L_p -space (see [RST17]).

In the second part of this thesis based on the second paper [MT17] presented by Mezrag and Tallab which call "*On Lipschitz p -summing operators*". The main focus of this article [MT17] is on Lipschitz version for Pietsch's notion of $\tau(p, q)$ -summing operators as well as so-called Cohen p -nuclear operators. The first class was introduced by Pietsch for $p = q = 1$.

In order to understand this (see the book "*operator ideals*" [Pie80]). This notion was generalized by X. Mujica to Multilinear case [Muj08].

The thesis is organized as follows.

In the chapter one, we remind some basic definitions and properties concerning the metric spaces, Lipschitz maps, non linear Hahn-Banach theorem, Lipschitz spaces, the Lipschitz dual of metric spaces, the predual of $\text{Lip}_0(X)$, adjoints of Lipschitz mappings, Arens Eels space and multilinear mappings.

In the chapter two, we introduced some basic definitions related to the spaces of sequences of vectors in Banach space E that are weakly p -summable and p -summable, some "*tensor norm*" in the sense that is used in the book of Defant and Floret [DF93], transpose of tensor norm, the left (right) injective and left (right) projective associate tensor norms. In the second section of this chapter we will introduced the extension of Defant and Floret idea [DF93] (i.e., T is absolutely p -summing if and only if $\widehat{T} : \ell_0^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_{\Delta_p} F$ is continuous) to the case of multilinear operators. In the third section we presented the bilinear multiple (γ, Δ_p, p) -summing operators as an application example on the second section. On the last one in this section we analyzed with our tools the case of the so called $\tau(p)$ -summing operators and their "*multiple version*" introduced by Rueda, Sánchez-Pérez and Tallab in [RST17]. They showing some applications relative to summing operators on L^p -spaces.

In the chapter three, we give a preliminaries concerning the Lipschitz p -summing operators, the domination and factorization theorem for this class. We introduce a new category of Lipschitz operators which we call Lipschitz $\tau(p, q)$ -summing operators. The notion and properties of Lipschitz tensor product of pointed metric space X and a Banach space E , which is defined by M.-G. Cabrera-Padilla and All in [CCJV15] will be introduced in order to study the ideal property of the class of Lipschitz $\tau(p)$ -summing operators. We give two domination theorems this category of operators by using the unified Pietsch domination theorem [BPR10] and Ky Fan's lemma. Also, we give some properties concerning this class.

In the chapter four, we extend the class of Cohen p -nuclear operators introduced in [Coh73] to Lipschitz operators and we presented the ideal property, and the domination

theorem which is essentially proved in previous articles (see [Cha11], [Saa15]). This class was studied by [AMS09] in sublinear operators and by [AA10] in multilinear operators. Also, we recall the class of strongly Lipschitz p -summing operators introduced by [YAR15] and [Saa15]. Finally, we end this chapter by studying the relations between all these classes.

Chapter 1

Preliminaries

1.1 Lipschitz spaces

Definition 1.1.1 *The space X equipped with d is called metric space (X, d) .*

Let (X, d_X, e) be a pointed metric space, i.e., a metric space (X, d_X) with a distinguished or neutral element e (a fixed point in X which is taken to be the zero element if X is a normed space). We denote by \mathcal{M}_0 the class of all complete pointed metric spaces.

1.1.1 Lipschitz functions

The natural morphism between metric spaces are Lipschitz functions like linear operators between Banach spaces.

Definition 1.1.2 *A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that*

$$\forall x, y \in X, \quad d_Y(f(x), f(y)) \leq C d_X(x, y). \quad (1.1.1)$$

If $C = 1$, the map is called nonexpansive (and contraction if $C < 1$). For a Lipschitz map f , we define its Lipschitz constant by

$$\text{Lip}(f) := \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf \{C : C \text{ verifying 1.1.1}\}.$$

If $(X, e_X, d_X), (Y, e_Y, d_Y)$ be pointed metric spaces. We say a map $f : (X, e_X, d_X) \longrightarrow (Y, e_Y, d_Y)$ preserves distinguished point if $f(e_X) = e_Y$.

Definition 1.1.3 *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called bi-Lipschitz or quasi-isometry, if f is bijective and both f, f^{-1} are Lipschitz.*

In this case X and Y are called Lipschitz isomorphic or Lipschitz homeomorphic (Kalton) and quasi-isometric (Weaver [Wea99]).

A bi-Lipschitz function f is an isometry if

$$\forall x, y \in X, \quad d_Y(f(x), f(y)) = d_X(x, y).$$

Let X, Y be (finite) metric spaces, ($|X| = |Y|$). The Lipschitz distance between X, Y is

$$d(X, Y) = \inf \{\text{Lip}(f) \text{Lip}(f^{-1}) ; f \text{ one-to-one from } X \text{ onto } Y\}$$

where $|X|$ denotes the cardinal of X .

In the theory of the nonlinear geometry of Banach spaces, the linear isomorphisms are replaced by bi-Lipschitz maps, the isometric isomorphism correspond exactly isometric and the Banach-Mazur distance by the Lipschitz distance or distortion.

1.1.2 Properties

We are interested in this subsection to study some properties concerning the composition and the extention of Lipschitz operators.

Proposition 1.1.1 *Let X, Y and Z be metric spaces and let $f : (X, d_X) \longrightarrow (Y, d_Y), g : (Y, d_Y) \longrightarrow (Z, d_Z)$ be Lipschitz maps. Then $g \circ f : (X, d_X) \longrightarrow (Z, d_Z)$ is Lipschitz and $\text{Lip}(g \circ f) \leq \text{Lip}(g) \text{Lip}(f)$.*

Proof. For x, y in X , we have

$$\begin{aligned} d_Z(g \circ f(x), g \circ f(y)) &\leq \text{Lip}(g) d_Y(f(x), f(y)) \\ &\leq \text{Lip}(g) \text{Lip}(f) d_X(x, y) \end{aligned}$$

and this shows the proposition. ■

Proposition 1.1.2 [Wea99] *Let X_0 and Y_0 be metric spaces and let X and Y be their completions. Consider a Lipschitz map $f_0 : X_0 \rightarrow Y_0$. Then f_0 has a unique Lipschitz extension $f : X \rightarrow Y$, and furthermore $\text{Lip}(f) = \text{Lip}(f_0)$.*

Proposition 1.1.3 *Let $(X, d_X), (Y, d_Y)$ be metric spaces and let f and $\{f_n\}_{n \in \mathbb{N}}$ be Lipschitz functions from X to Y . Suppose that $f_n \rightarrow f$ pointwise. Then*

$$\text{Lip}(f) \leq \sup_n \text{Lip}(f_n).$$

Proof. Let x, y be in X . We have

$$d_Y(f(x), f(y)) = \lim_{n \rightarrow \infty} d_Y(f_n(x), f_n(y)).$$

Dividing by $d_X(x, y)$ and taking the supremum over x and y . ■

We will present in the next proposition the non linear Hahn-Banach theorem. There is two proofs for this theorem, the first presented by Weaver in [Wea99] which used Zorn's Lemma and the recent presented by Assaf in [Ass15]. We choose the second one.

Theorem 1.1.1 (Non linear Hahn-Banach theorem) [Wea99] *Let E be a subset of a metric space (X, d) and let $f : E \rightarrow l_\infty(I)$ be a Lipschitz function. Then f can be extended to a Lipschitz function $\tilde{f} : X \rightarrow l_\infty(I)$ with the same Lipschitz constant (we say that $l_\infty(I)$ is 1-injective).*

Proof. Define the function $\tilde{f} : X \rightarrow \mathbb{R}$ by the formula

$$\tilde{f}(z) = \inf_{x \in E} (f(x) + \text{Lip}(f)d(x, z)), \quad z \in X.$$

To see that this function satisfies the results, fix an arbitrary $x_0 \in E$. Then, for any $x \in E$

$$\begin{aligned} f(x_0) - f(x) &\leq \text{Lip}(f)d(x_0, x), \\ &\leq \text{Lip}(f)(d(x_0, z) + d(z, x)). \end{aligned}$$

This implies (that $f(x) + \text{Lip}(f)d(x, z)$ is bounded below)

$$f(x_0) - \text{Lip}(f)d(x_0, z) \leq f(x) + \text{Lip}(f)d(x, z).$$

So $\tilde{f}(z)$ is well-defined. Also, if $z \in E$, the above shows that $\tilde{f}(z) = f(z)$. Finally (by definition of the inf), for $z, y \in X$ and $\epsilon > 0$, choose $x_z \in E$ such that

$$\begin{aligned} \tilde{f}(z) &\geq f(x_z) + \text{Lip}(f)d(z, x_z) - \epsilon \\ -\tilde{f}(z) &\leq -f(x_z) - \text{Lip}(f)d(z, x_z) + \epsilon \end{aligned}$$

Then

$$\begin{aligned} \tilde{f}(y) - \tilde{f}(z) &\leq f(x_z) + \text{Lip}(f)d(y, x_z) - f(x_z) - \text{Lip}(f)d(z, x_z) + \epsilon \\ &\leq \text{Lip}(f)d(y, z) + \epsilon. \end{aligned}$$

Thus, we see that \tilde{f} is indeed $\text{Lip}(f)$ -Lipschitz. ■

1.1.3 Lipschitz spaces

Definition 1.1.4 (a) Let (X, d_X) be metric space. Then $\text{Lip}(X)$ is the space of all bounded scalar valued Lipschitz functions on X with the norm

$$\|f\|_L = \max \{ \|f\|_\infty, \text{Lip}(f) \}.$$

(b) Let $(X, d_X), (Y, d_Y)$ be pointed metric spaces. We denote by $\text{Lip}_0(X, Y)$ the set of all base-point preserving Lipschitz maps from X to Y with the norm

$$\text{Lip}(f) := \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

If E is a Banach space, $\text{Lip}_0(X, E)$ is a Banach space under the Lipschitz norm given by

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)}, x \neq y \right\}.$$

For $E = \mathbb{K}$, we designate $\text{Lip}_0(X, \mathbb{K}) = \text{Lip}_0(X) = X^\#$. The Banach space $X^\#$ is called also Lipschitz dual of X . It has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case.

Notation. Designated by $\mathcal{B}_{X^\#}$ the unit ball of $X^\#$. Then $\mathcal{B}_{X^\#}$ is a compact Hausdorff space in the topology of pointwise convergence on X (see [Wea99, Page 39]).

Example 1.1.1 *Let X be a set. We denote by*

$$l_\infty(X) = \left\{ f : X \longrightarrow \mathbb{K} \text{ such that } \sup_{x \in X} |f(x)| < \infty \right\}.$$

Let X be a pointed metric space of finite diameter, i.e., $\sup_{x, y \in X} d(x, y) < \infty$. We have $\text{Lip}_0(X) \subset l_\infty(X)$. Indeed, $\text{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}$. This implies that by taking $y = 0$, $|f(x)| \leq \text{Lip}(f) d(x, 0)$. Consequently, $f \in l_\infty(X)$.

1.2 The predual of $\text{Lip}_0(X)$ and adjoint of Lipschitz mapping

1.2.1 The predual of $\text{Lip}_0(X)$

We shall present first the construction of Arens and Eells [AE56] (see also [Wea99, p. 38]) of the space X for which $\text{Lip}_0(X)$ is the dual space. Remark that another, less explicit, realization of $\text{Lip}_0(X)$ as a dual space was given by de Leeuw [Lee61] (see also [Wea99, p. 33]). It was shown by Arens and Eells [AE56] (see also [Wea99]) that $\text{Lip}_0(X)$ is even a dual Banach space (but not reflexive if X is infinite and does not have constant functions in general), i.e., there exists a Banach space Z such that $\text{Lip}_0(X)$ is isometrically isomorphic to Z^* .

This canonical space is known as the Arens-Eells space in [Wea99, p. 33] noted by $\mathcal{A}(X)$, and the Lipschitz-free space on X in [Kal04] noted by $\mathcal{F}(X, d_X)$.

Construction of this space

Let (X, e, d) be a metric space. A molecule on X is a real valued function m on X with finite support (i.e., the set where m has non-zero values) and satisfies $\sum_{x \in \text{supp}(m)} m(x) = 0$.

Denote by $\mathcal{M}(X)$ the real linear space of molecules on X . We can write

$$\begin{aligned} m &= \sum_{x \in \text{supp}(m)} m(x) \mathbf{1}_{\{x\}} \\ &= \sum_{i=1}^n m(x_i) \mathbf{1}_{\{x_i\}}. \end{aligned}$$

where $\text{supp}(m) = \{x_1, \dots, x_n\}$ and $\mathbf{1}_{\{x\}}$ denotes the characteristic function of the set $\{x\}$. For $x, y \in X$ we define the basic molecule $m_{xy} = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}$. It is easy to see that every molecule m can be written as a (non unique) finite linear combination of basic molecule. We have

$$\begin{aligned} m &= \sum_{j=1}^l \lambda_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}} \right) \\ &= \sum_{j=1}^l \lambda_j m_{x'_j x_j}. \end{aligned}$$

The condition $\sum_{i=1}^n m(x_i) = 0$ insures that such representations of m exist. Put

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^n |\lambda_j| d_X(x_j, x'_j) \right\},$$

over all representation of $m = \sum_{j=1}^n \lambda_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}} \right)$.

It follows that $\|\cdot\|_{\mathcal{M}(X)}$ is a norm on the vector space $\mathcal{M}(X)$. Arens Eells space is the completion of the normed space $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)})$ and denote by $\mathcal{A}(X, d_X)$. This space was first introduced by Arens and Eells [AE56] in 1956. Originally, the basic idea goes back to Kantorovich [Kan42]. The terminology Arens-Eells space $\mathcal{A}(X, d_X)$ is due to Weaver [Wea99]. A different notation was used in [KG03] by Godefroy and Kalton. It is the

Lipschitz-free space denoted by $\mathcal{F}(X, d_X)$ (see [Kal04]) and is defined to be the canonical predual of $\text{Lip}_0(X)$, i.e., the closed linear span of the points evaluations

$$\delta_X(x)(f) = \langle \delta_X(x), f \rangle = f(x), \quad x \in X$$

in $\text{Lip}_0(X)^*$ $\overline{\{\delta_X(x)\}_{x \in X}}^{\text{Lip}_0(X)^*} = \mathcal{F}(X, d_X)$.

The application $\delta_X : X \longrightarrow \mathcal{F}(X, d_X)$ is an isometric embedding. We can see $\mathcal{F}(X, d_X)$ as the completion of the set of all measures μ of finite support under the norm

$$\|\mu\| = \sup \left\{ \int f d\mu : \text{Lip}(f) \leq 1 \right\}.$$

Properties

The Banach space $\mathbb{A}(X, d_X)$ has some remarkable properties, as follows.

Theorem 1.2.1 ([Wea99]) *Let (X, e, d) be a pointed metric space. Then $\mathbb{A}^*(X, d_X) \equiv \text{Lip}_0(X)$.*

Proof. The application $i_X : X \longrightarrow \mathbb{A}(X, d_X)$ defined by

$$i_X(x) = (\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}})$$

is an isometric embedding of X into $\mathbb{A}(X, d_X)$. Indeed, we have

$$\|i_X(x) - i_X(y)\|_{\mathbb{A}} = \|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\|_{\mathbb{A}} = d(x, y).$$

Define

$$S : \mathbb{A}^*(X, d_X) \longrightarrow \text{Lip}_0(X)$$

by

$$(S\varphi)(x) = \varphi((\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}})).$$

Since $\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}\|_{\mathbb{A}(X, d_X)} \leq d(x, x')$ for all $x, x' \in X$, we have

$$\begin{aligned}
 |(S\varphi)(x) - (S\varphi)(x')| &= |\varphi((\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}})) - \varphi((\mathbf{1}_{\{x'\}} - \mathbf{1}_{\{e\}}))| \\
 &= |\varphi((\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}))| \\
 &\leq \|\varphi\| d(x, x').
 \end{aligned}$$

Also $(S\varphi)(e) = \varphi(0)$, so indeed $S\varphi \in \text{Lip}_0(X)$. It follows that S is a nonexpansive linear mapping from $\mathcal{A}^*(X, d_X)$ to $\text{Lip}_0(X)$.

Define now $R : \text{Lip}_0(X) \longrightarrow \mathcal{A}^*(X, d_X)$ by

$$(Rf)(m) = \sum_x m(x) f(x).$$

For $f \in \text{Lip}_0(X)$ and m a molecule. If $m = \sum_{j=1}^m \lambda_j (\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}})$, we have

$$\begin{aligned}
 |(Rf)(m)| &= \left| (Rf) \left(\sum_x m(x) f(x) \right) \right| \\
 &\leq \sum_{j=1}^m |\lambda_j| |f(x_j) - f(x'_j)| \\
 &\leq \text{Lip}(f) \sum_{j=1}^m |\lambda_j| d(x, x').
 \end{aligned}$$

Hence $|(Rf)(m)| \leq \text{Lip}(f) \|m\|_{\mathcal{M}(X)}$, which uniquely extends to a continuous linear functional on the completion $\mathcal{A}(X, d_X)$ of $\mathcal{M}(X)$, denoted by the same symbol Rf . Thus $Rf \in \mathcal{A}^*(X, d_X)$ and $\|Rf\| \leq \text{Lip}(f)$. Straightforward calculations show that R and S are inverses, so that $\text{Lip}_0(X)$ is isometrically isomorphic to $\mathcal{A}^*(X, d_X)$. ■

Corollary 1.2.1 ([Wea99]) *Let (X, e, d) be a pointed metric space.*

- (a) *For any molecule m we have $\|m\|_{\mathcal{E}} = \sup_{f \in \mathcal{B}_{X^\#}} |\langle m, f \rangle|$ and there exists $f \in \mathcal{B}_{X^\#}$ with $|\langle m, f \rangle| = \|m\|_{\mathcal{E}}$.*
- (b) *$\|\cdot\|_{\mathcal{E}}$ is a norm on the space of molecules and $\|m_{xy}\|_{\mathcal{E}} = d(x, y)$ for all $x, y \in X$.*
- (c) *$\|\cdot\|_{\mathcal{E}}$ is the largest seminorm on the space of molecules which satisfies $\|m_{xy}\|_{\mathcal{E}} \leq d(x, y)$ for all $x, y \in X$.*

Lemma 1.2.1 *Let $(X_1, e_1, d_{X_1}), (X_2, e_2, d_{X_2})$ be two pointed metric spaces. Let $T : X_1 \rightarrow X_2$ be a Lipschitz map such that $T(e_1) = e_2$. Then, there is a unique map*

$$\widehat{T} : \mathcal{A}(X_1, d_{X_1}) \longrightarrow \mathcal{A}(X_2, d_{X_2})$$

such that $\widehat{T}i_{X_1} = i_{X_2}T$, i.e., the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{T} & X_2 \\ \downarrow i_{X_1} & & \downarrow i_{X_2} \\ \mathcal{A}(X_1, d_{X_1}) & \xrightarrow{\widehat{T}} & \mathcal{A}(X_2, d_{X_2}) \end{array}$$

and $\|\widehat{T}\| = \text{Lip}(T)$.

Theorem 1.2.2 ([Wea99, Theorem 2.2.4]) *Let (X, e, d) be a pointed metric space.*

(a) *The map $i_X : X \rightarrow \mathcal{A}(X, d_X)$ defined by $i_X(x) = m_{xe} = (\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}})$ is an isometric embedding of X into $\mathcal{A}(X, d_X)$.*

(b) *Let $T : X \rightarrow E$ be a Lipschitz map which preserves base point (i.e., $T(e) = 0$). Then there is a unique bounded linear operator $T_L : \mathcal{A}(X, d_X) \rightarrow E$ such that $T = T_L \circ i_X$ and $\|T_L\| = \text{Lip}(T)$. The linear operator T_L is called the linearization of T .*

$$\begin{array}{ccc} \mathcal{A}(X) & & \\ i_X \downarrow & \searrow T_L & \\ X & \xrightarrow{T} & E \end{array}$$

Proof. (a) By Corollary 1.2.1, we have for all $x, y \in X$

$$\|m_{xy}\|_{\mathcal{E}} = \|m_{xe} - m_{ye}\|_{\mathcal{E}} = \|i_X(x) - i_X(y)\|_{\mathcal{E}} = \|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\|_{\mathcal{E}} = d(x, y).$$

So i_X is an isometry.

(b) Every molecule m is uniquely expressible in the form

$$\sum_{j=1}^n \lambda_j m_{x_j e} = \sum_{j=1}^l \lambda_j (\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{e\}}).$$

Where the points x_j are all distinct and none equals e . We then define T_L by

$$T_L(m) = \sum_{j=1}^n \lambda_j T(x_j).$$

Since T_L is essentially an extension of T that is, $T = T_L \circ i_X$ we automatically have $\|T_L\| \geq \text{Lip}(T)$. For the rest it will suffice to show that $\|T_L\| \geq \text{Lip}(T)$ (in particular, this implies that T is bounded, hence it extends to all of $\mathcal{E}(X)$). Define now a seminorm $\|\cdot\|_0$ on the space of molecules by setting $\|m\|_0 = \frac{\|T_L\|}{\text{Lip}(T)}$. Then

$$\|m_{xy}\|_0 = \|\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{y\}}\|_0 = \frac{\|T(x) - T(y)\|}{\text{Lip}(T)} \leq d(x, y)$$

for all $x, y \in X$, so Corollary 1.2.1 implies that $\|\cdot\|_0 \leq \|\cdot\|_{\mathcal{E}}$. Thus

$$\|T_L(m)\| \leq \text{Lip}(T) \cdot \|m\|_{\mathcal{E}},$$

which shows that $\|T_L(m)\| \leq \text{Lip}(T) \cdot \|m\|_{\mathcal{E}}$, as desired. ■

Remark 1.2.1 *On bounded subsets of $\text{Lip}_0(X)$ its weak* topology (i.e., $(\text{Lip}_0(X), \sigma(\text{Lip}_0(X), \mathcal{E}(X, d_X)))$) agrees with the topology of pointwise convergence. Indeed, If $T_i \rightarrow T$ for the weak* topology in $\text{Lip}_0(X)$, then*

$$T_i(x) = (RT_i)(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}) \rightarrow (RT)(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}) = T(x)$$

for all x in X . Hence weak* convergence implies pointwise convergence.

1.2.2 Adjoint of Lipschitz mapping

Let X, Y be pointed metric spaces. Sawashima in [Saw75] defined the Lipschitz adjoint (or dual) $T^\# : \text{Lip}_0(Y) \rightarrow \text{Lip}_0(X)$ of a Lipschitz map $T \in \text{Lip}_0(X, Y)$ by the formula

$$T^\#(g) = g \circ T, \quad g \in \text{Lip}_0(Y).$$

He showed that $T^\#$ is a continuous linear operator and that

$$\|T^\# \| = \text{Lip}(T) = \|T^\# |_{Y^*} \| \text{ (if } Y \text{ is a Banach space).}$$

We shall show that $T^\#$ corresponds in a canonical way to the usual adjoint of the linear operator attached to T by the theorem of linearization i.e., $T^\# |_{Y^*} = T_L^*$.

$$\begin{array}{ccc} \text{Lip}_0(E) & \xrightarrow{T^\#} & \text{Lip}_0(X) \\ p \downarrow & & T_L^* \nearrow \\ E^* & & \end{array}$$

Proposition 1.2.1 *Consider X, Y in \mathcal{M}_0 and let $T : X \rightarrow Y$ be a Lipschitz map which preserves base point. Then $T^\#$ is a bounded linear map and $\|T^\# \| = \text{Lip}(T) = \|T^\# |_{Y^*} \|$. The map $T^\#$ is compatible with products and preserves order.*

Proof. We have

$$\text{Lip}(T^\#(g)) = \text{Lip}(g \circ T) \leq \text{Lip}(g) \text{Lip}(T)$$

so $\|T^\# \| \leq \text{Lip}(T)$. For the converse inequality, fix $p, q \in Y$. Let $g = d_Y(\cdot, q) - d_Y(e_Y, q)$, then $\text{Lip}(g) = 1$ and

$$\begin{aligned} \|T^\# \| &\geq \text{Lip}(T^\#(g)) \\ &\geq \frac{|T^\#(g)(x) - T^\#(g)(y)|}{d_X(x, y)} \\ &\geq \frac{|gT(x) - gT(y)|}{d_X(x, y)} \\ &\geq \frac{|gT(x) - gT(y)| d_Y(T(x), T(y))}{d_Y(T(x), T(y)) d_X(x, y)}. \end{aligned}$$

Taking the supremum over x and y , we find $\|T^\# \| \geq \|T\|$. ■

Remark 1.2.2 *We have $(T_1 T_2)^\# = T_2^\# T_1^\#$.*

We will use the following classical results.

Proposition 1.2.2 Consider X, Y in \mathcal{M}_0 and let $T : X \longrightarrow Y$ be a Lipschitz map which preserves base point. Then

- (a) $T^\#$ is surjective if, and only if, $T : X \longrightarrow T(X)$ is a quasi-isometry.
- (b) $T^\#$ is injective if, and only if, $T(X)$ is dense in Y .
- (c) $T^\#$ is an isomorphism if, and only if, T is a quasi-isometry.

Proof. (a) Suppose that $T^\#$ is surjective. By the open mapping theorem, for some $r > 0$ the set $T^\#(B_{\text{Lip}_0(X)}(0, r))$ contains $B_{\text{Lip}_0(Y)}(0, 1)$. Thus, for any $x \in X$ there exists $g \in \text{Lip}_0(Y)$ with $\text{Lip}(g) \leq r$ such that $T^\#(g) = g = d_Y(\cdot, y) - d_Y(e_Y, y)$, and

$$d(x, y) = |g(x) - g(y)| = |T(g(x)) - T(g(y))| \leq rd(g(x), g(y))$$

for all $y \in X$.

(b) and (c) easy. ■

If $Y = E$ is a Banach space, the restriction of $T^\#$ to E^* is called the Lipschitz transpose map of T and is denoted here by T^t . The correspondence

$$T \longleftrightarrow T^t$$

establishes an isomorphism between the vector spaces $\text{Lip}_0(X, E)$ and $\mathcal{L}((E^*, w^*), (X^\#, w^*))$, where w^* denote the weak* topology (see [JSV14, Theorem 3.1]).

Let $T \in \text{Lip}_0(X, Y)$ and let i_X, i_Y be the isometric embeddings of X, Y into $\text{Lip}_0(X), \text{Lip}_0(Y)$, respectively). Let $\Psi(T) : \mathcal{A}(X, d_X) \longrightarrow Y$ be the bounded linear operator attached to T by Theorem 1.2.2, and let $\phi = i_Y \circ \Psi$. Let S, R be the linear isometrics between the spaces $\text{Lip}_0(X)$ and $\mathcal{A}(X, d_X)$, and $\text{Lip}_0(Y)$ and $\mathcal{A}(Y, d_Y)$.

Theorem 1.2.3 [Cob03] We have $T^\# = S_1 \circ \phi(T)^* \circ R_2$ or equivalently $\phi(T)^* = R_1 \circ T^\# \circ S_2$, i.e., the following diagrams are commutative

$$\begin{array}{ccc} \mathcal{A}(Y, d_Y)^* & \xrightarrow{\phi(T)^*} & \mathcal{A}(X, d_X)^* \\ R_2 \uparrow & & S_1 \downarrow \\ \text{Lip}_0(Y) & \xrightarrow{T^\#} & \text{Lip}_0(X) \end{array}$$

or equivalently

$$\begin{array}{ccc} \mathcal{A}(Y, d_Y)^* & \xrightarrow{\phi(T)^*} & \mathcal{A}(X, d_X)^* \\ S_2 \downarrow & & R_1 \uparrow \\ \text{Lip}_0(Y) & \xrightarrow{T^\#} & \text{Lip}_0(X) \end{array}$$

Proof. We have

$$\phi(m_{x\epsilon}) = i_Y(\Psi(T))(m_{x,0}) = i_Y(T(x)) = m_{T(x),0}. \quad (1.2.1)$$

Put

$$F = S_1 \circ \phi(T)^* \circ R_2.$$

Therefore

$$\begin{aligned} (S_1\varphi)(x) &= \varphi(M_{x,0}), & x \in X, \varphi \in \mathcal{A}(X)^* \\ \phi(T)^*(\psi) &= \psi \circ \phi(T), & \psi \in \mathcal{A}(Y)^* \\ (R_2g)(m) &= \sum_{y \in Y} m(y)g(y), & g \in \text{Lip}_0(Y), m \in M(Y). \end{aligned}$$

Taking into account these formula, the definitions of the operators R and S , and equality 1.2.1, we obtain successively

$$\begin{aligned} (Fg)(x) &= (S_1 \circ \phi(T)^* \circ R_2)(g)(x) = S_1(\phi(T)^*(R_2(g)))(x) \\ \phi(T)^*(\psi) &= S_1(R_2(g) \circ \phi(T))(x) \\ (R_2g)(m) &= S_1(R_2(g) \circ \phi(T))(m_{x,0}) \\ &= R_2(g)(m_{x,0}) \\ &= g \circ T(x) = T^\#(g)(x). \end{aligned}$$

And Theorem 1.2.3 is proved. ■

1.3 Continuous multilinear mappings

Let $m \in \mathbb{N}^*$ and E_1, \dots, E_m, F be Banach spaces over \mathbb{K} . We will denote by $\mathcal{L}(E_1, \dots, E_m; F)$ the space of all continuous m -linear operators from $E_1 \times \dots \times E_m$ into F i.e., there is a constant $C \geq 0$ such that

$$\|T(x^1, \dots, x^m)\| \leq C \|x^1\| \dots \|x^m\|. \quad (1.3.1)$$

If $F = \mathbb{K}$, we write $\mathcal{L}(E_1, \dots, E_m)$. In the case $E_1 = \dots = E_m = X$, we will use Nachbin's notation $\mathcal{L}({}^m E; F)$ (for the general theory of multilinear operators we refer to Dineen [Din99]). As usual, $E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_m$ stands for the (complete) projective tensor product of the Banach spaces E_1, \dots, E_m . If $T \in \mathcal{L}(E_1, \dots, E_m; F)$, we denote by \widetilde{T} the linearization of T , which is the linear map $\widetilde{T} : E_1 \otimes \dots \otimes E_m \rightarrow F$ given by

$$\widetilde{T}(\sum_{i=1}^n x_i^1 \otimes \dots \otimes x_i^m) = \sum_{i=1}^n T(x_i^1, \dots, x_i^m)$$

for all $x_i^j \in E_j$ ($n \in \mathbb{N}, 1 \leq i \leq n, 1 \leq j \leq m$). The correspondence $T \longleftrightarrow \widetilde{T}$ is an isometric isomorphism between the Banach spaces $\mathcal{L}(X_1, \dots, X_m; Y)$ and $\mathcal{B}(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_m; F)$ ($\|T\| = \|\widetilde{T}\|$), see for example [Ray01, Theorem 2.9]. We let

$$\|T\| = \inf \{C \geq 0, \text{ verifying the inequality (1.3.1)}\}.$$

This defines a norm on $\mathcal{L}(E_1, \dots, E_m; F)$ which is complete norm when $\|\cdot\|_F$ is complete.

Chapter 2

Traced tensor norms and multiple summing multilinear operators

2.1 Introduction and basic definitions

A considerable effort has been made in recent years to set a unified theory for multilinear operator ideals that are defined by summability properties. Generalizing the linear case, a broad family of multi-ideals that are defined by a vector norm inequality involving any kind of summability have been introduced in the mathematical literature (see [Gei84, Ma93, Pie83] for early attempts). Some of them are for instance the p -summing multilinear operators, on the Cohen strongly p -summing multilinear operators, absolutely continuous multilinear operators, $(p_1, \dots, p_n; p)$ -dominated multilinear operators, multiple p -summing operators and factorable p -summing operators, among others ([ADRS12, BBJP09, BP09, BPR10, DAS12, Dim03, PRS, PS11a, PS11b, PSS12a, PV03]).

The tensor product point of view is a powerful approach for the study of operator ideals. In particular, the comparison of different topologies on tensor products allows to prove results on their structure and provides characterizations of the most common ideals. This methodological approach was presented in the classical work by Defant and Floret [DF93]. As far as we know, some beautiful ideas appearing in Ch.19 of this book have not been used yet in the multilinear context. In particular, the advised reader may find there that a great

part of the linear operator ideals can be described in terms of continuity of the canonically defined tensor product operator

$$id \otimes T : \ell^p \otimes_\beta E \rightarrow \ell^p \otimes_\alpha F \tag{2.1.1}$$

for adequate p and tensor norms β and α (see [DF93, Theorem 29.4]). For instance, the linear operator $T : E \rightarrow F$ is absolutely p -summing if and only if, $id \otimes T : \ell^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_{\Delta_p} F$ is continuous, where Δ_p satisfies $\Delta_p(\sum_{k=1}^n e_k \otimes x_k) = (\sum_{k=1}^n \|x_k\|^p)^{1/p}$ and ε is the injective tensor norm; or $T : E \rightarrow F$ is p -dominated if and only if $id \otimes T : \ell^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_\pi F$ is continuous, where π is the projective tensor norm.

Also, the so called calculus of traced tensor norms ([Def86]) will be a fundamental tool in this chapter (see [RST17]). It is a very useful tool when dealing with topological tensor products —and so with operator ideals— that surprisingly enough has not been used very often, although it provides a clear point of view for the study of composition and quotients of operator ideals. The reader can find in [DP08] one of the rare applications of this technique in a similar context. We use the presentation of this theory that is given in [DF93, Chapter 29].

We show also that some multi-ideals can be understood using a unified tensor product point of view for the description of the ideals of multilinear operators in the same way that it is shown in the linear case. The main idea is to consider general classes of summing n -linear operators with respect to different tensor norms and characterize their summability properties in terms of an associated $(n - 1)$ -linear operator. This leads to an order reduction procedure that is faced in the first part of the chapter. The order reduction theorem (Theorem 2.2.1) is our main result. Afterwards, we show how our general results look like when we restrict the attention to the bilinear case. Some applications are given to several ideals of summing operators, as multiple p -summing bilinear operators or p -factorable linear operators. These applications provide some tools that will be used in the last part where $\tau(p)$ -summing linear operators are considered. As an example, we will use in this last part our ideas in the bilinear setting to get information for absolutely p -summing linear operators, showing that sometimes the multilinear point of view allows a better understanding of the linear problems: we will provide a —as far as we know— new characterization of absolutely

p -summing linear operators having values in a subspace or a quotient of an L^p -space. In order to do that, we use our tools to describe the so called $\tau(p)$ -summing linear operators and the corresponding “multiple version”.

Let us give now some background informations. Let E, F be (real or complex) Banach spaces and write F^* for the dual of F . Let \mathbb{K} be the real or complex scalar field. As usual, $\mathcal{L}(E, F)$ denotes the space of continuous linear operators from E to F endowed with the sup norm. For $1 \leq p < \infty$, let p^* be the conjugate of p , that is, $1/p + 1/p^* = 1$. We write $\ell_w^p(E)$ and $\ell^p(E)$ for the spaces of sequences of vectors in E that are weakly p -summable and p -summable, respectively. If $(a_i)_{i=1}^\infty$ is such a sequence, we write $\|(a_i)_{i=1}^\infty\|_{w,p}$ and $\|(a_i)_{i=1}^\infty\|_p$ for the corresponding p -weak and p -strong sums. We denote by $\ell_w^{p,0}(E) := \{(x_i)_{i=1}^\infty \in \ell_w^p(E) : \|(x_i)_{i=N}^\infty\|_{w,p} \rightarrow 0 \text{ as } N \rightarrow \infty\}$, and let ℓ_0^p denote the subspace of ℓ^p of all sequences with only a finite number of nonzero “coordinates”. The canonical unit vector basis of ℓ^p is denoted by $(e_i)_{i=1}^\infty$.

Our notation regarding tensor products and operator ideals is standard. We will use the term “tensor norm” in the sense that is used in [DF93], but including the norm Δ_p for the case of tensor products involving $L^p(\mu)$ spaces. As usual, $E \otimes_\alpha F$ denotes the tensor product $E \otimes F$ endowed with the tensor norm α and $E \widehat{\otimes}_\alpha F$ is its completion. The norm Δ_p on $L^p \otimes F$ is the one that comes from the Bochner space $L^p(F)$ such that $\Delta_1 = \pi$ on $L^1 \otimes F$ and $\Delta_\infty = \varepsilon$ on $L^\infty \otimes F$. We will consider several tensor norms on a tensor product $E \otimes F$. The most common ones are the projective norm π or the injective norm ε . It is well known that the space $\ell_w^{p,0}(F)$ is isometrically isomorphic to $\ell^p \widehat{\otimes}_\varepsilon F$ whereas $\ell^p(F)$ is isometrically isomorphic to $\ell^p \widehat{\otimes}_{\Delta_p} F$ (see [DF93, 12.9]), Such that Cohen in [Coh73], also they showed that $\ell^1 \widehat{\otimes} F$ is isometrically to $\ell^1(F)$.

If α is a tensor norm, we will write as usual α^t for its transpose, such that α^t is defined by $\alpha^t(z; E, F) = \alpha^t(z^t; F, E)$. It is obvious that α^t is a tensor norm and $\alpha^{tt} = \alpha$. The tensor norms ε and π are symmetric, i.e., $\varepsilon^t = \varepsilon$ and $\pi^t = \pi$, α' for its dual and α^* for its conjugate tensor norms ([DF93, Ch.12]). Moreover, we will write $\backslash\alpha$ and $/\alpha$ for the left injective and left projective associate tensor norms, respectively, and $\alpha\backslash$ and $\alpha/$ for the right projective and the right injective associate (see [DF93, Ch.20]). A tensor norm α on normed spaces

(or on finite dimensional Banach spaces, or other classes) is called right-injective on normed spaces (or on finite dimensional Banach spaces, or ...) if for all metric injections $I : F \hookrightarrow G$ the operator $id_E \otimes I : E \otimes_\alpha F \longrightarrow E \otimes_\alpha G$ is metric injection for E, F, G are normed spaces (or on finite dimensional Banach spaces, or...) and α is right-projective on normed spaces (or on finite dimensional Banach spaces) if for all metric surjections $Q : F \twoheadrightarrow G$ the operator $id_E \otimes Q : E \otimes_\alpha F \longrightarrow E \otimes_\alpha G$ is metric surjection for E, F, G are normed spaces (or on finite dimensional Banach spaces). If α^t is right-injective (resp. right-projective), then the tensor norm α is called left-injective (resp. left-projective) on normed spaces (or on finite dimensional Banach spaces); if α is right- and left-injective (resp. right- and left-projective), it is called injective (resp. projective) on normed spaces (or on finite dimensional Banach spaces). Clearly, ε is injective and π is projective. The related operator ideal notions are the injective hull \mathcal{U}^{inj} and the surjective hull \mathcal{U}^{sur} of a given operator ideal \mathcal{U} ([DF93, Ch.9]). If $1 \leq p \leq \infty$ and $z \in E \otimes F$, recall that

$$g_p(z) := \inf \{ \| (x_i)_i \|_p \| (y_i)_i \|_{w,p^*} : z = \sum_{i=1}^n x_i \otimes y_i \}$$

$$d_p(z) := \inf \{ \| (x_i)_i \|_{w,p^*} \| (y_i)_i \|_p : z = \sum_{i=1}^n x_i \otimes y_i \}.$$

Our main reference for the theory of tensor norms is [DF93], and for operator ideals that book and also [DJT95]. A lot of rather technical notions regarding tensor norms will be used in this chapter; we have tried to explain all of them, but sometimes it is not easy in the context of a chapter. By this reason, we will refer to the corresponding chapter of [DF93] when some new definition is introduced.

Let $T : E \rightarrow F$ be a continuous linear operator. We say that the bilinear operator $B_T : E \times F^* \rightarrow \mathbb{R}$ given by $B_T(a, b^*) := \langle T(a), b^* \rangle$, $(a, b^*) \in E \times F^*$, is the *bilinear map associated to T* . Given T , the operator $\widehat{T} : \ell_w^p(E) \rightarrow \ell_w^p(F)$ given by $\widehat{T}((a_i)_{i=1}^\infty) := (T(a_i))_{i=1}^\infty$, $(a_i)_{i=1}^\infty \in \ell_w^p(E)$ is well defined and continuous, and we refer to it as the *associated sequential operator* of T .

Let $\Pi_p(E, F)$ denote the space of all absolutely p -summing operators from E to F , endowed with its usual norm π_p . It is well known that T is p -summing if and only if \widehat{T} takes its values in $\ell^p(F)$ and $\widehat{T} : \ell_w^p(E) \rightarrow \ell^p(F)$ is continuous.

We will also make use of the ideal of $\tau(p)$ -summing linear operators. Since this operator ideal is not known for many researchers, let us explain its definition and its multiple version here. A linear operator $T : E \rightarrow F$ is $\tau(p)$ -summing if there is a constant $C > 0$ such that

$$\left(\sum_{i=1}^n |\langle T(a_i), b_i^* \rangle|^p \right)^{1/p} \leq C \sup_{\|a^*\| \leq 1, \|b^{**}\| \leq 1} \left(\sum_{i=1}^n |\langle a_i, a^* \rangle \langle b_i^*, b^{**} \rangle|^p \right)^{1/p}$$

for all $a_1, \dots, a_n \in E$, $b_1^*, \dots, b_n^* \in F^*$, $n \in \mathbb{N}$. The infimum of all constants $C > 0$ is denoted by $\pi_{\tau(p)}(T)$.

Since the definition involves a vector norm inequality for the associate bilinear map $B_T : E \times F^* \rightarrow \mathbb{K}$ that cannot be reduced to a norm inequality for the linear map T , it seems natural to expect that the "multiple linear version" of this ideal provides a new class of operators. We say that a linear operator $T : E \rightarrow F$ is *multiple $\tau(p)$ -summing* if for $a_1, \dots, a_m \in E$ and $b_1^*, \dots, b_n^* \in F^*$,

$$\left(\sum_{j=1}^m \sum_{i=1}^n |\langle T(a_i), b_j^* \rangle|^p \right)^{1/p} \leq C \sup_{\|a^*\| \leq 1, \|b^{**}\| \leq 1} \left(\sum_{j=1}^m \sum_{i=1}^n |\langle a_j, a^* \rangle \langle b_i^*, b^{**} \rangle|^p \right)^{1/p}. \quad (2.1.2)$$

Clearly, the above inequality is equivalent to

$$\left(\sum_{j=1}^m \sum_{i=1}^n |\langle T(a_i), b_j^* \rangle|^p \right)^{1/p} \leq C \|(a_j)_{j=1}^m\|_{w,p} \|(b_i^*)_{i=1}^n\|_{w,p}.$$

The infimum of all constants $C > 0$ is denoted by $\pi_{m,\tau(p)}(T)$.

The class of $\tau(p)$ -summing operators is a subclass of all p -summing operators and so a nicer behavior of the associated sequential operator is expected. Indeed, taking $a_i = a$ for all $i = 1, \dots, n$ in the definition we get for any $\tau(p)$ -summing operator $T : E \rightarrow F$ that

$$\sum_{j=1}^n |\langle T(a), b_j^* \rangle|^p \leq \pi_{\tau(p)}(T)^p \sup_{\|b^{**}\| \leq 1} \left(\sum_{j=1}^n |\langle b_j^*, b^{**} \rangle|^p \right) \|a\|^p.$$

If we apply such an inequality for $a_1, \dots, a_m \in E$ and do the sum, it follows

$$\sum_{i=1}^m \sum_{j=1}^n |\langle T(a_i), b_j^* \rangle|^p \leq \pi_{\tau(p)}(T)^p \sum_{i=1}^m \|a_i\|^p \sup_{\|b^{**}\| \leq 1} \left(\sum_{j=1}^n |\langle b_j^*, b^{**} \rangle|^p \right).$$

This shows that $\widehat{T}((a_i)_{i=1}^m) : F^* \rightarrow \ell^p$ is in fact a p -summing operator and that

$$\pi_p(\widehat{T}((a_i)_{i=1}^m)) \leq \pi_{\tau(p)}(T) \|(a_i)_{i=1}^m\|_p.$$

Consequently, $\widehat{T} : \ell^p(E) \rightarrow \Pi_p(F^*, \ell^p)$ is continuous and $\|\widehat{T}\| \leq \pi_{\tau(p)}(T)$.

A similar calculation easily shows that T is multiple $\tau(p)$ -summing if, and only if, \widehat{T} not only takes $\ell^p(E)$ to $\Pi_p(F^*, \ell^p)$ but also the bigger space $\ell_w^p(E)$ is mapped into $\Pi_p(F^*, \ell^p)$ in a continuous way, and that $\|\widehat{T}\| = \pi_{m, \tau(p)}(T)$. Although (multiple) $\tau(p)$ -summing operators are defined in a linear context, their definition is essentially of a bilinear nature because the inequality (2.1.2) depends on the operator T via the bilinear map $B_T(a, b^*) = \langle T(a), b^* \rangle$, $(a, b^*) \in E \times F^*$, which clearly is absolutely p -summing. Let us pay attention to this bilinear map B_T . Note that T is nothing but the operator $(B_T)_2(a)(b^*) := B_T(a, b^*)$. Therefore, the summability of the bilinear map B_T can be characterized by means of the summability of the associated sequential operator $\widehat{T} = \widehat{(B_T)_2}$. Our main aim in this chapter is to show that this is a particular case of a more general situation: several classes of distinguished summing bilinear (indeed, multilinear) operators can be characterize in terms of its associated sequential operator. Having in mind that $\ell^p(E) = \ell^p \widehat{\otimes}_{\Delta_p} E$ and that $\ell_w^{p,0}(E) = \ell_0^p \widehat{\otimes}_\varepsilon E$, these distinguished classes are determined by considering an arbitrary reasonable tensor norm γ : the (multiple) (γ, Δ_p, p) -summing multilinear operators. This is done in Section 2, where our main result (Theorem 2.2.1) is proved: it characterizes multiple (γ, Δ_p, p) -summing multilinear operators by means of the associated sequential operators. As a consequence of the Chevet-Persson-Saphar inequalities and some well known equivalences for different tensor norms, in Section 3 we provide the first examples where Theorem 2.2.1 applies: multiple p -summing bilinear operators and p -factorable linear operators. In Section 4 of this chapter we get back on (multiple) $\tau(p)$ -summing operators, that were our initial motivation. We analyze them and we end this section with an application to tensor products involving L^p -spaces that will give the coincidence of p -summing operators and multiple $\tau(p)$ -summing operators whenever they take values on a L^p -space.

2.2 Multilinear summing and multiple summing operators

In this section we are interested in analyzing classes of ideals of multilinear operators that can be characterized by means of abstract summability properties associated to the continuity of the tensor product operators defined between spaces of vector valued sequence spaces. Our main reference is the class of (linear) p -summing operators between Banach spaces E and F , that can be described as the class of operators $T : E \rightarrow F$ such that the associated tensor product map $id \otimes T : \ell^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_{\Delta_p} F$ is continuous. If we identify the elements of the algebraic tensor product $\ell_0^p \otimes E$ with finite sequences of elements of E via the identification $\sum_{i=1}^n e_i \otimes x_i \leftrightarrow (x_i)_{i=1}^n$ then, the restriction of the associated sequential operator \widehat{T} of T to finite sequences can also be identified with the restriction $id \otimes T : \ell_0^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_{\Delta_p} F$. Under these identifications, T is absolutely p -summing if and only if $\widehat{T} : \ell_0^p \otimes_\varepsilon E \rightarrow \ell^p \otimes_{\Delta_p} F$ is continuous. We will introduce the extension of this idea to the case of multilinear operators. Two natural extensions of absolutely p -summing linear operators to the multilinear context are required for our purposes: p -summing and multiple p -summing multilinear operators. Both classes have been intensively studied and are well-known, classical references to these notions are [Mat03], [PV03], [AM89]. And we use some basic definitions as follows.

Definition 2.2.1 *Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$ and E_1, \dots, E_n, F be Banach spaces. Recall that an n -linear operator $T : E_1 \times \dots \times E_n \rightarrow F$ is said to be p -summing if there is a constant $C > 0$ such that for every choice of vectors $x_1^i, \dots, x_m^i \in E_i$, $i = 1, \dots, n$, we have*

$$\left(\sum_{i=1}^m \|T(x_1^i, \dots, x_n^i)\|^p \right)^{1/p} \leq C \|(x_1^i)_i\|_{w,p} \cdots \|(x_n^i)_i\|_{w,p}. \quad (2.2.1)$$

It is said to be multiple p -summing if there is a constant $C > 0$ such that for $x_1^k, \dots, x_{m_k}^k \in E_k$, $k = 1, \dots, n$, we have

$$\left(\sum_{i_1=1}^m \cdots \sum_{i_n=1}^m \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^p \right)^{1/p} \leq C \|(x_{i_1}^1)_{i_1=1}^{m_1}\|_{w,p} \cdots \|(x_{i_n}^n)_{i_n=1}^{m_n}\|_{w,p}.$$

In that case, we define the multiple p -summing norm of T by

$$\pi_p(p) = \{\inf C : C \text{ satisfies 2.2.1}\}.$$

The class $\Pi_p(E_1 \times \cdots \times E_n, F)$ of multiple p -summing n -linear operators is a Banach space with the norm $\pi_p(\cdot)$.

In terms of sequence spaces, these definitions can be rewritten as follows.

An n -linear operator $T : E_1 \times \cdots \times E_n \rightarrow F$ is p -summing if the n -linear map \bar{T} defined as

$$\bar{T} : (\ell_0^p \otimes_\varepsilon E_1) \times \cdots \times (\ell_0^p \otimes_\varepsilon E_n) \rightarrow \ell^p \otimes_{\Delta_p} F$$

is continuous.

On the other hand, T is multiple p -summing if the n -linear map

$$\tilde{T} : (\ell_0^p \otimes_\varepsilon E_1) \times \cdots \times (\ell_0^p \otimes_\varepsilon E_n) \rightarrow (\ell^p \otimes_{\Delta_p} \cdots \otimes_{\Delta_p} \ell^p) \otimes_{\Delta_p} F$$

given by

$$((x_{i_1}^1)_{i_1}, \dots, (x_{i_n}^n)_{i_n}) \rightsquigarrow \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(x_{i_1}^1, \dots, x_{i_n}^n)$$

is continuous.

Note that both \bar{T} and \tilde{T} coincide with the restriction to finite sequences of the associated sequential operator \hat{T} whenever $n = 1$. These associated sequential multilinear operators are the key for the study, from an unified point of view, of those multilinear operators that improve the summability of sequences by means of all tensor tools described in the introduction of this chapter. They are the inspiration for the following definitions.

2.2.1 (γ, α, p) -summing multilinear operators

Definition 2.2.2 Let α and γ be two reasonable tensor norms for the tensor product $\ell^p \otimes F$ —including the case $\alpha = \Delta_p$ —, that is, $\varepsilon \leq \alpha, \gamma \leq \pi$. Let $T : E_1 \times \cdots \times E_n \rightarrow F$ be a continuous n -linear operator. Consider the n -linear map

$$\bar{T} : (\ell_0^p \otimes E_1) \times (\ell_0^p \otimes E_2) \times \cdots \times (\ell_0^p \otimes E_n) \rightarrow \ell^p \otimes F$$

defined as

$$\bar{T}\left(\sum_{i=1}^m e_i \otimes x_i^1, \dots, \sum_{i=1}^m e_i \otimes x_i^n\right) := \sum_{i=1}^m e_i \otimes T(x_i^1, \dots, x_i^n).$$

We say that $T : E_1 \times \cdots \times E_n \rightarrow F$ is (γ, α, p) -summing if

$$\bar{T} : (\ell_0^p \otimes_\gamma E_1) \times (\ell_0^p \otimes_\gamma E_2) \times \cdots \times (\ell_0^p \otimes_\gamma E_n) \rightarrow \ell^p \otimes_\alpha F$$

is continuous.

2.2.2 Multiple (γ, α, p) -summing multilinear operators

The "multiple" case can be defined in an analogous way.

Definition 2.2.3 An n -linear operator $T : E_1 \times \cdots \times E_n \rightarrow F$ is multiple (γ, α, p) -summing if the associated operator

$$\tilde{T} : (\ell_0^p \otimes_\gamma E_1) \times (\ell_0^p \otimes_\gamma E_2) \times \cdots \times (\ell_0^p \otimes_\gamma E_n) \rightarrow \ell^p \otimes_{\Delta_p} \cdots \otimes_{\Delta_p} \ell^p \otimes_\alpha F$$

given by

$$\tilde{T} \left(\sum_{i_1=1}^{m_1} e_{i_1} \otimes x_{i_1}^1, \dots, \sum_{i_n=1}^{m_n} e_{i_n} \otimes x_{i_n}^n \right) := \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(x_{i_1}^1, \dots, x_{i_n}^n)$$

is continuous.

Obviously, the notions of (γ, α, p) -summing and multiple (γ, α, p) -summing coincide for $n = 1$. Several well known multi-ideals are particular cases of these definitions. In the next sections we will analyze some of them, mainly the ones associated to important notions of summability. To start with, the class of (multiple) p -summing multilinear operators clearly coincides with the class of (resp. multiple) $(\varepsilon, \Delta_p, p)$ -summing operators.

Let us write now two linear examples. The first one is given by the so called (q, p) -mixing linear operators.

Definition 2.2.4 Recall that a linear operator $T : E \rightarrow F$ is (q, p) -mixing ($1 \leq p, q \leq \infty$), if for each Banach space valued q -summing operator S , the composition $S \circ T$ is p -summing. It is well-known that they can be characterized by means of the following summing inequality,

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n | \langle T(x_j), y_k^* \rangle |^q \right)^{p/q} \right)^{1/p} \leq C \| (x_j)_{j=1}^m \|_{p,w} \cdot \| (y_k^*)_{k=1}^n \|_q,$$

for a certain constant $C > 0$ and for all $x_1, \dots, x_m \in E$ and $y_1^*, \dots, y_n^* \in F^*$ (see for example [DF93, Proposition 32.4]). In [DF93, Proposition 32.3], it can be found that this ideal can be characterized as in the follows proposition.

Proposition 2.2.1 *An operator $T : E \rightarrow F$ is (q, p) -mixing if, and only if,*

$$id \otimes T : \ell^{q^*} \otimes_{g_{p^*}} E \rightarrow \ell^{q^*} \otimes_{\Delta_{q^*}} F$$

is continuous. In this case, $id \otimes T = \overline{T} = \widetilde{T}$ and so, $T : E \rightarrow F$ is (q, p) -mixing if, and only if, it is (multiple) $(g_{p^}, \Delta_{q^*}, q^*)$ -summing.*

The second example is given by the so called p -dominated (linear) operators or Cohen p -nuclear operators (see [Coh73]).

Definition 2.2.5 *Let $T : E \rightarrow F$ be a linear operator. We call that T is p -dominated if, and only if, the inequality is hold*

$$\sum_{i=1}^n |\langle T(x_i), y_i^* \rangle| \leq C \| (x_i)_{i=1}^n \|_{p,w} \cdot \| (y_i^*)_{i=1}^n \|_{p^*,w}$$

for a certain constant $C > 0$ and for all $x_1, \dots, x_m \in E$ and $y_1^, \dots, y_n^* \in F^*$.*

The following proposition characterized p -dominated (linear) operators [DF93, Proposition 29.5] or Cohen p -nuclear operators [Coh73, Theorem 2.1.3].

Proposition 2.2.2 *An operator T is p -dominated if, and only if, the map*

$$id \otimes T : \ell^p \otimes_{\varepsilon} E \rightarrow \ell^p \otimes_{\pi} F$$

is continuous.

That is if, and only if, it is (ε, π, p) -summing, or multiple (ε, π, p) -summing.

The next theorem is the main result in this chapter. It gives some sort of *order reduction procedure* for factorizations of multilinear maps. We define the tensor norm α appearing in it using the so called "calculus of traced tensor norms", that can be found in [DF93, Ch. 29]. Let us explain first this construction in the particular case that we are dealing with sort.

Definition 2.2.6 Let β and γ be tensor norms on $E \otimes \ell^{p^*}$ and $\ell^p \otimes F$ respectively and write $C : (E \otimes \ell^{p^*}) \otimes (\ell^p \otimes F) \rightarrow E \otimes F$ for the tensor contraction given by

$$C((a \otimes u) \otimes (v \otimes b)) = \langle u, v \rangle a \otimes b.$$

The traced tensor norm $\gamma \otimes_{\ell^p} \beta$ is defined as the quotient norm defined by the projective tensor product, that is, for $z \in E \otimes F$,

$$\gamma \otimes_{\ell^p} \beta(z) = \inf (\pi(w) : w \in (E \otimes_{\beta} \ell^{p^*}) \otimes_{\pi} (\ell^p \otimes_{\gamma} F) \text{ such that } C(w) = z).$$

(In this notation, notice the change of the order of β and γ with respect to the original order in the cartesian product). In the particular case of the usual tensor norms (as Δ_p and ε), it is easy to prove that for example $d_p = \Delta_p \otimes_{\ell^p} \varepsilon$ (see [DF93, Proposition 1]).

Theorem 2.2.1 Let $n \geq 2$, $1 \leq p < \infty$ and let γ be a tensor norm. Let $\alpha = \gamma \otimes_{\ell^p} \Delta_{p^*}^t$. If $T : E_1 \times \cdots \times E_n \rightarrow F$ is an n -linear operator then, the following assertions are equivalent.

- (i) T is multiple (γ, Δ_p, p) -summing.
- (ii) The $(n-1)$ -linear map

$$\tilde{T}_n : (\ell_0^p \otimes_{\gamma} E_1) \times \cdots \times (\ell_0^p \otimes_{\gamma} E_{n-1}) \rightarrow \left(\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right) \otimes_{\alpha} E_n \right)^* \quad (2.2.2)$$

given by

$$\begin{aligned} & \left\langle \tilde{T}_n \left((a_{i_1}^1)_{i_1=1}^{m_1}, \dots, (a_{i_{n-1}}^{n-1})_{i_{n-1}=1}^{m_{n-1}} \right), \sum_{i_n=1}^{m_n} \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_{n-1}}^* \right) \otimes a_{i_n}^n \right\rangle \\ & = \\ & \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \langle T(a_{i_1}^1, \dots, a_{i_{n-1}}^{n-1}), b_{i_1, \dots, i_{n-1}}^* \rangle \end{aligned}$$

is well-defined and continuous.

Proof. First let us show that the map \tilde{T}_n is well-defined. Note that each element of the tensor product $\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right)$ can be uniquely represented as

$$\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_{n-1}}^*.$$

Indeed, clearly each element of the tensor product can be written in this way. Take now two representations

$$\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_{n-1}}^*$$

and

$$\sum_{i_1=1}^{m'_1} \cdots \sum_{i_{n-1}=1}^{m'_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes (b^*)'_{i_1, \dots, i_{n-1}}$$

of the same tensor. Completing with zeros when necessary, we can assume that $m_k = m'_k$ for all $k = 1, \dots, n-1$. For each j_1, \dots, j_{n-1} with $1 \leq j_k \leq m_k, k = 1, \dots, n-1$, and $b \in F$ consider $S_{j_1, \dots, j_{n-1}; b} := e_{j_1} \otimes \cdots \otimes e_{j_{n-1}} \otimes b \in \ell_0^p \otimes \cdots \otimes \ell_0^p \otimes F$ as an element in the dual $\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right)^*$. Then,

$$\begin{aligned} & S_{j_1, \dots, j_{n-1}; b} \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_{n-1}}^* \right) \\ &= \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} S_{j_1, \dots, j_{n-1}; b} (e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_{n-1}}^*) \\ &= \langle b, b_{j_1, \dots, j_{n-1}}^* \rangle. \end{aligned}$$

Then

$$\begin{aligned} & S_{j_1, \dots, j_{n-1}; b} \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes (b_{i_1, \dots, i_{n-1}}^* - (b^*)'_{i_1, \dots, i_{n-1}}) \right) \\ &= \langle b, b_{j_1, \dots, j_{n-1}}^* - (b^*)'_{j_1, \dots, j_{n-1}} \rangle \\ &= 0. \end{aligned}$$

Since this can be done for every $b \in E$, this gives that $b_{j_1, \dots, j_{n-1}}^* = (b^*)'_{j_1, \dots, j_{n-1}}$ for every set of indexes j_1, \dots, j_{n-1} .

Identifying $\ell_0^p \otimes E_j$ with a linear subspace of $\ell^p(E_j)$, fix $(a_{i_j}^j)_{i_j=1}^{m_j}$ in $\ell^p \otimes E_j, j = 1, \dots, n-1$, and write $\mathbf{a} := \left((a_{i_1}^1)_{i_1=1}^{m_1}, \dots, (a_{i_{n-1}}^{n-1})_{i_{n-1}=1}^{m_{n-1}} \right)$ for short. Let us see that

$$\tilde{T}_n(\mathbf{a}) = \tilde{T}_n \left((a_{i_1}^1)_{i_1=1}^{m_1}, \dots, (a_{i_{n-1}}^{n-1})_{i_{n-1}=1}^{m_{n-1}} \right)$$

defines an element of the dual space $\left(\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right) \otimes_{\alpha} E_n \right)^*$. Observe that given an element of the product $\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right) \times E_n$ of the form $\left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_{n-1}}^*, a^n \right)$, the formula

$$\begin{aligned} & B_{\tilde{T}_n(\mathbf{a})} \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_{n-1}}^*, a^n \right) \\ &= \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \langle T(a_{i_1}^1, \dots, a_{i_{n-1}}^{n-1}, a^n), b_{i_1, \dots, i_{n-1}}^* \rangle \end{aligned}$$

defines a bilinear form whose linearization takes the same values as does the form $\tilde{T}_n(\mathbf{a})$.

Since the linearization of $B_{\tilde{T}_n(\mathbf{a})}$ is defined on the tensor product space

$$\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right) \otimes E_n,$$

the form $\tilde{T}_n(\mathbf{a})$ can be considered an element of its algebraic dual space. Assuming (i),

let us prove that it actually belongs to the topological dual when we consider the norm $\alpha = \gamma \otimes_{\ell^p} \Delta_{p^*}^t$, that is, $\tilde{T}_n(\mathbf{a}) \in \left(\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right) \otimes_{\alpha} E_n \right)^*$. Take

$$\sum_{i_n=1}^{m_n} \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^* \otimes a_{i_n}^n \right)$$

in $\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right) \otimes E_n$ and define

$$S := \sum_{j_1=1}^{m_1} \cdots \sum_{j_n=1}^{m_n} e_{j_1} \otimes \cdots \otimes e_{j_{n-1}} \otimes T(a_{i_1}^1, \dots, a_{i_n}^n) \in \ell^p \otimes \cdots \otimes \ell^p \otimes F$$

as an element in $\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right)^*$. Then,

$$\begin{aligned} & \left| \left\langle \tilde{T}_n(\mathbf{a}), \sum_{i_n=1}^{m_n} \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^* \otimes a_{i_n}^n \right) \right\rangle \right| \\ &= \left| \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} \left\langle T(a_{i_1}^1, \dots, a_{i_{n-1}}^{n-1}, a_{i_n}^n), b_{i_1, \dots, i_n}^* \right\rangle \right| \\ &= \left| \left\langle \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^*, S \right\rangle \right| \\ &\leq \|S\| \Delta_{p'} \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^* \right). \end{aligned} \tag{2.2.3}$$

As we are assuming that T is multiple (γ, Δ_p, p) -summing, we have that

$$\begin{aligned} \|S\| &\leq \left\| (T(a_{i_1}^1, \dots, a_{i_n}^n))_{i_1, \dots, i_n} \right\|_p \\ &= \Delta_p \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(a_{i_1}^1, \dots, a_{i_n}^n) \right) \\ &\leq \left\| \tilde{T} \right\| \prod_{1 \leq k \leq n} \gamma \left(\sum_{i_k=1}^{m_k} e_{i_k} \otimes a_{i_k}^k \right). \end{aligned} \tag{2.2.4}$$

Combining (2.2.3) with (2.2.4) we get

$$\tilde{T}_n(\mathbf{a}) \in \left(\left(\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^* \right) \otimes_{\alpha} E_n \right)^*$$

and

$$\|\tilde{T}_n(\mathbf{a})\| \leq \|\tilde{T}\| \prod_{1 \leq k \leq n-1} \gamma \left(\sum_{i_k=1}^{m_k} e_{i_k} \otimes a_{i_k}^k \right)$$

that is, \tilde{T}_n is continuous and $\|\tilde{T}_n\| \leq \|\tilde{T}\|$.

We now assume (ii) and let us prove that T is multiple (γ, Δ_p, p) -summing.

Fix sequences $(a_{i_1}^1)_{i_1=1}^{m_1}, \dots, (a_{i_n}^n)_{i_n=1}^{m_n}$ in the corresponding spaces and take an element

$$\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^*$$

in the unit ball of $\ell_0^{p^*} \otimes_{\Delta_{p^*}} \cdots \otimes_{\Delta_{p^*}} \ell_0^{p^*} \otimes_{\Delta_{p^*}} F^*$ satisfying

$$\begin{aligned} & \Delta_p \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(a_{i_1}^1, \dots, a_{i_n}^n) \right) \\ &= \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(a_{i_1}^1, \dots, a_{i_n}^n)\|^p \right)^{1/p} \\ &= \left| \left\langle \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(a_{i_1}^1, \dots, a_{i_n}^n), \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^* \right\rangle \right| \\ &= \left| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \left\langle T(a_{i_1}^1, \dots, a_{i_{n-1}}^{n-1}, a_{i_n}^n), b_{i_1, \dots, i_n}^* \right\rangle \right| \end{aligned}$$

Using the continuity of \tilde{T}_n in the second inequality, we get

$$\begin{aligned} & \Delta_p \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(a_{i_1}^1, \dots, a_{i_n}^n) \right) \\ &= \left| \left\langle \tilde{T}_n \left((a_{i_1}^1)_{i_1=1}^{m_1}, \dots, (a_{i_{n-1}}^{n-1})_{i_{n-1}=1}^{m_{n-1}} \right), \sum_{i_n=1}^{m_n} \left(\sum_{i_1=1, \dots, i_{n-1}=1}^{m_1, \dots, m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^* \otimes a_{i_n}^n \right) \right\rangle \right| \\ &\leq \|\tilde{T}\| \prod_{1 \leq k \leq n-1} \gamma \left(\sum_{i_k=1}^{m_k} e_{i_k} \otimes a_{i_k}^k \right) \alpha \left(\sum_{i_n=1}^{m_n} \left(\sum_{i_1=1, \dots, i_{n-1}=1}^{m_1, \dots, m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^* \otimes a_{i_n}^n \right) \right) \\ &\leq \|\tilde{T}\| \prod_{1 \leq k \leq n} \gamma \left(\sum_{i_k=1}^{m_k} e_{i_k} \otimes a_{i_k}^k \right) \Delta_{p^*} \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1, \dots, i_n}^* \right). \end{aligned}$$

This proves that T is multiple (γ, Δ_p, p) -summing. ■

2.3 Bilinear multiple (γ, Δ_p, p) -summing operators

In this section several applications of Theorem 2.2.1 are provided for multiple (γ, Δ_p, p) -summing bilinear operators, with respect to classical tensor norms γ that are relevant for the general theory of operator ideals. The aim is to characterize the classes formed by these bilinear operators by means of the associated sequential operator.

In this section we show the characterization in terms of summability of sequences of the main cases of bilinear maps that are multiple (γ, Δ_p, p) -summing in some particular sense, according to the election of classical tensor norms that are relevant for the general theory of operator ideals. All of them are direct applications of our "reduction procedure". We start with the most known case.

Definition 2.3.1 *Let $1 \leq p < \infty$. A bilinear operator $T : E_1 \times E_2 \rightarrow F$ is multiple p -summing if there is a constant $C > 0$ such that for $x_1, \dots, x_m \in E_1$ and $y_1, \dots, y_n \in E_2$,*

$$\begin{aligned} & \left(\sum_{j=1}^m \sum_{i=1}^n \|T(x_j, y_i)\|^p \right)^{1/p} \\ & \leq C \| (x_j) \|_{w,p} \| (y_i) \|_{w,p} \\ & = C \sup_{x^* \in B_{E_1^*}, y^* \in B_{E_2^*}} \left(\sum_{j=1}^m \sum_{i=1}^n |\langle x_j, x^* \rangle|^p |\langle y_i, y^* \rangle|^p \right)^{1/p}. \end{aligned}$$

We will write $\pi_{m,p}(T)$ for the infimum of all constants satisfying the inequality above. As we said before, multiple p -summing operators coincide with multiple $(\varepsilon, \Delta_p, p)$ -summing operators. We write now the characterization of this class of bilinear maps in terms of sequential properties.

Given a continuous bilinear operator $T : E_1 \times E_2 \rightarrow F$, the associated sequential operator associated to the linear operator $T_2 : E_1 \rightarrow \mathcal{L}(E_2, F)$, $T_2(x_1)(x_2) := T(x_1, x_2)$, is the operator $\widehat{T}_2 : \ell^p(E_1) \rightarrow \mathcal{L}(E_2, \ell^p(F))$ given by $\widehat{T}_2((x_i)_i)(y) = (T(x_i, y))_i \in \ell^p(F)$, $y \in E_2$, $(x_i)_i \in \ell^p(E_1)$. It is easy to see that it is continuous, but this continuity does not characterize to be multiple p -summing. Next our result does it.

Corollary 2.3.1 *Let $T : E_1 \times E_2 \rightarrow F$ be a continuous bilinear operator. The following statements are equivalent.*

(i) T is multiple p -summing, that is, T is multiple $(\varepsilon, \Delta_p, p)$ -summing.

(ii) \widehat{T}_2 maps $\ell_w^{p,0}(E_1)$ into $\Pi_p(E_2, \ell^p(F))$ continuously.

Moreover, $\pi_{m,p}(T) = \|\widehat{T}_2\|$.

Proof. Recall that the bilinear map $T : E_1 \times E_2 \rightarrow F$ is multiple $(\varepsilon, \Delta_p, p)$ -summing if

$$\widetilde{T} : (\ell_0^p \otimes_\varepsilon E_1) \times (\ell_0^p \otimes_\varepsilon E_2) \rightarrow (\ell^p \otimes_{\Delta_p} \ell^p) \otimes_{\Delta_p} F$$

is continuous. Consider

$$\alpha = \gamma \otimes_{\ell^p} \Delta_{p^*}^t = \varepsilon \otimes_{\ell^p} \Delta_{p^*}^t = g_{p^*}$$

(see [DF93, S.29.3] for the last equality). Taking $\gamma = \varepsilon$ in Theorem 2.2.1, we get that the continuity of \widetilde{T} is equivalent to the continuity of

$$\widetilde{T}_2 : \ell_0^p \otimes_\varepsilon E_1 \rightarrow \left((\ell_0^{p^*} \otimes_{\Delta_{p^*}} F^*) \otimes_{g_{p^*}} E_2 \right)^*.$$

Since

$$\begin{aligned} & \left((\ell_0^{p^*} \otimes_{\Delta_{p^*}} F^*) \otimes_{g_{p^*}} E_2 \right)^* \\ &= \left(\ell_0^{p^*}(F^*) \otimes_{g_{p^*}} E_2 \right)^* \\ &= \left(E_2 \otimes_{(g_{p^*}^*)} \ell_0^{p^*}(F^*) \right)^* \\ &= \Pi_p(E_2, (\ell_0^{p^*}(F^*))^*) \end{aligned}$$

(see [DF93, S.17.12] for the last equality) and taking into account that by its definition the evaluations of $\widehat{T}_2 = \widetilde{T}_2$ takes values in $\ell_0^p(F)$, we get the result. ■

Inspired by the Chevet-Persson-Saphar inequalities

$$d_{p^*}^* \leq d_p \leq \Delta_p \leq g_{p^*}^* \leq g_p \tag{2.3.1}$$

on $\ell^p \otimes E$ (see 25.10 in [DF93]), let us write the results for the natural tensor norms associated to Δ_{q^*} that provide weaker topologies, that are d_q^* and d_{q^*} . We start giving the specific definitions of multiple (γ, Δ_p, p) -summing operators for $\gamma = d_q^*, d_{q^*}$ in terms of inequalities.

Definition 2.3.2 *Let the continuous bilinear map $T : E_1 \times E_2 \rightarrow F$.*

1) T is multiple (d_q^*, Δ_p, p) -summing if there is a constant $k > 0$ such that for all finite sequences $(x_j)_{j=1}^m$ and $(y_i)_{i=1}^n$ in E_1 and E_2 , respectively, we have

$$\left(\sum_{j=1}^m \sum_{i=1}^n \|T(x_j, y_i)\|^p \right)^{1/p} \leq k d_q^*((x_j)_{j=1}^m) d_q^*((y_i)_{i=1}^n).$$

The norm of T as a multiple (d_q^*, Δ_p, p) -summing operator is the infimum of all such constants $k > 0$.

2) T is multiple (d_{q^*}, Δ_p, p) -summing if there is a constant $k > 0$ such that for all finite sequences $(x_j)_{j=1}^m$ and $(y_i)_{i=1}^n$ in E_1 and E_2 , respectively, we have

$$\left(\sum_{j=1}^m \sum_{i=1}^n \|T(x_j, y_i)\|^p \right)^{1/p} \leq k d_{q^*}((x_j)_{j=1}^m) d_{q^*}((y_i)_{i=1}^n).$$

The norm of T as a multiple (d_{q^*}, Δ_p, p) -summing operator is the infimum of all such constants $k > 0$.

As a consequence of (2.3.1) we have that multiple p -summing bilinear operators are multiple (d_q^*, Δ_p, p) -summing, and these ones are (d_{q^*}, Δ_p, p) -summing.

Corollary 2.3.2 *Let $T : E_1 \times E_2 \rightarrow F$ be a continuous bilinear operator and $1 \leq q < \infty$. The following statements are equivalent.*

(i) T is multiple (d_q^*, Δ_p, p) -summing.

(ii) \tilde{T}_2 maps $\ell_0^p \otimes_{d_q^*} E_1$ into $\mathcal{L}_{p,q}^{inj}(E_2, \ell^p(F))$ continuously.

Moreover, the norm as multiple (d_q^*, Δ_p, p) -summing operator coincides with $\|\tilde{T}_2\|$.

Proof. The proof is again an application of Theorem 2.2.1, but in this case $\gamma = d_q^*$, and

$$\tilde{T}_2 : \ell_0^p \otimes_{d_q^*} E_1 \rightarrow ((\ell_0^{p^*} \otimes_{\Delta_{p^*}} F^*) \otimes_{\alpha} E_2)^*$$

for

$$\alpha = d_q^* \otimes_{\ell^p} \Delta_{p^*}^t = \setminus \alpha_{p,q}^*,$$

where the last equality follows from [DF93, Proposition 29.9]. There, it is also shown that $\alpha_{p,q} \setminus \sim \mathcal{L}_{p,q}^{inj}$, that is, $\alpha_{p,q} \setminus$ is the associated tensor norm to the operator ideal of the injective hull of the (p, q) -factorable operators. Since $\setminus \alpha_{p,q}^* = (\alpha_{p,q} \setminus)^* = ((\alpha_{p,q} \setminus)')^t$, by

the representation theorem for the injective hull of the p, q -factorable operators (see [DF93, Theorem 17.5]) we get

$$\left(\ell_0^{p^*}(F^*) \otimes_{(\alpha_{p,q})^*} E_2 \right)^* = \left(E_2 \otimes_{(\alpha_{p,q})'} \ell_0^{p^*}(F^*) \right)^* = \mathcal{L}_{p,q}^{inj} \left(E_2, (\ell_0^{p^*}(F^*))^* \right),$$

which, again by the definition of $\widehat{T}_2 = \widetilde{T}_2$, gives the result. ■

Let us show now the case $\gamma = d_{q^*}$. In this case $\alpha = d_{q^*} \otimes_{\ell^p} \Delta_{p^*}^t = \backslash \alpha_{p,q}^* /$, where the last equality follows from [DF93, Proposition 29.9]. The surjective hull of the injective hull of the ideal of p, q -factorable operators is the associated operator ideal to $(\backslash \alpha_{p,q}^* /)^*$, that is, $\mathcal{L}_{p,q}^{inj\ sur} \sim / \alpha_{p,q} \backslash = (\backslash \alpha_{p,q}^* /)^*$ (see [DF93, S.29.9]). Since $\backslash \alpha_{p,q}^* /$ is the projective associate to $\alpha_{p,q}$, it is finitely generated ([DF93, Corollary 20.6.2]) and so $\backslash \alpha_{p,q}^* / = (\backslash \alpha_{p,q}^* /)''$ by [DF93, S. 15.3]. Then, by the representation theorem [DF93, Theorem 17.5] we get

$$\mathcal{L}_{p,q}^{inj\ sur} \left(E_2, (\ell_0^{p^*}(F^*))^* \right) = \left(E_2 \otimes_{(\backslash \alpha_{p,q}^* /)^*} \ell_0^{p^*}(F^*) \right)^* = \left(\ell_0^{p^*}(F^*) \otimes_{\backslash \alpha_{p,q}^* /} E_2 \right)^*.$$

This gives the following corollary of Theorem 2.2.1.

Corollary 2.3.3 *Let $T : E_1 \times E_2 \rightarrow F$ be a continuous bilinear operator and $1 \leq q < \infty$. The following statements are equivalent.*

- (i) T is multiple (d_{q^*}, Δ_p, p) -summing.
- (ii) \widetilde{T}_2 maps $\ell_0^p \otimes_{d_{q^*}} E_1$ into $\mathcal{L}_{p,q}^{inj\ sur}(E_2, \ell_0^p(F))$ continuously.

Moreover, the norm as multiple (d_{q^*}, Δ_p, p) -summing operator coincides with $\|\widetilde{T}_2\|$.

For the simplest case $q = p^*$ the operator ideals involved are injective and surjective hulls of the ideal of the p -factorable operators ([DF93, Ch.18]). In particular, an operator belongs to \mathcal{L}_p^{inj} if, and only if, it factors through a subspace of some $L^p(\mu)$, and the factorization norm given by the infimum of the products of the norms of the operators involved in each factorization for each suitable subspace coincides with the ideal norm (see the comments after Proposition 25.9 in [DF93]). In the same way, the ideal $\mathcal{L}_p^{inj\ sur}$ is characterized by factorizations through a subspace of a quotient of some L^p -space. This provides the following results.

1) T is multiple (d_{p^*}, Δ_p, p) -summing if and only if there is a constant $k > 0$ such that for each sequence (x_i) in $\ell_0^p \otimes E_1$, the factorization norm $\inf \|A\| \cdot \|B\|$ of $\tilde{T}_2((x_i)) = A \circ B$ through a subspace of an L^p -space satisfies

$$\inf_{\tilde{T}_2((x_i))=A \circ B} \|A\| \cdot \|B\| \leq k d_{p^*}((x_i)).$$

2) T is multiple (d_p, Δ_p, p) -summing if and only if there exists a constant $k > 0$ such that for each sequence (x_i) in $\ell_0^p \otimes E_1$, the factorization norm $\inf \|A\| \cdot \|B\|$ of $\tilde{T}_2((x_i)) = A \circ B$ through a subspace of a quotient of an L^p -space (equivalently, a quotient of a subspace of some L^p -space) satisfies

$$\inf_{\tilde{T}_2((x_i))=A \circ B} \|A\| \cdot \|B\| \leq k d_p((x_i)).$$

2.4 Applications. The linear case again: multiple $\tau(p)$ and p -summing linear operators

In this section we will analyze with our tools the case of the so called $\tau(p)$ -summing operators and their "multiple version". As we explained in subsection 2.2.2, the multiple-type ideals of the usual classes of summing operators make sense in the multilinear case rather than in the linear one, since *linear* p -summing operators and *linear* multiple p -summing operators are the same thing. However, we will see in this section that $\tau(p)$ -summing (linear) operators are naturally defined in the multilinear context rather than in the linear one, as happens for instance in the case of linear (p, q) -dominated operators (see for example [DF93, Ch.19]). This is so because the inequality in the definition involves the evaluation of the associated bilinear form $B_T(a, b') = \langle T(a), b' \rangle$ instead of $\|T(a)\|$. In fact, multiple $\tau(p)$ -summing linear operators are the ones that satisfy that this bilinear form is p -summing. This makes natural to analyze both the original and the multiple cases. We will finish this section by showing some applications to summing operators on L^p -spaces.

2.4.1 The $\tau(p)$ -summing operators

The following definition was studied by X. Mujica in [Muj08] for multilinear operators, which generalizes absolutely τ -summing linear operators introduced by A. Pietsch in [Pie80].

Definition 2.4.1 Let $T \in \mathcal{L}(E, F)$ and $1 \leq q \leq p < \infty$. We say that T is absolutely $\tau(p, q)$ -summing if there is a positive constant C such that, for all $n \in \mathbb{N}$, $(a_i)_{1 \leq i \leq n} \subset E$ and $(b_i^*)_{1 \leq i \leq n} \subset F^*$, we have

$$\left(\sum_{i=1}^n |\langle T(a_i), b_i^* \rangle|^p \right)^{\frac{1}{p}} \leq C \sup_{\substack{\|a^*\|_{E^*} \leq 1 \\ \|b\|_F \leq 1}} \left(\sum_{i=1}^n |\langle a_i, a^* \rangle \langle b_i^*, b \rangle|^q \right)^{\frac{1}{q}}. \quad (2.4.1)$$

We denote by $\Pi_{\tau(p,q)}(E, F)$ the vector space of all $\tau(p, q)$ -summing linear operators T from E into F , which is a Banach space if we consider the norm $\pi_{\tau(p,q)}(T)$, the infimum of all C verifying the above inequality. When $p = q$, we write $\Pi_{\tau(p)}$ and $\pi_{\tau(p)}(T)$ instead of $\Pi_{\tau(p,p)}$ and $\pi_{\tau(p,p)}(T)$ respectively. In this case, we say that T is $\tau(p)$ -summing. If $p = q = 1$, we simply write Π_{τ} and π_{τ} and we say that T is τ -summing.

The following theorem, due to Pietsch for $p = 1$ and Mujica for multilinear operators when $1 \leq p < \infty$.

Theorem 2.4.1 Consider $1 \leq p < \infty$. A mapping $T \in \mathcal{L}(E, F)$ is $\tau(p)$ -summing if, and only if, there is a positive constant C and a Radon-Borel probability measure μ on $B_{E^*} \times B_{F^{**}}$, such that

$$|\langle T(a), b^* \rangle| \leq C \left(\int_{B_{E^*} \times B_{F^{**}}} |\langle a, a^* \rangle \langle b^*, b^{**} \rangle|^p d\mu(a^*, b^{**}) \right)^{\frac{1}{p}} \quad (2.4.2)$$

for all $a \in E$, $b^* \in F^*$. In this case, $\pi_{\tau(p)}(T) = \inf C$.

Let us consider now the $\tau(p)$ -summing operators. Mujica extended in [Muj08] the τ -summing linear operators introduced by Pietsch in [Pie80] to the context of the multilinear operators, generalizing also the class to the corresponding case $1 \leq p < \infty$. We shall consider only the linear case.

There is second domination theorem for $\tau(p)$ -summing operators, that is a particular case of the characterization of Mujica for multilinear operators given in [Muj08, Theorem 2.1] when $1 \leq p < \infty$.

Theorem 2.4.2 ([Muj08, Theorem2.1]) *Let $T \in \mathcal{L}(E, F)$. Then there is a positive constant C such that the following assertions are equivalent.*

(i) *The operator T is $\tau(p)$ -summing and $\pi_{\tau(p)}(T) \leq C$.*

(ii) *There exist Borel probability measures μ on B_{E^*} and ν on $B_{F^{**}}$, such that for all a in E and b^* in F^* , we have*

$$|\langle T(a), b^* \rangle| \leq C \left(\int_{B_{E^*}} \int_{B_{F^{**}}} |\langle a, a^* \rangle \langle b^*, b^{**} \rangle|^p d\nu(b^{**}) d\mu(a^*) \right)^{\frac{1}{p}}. \quad (2.4.3)$$

Moreover, $\pi_{\tau(p)}(T) = \inf C$, where the infimum is taken over all constants $C > 0$ fulfilling (2.4.3).

If $T : E \rightarrow F$ is a continuous linear operator, its transpose is denoted by $T^* : F^* \rightarrow E^*$. The proof of the following lemma is straightforward.

Lemma 2.4.1 *Let $T : E \rightarrow F$ be a continuous linear operator. If T is $\tau(p)$ -summing then T and T^* are p -summing.*

Proof. Let be T is $\tau(p)$ -summing. Firstly we show that T is p -summing. For all $a \in E$, $b^* \in F^*$ we have by 2.4.3

$$\begin{aligned} \|T(a)\| &= \sup_{b^* \in F^*} |\langle T(a), b^* \rangle| \\ &\leq \pi_{\tau(p)}(T) \sup_{b^* \in F^*} \left(\int_{B_{E^*}} |\langle a, a^* \rangle|^p d\mu(a^*) \int_{B_{F^{**}}} |\langle b^*, b^{**} \rangle|^p d\nu(b^{**}) \right)^{\frac{1}{p}} \\ &\leq \pi_{\tau(p)}(T) \left(\int_{B_{E^*}} |\langle a, a^* \rangle|^p d\mu(a^*) \right)^{\frac{1}{p}}. \end{aligned}$$

This implies that, T is p -summing and $\pi_p(T) \leq \pi_{\tau(p)}(T)$.

Secondly, we put T is $\tau(p)$ -summing and we prove that T^* is p -summing. Let be $T^* : F^* \rightarrow E^*$ and $b^* \in F^*$, $a^{**} \in E^{**}$, we have

$$\begin{aligned} \|T^*(b^*)\| &= \sup_{a^{**} \in \mathcal{B}_{E^{**}}} |\langle T^*(b^*), a^{**} \rangle| \\ &\leq \sup_{a \in \mathcal{B}_E} |\langle T^*(b^*), a \rangle| \\ &\leq \sup_{a \in \mathcal{B}_E} |\langle b^*, T(a) \rangle| \\ &\leq \pi_{\tau(p)}(T) \sup_{a \in \mathcal{B}_E} \left(\int_{B_{E^*}} |\langle a, a^* \rangle|^p d\mu(a^*) \int_{B_{F^{**}}} |\langle b^*, b^{**} \rangle|^p d\nu(b^{**}) \right)^{\frac{1}{p}} \\ &\leq \pi_{\tau(p)}(T) \left(\int_{B_{F^{**}}} |\langle b^*, b^{**} \rangle|^p d\nu(b^{**}) \right)^{\frac{1}{p}}. \end{aligned}$$

Then, T^* is p -summing and $\pi_p(T^*) \leq \pi_{\tau(p)}(T)$. ■

Consider the weak star topology on B_{X^*} and a probability Radon measure μ on B_{X^*} . Given the canonical evaluation mapping $e_X : X \rightarrow C(B_{X^*})$ and the formal identity mapping $j_p : C(B_{X^*}) \rightarrow L^p(\mu)$, define the linear subspace $S_E^\mu := j_p(e_X(X)) \subset L^p(\mu)$. For short we just denote $i_E^\mu : E \rightarrow S_E^\mu$ the composition $j_p \circ e_X : X \rightarrow S_E^\mu$.

Theorem 2.4.3 *Let $T : E \rightarrow F$ be a continuous linear operator. Let $B_T : E \times F^* \rightarrow \mathbb{K}$ be its associated bilinear map. The following statements are equivalent.*

(i) T is $\tau(p)$ -summing.

(ii) The (non-linear, non-bilinear) map $T_\times : \ell_w^{p,0}(E \times F^*) \rightarrow \ell^p$ given by $T_\times((a_i, b_i^*)_i) := (\langle T(a_i), b_i^* \rangle)_i$, $(a_i, b_i^*) \in E \times F^*$ for all i , is bounded with bound C .

(iii) There are Borel probability measures η and ν on B_{E^*} and $B_{F^{**}}$ respectively and a continuous bilinear form $S : S_E^\mu \times S_{F^*}^\nu \rightarrow \mathbb{K}$ such that the associated bilinear map $B_T : E \times F^* \rightarrow \mathbb{K}$ factors as

$$\begin{array}{ccc} E \times F^* & \xrightarrow{B_T} & \mathbb{K} \\ i_E^\mu \times i_{F^*}^\nu \downarrow & \nearrow S & \\ S_E^\mu \times S_{F^*}^\nu & \subseteq & L^p(\eta) \times L^p(\nu) \end{array}$$

Moreover, $\pi_{\tau(p)}(T)$ coincides with the infimum of the bounds $C > 0$.

Proof. The equivalence between (i) and (ii) is clear: it follows from the inequality

$$\begin{aligned} \|(\langle T(a_i), b_i^* \rangle)_{i=1}^n\|_{\ell^p} &= \left(\sum_{i=1}^n |\langle T(a_i), b_i^* \rangle|^p \right)^{\frac{1}{p}} \\ &\leq C \sup_{\substack{\|a^*\|_{E^*} \leq 1 \\ \|b\|_{F^*} \leq 1}} \left(\sum_{i=1}^n |\langle a_i, a^* \rangle \langle b_i^*, b \rangle|^p \right)^{\frac{1}{p}} \\ &= C \|((a_i, b_i^*))_{i=1}^n\|_{p,w}, \end{aligned}$$

where $(a_i)_{i=1}^n$ is a sequence in E and $(b_i^*)_{i=1}^n$ is a sequence in F^* .

(i) \Rightarrow (iii). Using Lemma 2.4.1, by the classical Pietsch factorization theorem there are Borel probabilities measures μ and ν on B_{E^*} and $B_{F^{**}}$ respectively and continuous linear operators $u : S_E^\mu \rightarrow F$ and $v : S_{F^*}^\nu \rightarrow E^*$ such that $T = u \circ i_E^\mu$ and $T^* = v \circ i_{F^*}^\nu$. The bilinear mapping $R : u(S_E^\mu) \times v(S_{F^*}^\nu) \rightarrow \mathbb{K}$ given by $R(T(a), T^t(b^*)) = \langle T(a), b^* \rangle$ is well defined and the composition $S := R \circ (u \times v) : S_E^\mu \times S_{F^*}^\nu \rightarrow \mathbb{K}$ is a bilinear form. The continuity

of S follows from Theorem 2.4.2. A simple calculation shows that $S \circ (i_E^\mu \times i_{F^*}^\nu) = B_T$. Conversely, the continuity of S gives the inequality (2) in the domination Theorem 2.4.2, which finishes the proof. ■

2.4.2 Multiple $\tau(p)$ -summing operators

As it happens with the examples in this chapter, multiple $\tau(p)$ -summing operators are closely related to $\tau(p)$ -summing operators, but the inequality is required for matrices $(a_j)_{j=1}^m \times (b_i^*)_{i=1}^n$ and not just for the diagonals $((a_i, b_i^*))_i \subseteq E \times F^*$. This class can be defined just by considering the linear operators that satisfy that the associated bilinear form is multiple p -summing. This our new definition as follows.

Definition 2.4.2 *Let $1 \leq p < \infty$. We will say that a linear operator $T : E \rightarrow F$ is multiple $\tau(p)$ -summing if there is a constant $C > 0$ such that for $a_1, \dots, a_m \in E$ and $b_1^*, \dots, b_n^* \in F^*$,*

$$\sum_{j=1}^m \sum_{i=1}^n |\langle T(a_j), b_i^* \rangle|^p \leq C^p \sup_{a^* \in B_{E^*}, b^{**} \in B_{F^{**}}} \sum_{j=1}^m \sum_{i=1}^n |\langle a_j, a^* \rangle|^p |\langle b_i^*, b^{**} \rangle|^p.$$

We will write $\pi_{m, \tau(p)}(T)$ for the infimum of all constants satisfying the inequality above. As the reader can notice, multiple $\tau(p)$ -summing operators are just operators that satisfy that its associated bilinear form $(a, b^*) \mapsto \langle T(a), b^* \rangle$ is multiple p -summing.

Note also that in this case, given a continuous linear operator $T : E \rightarrow F$, the associated sequential operator \widehat{T} can be defined from $\ell_w^p(E)$ to $\ell_w^p(F)$. In this case, Corollary 2.3.1 reads as follows, which is an alternative straightforward way of getting the characterization obtained at the end of introduction of third chapter.

Corollary 2.4.1 *Let $T : E \rightarrow F$ be a continuous linear operator. Let $\widehat{T} : \ell_w^p(E) \rightarrow \ell_w^p(F)$ be the associated sequential operator induced by T . The following statements are equivalent.*

- (i) T is multiple $\tau(p)$ -summing.
- (ii) \widehat{T} maps $\ell_w^{p,0}(E)$ into $\Pi_p(F^*, \ell^p)$ continuously.

Moreover, $\pi_{m, \tau(p)}(T) = \|\widehat{T}\|$.

Proof. Recall, if $\xi = (a_j)_{1 \leq j \leq m} \in \ell_w^p(E)$ then $\|\xi\|_{\ell_w^p(E)} = \sup_{a^* \in \mathcal{B}_{E^*}} \left(\sum_{j=1}^m |\langle a_j, a^* \rangle|^p \right)^{\frac{1}{p}}$.

(i) \implies (ii) Let $\xi = (a_j)_{1 \leq j \leq m} \in \ell_w^p(E)$. Then $\widehat{T}_\xi : F^* \rightarrow \ell^p$ is given by $\widehat{T}_\xi(b^*) = (\langle T(a_1), b^* \rangle, \dots, \langle T(a_m), b^* \rangle, 0, \dots) = \sum_{j=1}^m |\langle T(a_j), b^* \rangle| e_j$ and we must prove that \widehat{T}_ξ is p -summing. Now note that $\left\| \widehat{T}_\xi(b^*) \right\|_p^p = \sum_{j=1}^m |\langle T(a_j), b^* \rangle|^p$ for $b^* \in F^*$. Let $(b_i^*)_{1 \leq i \leq n} \in F^*$.

Then, by (i)

$$\begin{aligned} \sum_{i=1}^n \left\| \widehat{T}_\xi(b_i^*) \right\|_p^p &= \sum_{i=1}^n \sum_{j=1}^m |\langle T(a_j), b_i^* \rangle|^p \\ &\leq \pi_{m, \tau(p)}^p(T) \sup_{a^* \in \mathcal{B}_{E^*}} \left(\sum_{j=1}^m |\langle a_j, a^* \rangle|^p \right) \sup_{b^{**} \in \mathcal{B}_{F^{**}}} \left(\sum_{j=1}^n |\langle b_i^*, b^{**} \rangle|^p \right). \end{aligned}$$

By the definition of p -summing operators and their norm we deduce that $\widehat{T}_\xi : F^* \rightarrow \ell^p$ is p -summing and

$$\pi_p(\widehat{T}_\xi) \leq \pi_{m, \tau(p)}(T) \sup_{a^* \in \mathcal{B}_{E^*}} \left(\sum_{j=1}^m |\langle a_j, a^* \rangle|^p \right)^{\frac{1}{p}} = \pi_{m, \tau(p)}(T) \|\xi\|_{\ell_w^p(E)}.$$

This means that \widehat{T} maps $\ell_w^p(E)$ into $\Pi_p(F^*, \ell^p)$ continuously, $\|\widehat{T} : \ell_w^p(E) \rightarrow \Pi_p(F^*, \ell^p)\| \leq \pi_{m, \tau(p)}(T)$, which (ii).

(ii) \implies (i). Let $(a_j)_{1 \leq j \leq m} \in E$ and $(b_i^*)_{1 \leq i \leq n} \in F^*$. Write $\xi = (a_1, \dots, a_m, 0, \dots) \in \ell_w^p(E)$. Since \widehat{T} maps $\ell_w^p(E)$ into $\Pi_p(F^*, \ell^p)$ we have

$$\left(\sum_{i=1}^n \left\| \widehat{T}_\xi(b_i^*) \right\|_p^p \right)^{\frac{1}{p}} \leq \pi_p(\widehat{T}_\xi) \sup_{b^{**} \in \mathcal{B}_{F^{**}}} \left(\sum_{j=1}^n |\langle b_i^*, b^{**} \rangle|^p \right)^{\frac{1}{p}}.$$

But, \widehat{T} maps $\ell_w^p(E)$ into $\Pi_p(F^*, \ell^p)$ continuously, that is

$$\pi_p(\widehat{T}_\xi) \leq \|\widehat{T} : \ell_w^p(E) \rightarrow \Pi_p(F^*, \ell^p)\| \|\xi\|_{\ell_w^p(E)}.$$

We deduce

$$\left(\sum_{i=1}^n \left\| \widehat{T}_\xi(b_i^*) \right\|_p^p \right)^{\frac{1}{p}} \leq \|\widehat{T} : \ell_w^p(E) \rightarrow \Pi_p(F^*, \ell^p)\| \|\xi\|_{\ell_w^p(E)}$$

or

$$\begin{aligned} &\left(\sum_{j=1}^m \sum_{i=1}^n |\langle T(a_j), b_i^* \rangle|^p \right)^{\frac{1}{p}} \\ &\leq \|\widehat{T} : \ell_w^p(E) \rightarrow \Pi_p(F^*, \ell^p)\| \sup_{a^* \in \mathcal{B}_{E^*}} \left(\sum_{j=1}^m |\langle a_j, a^* \rangle|^p \right) \sup_{b^{**} \in \mathcal{B}_{F^{**}}} \left(\sum_{j=1}^n |\langle b_i^*, b^{**} \rangle|^p \right) \end{aligned}$$

that is (i). ■

This result allows the comparison of our new space of operators with some classical ones. Let us show some direct consequences, all of them based in the continuous inclusions

$$\ell^p(F) = F \widehat{\otimes}_{\Delta_p} \ell^p \hookrightarrow F \widehat{\otimes}_{d'_{p^*}} \ell^p \hookrightarrow F \widehat{\otimes}_{\varepsilon} \ell^p = \ell_w^{p,0}(F),$$

that we explain in what follows.

First of all, note that \widehat{T} takes finite sequences to elements in the space $F \otimes \ell^p$. In the following formulas, the space ℓ^{p^*} should be replaced by c_0 whenever $p^* = \infty$. Since the tensor norm d'_{p^*} is totally accesible (see Theorem 21.5 in [DF93]), the natural map

$$F \widehat{\otimes}_{d'_{p^*}} \ell^p \hookrightarrow (F^* \otimes_{d_{p^*}} \ell^{p^*})^*$$

is in fact a metric injection by the duality theorem for tensor norms (see also 15.7 in [DF93], or [Sap70]). Therefore, we have that

$$\Pi_p^0(F^*, \ell^p) := F \widehat{\otimes}_{d'_{p^*}} \ell^p \hookrightarrow (F^* \otimes_{d_{p^*}} \ell^{p^*})^* = \Pi_p(F^*, \ell^p). \quad (2.4.4)$$

Since $\varepsilon \leq d'_{p^*}$ —it is a reasonable tensor norm— and $F \widehat{\otimes}_{\varepsilon} \ell^p = \ell_w^{p,0}(F)$ we have that the identification map i is continuous. This closes the factorization. The converse use the same identifications and is also easy to see. This proves the following.

Corollary 2.4.2 *Let $1 \leq p < \infty$. The operator T is multiple $\tau(p)$ -summing if and only if there is a factorization for the associated sequential operator \widehat{T} as*

$$\begin{array}{ccc} \ell_p^w(E) & \xrightarrow{\widehat{T}} & \ell_p^w(F) \\ \widehat{T} \searrow & & \nearrow i \\ & \Pi_p^0(F^*, \ell^p) & \end{array}$$

where the space $\Pi_p^0(F^*, \ell^p) := F \widehat{\otimes}_{d'_{p^*}} \ell^p$ is (isometrically isomorphic to) a subspace of $\Pi_p(F^*, \ell^p)$.

We can also see that if an operator $T : E \rightarrow F$ is p -summing, then it is multiple $\tau(p)$ -summing as a consequence of the factorization; it can also be proved by a direct calculation. Indeed, if T is p -summing \widehat{T} is defined from $\ell_w^{p,0}(E)$ to $\ell^p(F)$. By the Chevet-Persson-Saphar

inequalities (see 15.10 in [DF93]), we have that $d_{p^*}^* \leq \Delta_p$ on $\ell^p \otimes F$, and so the continuity of the identification map

$$\ell^p(F) = \ell^p \widehat{\otimes}_{\Delta_p} F \rightarrow \ell^p \widehat{\otimes}_{d_{p^*}^*} F = F \widehat{\otimes}_{d_{p^*}^*} \ell^p = \Pi_p^0(F^*, \ell^p).$$

Corollary 2.4.1 gives that T is multiple $\tau(p)$ -summing.

2.4.3 The p -summing and multiple $\tau(p)$ -summing operators on L^p -spaces

In this section we will present some applications to the representation of ideals of summing operators into L^p -spaces. One of the former papers that contributed to the success of the operator ideal theory and showed the central role of the p -summing operators in the modern functional analysis was the paper [LP68] by Lindenstrauss and Pełczyński. Closely related to the applications of Grothendieck's inequality, the results on the coincidence of p -summing operators with other operator ideals for particular —but relevant— spaces opened the door to some fruitful applications that are nowadays well-known (see for example Ch.1 in [DJT95] and the references therein). In the same direction and from the tensorial point of view, the Chevet-Persson-Saphar inequalities (see 15.10 in [DF93]) provide different descriptions of the ideal of p -summing operators when the spaces involved are L^p -spaces (see [Sap70]). In what follows we show one of them, that holds as a consequence of the results of the previous section when F is an L^p -space.

Corollary 2.4.3 *Let $1 \leq p \leq \infty$. Let μ be a measure and let $T : E \rightarrow L^p(\mu)$ be a continuous linear operator. Then T is multiple $\tau(p)$ -summing if, and only if, T is p -summing.*

Proof. This is a consequence of Theorem 2.4.1 when $F = L^p(\mu)$. The right-to-left implication is a direct consequence of the comments at the end of the last subsection, and is true for any range space F . Assume now that T is multiple $\tau(p)$ -summing. Note that $\ell^p \widehat{\otimes}_{\Delta_p} L^p(\mu) = \ell^p(L^p(\mu))$ and $\Delta_p = d_{p^*}^*$ on $\ell^p \otimes L^p(\mu)$ (see Corollary 2 in 15.10 of [DF93], see also [DF93, 20.5] and [DF93, 25.10]). Thus,

$$\Pi_p^0(L^p(\mu), \ell^p) = L^p(\mu) \widehat{\otimes}_{d_{p^*}^*} \ell^p = \ell^p \widehat{\otimes}_{d_{p^*}^*} L^p(\mu) = \ell^p \widehat{\otimes}_{\Delta_p} L^p(\mu) = \ell^p(L^p(\mu)).$$

Corollary 2.4.2 yields then the following diagram

$$\begin{array}{ccc} \ell_p^w(L^p(\mu)) & \overset{\hat{T}}{\dashrightarrow} & \ell_p^w(L^p(\mu)) \\ \hat{T} \searrow & & \nearrow i \\ \Pi_p^0(L^{p^*}(\mu), \ell^p) & & \ell^p(L^p(\mu)) \end{array}$$

Therefore, T is p -summing. ■

Another direct application can be obtained for the multilinear versions of $\tau(p)$ -summing operators and multiple $\tau(p)$ -summing operators by using the arguments if Ch.25 of [DF93]. In [DF93, Ch.25.10] coincidence results of the form $\ell^p \widehat{\otimes}_\alpha F = \ell^p(F)$ are provided for some tensor norms α whenever F is a subspace of a quotient of some L^p -space. Thus, we can extend the previous arguments for subspaces and quotients of L^p -spaces and for (γ, α, p) -summing operators and their multiple version to our broader class of multilinear operators. Let us write the (γ, α, p) -summing case; the result for the multiple case is similar. Recall that an n -linear operator $T : E_1 \times \cdots \times E_n \rightarrow F$ is (γ, α, p) -summing if the associated n -linear operator

$$\bar{T} : (\ell_0^p \otimes_\gamma E_1) \times (\ell_0^p \otimes_\gamma E_2) \times \cdots \times (\ell_0^p \otimes_\gamma E_n) \rightarrow \ell^p \otimes_\alpha F$$

defined in Subsection 2.2.1, is continuous.

As an application of [DF93, Proposition 25.10] we obtain the following result.

Corollary 2.4.4 *Let E_1, \dots, E_n and F be Banach spaces, let $1 \leq p < \infty$, and consider an n -linear operator $T : E_1 \times \cdots \times E_n \rightarrow F$. If*

(i) *F is isomorphic to a subspace of some $L^p(\mu)$, and $\alpha = d_{p^*}^*$ or $\alpha = g_{p^*}^*$, or*

(ii) *F^{**} (or F if $1 < p < \infty$) is isomorphic to a quotient of some $L^p(\mu)$ and $\alpha = d_p$ or*

$\alpha = g_p$,

then T is (γ, α, p) -summing if, and only if, it is (γ, Δ_p, p) -summing.

Chapter 3

On the Lipschitz $\tau(p)$ -summing operators

3.1 Introduction

The notion of p -summing linear operators goes back to Grothendieck in the 1950s, but just in 1967 and 1968, the classical works of Pietsch [Pie67] and Lindenstrauss-Pełczyński [LP68] clarified Grothendieck's precious ideas and contributed clearly to the vigorous development of this notion. Recently, Lipschitz versions of different types of linear operators were investigated by several authors such as [Cha11], [CZ11], [CZ12], [FJ09], [Saa15] and [YAR15] among others. The first paper is due to Farmer and Johnson [FJ09]. They introduced the notion of Lipschitz p -summing operators and showed that is really a good generalization of the concept of linear p -summing operators [FJ09, Theorem 2]. This notion marked the beginning of the theory of nonlinear summability. Motivated by the importance of this theory, several authors have developed and studied many concepts relating to summability. Chen and Zheng introduced in [CZ12] (strongly) Lipschitz p -integral and p -nuclear operators. In [Cha11] Chávez-Domínguez introduced the notion of Lipschitz (r, p, q) -summing operators and Lipschitz (q, p) -mixing in [Cha12]. The latest papers in this domain are due to Yahi, Achour and Rueda [YAR15] and Saadi [Saa15]. Independently, they introduced and studied the class of Lipschitz strongly p -summing operators. The first authors introduced also summing Lipschitz conjugates and (p, σ) -summability with an appropriate factoriza-

tion. They characterized also those Lipschitz operators whose Lipschitz conjugates are absolutely p -summing.

In his book [Pie80] "Operator Ideals", Pietsch introduced the notion of τ -summing linear operators. Mujica, in [Muj08] extends this notion to multilinear operators. In this chapter, we introduce the generalization of this class of linear operators to Lipschitz operators. We give some characterizations and properties of this classe.

3.2 Some basic definitions

3.2.1 Lipschitz p -summing operators

The nonlinear version of p -summing operators was introduced by J.-D. Farmer and W.-B. Johnson in [FJ09]. They called it Lipschitz p -summing operator. Let $(X, d_X), (Y, d_Y)$ be metric spaces. In the sequel, we consider (X, d_X) a pointed metric space, i.e., a metric space (X, d_X) with a distinguished element noted 0. We denote by $X^\# = \text{Lip}_0(X, \mathbb{R})$ the space of all Lipschitz mappings $T : X \rightarrow \mathbb{R}$ vanishing at 0. The space $(X^\#, \text{Lip}(\cdot))$ equipped with the norm $\text{Lip}_0(\cdot)$ is a Banach space and $\mathcal{B}_{X^\#}$ (here $\mathcal{B}_{X^\#}$ is the unit ball of $X^\#$) is a compact Hausdorff space in the topology of pointwise convergence on X .

Definition 3.2.1 *A Lipschitz map $T : X \rightarrow Y$ is called Lipschitz p -summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all $x_1, \dots, x_n; x'_1, \dots, x'_n$ in X and all non negative reals $\lambda_1, \dots, \lambda_n$ we have*

$$\left(\sum_{i=1}^n \lambda_i d_Y(T(x_i), T(x'_i))^p \right)^{\frac{1}{p}} \leq C \sup_{f \in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x'_i))|^p \right)^{\frac{1}{p}}.$$

The infimum of such constants is denoted by $\pi_p^L(T)$. This is a true generalization of the concept of linear p -summing operators, since it is shown in [FJ09] that the Lipschitz p -summing norm of a linear operator is the same as its p -summing norm. In the sequel, it will be useful to note that the above definition is the same if we restrict to $\lambda_i = 1$ (see [FJ09] for an implicit proof).

The set of all Lipschitz p -summing maps from X to Y is denoted by $\Pi_p^L(X, Y)$.

Proposition 3.2.1 (Ideal property) *Let X_1, Y_1, X_2, Y_2 be metric spaces. Let $v : X_2 \rightarrow X_1$, $W : Y_1 \rightarrow Y_2$ be Lipschitz functions and $T : X_1 \rightarrow Y_1$ be a Lipschitz p -summing operator. Then wTv is Lipschitz p -summing operator and $\pi_p^L(wTv) \leq \text{Lip}(w)\pi_p^L(T)\text{Lip}(v)$.*

Proof. This composition make sens by Proposition 1.1.1.

$$\begin{aligned} \sum_{i=1}^n d_F(wTv(x_i), wTv(y_i))^p &\leq \text{Lip}(w)^p \sum_{i=1}^n d_Y(Tv(x_i), Tv(y_i))^p \\ &\leq \text{Lip}(w)^p \pi_p^L(T)^p \sup_{f \in \mathcal{B}_{X^\#}} \sum_{i=1}^n |f(v(x_i)) - f(v(y_i))|^p \\ &\leq \text{Lip}(w)^p \pi_p^L(T)^p \text{Lip}(v)^p \sup_{f \in \mathcal{B}_{X^\#}} \sum_{i=1}^n \left| \frac{f(v(x_i)) - f(v(y_i))}{\text{Lip}(v)} \right|^p \\ &\leq \text{Lip}(w)^p \pi_p^L(T)^p \text{Lip}(v)^p \sup_{f \in \mathcal{B}_{X^\#}} \sum_{i=1}^n |f(x_i) - f(y_i)|^p. \end{aligned}$$

This proves the proposition. ■

If Y is a Banach space, the space $\Pi_p^L(X, Y)$ of Lipschitz p -summing functions from any metric space into Y is a Banach space under the norm $\pi_p^L(\cdot)$.

We next state the domination and factorization theorem for Lipschitz p -summing operators [FJ09].

Theorem 3.2.1 (Domination and factorization theorem) [FJ09]. *Let $1 \leq p < \infty$. The following properties are equivalent for a mapping $T : X \rightarrow Y$ between metric spaces and a positive constant C .*

- (a) *The mapping T is Lipschitz p -summing and $\pi_p^L(T) \leq C$.*
- (b) *There is a probability μ on $\mathcal{B}_{X^\#}$ such that*

$$\|T(x) - T(y)\| \leq C \left(\int_{\mathcal{B}_{X^\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}}.$$

- (c) *For any isometric embedding j of Y into a 1-injective space Z , the following diagram commutte*

$$\begin{array}{ccc}
 L_\infty(\mathcal{B}_{X^\#}, \mu) & \xrightarrow{i_p} & L_p(\mathcal{B}_{X^\#}, \mu) \\
 i \uparrow & & \downarrow \tilde{T} \\
 X \xrightarrow{T} Y & \xrightarrow{j} & Z
 \end{array}$$

with $\text{Lip}(\tilde{T}) \leq C$.

The domination theorem of Lipschitz p -summing operators yields the following theorem.

Theorem 3.2.2 *If $1 \leq p \leq q < \infty$, then $\Pi_p^L(X, Y) \subset \Pi_q^L(X, Y)$. Moreover, $\pi_p^L(T) \leq \pi_q^L(T)$.*

For a linear operator $T \in \mathcal{L}(E, F)$ it is clear that $\pi_p^L(T) \leq \pi_p(T)$. J.-D. Farmer and W.-B. Johnson proved that the reverse inequality is true. This justifies that the notion of Lipschitz p -summing operator is really a generalization of the concept of linear p -summing operator.

Theorem 3.2.3 [FJ09, Theorem2] *Let T be a bounded linear operator from E into F and $1 \leq p < \infty$. Then $\pi_p^L(T) = \pi_p(T)$.*

3.2.2 Lipschitz tensor product $X \boxtimes Y$

In this subsection we based on the article of [CCJV15]. Let X be a pointed metric space and let E be a Banach space. The Lipschitz tensor product $X \boxtimes E$ is defined as the vector subspace of $\text{Lip}_0(X, E^*)'$ (the algebraic dual of $\text{Lip}_0(X, E^*)$) spanned by the set

$$\{\delta_{(x,y)} \boxtimes e : (x, y) \in X^2, e \in E\}$$

where

$$(\delta_{(x,y)} \boxtimes e)(f) = \langle f(x) - f(y), e \rangle, \forall f \in \text{Lip}_0(X, E^*).$$

We say that, $\delta_{(x,y)} \boxtimes e$ is an elementary Lipschitz tensor. We have $u \in X \boxtimes E$ if, and only if, $u = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \boxtimes e_i$ and this representation is not unique. Note that each element

$u \in X \boxtimes E$ can be represented as $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ since $\lambda \delta_{(x,y)} \boxtimes e = \delta_{(x,y)} \boxtimes \lambda e$. We have also

$$u(f) = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle \text{ for all } f \text{ in } \text{Lip}_0(X, E^*).$$

Denote by $\mathcal{F}((X^\#, \tau_p); E)$ the space of all finite rank continuous linear operators from $(X^\#, \tau_p)$ to E , where τ_p denotes the topology of pointwise convergence of $X^\#$.

Theorem 3.2.4 [CCJV15] *The map $J : X \boxtimes E \longrightarrow \mathcal{F}((X^\#, \tau_p); E)$, given by*

$$J(u)(f) = \sum_{i=1}^n (f(x_i) - f(y_i)) e_i$$

for $u \in X \boxtimes E = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \boxtimes e_i$ and $f \in X^\#$, is a linear isomorphism.

Lipschitz injective norm

For a Banach space E and a metric space X , we give in then following the Lipschitz injective norm on $X \boxtimes E$.

Definition 3.2.2 [CCJV15] *For each $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, define*

$$\varepsilon(u) = \sup \left\{ \left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle \right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*} \right\}.$$

Notice that the supremum on the right side in the previous definition exists since

$$\begin{aligned} \left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle \right| &\leq \sum_{i=1}^n |(f(x_i) - f(y_i)) \langle \varphi, e_i \rangle| \\ &\leq \sum_{i=1}^n \text{Lip}(f) d(x_i, y_i) \|\varphi\| \|e_i\| \\ &\leq \sum_{i=1}^n d(x_i, y_i) \|e_i\| \end{aligned}$$

for all $f \in \mathcal{B}_{X^\#}$ and $\varphi \in B_{E^*}$. Note that

$$\sum_{i=1}^n |(f(x_i) - f(y_i)) \langle \varphi, e_i \rangle| = (f \boxtimes \varphi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$$

and, consequently, $\varepsilon(u)$ does not depend on the representation of u by [CCJV15, Lemma2.2], so ε defines a map from $X \boxtimes E$ to \mathbb{R} .

ε is a dualizable Lipschitz cross-norm on $X \boxtimes E$.

Cross-norm, i.e., $\varepsilon(\delta_{(x,y)} \boxtimes e) = d(x, y) \|e\|$, $\forall x, y \in X, \forall e \in E$.

Dualizable, i.e.,

$$\left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle \right| \leq \text{Lip}(f) \|\varphi\| \varepsilon \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right), \forall f \in X^\#, \forall \varphi \in E^*,$$

ε is the least dualizable Lipschitz cross-norm on $X \boxtimes E$.

The completion $X \widehat{\boxtimes}_\varepsilon E$ of $X \boxtimes_\varepsilon E$ is called the injective Lipschitz tensor product of X and E . Next we justify this terminology in the case $K = \mathbb{R}$. We can identify $X \widehat{\boxtimes}_\varepsilon E$ with the space of approximable bounded linear operators of $(X^\#, \tau_p)$ ($= (X^\#, \sigma(X^\#, X))$).

Proposition 3.2.2 [CCJV15] *The map $J : X \boxtimes_\varepsilon E \rightarrow \mathcal{F}((X^\#, \tau_p); E)$, defined by*

$$J(u)(f) = \sum_{i=1}^n (f(x_i) - f(y_i)) e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ and $f \in X^\#$ is an isometric isomorphism. As a consequence, $X \widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to the closure in the operator norm topology of $\mathcal{F}((X^\#, \tau_p); E)$.

Proof. By Theorem 3.2.4, J is a linear bijection. If $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$, we have

$$\begin{aligned} \|J(u)\| &= \sup\{\|J(u)(f)\| : f \in \mathcal{B}_{X^\#}\} \\ &= \sup\left\{ \left| \varphi \left(\sum_{i=1}^n (f(x_i) - f(y_i)) e_i \right) \right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*} \right\} \\ &= \sup\left\{ \left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle \right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*} \right\} \\ &= \varepsilon(u). \end{aligned}$$

The consequence is immediate. ■

Let $\mathcal{F}(X)$ be the Lipschitz-free Banach space over a pointed metric space X . Let us recall that $\mathcal{F}(X)$ is the closed linear subspace of $(X^\#)^*$ spanned by the set $\{\delta_x : x \in X\}$, where for each $x \in X$, δ_x is the evaluation functional at the point x defined on $X^\#$. It was proved that if X is a pointed metric space, then $X \widehat{\boxtimes}_\varepsilon \mathbb{K}$ is isometrically isomorphic to $\mathcal{F}(X)$;

in fact, much more is true. We show below that the space $X\widehat{\boxtimes}_\varepsilon E$ can be identified with the injective Banach-space tensor product $\mathcal{F}(X)\widehat{\otimes}_\varepsilon E$.

Proposition 3.2.3 [CCJV15] *The map $I : X \boxtimes_\varepsilon E \rightarrow \mathcal{F}(X) \otimes_\varepsilon E$, defined by*

$$I(u) = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ is a linear isometry. As a consequence, $X\widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to $\mathcal{F}(X)\widehat{\otimes}_\varepsilon E$.

Proof. Let $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$. Since $\mathcal{F}(X) = X^\#$, note that the norm of $\sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$ in $\mathcal{F}(X) \otimes_\varepsilon E$ is given by

$$\sup \left\{ \left| \sum_{i=1}^n \langle f, \delta_{x_i} - \delta_{y_i} \rangle \langle \varphi, e_i \rangle \right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*} \right\}.$$

Since $\langle f, \delta_{x_i} - \delta_{y_i} \rangle$ is precisely $f(x_i) - f(y_i)$, [CCJV15, Proposition 1.7] shows that I is well defined (and thus linear) and moreover a quick glance at Definition 3.2.2 shows that I is an isometry. ■

Recall that the linear span of $\{\delta_x\}_{x \in X}$ is dense in $\mathcal{F}(X)$, hence the tensors of the form $\sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$, with $x_i, y_i \in X$ and $e_i \in E$ are dense in $\mathcal{F}(X)\widehat{\otimes}_\varepsilon E$. This shows that the map I has dense range, and thus $X\widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to $\mathcal{F}(X)\widehat{\otimes}_\varepsilon E$.

Lipschitz projective norm

We introduce the Lipschitz projective norm on $X \boxtimes E$.

Definition 3.2.3 [CCJV15] *For each $u \in X \boxtimes E$, define*

$$\pi(u) = \inf \left\{ \sum_{i=1}^n d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right\}$$

the infimum being taken over all representations of u .

Theorem 3.2.5 [CCJV15] *The Lipschitz projective norm π is a dualizable Lipschitz cross-norm on $X \boxtimes E$ such that $L \leq \pi$.*

The completion $X \widehat{\boxtimes}_{\pi} E$ of $X \boxtimes_{\pi} E$ is called the projective Lipschitz tensor product of X and E .

Theorem 3.2.6 [CCJV15] *The Lipschitz projective norm π is the greatest Lipschitz cross-norm on $X \boxtimes E$.*

Proof. We have seen in the precedent Theorem that π is a Lipschitz cross-norm on $X \boxtimes E$. Now, let α be a Lipschitz cross-norm on $X \boxtimes E$ and let $u \in X \boxtimes E$. If $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ is a representation of u , we have

$$\alpha(u) = \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \leq \sum_{i=1}^n \alpha(\delta_{(x_i, y_i)} \boxtimes e_i) = \sum_{i=1}^n d(x_i, y_i) \|e_i\|.$$

Now the very definition of π gives $\alpha(u) \leq \pi(u)$. ■

3.3 Lipschitz $\tau(p)$ -summing operators

Definition 3.3.1 *Let T be in $\text{Lip}_0(X, E)$ and consider $1 \leq q \leq p < \infty$. We say that T is Lipschitz $\tau(p, q)$ -summing if there is a positive constant C such that, for all $n \in \mathbb{N}$; $(x_i), (x'_i) \subset X$; $(a_i^*) \subset E^*$ and $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}_+$, we have*

$$\begin{aligned} & \left(\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), a_i^* \rangle|^p \right)^{\frac{1}{p}} \\ & \leq C \sup_{\substack{\|f\| \leq 1 \\ \|a\| \leq 1}} \left(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x'_i)) \langle a_i^*, a \rangle|^q \right)^{\frac{1}{q}} \end{aligned} \quad (3.3.1)$$

where $f \in X^{\#}$ and $a \in E$. We will denote this class of mappings by $\Pi_{\tau(p, q)}^L(X, E)$ and we equip it with the norm $\pi_{\tau(p, q)}^L(T) = \inf C$, for the constants that appear in the above expression, for which it becomes a Banach space. When $p = q$, we write $\Pi_{\tau(p)}^L$ and $\pi_{\tau(p)}^L$

instead of $\Pi_{\tau(p,p)}^L$ and $\pi_{\tau(p,p)}^L$ respectively and we say that T is Lipschitz $\tau(p)$ -summing. If $p = q = 1$, we simply write Π_{τ}^L and π_{τ}^L and we say that T is Lipschitz τ -summing. Like in the linear case, if $1 \leq s \leq r \leq q \leq p$, then $\Pi_{\tau(q,r)}^L \subset \Pi_{\tau(p,s)}^L$ and $\pi_{\tau(p,s)}^L(T) \leq \pi_{\tau(q,r)}^L(T)$ for all T in $\Pi_{\tau(q,r)}^L$. Moreover, it follows that

$$\Pi_{\tau(q,r)}^L \subset \Pi_{\tau(p,r)}^L \text{ and } \pi_{\tau(p,r)}^L(T) \leq \pi_{\tau(q,r)}^L(T) \text{ for all } T \text{ in } \Pi_{\tau(q,r)}^L$$

and

$$\Pi_{\tau(q,r)}^L \subset \Pi_{\tau(q,s)}^L \text{ and } \pi_{\tau(q,s)}^L(T) \leq \pi_{\tau(q,r)}^L(T) \text{ for all } T \text{ in } \Pi_{\tau(q,r)}^L.$$

Remark 3.3.1 (1) *The definition is the same if we restrict to $\lambda_i = 1$ (by the same argument cited implicitly in [FJ09]).*

(2) *By Goldstine's theorem, we can replace a by $a^{**} \in E^{**}$ in the inequality 3.3.1.*

(3) *If T is linear then T is $\tau(p, q)$ -summing implies that T is Lipschitz $\tau(p, q)$ -summing and $\pi_{\tau(p,q)}^L(T) \leq \pi_{\tau(p,q)}(T)$. We do not know if the converse is true. Because we do not have now a factorization theorem and $\mathcal{B}_{X\#}$ is also difficult to handle. Is it a good generalization?*

Lemma 3.3.1 *Let $1 \leq p < \infty$. For $n \in \mathbb{N}$, $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n} \subset X$, $(a_i^*)_{1 \leq i \leq n} \subset E^*$ and $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{R}_+$; let $v : \ell_n^{p*} \rightarrow X \boxtimes_{\varepsilon} E^*$ be a linear operator such that $v(e_i) = \delta_{(x_i, x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$; where (e_i) denotes the unit vector basis of ℓ_n^{p*} and \boxtimes denotes the Lipschitz tensor product as introduced in [CCJV15]. We have*

$$\|v\| = \sup_{\substack{\|f\|_{X\#}=1 \\ \|a\|_E=1}} \left(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x'_i)) \langle a_i^*, a \rangle|^p \right)^{\frac{1}{p}}.$$

Proof. We have

$$\begin{aligned}
 \|v\| &= \sup_{\|\alpha\|_{\ell_n^{p^*}}=1} \|v(\alpha)\|_{X \boxtimes_{\varepsilon} E^*} \\
 &= \sup_{\|\alpha\|_{\ell_n^{p^*}}=1} \left\| \sum_{i=1}^n \alpha_i v(e_i) \right\|_{X \boxtimes_{\varepsilon} E^*} \quad \left(\alpha = \sum_{i=1}^n \alpha_i e_i \right) \\
 &= \sup_{\|\alpha\|_{\ell_n^{p^*}}=1} \left\| \sum_{i=1}^n \alpha_i \delta_{(x_i, x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^* \right\|_{X \boxtimes_{\varepsilon} E^*} \\
 &= \sup_{\|\alpha\|_{\ell_n^{p^*}}=1} \sup_{\substack{\|f\|_{X^{\#}}=1 \\ \|\alpha\|_{E^*}=1}} \left(\sum_{i=1}^n \alpha_i \lambda_i^{\frac{1}{p}} |(f(x_i) - f(x'_i)) \langle a_i^*, a \rangle| \right) \\
 &= \sup_{\substack{\|f\|_{X^{\#}}=1 \\ \|\alpha\|_{E^*}=1}} \left(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x'_i)) \langle a_i^*, a \rangle|^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

This proves the Lemma. ■

Proposition 3.3.1 *Let T be in $\text{Lip}_0(X, E)$. The operator T is Lipschitz $\tau(p)$ -summing if, and only if, for all $n \in \mathbb{N}$, $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n} \subset X$, $(a_i^*)_{1 \leq i \leq n} \subset E^*$, $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}_+$ and all linear operators $v : \ell_n^{p^*} \rightarrow X \boxtimes_{\varepsilon} E^*$ such that $v(e_i) = \delta_{(x_i, x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$, we have*

$$\left(\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), a_i^* \rangle|^p \right)^{\frac{1}{p}} \leq C \|v\|. \quad (3.3.2)$$

We now give the left ideal property in "Pietsch's sense".

Proposition 3.3.2 *Consider T in $\text{Lip}_0(Y, E)$ and R in $\text{Lip}_0(X, Y)$. If T is Lipschitz $\tau(p)$ -summing operator, then $T \circ R$ is Lipschitz $\tau(p)$ -summing and $\pi_{\tau(p)}^L(T \circ R) \leq \pi_{\tau(p)}^L(T) \text{Lip}(R)$.*

Proof. Let $n \in \mathbb{N}$, $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n} \subset X$, $(a_i^*)_{1 \leq i \leq n} \subset E^*$ and $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}_+$. It suffices by 3.3.2 to show that $\left(\sum_{i=1}^n \lambda_i |\langle T \circ R(x_i) - T \circ R(x'_i), a_i^* \rangle|^p \right)^{\frac{1}{p}} \leq \pi_{\tau(p)}^L(T) \text{Lip}(R) \|w\|$ where $w : \ell_n^{p^*} \rightarrow X \boxtimes_{\varepsilon} E^*$ such that $w(e_i) = \delta_{(x_i, x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$.

Consider the following commutative diagram

$$\begin{array}{ccc}
 \ell_n^{p^*} & \xrightarrow{v} & Y \boxtimes_{\varepsilon} E^* \\
 w \downarrow & & R \boxtimes id_{E^*} \nearrow \\
 X \boxtimes_{\varepsilon} E^* & &
 \end{array}$$

where

$$v(e_i) = \delta_{(R(x_i), R(x'_i))} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$$

and

$$R \boxtimes id_{E^*} \left(\delta_{(x_i, x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^* \right) = \delta_{(R(x_i), R(x'_i))} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*.$$

The Lipschitz injective norm ε is uniform by [CCJV15, Theorem 7.1] and by [CCJV15, Proposition 4.2], we have

$$\begin{aligned} \left(\sum_{i=1}^n \lambda_i |\langle T \circ R(x_i) - T \circ R(x'_i), a_i^* \rangle|^p \right)^{\frac{1}{p}} &\leq \pi_{\tau(p)}^L(T) \|v\| \\ &\leq \pi_{\tau(p)}^L(T) \|w\| \|R \boxtimes id_{E^*}\| \\ &\leq \pi_{\tau(p)}^L(T) \text{Lip}(R) \|w\|. \end{aligned}$$

This implies by 3.3.2 that $T \circ R$ is Lipschitz $\tau(p)$ -summing and $\pi_{\tau(p)}^L(T \circ R) \leq \pi_{\tau(p)}^L(T) \text{Lip}(R)$ and this ends the proof. ■

The dual E^* of E is naturally isometric to a subspace of $\text{Lip}_0(E)$. By [BL00, p. 37], there is a norm one projection, $p : \text{Lip}_0(E) \longrightarrow E^*$.

We notice that $T^t = T^\# |_{E^*}$ corresponds in a canonical way to the usual adjoint of the linear operator attached to T by Theorem 1.2.2, i.e., $T^t = T_L^*$.

$$\begin{array}{ccc} \text{Lip}_0(E) & \xrightarrow{T^\#} & \text{Lip}_0(X) \\ p \downarrow & & T_L^* \nearrow \\ & & E^* \end{array}$$

Now, we can show the right ideal property for Banach spaces.

Proposition 3.3.3 *Consider T in $\text{Lip}_0(Y, E)$ and S in $\text{Lip}_0(E, F)$. If T is Lipschitz $\tau(p)$ -summing operator, then $S \circ T$ is Lipschitz $\tau(p)$ -summing and $\pi_{\tau(p)}^L(S \circ T) \leq \text{Lip}(S) \pi_{\tau(p)}^L(T)$.*

Proof. Let $(y_i)_{1 \leq i \leq n}, (y'_i)_{1 \leq i \leq n} \subset Y$, $(b_i^*)_{1 \leq i \leq n} \subset F^*$ and $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}_+$, we have

$$\begin{aligned} &\left(\sum_{i=1}^n \lambda_i |\langle S \circ T(y_i) - S \circ T(y'_i), b_i^* \rangle|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n \lambda_i |\langle T(y_i), S^\#(b_i^*) \rangle - \langle T(y'_i), S^\#(b_i^*) \rangle|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n \lambda_i |\langle T(y_i), S^t(b_i^*) \rangle - \langle T(y'_i), S^t(b_i^*) \rangle|^p \right)^{\frac{1}{p}} \end{aligned}$$

(S^t is the transposed of the linear operator attached to S)

$$\begin{aligned}
 &= \left(\sum_{i=1}^n \lambda_i |\langle T(y_i) - T(y'_i), S^t(b_i^*) \rangle|^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{\tau(p)}^L(T) \sup_{\substack{\|f\|_{Y^\#} \leq 1 \\ \|a\|_E = 1}} \left(\sum_{i=1}^n \lambda_i |(f(y_i) - f(y'_i)) \langle S^t(b_i^*), a \rangle|^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{\tau(p)}^L(T) \|S_L\| \sup_{\substack{\|f\|_{Y^\#} \leq 1 \\ \|a\|_E = 1}} \left(\sum_{i=1}^n \lambda_i \left| (f(y_i) - f(y'_i)) \left\langle \frac{S^t(b_i^*)}{\|S_L^*\|}, a \right\rangle \right|^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{\tau(p)}^L(T) \|S_L\| \sup_{\substack{\|f\|_{Y^\#} \leq 1 \\ \|a\|_E = 1}} \left(\sum_{i=1}^n \lambda_i \left| (f(y_i) - f(y'_i)) \left\langle b_i^*, \frac{S_L(a)}{\|S_L\|} \right\rangle \right|^p \right)^{\frac{1}{p}} \\
 &\leq \text{Lip}(S) \pi_{\tau(p)}^L(T) \sup_{\substack{\|f\|_{B_{Y^\#}} \leq 1 \\ \|b\|_E = 1}} \left(\sum_{i=1}^n \lambda_i |(f(y_i) - f(y'_i)) \langle b_i^*, b \rangle|^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

Therefore, $S \circ T$ is Lipschitz $\tau(p)$ -summing operator and $\pi_{\tau(p)}^L(S \circ T) \leq \pi_{\tau(p)}^L(T) \text{Lip}(S)$. ■

By the left ideal property, we have the following proposition. For the converse, we can see [Saa15, Remark 3.3].

Proposition 3.3.4 *Consider $1 \leq p < \infty$. Let $T : X \rightarrow E$ be a Lipschitz map. If T_L is $\tau(p)$ -summing, then T is Lipschitz $\tau(p)$ -summing. The converse is false.*

We will give "an alternative reciprocal" after the domination theorem.

Now, we characterize this type of operators by giving the Pietsch Domination Theorem. Recently, a general version of the Pietsch Domination Theorem was proved in [PSS12b], which is an improved version of a similar result in [BPR10] (see also [PSS12a]). Let X, Y and V be (arbitrary) non-void sets, $\mathcal{H}(X; Y)$ be a non-void family of mappings from X to Y , G be a Banach space and K be a compact Hausdorff topological space. Let

$$R: K \times V \times G \rightarrow [0, \infty) \text{ and } S: \mathcal{H}(X; Y) \times V \times G \rightarrow [0, \infty)$$

be arbitrary mappings and $1 \leq q < \infty$. According to [PSS12b], [BPR10] a mapping $f \in \mathcal{H}(X; Y)$ is RS -abstract q -summing if there is a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^n S(f, x_i, b_i)^q \right)^{\frac{1}{q}} \leq C \sup_{\varphi \in K} \left(\sum_{i=1}^n R(\varphi, x_i, b_i)^q \right)^{\frac{1}{q}} \quad (3.3.3)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in V$ and $b_1, \dots, b_n \in G$. We define

$$\mathcal{H}_{RS,q}(X; Y) = \{f \in \mathcal{H}(X; Y) : f \text{ is } RS\text{-abstract } q\text{-summing}\}.$$

Suppose that R is such that the mapping

$$R_{x,b}: K \longrightarrow [0, \infty) \text{ defined by } R_{x,b}(\varphi) = R(\varphi, x, b) \quad (3.3.4)$$

is continuous for every $x \in V$ and $b \in G$. The Pietsch Domination Theorem from [PSS12b] reads as follows.

Theorem 3.3.1 . *Suppose that S is arbitrary, R satisfies 3.3.4 and let $1 \leq q < \infty$. A map $f \in \mathcal{H}(X; Y)$ is RS -abstract q -summing if, and only if, there is a constant $C \geq 0$ and a Borel probability measure μ on K such that*

$$S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^q d\mu(\varphi) \right)^{1/q} \quad (3.3.5)$$

for all $x \in V$ and $b \in G$.

We give the domination theorem of the Lipschitz $\tau(p)$ -summing operators by using the technique above.

Theorem 3.3.2 *An operator T in $\text{Lip}_0(X, E)$ is Lipschitz $\tau(p)$ -summing if, and only if, there exists a positive constant C and a probability μ on $K = \mathcal{B}_{X^\#} \times B_{E^{**}}$ such that*

$$|\langle T(x) - T(x'), a^* \rangle| \leq C \left(\int_K |(f(x) - f(x')) \langle a^{**}, a^* \rangle|^p d\mu(f, a^{**}) \right)^{\frac{1}{p}} \quad (3.3.6)$$

for all $x, x' \in X$ and $a^* \in E^*$. In this case, $\pi_{\tau(p)}^L(T) = \inf C$.

Proof. Note that T is Lipschitz $\tau(p)$ -summing if, and only if, it is RS -abstract p -summing with

$$\begin{aligned} V &= X \times X, \\ G &= E^* \end{aligned}$$

and $K = \mathcal{B}_{X^\#} \times B_{E^{**}}$, which is a compact Hausdorff space in the topology of pointwise convergence on K , $\mathcal{H} = \text{Lip}_0(X, E)$ and R, S are defined as follows

$$R: (\mathcal{B}_{X^\#} \times B_{E^{**}}) \times (X \times X) \times E^* \longrightarrow [0, \infty)$$

such that $R((f, a^{**}), (x, x'), a^*) = \lambda^{\frac{1}{p}} |(f(x) - f(x')) \langle a^{**}, a^* \rangle|$ and

$$S: \mathcal{H} \times (X \times X) \times E^* \longrightarrow [0, \infty)$$

such that

$$S(T, (x, x'), a^*) = \lambda^{\frac{1}{p}} |\langle T(x) - T(x'), a^* \rangle|.$$

So we have, as a consequence of Theorem 3.3.1.

$$\begin{aligned} & |\langle T(x) - T(x'), a^* \rangle| \\ & \leq C \left(\int_{\mathcal{B}_{X^\#} \times B_{E^{**}}} |(f(x) - f(x')) \langle a^{**}, a^* \rangle|^p d\mu(f, a^{**}) \right)^{\frac{1}{p}} \end{aligned}$$

and the proof is complete. ■

As an immediate corollary of our theorem.

Corollary 3.3.1 $\Pi_{\tau(p)}^L \subseteq \Pi_{\tau(q)}^L$, when $1 \leq p \leq q < \infty$ and $\Pi_{\tau(p)}^L \subseteq \Pi_p^L$, for all $1 \leq p < \infty$.

We will present another characterization (*Pietsch's domination theorem*) concerning this class of Lipschitz operators. For the proof, we use the same idea as used for example in [AMS09] and [Muj08, Theorem 3.6]. Before this, we first state Ky Fan's lemma. The proof can be consulted in [DJT95, p. 190].

Lemma 3.3.2 *Let K be a Hausdorff topological vector space and let C be a compact convex subset of K . Let \mathcal{M} be a set of functions on C with values in $(-\infty, \infty]$ having the following properties:*

- (a) *each $f \in \mathcal{M}$ is convex and lower semicontinuous;*
- (b) *if $g \in \text{conv}(\mathcal{M})$, there is an $f \in \mathcal{M}$ with $g(x) \leq f(x), \forall x \in C$;*

(c) there is an $r \in \mathbb{R}$ such that each $f \in \mathcal{M}$ has a value $\leq r$.

Then there is an $x_0 \in C$ such that $f(x_0) \leq r$ for all $f \in \mathcal{M}$.

Theorem 3.3.3 Consider $T \in \text{Lip}_0(X, E)$ and C a positive constant.

(1) The operator T is Lipschitz $\tau(p)$ -summing and $\pi_{\tau(p)}^L(T) \leq C$.

(2) There exist Radon probability measures μ_1 on $\mathcal{B}_{X^\#}$ and μ_2 on $B_{E^{**}}$, such that for all x, x' in X and a^* in E^* , we have

$$\begin{aligned} & |\langle T(x) - T(x'), a^* \rangle| \\ & \leq C \left(\int_{\mathcal{B}_{X^\#}} \int_{B_{E^{**}}} |(f(x) - f(x')) \langle a^*, a^{**} \rangle|^p d\mu_1(f) d\mu_2(a^{**}) \right)^{\frac{1}{p}}. \end{aligned} \quad (3.3.7)$$

Moreover, in this case

$$\pi_{\tau(p)}^L(T) = \inf \{ C > 0 : \text{for all } C \text{ verifying the inequality 3.3.7} \}.$$

Proof. We only need to show the implication (1) \implies (2) because using 3.3.7, one easily shows that T is Lipschitz $\tau(p)$ -summing and $\pi_{\tau(p)}^L(T) \leq C$.

Consider the sets $\mathcal{P}(\mathcal{B}_{X^\#})$ and $\mathcal{P}(B_{E^{**}})$ of probability measures in $\mathcal{C}(\mathcal{B}_{X^\#})^*$ and $\mathcal{C}(B_{E^{**}})^*$ respectively endowed with their weak* topologies. These sets are compact and convex. We are going now to apply Ky Fan's lemma with $K = \mathcal{C}(\mathcal{B}_{X^\#})^* \times \mathcal{C}(B_{E^{**}})^*$ and $C = \mathcal{P}(\mathcal{B}_{X^\#}) \times \mathcal{P}(B_{E^{**}})$ which is convex and compact.

Let \mathcal{M} be the set of all functions φ from C with values in \mathbb{R} of the form

$$\begin{aligned} & \varphi_{((x_i), (x'_i), (a_i^*), (\lambda_i))}(\mu_1, \mu_2) \\ & = \sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), a_i^* \rangle|^p - C \int_{\mathcal{B}_{X^\#}} \int_{B_{E^{**}}} \lambda_i |(f(x_i) - f(x'_i)) \langle a_i^*, a^{**} \rangle|^p d\mu_1(f) d\mu_2(a^{**}) \end{aligned}$$

where $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n} \subset X$; $(a_i^*)_{1 \leq i \leq n} \subset E^*$ and $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}_+$. These functions are continuous and convex. The set \mathcal{M} is a convex cone. We now apply Ky Fan's lemma (the conditions (a) and (b) are satisfied). For the condition (c), since $\mathcal{B}_{X^\#}$ is a compact Hausdorff space in the topology of pointwise convergence on X and $B_{E^{**}}$ are weak* compact and norming sets, using the fact that X is isometrically embedded into $\mathcal{B}_{X^\#}$ and by the classical Goldstine's theorem there exist for $\varphi \in \mathcal{M}$ two elements, f_0 in $\mathcal{B}_{X^\#}$ and a_0^{**} in $B_{E^{**}}$ such that

$$\sup_{\substack{\|a^{**}\|_{E^{**}}=1 \\ \|f\|_{X^\#}=1}} \left\| \left(\lambda_i^{\frac{1}{p}} (f(x_i) - f(x'_i)) \langle a_i^*, a^{**} \rangle \right) \right\|_{l_p^n}^p = \sum_{i=1}^n \lambda_i |f_0(x_i) - f_0(x'_i) \langle a_i^*, a_0^{**} \rangle|^p.$$

If δ_{f_0} and $\delta_{a_0^{**}}$ denote the Dirac measures supported by f_0 and a_0^{**} respectively, we have

$$\begin{aligned} & \varphi_{((x_i), (x'_i), (a_i^*), (\lambda_i))}(\delta_{f_0}, \delta_{a_0^{**}}) \\ &= \sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), a_i^* \rangle|^p - C^p \sum_{i=1}^n \lambda_i |f_0(x_i) - f_0(x'_i) \langle a_i^*, a_0^{**} \rangle|^p \\ &\leq 0 \end{aligned}$$

Hypothesis (1) yields

$$\sup \left\{ \varphi_{((x_i), (x'_i), (a_i^*), (\lambda_i))}(\mu_1, \mu_2) : (\mu_1, \mu_2) \in K \right\} \leq 0.$$

By the conclusion of Ky Fan's lemma, there is $\mu = (\mu_1, \mu_2) \in C$ such that $\mu(\varphi) \leq 0$ for all φ in \mathcal{M} . If φ is generated by the simple elements $x, x' \in X$, $a^* \in E^*$ and $\lambda = 1$, we find

$$\begin{aligned} & \varphi_{(x, x', a^*, 1)}(\mu_1, \mu_2) \\ &= |\langle T(x) - T(x'), a^* \rangle|^p - C^p \\ & \int_{\mathcal{B}_{X\#}} \int_{\mathcal{B}_{E^{**}}} |(f(x) - f(x') \langle a^*, a^{**} \rangle)|^p d\mu_1(f) d\mu_2(a^{**}) \leq 0. \end{aligned}$$

It follows that

$$\begin{aligned} & |\langle T(x) - T(x'), a^* \rangle| \\ & \leq C \left(\int_{\mathcal{B}_{X\#}} \int_{\mathcal{B}_{E^{**}}} |(f(x) - f(x') \langle a^*, a^{**} \rangle)|^p d\mu_1(f) d\mu_2(a^{**}) \right)^{\frac{1}{p}} \end{aligned}$$

and this completes the proof. ■

As a corollary, we get.

Proposition 3.3.5 *Consider T in $\text{Lip}_0(X, E)$. If the operator T is Lipschitz $\tau(p)$ -summing then, its linearization T_L is in $\mathcal{D}_{p^*}(\mathcal{E}(X), E)$ in the sense of [Coh73] and consequently T_L^* is p -summing.*

Proof. Suppose that $T \in \Pi_{\tau(p)}^L(X, E)$. Let $m = \sum_{j=1}^l \lambda_j m_{x_j, x'_j}$ be in $\mathcal{M}(X)$ and $a^* \in E^*$.

By Theorem 3.3.3, we have

$$\begin{aligned} & |\langle T_L(m), a^* \rangle| \\ & \leq \sum_{j=1}^l |\lambda_j| |\langle T(x_j) - T(x'_j), a^* \rangle| \\ & \leq \pi_{\tau(p)}^L(T) \sum_{j=1}^l |\lambda_j| \left(\int_{\mathcal{B}_{X\#}} \int_{\mathcal{B}_{E^{**}}} |(f(x_j) - f(x'_j) \langle a^*, a^{**} \rangle)|^p d\mu_1(f) d\mu_2(a^{**}) \right)^{\frac{1}{p}} \\ & \leq \pi_{\tau(p)}^L(T) \sum_{j=1}^l |\lambda_j| d(x_j, x'_j) \left(\int_{\mathcal{B}_{E^{**}}} |\langle a^*, a^{**} \rangle|^p d\mu_2(a^{**}) \right)^{\frac{1}{p}} \\ & \leq \pi_{\tau(p)}^L(T) \|m\|_{\mathcal{E}(X)} \left(\int_{\mathcal{B}_{E^{**}}} |\langle a^*, a^{**} \rangle|^p d\mu_2(a^{**}) \right)^{\frac{1}{p}}. \end{aligned}$$

By density, this implies that $T_L \in \mathcal{D}_{p^*}(\mathcal{A}(X), E)$ and $d_{p^*}(T_L) \leq \pi_{\tau(p)}^L(T)$, hence $\pi_p(T_L^*) \leq \pi_{\tau(p)}^L(T)$ by [Coh73, Theorem 2.2.2]. ■

Chapter 4

On the Cohen Lipschitz p -nuclear operators

4.1 Preliminaries and definitions

Analogously for sequence $(\lambda_i)_i$ of real numbers and $(x_i)_i ; (x'_i)_i$ of points in metric space X , we denote their weak Lipschitz p -norm by

$$\omega_p^L(\lambda_i, (x_i)_i, (x'_i)_i) = \sup_{f \in \mathcal{B}_{X^\#}} \|\lambda_i |f(x_i) - f(x'_i)|\|_p.$$

J. A. Chàvez-Doménguez defined in [Cha11] the Lipschitz (r, p, q) -summing operators as follows.

Definition 4.1.1 [Cha11] *An operator $T : X \rightarrow E$ is called Lipschitz (r, p, q) -summing ($1 \leq p, r < \infty, 1 \leq q \leq \infty$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$) if there is a positive constant C such that for all $n \in \mathbb{N}$, $(x_i)_{1 \leq i \leq n}; (x'_i)_{1 \leq i \leq n} \subset X$, $(a_i^*)_{1 \leq i \leq n} \subset E^*$ and $\lambda_i \geq 0, k_i > 0$, we have*

$$\|(\lambda_i \langle T(x_i) - T(x'_i), a_i^* \rangle)\|_r \leq C \omega_p^L(\lambda_i k_i^{-1}, (x_i)_i, (x'_i)_i) \omega_q((k_i a_i^*)_i). \quad (4.1.1)$$

The smallest constant C such that 4.1.1 holds is noted by $\pi_{r,p,q}^L(T)$ and $\Pi_{r,p,q}^L(X, E)$ will denote the set of all such operators which is a Banach space equipped with the norm $\pi_{r,p,q}^L(\cdot)$. When E is a dual we can consider the a_i^* in the predual. Also, the case (r, r, ∞) corresponds

to Lipschitz r -summing operators from X into E as in [FJ09], whereas the case (r, p, ∞) corresponds to the Lipschitz (r, p) -summing operators from X into E as in [JS09]. Moreover, by the same argument as in [FJ09], we may take $\lambda_i = 1$ in (1.1), for all i .

We will introduced the corresponding definition for spaces of molecules, which studied by J. A. Chàvez-Doménguez in [Cha11].

Definition 4.1.2 For all $\lambda_j, k_j > 0$ and $m \in \mathcal{M}(X, E)$ such that $m = \sum_{j=1}^n v_j m_{x_j x'_j}$, let

$$\begin{aligned} & \mu_{r,p,q}(m) \\ &= \inf \left\{ \left\| (\lambda_j)_{j=1}^n \right\|_r \sup_{f \in \mathcal{B}_{X^\#}} \left\| (k_j^{-1} \lambda_j^{-1} [f(x_j) - f(x'_j)])_{j=1}^n \right\|_p \left\| (k_j v_j)_{j=1}^n \right\|_{q,w} \right\}. \end{aligned}$$

$\mu_{r,p,q}(m)$ is β -norm on $\mathcal{M}(X, E)$ for $\frac{1}{\beta} = \frac{1}{p} + \frac{1}{r} + \frac{1}{q} \geq 1$ and $0 < \beta \leq 1$ see [Cha11, Theorem 5.2]. The β -normed space $(\mathcal{M}(X, E), \mu_{r,p,q}(\cdot))$ denoted by $\mathcal{M}_{r,p,q}(X, E)$. Such that for $0 < \beta \leq 1$, a non-negative positively homogeneous functional μ defined on a vector space U is called a β -seminorm if $\mu(u_1 + u_2)^\beta \leq \mu(u_1)^\beta + \mu(u_2)^\beta$ for all $u_1, u_2 \in U$. If in addition μ vanishes only at 0, it is called a β -norm.

Theorem 4.1.1 [Cha11, Theorem 5.2] The spaces $\Pi_{r^*,p,q}^L(X, E^*)$ and $\mathcal{M}_{r,p,q}(X, E)$ are isometrically isomorphic via the canonical pairing. Moreover, on the unit ball of $\Pi_{r^*,p,q}^L(X, E^*)$ the weak* topology coincides with the topology of pointwise $\sigma(E^*, E)$ -convergence.

The following theorem which states the Domination/Factorization theorem related to the space of (r, p, q) -summing operators and identifies the dual of $\mathcal{M}_{r,p,q}(X, E)$ in this special case.

Theorem 4.1.2 (Domination/Factorization) ([Cha11, Theorem 5.2]) Suppose $\frac{1}{p} + \frac{1}{r} + \frac{1}{q} = 1$ and let $T \in Lip_0(X, E^*)$, $C > 0$. The following are equivalent:

- (a) $|\langle T, m \rangle| \leq C \mu_{r,p,q}(m)$ for all $m \in \mathcal{M}(X, E)$.
- (b) There exist regular Borel probability measures μ and ν on the weak*-compact unit balls $\mathcal{B}_{X^\#}, \mathcal{B}_{E^*}$ such that for all $x, x' \in X$ and $v \in E$,

$$|\langle T(x) - T(x'), v \rangle| \leq C \left(\int_{\mathcal{B}_{X^\#}} (|f(x) - f(x')|)^r d\mu(f) \right)^{\frac{1}{p}} \left(\int_{\mathcal{B}_{E^*}} |(v^*(v))^q| d\nu(v^*) \right)^{\frac{1}{q}}.$$

(c) There exist a Banach space Z , a Lipschitz p -summing operator $R : X \longrightarrow Z^*$ and a linear q -summing operator $S : E \longrightarrow Z$ such that $\pi_p^L(S^*) \cdot \pi_q(S) \leq C$ and

$$\langle T(x), v \rangle = \langle R(x), S(v) \rangle \text{ for all } x \in X, v \in E;$$

that is the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{T} & E^* \\ R \searrow & & \nearrow S^* \\ & Z^* & \end{array}$$

The statement (a) of the above theorem means that the space $\Pi_{r^*,p,q}^L(X, E^*)$ has as a predual the space $\mathcal{M}_{r,p,q}(X, E)$.

Now we give the definition of Cohen Lipschitz p -nuclear operators.

4.2 Cohen Lipschitz p -nuclear operators

The results obtained in this chapter are in [MT17]. We introduce the following generalization to Lipschitz operators of the class of Cohen p -nuclear operators studied in [Coh73]. It is a particular case from that defined by J. A. Chávez-Domínguez in [Cha11] which called the Lipschitz (r, p, q) -summing operators if we take $(r, p, q) = (1, p, p^*)$ and $k_i = 1$ for all i . The notion of p -nuclear operators was introduced in [PP69] by A. Persson and A. Pietsch. Initially the definition of nuclear operators for Banach spaces, was given by Grothendieck in [Gro55]. J. S. Cohen has initiated another concept of p -nuclear operators in [Coh73] which is not the same as the preceding notion and have generalized to (p, q) -nuclear operators ($1 \leq q \leq \infty$) by H. Apiola in [Api76]. In [CZ12], D. Chen and B. Zheng have generalized this notion to Lipschitz operators. For distinguish these two notions, we say Cohen p -nuclear operators for that investigated by J. S. Cohen and we try to generalize this notion to Lipschitz operators.

Definition 4.2.1 *A Lipschitz operator $T : X \longrightarrow E$ is Cohen Lipschitz p -nuclear ($1 < p < \infty$), if there is a positive constant C such that for any n in \mathbb{N} ; $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n}$ in X ;*

$(a_i^*)_{1 \leq i \leq n}$ in E^* and $(\lambda_i)_{1 \leq i \leq n}$ in \mathbb{R}_+ , we have

$$\begin{aligned} & \left| \sum_{i=1}^n \lambda_i \langle T(x_i) - T(x'_i), a_i^* \rangle \right| \\ & \leq C \sup_{f \in \mathcal{B}_{X\#}} \left(\sum_{i=1}^n \lambda_i |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}} \sup_{\|a\|_E \leq 1} \left(\sum_{i=1}^n |\langle a, a_i^* \rangle|^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned} \quad (4.2.1)$$

The smallest constant C which is noted by $\eta_p^L(T)$, such that the above inequality holds, is called the Cohen Lipschitz p -nuclear norm on the space $\mathcal{N}_p^L(X, E)$ of all Cohen Lipschitz p -nuclear operators from X into E which is a Banach space. For $p = 1$ and $p = \infty$ we have as in the linear case $\mathcal{N}_1^L(X, E) = \Pi_1^L(X, E)$ and $\mathcal{N}_\infty^L(X, E) = \mathcal{D}_{st, \infty}^L(X, E)$ (see below). The definition remains the same if we restrict to $\lambda_i = 1$, as in [FJ09]. We use this definition with the λ_i only in the proof of Pietsch's domination theorem. Immediately $\mathcal{N}_p^L(X, E) = \Pi_{1,p,p^*}^L(X, E)$ for $k_i = 1$, this implies that $\mathcal{N}_p^L(X, E)$ and $\mathcal{M}_{1,p,p^*}(X, E)$ are isometrically isomorphic via the canonical pairing as a special case in Theorem 4.1.1

We know (see [DJT95]) that $\ell^p(E) \equiv \ell_w^p(E)$ (the symbol \equiv indicates that two Banach spaces are isometrically isomorphic) for some $1 \leq p < \infty$ if and only if $\dim(E)$ is finite. If $p = \infty$, we have $\ell^\infty(E) \equiv \ell_w^\infty(E)$. We have also if $1 < p \leq \infty$, $\ell_w^p(E) \equiv \mathcal{L}(\ell^{p^*}, E)$ isometrically. In other words, let $v : \ell^{p^*} \rightarrow E$ be a linear operator such that $v(e_i) = a_i$ (namely, $v = \sum_{i=1}^{\infty} e_i \otimes a_i$, e_i denotes the unit vector basis of ℓ^{p^*}) then,

$$\|v\| = \|(x_i)\|_{\ell_w^p(E)}. \quad (4.2.2)$$

Let T be a Lipschitz operator between X, E and $v : \ell_n^{p^*} \rightarrow E^*$ be a bounded linear operator. By 4.2.2, the Lipschitz operator T is Cohen Lipschitz p -nuclear if, and only if,

$$\begin{aligned} & \left| \sum_{i=1}^n \lambda_i \langle T(x_i) - T(x'_i), v(e_i) \rangle \right| \\ & \leq C \sup_{f \in \mathcal{B}_{X\#}} \left(\sum_{i=1}^n \lambda_i |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}} \|v\|. \end{aligned} \quad (4.2.3)$$

Proposition 4.2.1 Consider T in $\text{Lip}_0(X, E)$, R in $\text{Lip}_0(E, F)$ and S in $\text{Lip}_0(Y, X)$. If T is Cohen Lipschitz p -nuclear operator, then $R \circ T \circ S$ is Cohen Lipschitz p -nuclear operator and $\eta_p^L(R \circ T \circ S) \leq \text{Lip}(R)\eta_p^L(T)\text{Lip}(S)$.

Proof. Let $n \in \mathbb{N}$; $(y_i)_{1 \leq i \leq n}, (y'_i)_{1 \leq i \leq n} \subset Y$ and $(a_i^*)_{1 \leq i \leq n} \subset E^*$. By (3.3), we prove in the first part that

$$\left| \sum_{i=1}^n \langle TS(y_i) - TS(y'_i), a_i^* \rangle \right| \leq \eta_p^L(T)\text{Lip}(S) \sup_{f \in B_{Y^\#}} \left(\sum_{i=1}^n |f(y_i) - f(y'_i)|^p \right)^{\frac{1}{p}} \|v\|$$

where $v : E \rightarrow \ell_n^{p^*}$ defined by $v(a) = \sum_{i=1}^n a_i^*(a) e_i$. We have

$$\begin{aligned} & \left| \sum_{i=1}^n \langle TS(y_i) - TS(y'_i), a_i^* \rangle \right| \\ & \leq \eta_p^L(T) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n |f(S(y_i)) - f(S(y'_i))|^p \right)^{\frac{1}{p}} \|v\|, \\ & \leq \eta_p^L(T)\text{Lip}(S) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n \left| \frac{f(S(y_i))}{\text{Lip}(S)} - \frac{f(S(y'_i))}{\text{Lip}(S)} \right|^p \right)^{\frac{1}{p}} \|v\|, \\ & \leq \eta_p^L(T)\text{Lip}(S) \sup_{f \in B_{Y^\#}} \left(\sum_{i=1}^n |f(y_i) - f(y'_i)|^p \right)^{\frac{1}{p}} \|v\|. \end{aligned}$$

This implies that

$$\eta_p^L(T \circ S) \leq \eta_p^L(T)\text{Lip}(S).$$

Let $n \in \mathbb{N}$; $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n} \subset X$; $(b_i^*)_{1 \leq i \leq n} \subset F^*$. In the second part, By 4.2.3 we prove that

$$\left| \sum_{i=1}^n \langle RT(x_i) - RT(x'_i), b_i^* \rangle \right| \leq \eta_p^L(T)\text{Lip}(R) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}} \|w\|$$

where $w : F \rightarrow \ell_n^{p^*}$ is defined by $w(b) = \sum_{i=1}^n b_i^*(b) e_i$. We have

$$\begin{aligned} & \left| \sum_{i=1}^n \langle RT(x_i) - RT(x'_i), b_i^* \rangle \right| \\ & = \left| \sum_{i=1}^n \langle T(x_i) - T(x'_i), R^\#(b_i^*) \rangle \right|, \\ & \leq \eta_p^L(T) \sup_{f \in B_{E^\#}} \left(\sum_{i=1}^n |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}} \|w\|, \\ & \leq \eta_p^L(T)\text{Lip}(R) \sup_{f \in B_{E^\#}} \left(\sum_{i=1}^n |f(e_i) - f(e'_i)|^p \right)^{\frac{1}{p}} \|w\|. \end{aligned}$$

Where $u(y) = \sum_{i=1}^n \langle R^\#(b_i^*), y \rangle e_i = \sum_{i=1}^n \langle b_i^*, R(y) \rangle e_i$.

This implies that T is Cohen Lipschitz p -nuclear and $\eta_p^L(R \circ T) \leq \|R\| \eta_p^L(T)$. ■

Let us present the *Pietsch's domination theorem* concerning this class of Lipschitz operators. The proof is like that used in [AMS09]. In [Cha11], J. A. Chávez-Domínguez gives a domination theorem for Lipschitz (r, p, q) -summing operators from X into E^* such that $1/r + 1/p + 1/q = 1$.

Theorem 4.2.1 *Consider $T \in \text{Lip}_0(X, E)$ and C a positive constant. Then the following assertions are equivalent.*

- (1) *The operator T is Cohen Lipschitz p -nuclear and $\eta_p^L(T) \leq C$.*
- (2) *For any n in \mathbb{N} ; $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n}$ in X ; $(a_i^*)_{1 \leq i \leq n}$ in E^* and $(\lambda_i)_{1 \leq i \leq n}$ in \mathbb{R}_+ , we have*

$$\begin{aligned} & \sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), a_i^* \rangle| \\ \leq & C \sup_{f \in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^n \lambda_i |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}} \sup_{\|a\|_{E^*} \leq 1} \left(\sum_{i=1}^n |\langle a, a_i^* \rangle|^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned} \quad (4.2.4)$$

- (3) *There exist Radon probability measures μ_1 on $\mathcal{B}_{X^\#}$ and μ_2 on $B_{E^{**}}$, such that for all x, x' in X and a^* in E^* , we have*

$$\begin{aligned} & |\langle T(x) - T(x'), a^* \rangle| \\ \leq & C \left(\int_{\mathcal{B}_{X^\#}} |f(x) - f(x')|^p d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} d\mu_2(a^{**}) \right)^{\frac{1}{p^*}}. \end{aligned} \quad (4.2.5)$$

Moreover, in this case

$$\eta_p^L(T) = \inf \{ C > 0 : \text{for all } C \text{ verifying the inequality 4.2.5} \}.$$

4.3 The Lipschitz strongly p -summing operators

The following definition was introduced independently by [Saa15] and [YAR15]. For our convenience, we will adopt the notation of [YAR15].

Definition 4.3.1 *A Lipschitz map $T : X \rightarrow E$ is Lipschitz strongly p -summing ($1 < p \leq \infty$) if there is a constant $C > 0$, such that for all $n \in \mathbb{N}$, $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n}$ in X , $(a_i^*)_{1 \leq i \leq n}$*

in E^* and $(\lambda_i)_{1 \leq i \leq n}$ in \mathbb{R}_+ , we have

$$\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), a_i^* \rangle| \leq C \left(\sum_{i=1}^n \lambda_i d_X(x_i, x'_i)^p \right)^{\frac{1}{p}} \omega_{p^*}((a_i^*)_i). \quad (4.3.1)$$

We denote by $\mathcal{D}_{st,p}^L(X, E)$ the class of all Lipschitz strongly p -summing operators from X into E and $d_{st,p}^L(T)$ the smallest C such that inequality 4.3.1 holds. This generalizes the definition introduced by [Coh73] in the linear case. If T is linear, then in the absence of $\mathcal{B}_{X\#}$ we have $\mathcal{D}_{st,p}^L(X, E) = \mathcal{D}_p(X, E)$.

Let $T \in \text{Lip}_0(X; E)$ and $v : \ell_n^p \rightarrow E^*$ be a bounded linear operator. The Lipschitz operator T is strongly Lipschitz p -summing if, and only if,

$$\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), v(e_i) \rangle| \leq C \left(\sum_{i=1}^n \lambda_i d_X(x_i, x'_i)^p \right)^{\frac{1}{p}} \|v\|. \quad (4.3.2)$$

Now, we give the domination theorem for the strongly Lipschitz p -summing operators (see [Saa15] and [YAR15]).

Theorem 4.3.1 *A Lipschitz operator T from X into E is Lipschitz strongly p -summing ($1 < p < \infty$) if, and only if, there exist a positive constant C and Randon probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, we have*

$$|\langle T(x) - T(x'), a^* \rangle| \leq C d_X(x, x') \left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} d\mu(a^{**}) \right)^{\frac{1}{p^*}}. \quad (4.3.3)$$

Moreover, in this case

$$d_{st,p}^L(T) = \inf \{C > 0 : \text{for all } C \text{ verifying the inequality 4.3.3}\}.$$

From the above result, we deduce (see also [Saa15, Theorem 3.8] and [Cha11]) the following corollary.

Corollary 4.3.1 *Let C be a positive constant. A Lipschitz map $T : X \rightarrow E$ is Cohen Lipschitz p -nuclear and $\eta_p^L(T) \leq C$ if, and only if, there exist a Banach space Z , a Lipschitz p -summing operator $R : X \rightarrow Z^*$ and a linear p^* -summing operator $S : E^* \rightarrow Z$ (i.e., S^* is strongly p -summing) such that $T = S^* \circ R$ and $\pi_p^L(R)\pi_{p^*}(S) \leq C$.*

4.4 Relationships between $\Pi_p^L(X, E), \mathcal{D}_{st,p}^L(X, E), \Pi_{\tau(p)}^L(X, E)$ and $\mathcal{N}_p^L(X, E)$

In this section, we investigate as in the linear case, the relationships between the various classes of Lipschitz operators.

Theorem 4.4.1 *We have for a Lipschitz operator $T : X \rightarrow E$.*

- (1) $\mathcal{N}_p^L(X, E) \subseteq \mathcal{D}_{st,p}^L(X, E)$ and $d_{st,p}^L(T) \leq \eta_p^L(T)$ for $1 < p \leq \infty$.
- (2) $\mathcal{N}_p^L(X, E) \subseteq \Pi_p^L(X, E)$ and $\pi_p^L(T) \leq \eta_p^L(T)$ for $1 \leq p < \infty$.
- (3) $\Pi_{\tau(p)}^L(X, E) \subseteq \mathcal{D}_{st,p^*}^L(X, E)$ and $d_{st,p^*}^L(T) \leq \pi_{\tau(p)}^L(T)$ for $1 \leq p < \infty$.
- (4) $\Pi_{\tau}^L(X, E) \subset \mathcal{N}_p^L(X, E)$ and $\eta_p^L(T) \leq \pi_{\tau}^L(T)$ for $1 \leq p \leq \infty$.

Proof. (1) Let $T \in \mathcal{N}_p^L(X, E)$. Consider x, x' in X and $a^* \in E^*$. We have by 4.2.5

$$\begin{aligned} & |\langle T(x) - T(x'), a^* \rangle| \\ & \leq \eta_p^L(T) \left(\int_{\mathcal{B}_{X\#}} |f(x) - f(x')|^p d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} d\mu_2(a^{**}) \right)^{\frac{1}{p^*}} \\ & \leq \eta_p^L(T) \left(\int_{\mathcal{B}_{X\#}} d^p(x, x') d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} d\mu_2(a^{**}) \right)^{\frac{1}{p^*}} \\ & \leq \eta_p^L(T) d(x, x') \left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} d\mu_2(a^{**}) \right)^{\frac{1}{p^*}}. \end{aligned}$$

Hence by 4.3.3, T is Lipschitz strongly p -summing and $d_{st,p}^L(T) \leq \eta_p^L(T)$.

(2) Let T be an operator in $\mathcal{N}_p^L(X, E)$. We have by 4.2.5

$$\begin{aligned} & \|T(x) - T(x')\| \\ & = \sup_{a^* \in B_{E^*}} |\langle T(x) - T(x'), a^* \rangle| \\ & \leq \sup_{a^* \in B_{E^*}} \eta_p^L(T) \left(\int_{\mathcal{B}_{X\#}} |f(x) - f(x')|^p d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} d\mu_2(a^{**}) \right)^{\frac{1}{p^*}} \\ & \leq \eta_p^L(T) \left(\int_{\mathcal{B}_{X\#}} |f(x) - f(x')|^p d\mu_1(f) \right)^{\frac{1}{p}}. \end{aligned}$$

By Pietsch domination theorem [FJ09], T is Lipschitz p -summing and $\pi_p^L(T) \leq \eta_p^L(T)$.

(3) Let T be in $\Pi_{\tau(p)}^L(X, E)$. Consider $x, x' \in E$ and $a^* \in E^*$. We have by 3.3.7

$$\begin{aligned}
 & |\langle T(x) - T(x'), a^* \rangle| \\
 & \leq \pi_{\tau(p)}^L(T) \left(\int_{\mathcal{B}_{X^\#}} \int_{B_{E^{**}}} |f(x) - f(x') \langle a^*, a^{**} \rangle|^p d\mu_1(f) d\mu_2(a^{**}) \right)^{\frac{1}{p}} \\
 & \leq \pi_{\tau(p)}^L(T) d(x, x') \left(\int_{\mathcal{B}_{X^\#}} \int_{B_{E^{**}}} \left| \frac{f(x) - f(x')}{d(x, x')} \langle a^*, a^{**} \rangle \right|^p d\mu_1(f) d\mu_2(a^{**}) \right)^{\frac{1}{p}} \\
 & \leq \pi_{\tau(p)}^L(T) d(x, x') \left(\int_{\mathcal{B}_{X^\#}} \int_{B_{E^{**}}} \sup_{x \neq x'} \left| \frac{f(x) - f(x')}{d(x, x')} \langle a^*, a^{**} \rangle \right|^p d\mu_1(f) d\mu_2(a^{**}) \right)^{\frac{1}{p}} \\
 & \leq \pi_{\tau(p)}^L(T) d(x, x') \left(\int_{B_{E^{**}}} |\langle a^*, a^{**} \rangle|^p d\mu_2(a^{**}) \right)^{\frac{1}{p}}.
 \end{aligned}$$

This implies by 4.3.3 that T is Lipschitz strongly p^* -summing and $d_{st,p^*}^L(T) \leq \pi_{\tau(p)}^L(T)$.

(4) Let $T \in \Pi_\tau^L(X, E)$. For n in \mathbb{N} , $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n}$ in X and $(a_i^*)_{1 \leq i \leq n}$ in E^* , we have

$$\begin{aligned}
 & \sum_{i=1}^n |\langle T(x_i) - T(x'_i), a_i^* \rangle| \\
 & \leq \pi_\tau^L(T) \sup_{\substack{\|f\| \leq 1 \\ \|a^{**}\| \leq 1}} \left(\sum_{i=1}^n |f(x_i) - f(x'_i) \langle a_i^*, a_i^{**} \rangle| \right) \\
 & \leq \pi_\tau^L(T) \omega_p^L(1, (x_i), (x'_i)) \omega_{p^*}((a_i^*)_i) \text{ by Hölder inequality.}
 \end{aligned}$$

This proves that $T \in \mathcal{N}_p^L(X, E)$ and $\eta_p^L(T) \leq \pi_\tau^L(T)$. ■

From the results obtained above we get.

Theorem 4.4.2 Consider $1 \leq p \leq \infty$. Let $T \in \text{Lip}_0(X, E)$ and $L \in \text{Lip}_0(E, F)$. If L is Lipschitz strongly p -summing operator, and T is Lipschitz p -summing operator, then $L \circ T$ is Cohen Lipschitz p -nuclear operator and $\eta_p^L(L \circ T) \leq d_{st,p}^L(L) \pi_p^L(T)$.

Proof. Let $x, x' \in E$ and $b^* \in F^*$. By 4.3.3, we have

$$\begin{aligned}
 & |\langle L \circ T(x) - L \circ T(x'), b^* \rangle| \\
 & = |\langle L(T(x)) - L(T(x')), b^* \rangle| \\
 & \leq d_{st,p}^L(L) \|T(x) - T(x')\| \int_{B_{F^{**}}} |b^*(b^{**})|^{p^*} d\mu_2(b^{**})^{\frac{1}{p^*}},
 \end{aligned}$$

and by Pietsch domination theorem in [FJ09]

$$\begin{aligned}
 & |\langle L \circ T(x) - L \circ T(x'), b^* \rangle| \\
 & \leq d_{st,p}^L(L) \pi_p^L(T) \left(\int_{\mathcal{B}_{X^\#}} |f(x) - f(x')|^p d\mu_1(f) \right)^{\frac{1}{p}} \int_{B_{F^{**}}} |b^*(b^{**})|^{p^*} d\mu_2(b^{**})^{\frac{1}{p^*}}.
 \end{aligned}$$

This gives that $L \circ T \in \mathcal{N}_p^L(X, F)$ and $\eta_p^L(L \circ T) \leq d_{st,p}^L(L) \pi_p^L(T)$. ■

Corollary 4.4.1 If $p \geq 2$. Then $\pi_{\tau(p)}^L(L \circ T) \leq d_{st,p}^L(L) \pi_p^L(T)$.

Theorem 4.4.3 *Let $1 \leq r, p, q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $T \in \text{Lip}_0(X, E)$ and $L \in \text{Lip}_0(E, F)$. If L is Lipschitz $\tau(r)$ -summing and T is Lipschitz p -summing, then $L \circ T$ is Lipschitz (r, p, q) -summing operator and $\pi_{(r,p,q)}^L(L \circ T) \leq \pi_{\tau(r)}^L(L)\pi_p^L(T)$.*

Proof. Let $x, x' \in X$ and $b^* \in F^*$. We have by 3.3.7

$$\begin{aligned} & |\langle L \circ T(x) - L \circ T(x'), b^* \rangle| \\ & \leq \pi_{\tau(r)}^L(L) \left(\int_{\mathcal{B}_{E^\#}} \int_{\mathcal{B}_{F^{**}}} |(f(T(x)) - f(T(x'))) \langle b^*, b^{**} \rangle|^r d\mu_1(f) d\mu_2(b^{**}) \right)^{\frac{1}{r}}. \end{aligned}$$

Using general Hölder's inequality and the fact that T is Lipschitz p -summing, we get

$$\begin{aligned} & |\langle L \circ T(x) - L \circ T(x'), b^* \rangle| \\ & \leq \pi_{\tau(r)}^L(L) \left(\int_{\mathcal{B}_{E^\#}} |f(T(x)) - f(T(x'))|^p d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{\mathcal{B}_{F^{**}}} |\langle b^*, b^{**} \rangle|^q d\mu_2(b^{**}) \right)^{\frac{1}{q}} \\ & \leq \pi_{\tau(r)}^L(L) \|T(x) - T(x')\| \left(\int_{\mathcal{B}_{F^{**}}} |\langle b^*, b^{**} \rangle|^q d\mu_2(b^{**}) \right)^{\frac{1}{q}} \\ & \leq \pi_{\tau(r)}^L(L) \pi_p^L(T) \left(\int_{\mathcal{B}_{X^\#}} |f(x) - f(x')|^p d\mu(f) \right)^{\frac{1}{p}} \left(\int_{\mathcal{B}_{F^{**}}} |\langle b^*, b^{**} \rangle|^q d\mu_2(b^{**}) \right)^{\frac{1}{q}}. \end{aligned}$$

This implies that $L \circ T \in \Pi_{(r,p,q)}^L(X, F)$ and $\pi_{(r,p,q)}^L(L \circ T) \leq \pi_{\tau(r)}^L(L)\pi_p^L(T)$. ■

Corollary 4.4.2 *Let $1 < p < \infty$. Let $T \in \text{Lip}_0(X, E)$ and $L \in \text{Lip}_0(E, F)$. If L is Lipschitz τ -summing and T is Lipschitz p -summing, then $L \circ T$ is Cohen Lipschitz p -nuclear operator and $\eta_p^L(L \circ T) \leq \pi_\tau^L(L)\pi_p^L(T)$.*

Perspectives

We end this thesis by asking some problems, we hope to have a solution in the future.

Probleme 4.4.1 *If we define the Lipschitz multiple $\tau(p)$ -summing as follows. Let $1 \leq p < \infty$. let X be a pointed metric space and let E be a Banach space. We will say that a Lipschitz mapping $T : X \rightarrow F$ is Lipschitz multiple $\tau(p)$ -summing if there is a constant $C > 0$ such that for $x_1, \dots, x_m; x'_1, \dots, x'_m \in E$ and $a_1^*, \dots, a_m^* \in E^*$,*

$$\sum_{j=1}^m \sum_{i=1}^n |\langle T(x_j) - T(x'_j), a_i^* \rangle|^p \leq C^p \sup_{f \in B_{X^\#}, a^{**} \in B_{E^{**}}} \sum_{j=1}^m \sum_{i=1}^n |f(x_j) - f(x'_j)|^p |\langle a_i^*, a^{**} \rangle|^p.$$

We will write $\pi_{m, \tau(p)}^L(T)$ for the infimum of all constants satisfying the inequality above. The following corollary is true as in the linear case?

Corollary 4.4.3 *Let $1 \leq p \leq \infty$. Let μ be a measure and let $T : X \rightarrow L^p(\mu)$ be a Lipschitz mapping. Then T is Lipschitz multiple $\tau(p)$ -summing if and only if T is Lipschitz p -summing.*

Probleme 4.4.2 *Let X be a metric pointed space and let E, F be Banach spaces. Then every Lipschitz map $T : X \widehat{\boxtimes}_\varepsilon E \rightarrow F$ induces a Lipschitz map $T^\# : X \rightarrow \text{Lip}_0(E; F)$ define by*

$$T^\#(x)(e) = T(\delta_{(x,0)} \boxtimes e).$$

Probleme 4.4.3 *Are the following true?*

Proposition 4.4.1 *Let X be a metric pointed space and E, F be a Banach spaces, and let $T : X \widehat{\boxtimes}_\varepsilon E \rightarrow F$ be a Lipschitz map. Denote by $i : F \rightarrow F^{**}$ the isometric embedding of F into F^{**} . Then the following two properties are equivalent:*

(i) $T \in \Pi_{\tau}^L(X \widehat{\otimes}_{\varepsilon} E, F)$

(ii) $\hat{i} \circ T \in \Pi_{\tau}^L(X, \Pi_{\tau}^L(E, F^{**}))$, where $\hat{i} : \Pi_{\tau}^L(E, F) \rightarrow \Pi_{\tau}^L(E, F^{**})$ is defined by $\hat{i}(U) = i \circ U$ for each $U \in \Pi_{\tau}^L(E, F)$.

Proposition 4.4.2 *Let X be a metric pointed space and E, F be a Banach spaces, and let $T : X \widehat{\otimes}_{\varepsilon} E \rightarrow F$ be a Lipschitz map. If T is Lipschitz τ -summing, then $T^{\#} : X \rightarrow \Pi_{\tau}^L(E, F)$ is Lipschitz 1-summing.*

Bibliography

- [AA10] A. ACHOUR AND A. ALOUANI, *On multilinear generalizations of the concept of nuclear operators*, Colloq. Math. **120** (1) (2010), 85-102.
- [AE56] R. F. ARENS AND J. EELS, *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397-403.
- [ADRS12] D. ACHOUR, E. DAHIA, P. RUEDA AND E.A. SÁNCHEZ-PÉREZ, *Factorization of strongly (p, σ) -continuous multilinear operators*, Linear Multilinear Algebra. **62** (2014), 1649–1670.
- [AM89] R. ALENCAR, M. C. MATOS, *Some classes of multilinear mappings between Banach spaces*, Madrid:
Publicaciones del Departamento de Análisis Matemático **12**, Universidad Complutense Madrid. 1989.
- [AM07] D. ACHOUR AND L. MEZRAG, *On the Cohen strongly p -summing multilinear operators*, J. Math. Anal. Appl. **327** (2007), 55-63.
- [AMS09] D. ACHOUR, L. MEZRAG AND K. SAADI, *On the Cohen p -nuclear sublinear operators*, J. Inequal. Pure and App. Math. **10** (2) (2009), 1-13.
- [Api76] H. APIOLA, *Duality between spaces of p -summable sequences, (p, q) -summing operators and characterizations of nuclearity*, Math. Ann. **219** (1976), 53–64.
- [Ass15] ASSAF. N, *Metric Embeddings and Lipschitz Extensions*, Lecture Notes, Department of Mathematics, Princeton University, Spring 2015.

-
- [BBJP09] G. BOTELHO, H.-A. BRAUNSS, H. JUNEK AND D. PELLEGRINO, *Inclusions and coincidences for multiple summing multilinear mappings*, Proc. Amer. Math. Soc. **137** (2009), 991–1000.
- [BL00] Y. BENYAMINI AND J. LINDENSTRAUSS, *Geometric Nonlinear Functional Analysis*, **48**, Amer. Math. Soc., Providence, 2000.
- [Bou85] BOURGAIN, J., *On Lipschitz embedding of finite metric spaces in Hilbert space*, Israel J. Math. **52** (1-2) (1985), 46-52.
- [BP09] G. BOTELHO, AND D. PELLEGRINO, *When every multilinear mapping is multiple summing*, Math. Nach. **282**(10) (2009), 1414–1422.
- [BPR10] G. BOTELHO, D. PELLEGRINO AND P. RUEDA, *A unified Pietsch Domination Theorem*. J. Math. Anal, Appl. **365** (2010), 269-276.
- [CCJV15] M. G. CABRERA-PADILLA, J. A. CHÁVEZ-DOMÍNGUEZ, A. JIMÉNEZ-VARGAS AND M. VILLEGAS-VALLECILLOS, *Lipschitz tensor product*, Khayyam J. Math. **1** (2015), 185–218.
- [Cha11] J. A. CHÁVEZ-DOMÍNGUEZ, *Duality for Lipschitz p -summing operators*, J. Funct. Anal. **261** (2) (2011), 387–407.
- [Cha12] J. A. CHÁVEZ-DOMÍNGUEZ, *Lipschitz (q, p) -mixing operators*, Proc. Amer. Math. Soc. **140** (2012), 3101–3115.
- [Cob03] S. COBZAS, *Adjoint of Lipschitz mappings*, Studia Univ. “BABES–BOLYAI”, Mathematica, **48** (1) (2003).
- [Coh73] J. S. COHEN, *Absolutely p -summing, p -nuclear operators and their conjugates*. Math. Ann. **201** (1973), 177-200.
- [CZ11] D. CHEN AND B. ZHENG, *Remarks on Lipschitz p -summing operators*, **139** (8) (2011), 2891–2898.
- [CZ12] D. CHEN AND B. ZHENG, *Lipschitz p -integral operators and Lipschitz p -nuclear operators*, Nonlinear Analysis **75** (2012), 5270-5282.

- [DAS12] E. DAHIA, D. ACHOUR AND E.A. SÁNCHEZ-PÉREZ, Absolutely continuous multilinear operators, *J. Math. Anal. Appl.* **397** (2013), 205–224.
- [Def86] A. DEFANT, *Produkte von Tensornormen, Habilitationsschrift*, U. Oldenburg, 1986.
- [DF93] A. DEFANT; K. FLORET, *Tensor norms and operator ideals*, North-Holland Mathematics Studies, 176. North-Holland Publishing Co., Amsterdam, 1993.
- [Dim03] V. DIMANT, *Strongly p -summing multilinear operators*, *J. Math. Anal. Appl.* **278** (2003), 182–193
- [Din99] S. DINEEN, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London, 1999.
- [DJT95] J. DIESTEL, H. JARCHOW AND A. TONGE, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math., **43**, Cambridge University Press, Cambridge, 1995.
- [DP08] A. DEFANT, D. PÉREZ-GARCÍA, *A tensor norm preserving unconditionality for L_p -spaces*, *Trans. Amer. Math. Soc.* **360**(6), (2008) 3287–3306.
- [FJ09] J.-D. FARMER AND W.-B. JOHNSON, *Lipschitz p -summing operators*, *Proc. Amer. Math. Soc.* **137** (9) (2009), 2989–2995.
- [Gei84] S. GEISS, *Ideale multilinearer Abbildungen*, Diplomarbeit, Brandenburgische Landeshochschule, 1985.
- [Gre67] W. H. GREUB, *Multilinear algebra*, Springer-Verlag Berlin Heidelberg New York (1967).
- [Gro55] A. GROTHENDIECK, *Produits tensoriels topologiques et espaces nucléaires*, *Mem. Am. Math. Soc.* **16**, Amer. Math. Soc., Providence, R.I., 1955.
- [JS09] WILLIAM B. JOHNSON, GIDEON SCHECHTMAN, *Diamond graphs and super-reflexivity*, *J. Topol. Anal.* **1** (2) (2009), 177–189.

-
- [JSV14] A. JIMÉNEZ-VARGAS, J. M. SEPULCRE AND MOISÉS VILLEGAS-VALLECILLOS, *Lipschitz compact operators*, J. Math. Anal. Appl. **415** (2)(2014), 889–901.
- [Kan42] L. V. KANTOROVICH, *On the translocation of masses*, Dokl. Akad. Nauk SSSR **37** (1942), 227-229.
- [Kal04] N. J. KALTON, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math. **55**(2) (2004), 171-217.
- [KG03] N. J. KALTON AND G. GODEFORY, *Lipschitz-free Banach spaces*, Studia Math. **69** (2003), 121-141.
- [Lee61] K. DE LEEUW, *Banach space of Lipschitz functions*, Studia Math. **21** (1961), 55-66.
- [LP68] J. LINDENSTRAUSS, A. PELCZYŃSKI, *Absolutely summing operators in L_p -spaces and their applications*, Studia Math. **29** (1968), 275-326.
- [Ma93] M. C. MATOS, *On multilinear mappings of nuclear type*, Rev. Mat. Comput. **6** (1993), 61–81.
- [Mat03] M. C. MATOS, *Fully absolutely summing and Hilbert–Schmidt multilinear mappings*, Collect. Math. (**54**) (2003), 111–136.
- [MT17] L. MEZRAG AND A. TALLAB, *On Lipschitz $\tau(p)$ -summing operators*, Colloq. Math. **147** (1) (2017), 95-114.
- [Muj08] X. MUJICA, *$\tau(p; q)$ -summing mappings and the domination theorem*, Portugal. Math. (N. S.) **65** (2) (2008), 211–226.
- [Pie67] A. PIETSCH, *Absolut p -summierende Abbildungen in normierten Räumen*, Studia Math, **28** (1967), 333-353.
- [Pie80] A. PIETSCH, *Operator ideals*. North-Holland Math. Library 20, North Holland Publishing Co., Amsterdam 1980.

-
- [Pie83] A. PIETSCH, *Ideals of multilinear functionals (designs of a theory)*, Proceedings of the Second International Conference on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leipzig). Teubner-Texte (1983), 185–199.
- [PP69] A. PERSSON AND A. PIETSCH, *p-nukleare and p-integrale Abbildungen in Banachräumen*, Studia Math. **33** (1969), 213–222.
- [PRS] D. PELLEGRINO; P. RUEDA AND E. A. SÁNCHEZ PÉREZ, *Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials*, RACSAM. DOI: 10.1007/s13398-015-0224-8.
- [PSS12a] D. PELLEGRINO, J. SANTOS AND J.-B. SEOANE-SEPÚLVEDA, *Some techniques on nonlinear analysis and applications*, Adv. Math. **229** (2012), 1235–1265.
- [PSS12b] D. PELLEGRINO, J. SANTOS AND J.-B. SEOANE-SEPÚLVEDA, *A general Extrapolation theorem for Absolutely summing operators*, Bull. London Math. Soc. **44** (2012), 1292–1302.
- [PS11a] D. PELLEGRINO AND J. SANTOS, *Absolutely summing multilinear operators: a panorama*, Quaest. Math. **34**(4) (2011), 447–478.
- [PS11b] D. PELLEGRINO AND J. SANTOS, *A general Pietsch Domination Theorem*, J. Math. Anal. Appl. **375** (2011), 371–374.
- [PV03] D. PÉREZ GARCÍA AND I. VILLANUEVA, *Multiple summing operators on Banach spaces*. J. Math. Anal. Appl. **285**(1) (2003) 86–96.
- [Ray01] RYAN, R.A, *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. (2001).
- [RS14] P. RUEDA AND E. A. SÁNCHEZ PÉREZ, *Factorization of p-dominated polynomials through L_p -spaces*. Michigan Math. J. **63** (2) (2014), 345–353.

- [RST17] P. RUEDA, E.A. SÁNCHEZ-PÉREZ AND A.TALLAB, *Traced tensor norms and multiple summing multilinear operators*. Linear and multilinear Algebra, **65** (4) (2017), 768-786.
- [Saa15] K. SAADI, *Some Properties for Lipschitz strongly p -summing operators*, J. Math. Anal. App. **423** (2015), 1410-1426.
- [Sap70] P. SAPHAR, *Produits tensoriels d'espaces de Banach et classes d'applications linéaires*, Studia Math. **38** (1970) 71–100.
- [Saw75] I. SAWASHIMA, *Methods of Lipschitz duals*, Lecture Notes Ec. Math. Sust. **419** (1975), 247-259.
- [Wea99] N. WEAVER, *Lipschitz Algebras*, World Scientific, Singapore 1999.
- [YAR15] R. YAHI, D. ACHOUR AND P. RUEDA, *Absolutely summing Lipschitz conjugates*, Mediterr. J. Math. **13** (2016), 1949–1961.

Résumé.

L'idée de cette thèse nous a été inspirée de l'article de J.-D. Farmer et W.-B. Johnson intitulé, "les opérateurs Lipschitz p -sommants". Ils ont posé dans le problème six la question suivante: peut-on généraliser certaines propriétés des opérateurs linéaires p -sommants aux opérateurs Lipschitz p -sommants? Certains auteurs ont répondu à cette question par la généralisation de quelques notions dans le cas linéaire au cas lipschitzien, par exemple les notions de p -nucléaires, p -intégrals, opérateurs fortement lipschitziens p -sommants, les opérateurs (p, σ) -absolument lipschitziens, les opérateurs compacts et faiblement compacts parmi tant d'autres. Dans cette thèse, nous allons généraliser ce concept aux opérateurs $\tau(p)$ -sommants qui sont les opérateurs Lipschitz $\tau(p)$ -sommants et leur version "multiple $\tau(p)$ -sommant " suivi de quelques applications sur les espaces L_p . En utilisant l'idée de Defant et Floret (i.e., T est un

opérateur linéaire p -sommant si et seulement si $\hat{T}: l_0^p \otimes_\varepsilon E \rightarrow l^p \otimes_{\Delta_p} E$ est continu), nous avons généralisé aussi cette idée au cas des opérateurs multilinéaires. Deux extensions naturelles des opérateurs p -sommants ont été étudiées dans le cas multilinéaire qui sont les opérateurs multiples p -sommants et les opérateurs multilinéaires (γ, α, p) -sommants. Ces deux classes ont été intensivement étudiées et sont bien connues.

Mots-clés: opérateurs Lipschitz Cohen p -nucléaires, opérateurs Lipschitz p -sommants, opérateurs Lipschitz $\tau(p)$ -sommants, opérateurs Lipschitz fortement p -sommants, espace de Arens-Eels, opérateurs multiples p -sommants, opérateurs multilinéaires (γ, α, p) -sommants, norme tensorielle.

Abstract.

The idea of this thesis was inspired by the article of J.-D. Farmer and W.-B. Johnson entitled, "the Lipschitz p -summing operators". They posed in problem six the following question: what results about p -summing operators have analogues for Lipschitz p -summing operators? Some authors have answered this question by generalizing some concepts in the linear case to the Lipschitz case, for example p -nuclears, p -integrals, Lipschitz strongly p -summing operators, Lipschitz (p, σ) -absolutely operators, compact operators and weakly compact operators among many others. In this thesis, we will generalize this concept to $\tau(p)$ -summing operators which Lipschitz $\tau(p)$ -summing operators and their version "multiple $\tau(p)$ -summing" with some applications on L_p -spaces. Using the idea of Defant and Floret (i.e., T is absolutely p -summing if and only if $\hat{T}: l_0^p \otimes_\varepsilon E \rightarrow l^p \otimes_{\Delta_p} E$ is continuous). We have

also generalized this idea to the case of multilinear operators. Two natural extensions of p -summing operators have been studied in the multilinear case which are the multiple p -summing operators and (γ, α, p) -summing multilinear operators. These two classes have been intensively studied and are well known

Keywords: Cohen Lipschitz p -nuclear operators, Lipschitz p -summing operators, Lipschitz $\tau(p)$ -summing operators, Lipschitz strongly p -summing operators, Arens-Eells space, multiple summing operators, (γ, α, p) -summing multilinear operators, tensor norm.

الملخص بالعربية.

فكرة هذه الأطروحة مستمدة من المقال المقدم من طرف J.-D. Farmer و W.-B. Johnson تحت عنوان "المؤثرات الليبشيتزية p - جمعية" وقد قاموا في آخر هذا المقال بطرح مسائل من بينها المسألة السادسة المتمثلة في السؤال الآتي: "هل نستطيع تعميم خصائص المؤثرات p - جمعية إلى المؤثرات الليبشيتزية p - جمعية" بعض المؤلفين أجابوا على هذا السؤال بتعميم بعض المفاهيم في الحالة الخطية إلى الحالة الليبشيتزية على سبيل المثال p - نووية p - تكاملية، المؤثرات الليبشيتزية القوية p - جمعية، المؤثرات الليبشيتزية (p, σ) - مطلقا، المؤثرات المتراسة و المؤثرات الضعيفة التراص وهذه من بين مفاهيم أخرى. في هذه الأطروحة سنقوم بتعميم هذا المفهوم إلى $\tau(p)$ - جمعي والذي يسمى المؤثرات الليبشيتزية $\tau(p)$ - جمعية وكذلك نسخته المتعددة $\tau(p)$ - جمعية مع بعض التطبيقات على فضاءات L_p . مستعملين فكرة

Defant و Floret وهي (المؤثر I_p - جمعي إذا فقط إذا كان المؤثر $\hat{T}: l_0^p \otimes_\varepsilon E \rightarrow l^p \otimes_{\Delta_p} E$ مستمر) وقمنا أيضا بتعميم هذه الفكرة إلى المؤثرات المتعددة الخطية حيث درسنا تعميمان للمؤثرات p - جمعية في حالة المؤثرات المتعددة الخطية وهما المؤثرات المتعددة p - جمعية والمؤثرات (γ, α, p) - جمعية. الكلمات المفتاحية:

المؤثرات الليبشيتزية Cohen p - نووية، المؤثرات الليبشيتزية p - جمعية، المؤثرات الليبشيتزية القوية p - جمعية، فضاء Arens-Eells، المؤثرات المتعددة p - جمعية، المؤثرات (γ, α, p) - جمعية.