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Theme

On Picture Fuzzy Sets

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Introduction

The theory of fuzzy sets was presented by the Iranian Lutfi Zadeh in 1965, as an extension of the well-known traditional group within classical maths, the membership of elements during a very set is assessed in binary terms to remain with a bivalent condition, part either belongs or doesn't belong to the set. While fuzzy sets theory permits the gradual assessment of the membership of elements during a awfully set, this is often described with the help of a membership function within the 0, 1 unit interval [0, 1] or on a bounded poset. Classical sets are generalized by fuzzy set, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets if the latter only take the values 0 or 1. Within mathematics, classical bivalent sets are usually called crisp sets. Fuzzy sets are utilized in an exceedingly large range of domains within which the information are incomplete or imprecise, like mathematics, informatics, chemistry, economics, social sciences ... etc. In 1983, Atanassov has introduced the concept of "Intuitionistic Fuzzy Set" as a fuzzy set generalization. Atanassov added new components that determine the degree of nonmembership within the defnition of fuzzy set. The fuzzy sets are characterized by the degree of membership, while intuitionistic fuzzy sets are characterized by both the degree of membership and also the degree of non-membership. In 2013, Cuong has introduced a replacement notion named "picture fuzzy sets (PFS)", which could be an immediately extension of the fuzzy sets and also the intuitionistic fuzzy sets. This memory is divided into three chapters:

- the first chapter is preliminary on fuzzy sets. It provides generalities on fuzzy sets, basic operations, some properties of fuzzy sets, characteristics of fuzzy sets, fuzzy relations and fazzy lattices.
- The second chapter presents the essential concepts of the intuitionistic fuzzy sets theory, operations on IFS, operators of IFS, intuitionistic fuzzy relations and lattices.
- The third chapter is devoted to the picture fuzzy sets notion, where we recall the definition of the set D^* and some order relations defined on it. After that, we use the above proprieties to study picture fuzzy set, and related concepts.

Chapter 1

Preliminaries

1.1 Fuzzy sets

Zadeh [8] was the first to apply the concept of a fuzzy set (FS) to deal with uncertainty in a variety of disciplines. The range values of classical sets were generalized by fuzzy set theory in the integer 0 and 1 to the interval $[0, 1]$. In addition, this concept has proven to be very useful in many different fields [8, 9, 3].

1.1.1 Notations and definitions

Definition 1.1.1. [8] *If X be a universe of discourse. the fuzzy set A defined by $A = \{\langle x, \mu_A(x)\rangle | \, x \in X\}$. where $\mu_A : X \to [0, 1]$, is called membership function.*

Example 1.1.1. *Let $X = \{a, b, c\}$ be a universal set, $A = \{\langle a, 0.4\rangle, \langle b, 0.1\rangle, \langle c, 0.7\rangle\}$ and $B = \{\langle a, 0.5\rangle, \langle b, 0.0\rangle, \langle c, 0.3\rangle\}$ be fuzzy subsets in X.*

1.1.2 Properties of fuzzy sets

1. Commutativity

$$A \cap B = B \cap A, \; A \cup B = B \cup A.$$

2. Associativity

$$(A \cap B) \cap C = A \cap (B \cap C),$$

$$(A \cup B) \cup C = A \cup (B \cup C).$$

3. Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

4. Idempotency

$$A \cap A = A,\ A \cup A = A.$$

5. Identity

$$A \cup \emptyset = A,\ A \cap \emptyset = \emptyset,$$

$$A \cup X = X,\ A \cap X = A.$$

6. Transitivity

$$\text{If } A \subseteq B,\ B \subseteq C, \text{then } A \subseteq C.$$

7. Involution

$$(A^c)^c = A.$$

8. De Morgan " Laws "

$$(A \cup B)^c = A^c \cap B^c,$$

$$(A \cap B)^c = A^c \cup B^c.$$

9. Law of Extended Middle

$$A \cup A^c \neq X \text{ (this property does not hold good in fuzzy set theory).}$$

10. Law of contradiction

$$A \cap A^c \neq \emptyset \text{ (this property does not hold good in fuzzy set theory).}$$

11. Absorption

$$A \cup (A \cap B) = A,$$

$$A \cap (A \cup B) = A.$$

1.1.3 Some basic operations on fuzzy sets

The concept of a fuzzy set X is a generalization of the concept of a classical set X, these operations are selected to be equivalent to classical set theory operations when the membership function only accepts the values 0 or 1.

Given two sets A and B of X. For more details see [6].

- Inclusion: We say that A is included in B if and only if

$$\mu_A(x) \leqslant \mu_B(x), \text{ for any x in X }.$$

In symbols

$$A \subseteq B \Longleftrightarrow \mu_A(x) \leqslant \mu_B(x).$$

- Equality: We say that $A = B$, if and only if

$$\mu_A(x) = \mu_B(x), \text{for all } x \in X.$$

- Union: the union of A and B is

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

$$= \mu_A(x) \vee \mu_B(x).$$

- Intersection: the intersection of A and B is

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$$= \mu_A(x) \wedge \mu_B(x).$$

- Complement: the complement of a fuzzy set A is denoted $CA(orA^c)$ is defined by:

$$\mu_{A^c}(x) = 1 - \mu_A(x), \text{ for all } x \in X.$$

- Product $(A \cdot B)$:

$$A \cdot B = \{\langle x, \mu_A(x).\mu_B(x)\rangle \,|x \in X\}.$$

- The difference $(A - B)$: the difference of two fuzzy sets A and B, is a fuzzy set $A - B = A \cap CB$ whose the membership function is:

$$\mu_{A-B}(x) = \min\{\mu_A(x), 1 - \mu_B(x)\}.$$

Example 1.1.2. *Let* $X = \{a, b\}$, *let* $A = \{\langle a, 0.5\rangle, \langle b, 0.1\rangle\}$ *and* $B = \{\langle a, 0.2\rangle, \langle b, 0.3\rangle\}$ *then:*

1. $A \cap B = \{\langle a, 0.2\rangle, \langle b, 0.1\rangle\}.$

2. $A \cup B = \{\langle a, 0.5\rangle, \langle b, 0.3\rangle\}.$

3. $C(B) = \{\langle a, 0.8\rangle, \langle b, 0.7\rangle\}.$

4. $C(A) = \{\langle a, 0.5\rangle, \langle b, 0.9\rangle\}.$

1.1.4 Characteristics of fuzzy sets

Definition 1.1.2. [9] *(The support) the support of a fuzzy set A denoted by Supp(A). Supp(A) is a crisp set defined by:*

$$Supp(A) = \{x \in X : \mu_A(x) > 0\}.$$

Definition 1.1.3. [9] *(The Kernel) the kernel $Ker(A)$ of a fuzzy set A, is the set of elements of X for which the membership function of A is equal to* 1

$$Ker(A) = \{x \in X : \mu_A(x) = 1\}.$$

Definition 1.1.4. [9] *(The height) the height $H(A)$ of a fuzzy set A, is the largest membership grade obtained by any elament in that set.*

$$H(A) = \sup\{\mu_A(x) : x \in X\}.$$

Definition 1.1.5. [9] *(The Cardinal of A) For a finit fuzzy set A, the cardinality $|A|$ is defined by:*

$$|A| = \sum_{x \in X} \mu_A(x).$$

$||A|| = \dfrac{|A|}{|X|}$ *is called the relative cardinality of A.*

Example 1.1.3. *Let $X = \{a, b, c, d\}$, and $A = \{\langle a, 0.7\rangle, \langle b, 1\rangle, \langle c, 0\rangle, \langle d, 0.1\rangle\}$*
Supp$(A) = \{a, b, d\}$.
$Ker(A) = \{b\}$.
$H(A) = 1$.
$|A| = 1.8$.

1.1.5 α-cuts of a fuzzy set

Definition 1.1.6. [10] *Given a fuzzy set A in X and α-cut of A, denoted A_α, we mean all elements of X that belong to a degree of least α. That is A_α is a classical set defined by:*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}.$$

and A_α is an ordinary subset of characteristic function:

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & if\ \mu_A(x) \geq \alpha, \\ 0 & otherwise. \end{cases}$$

If A is a fuzzy set of a universe X with membership function μ_A we have:

$$\forall x \in X, \mu_A(x) = \sup_{\alpha \in]0,1]} (\alpha.\mu_{A_\alpha}(x))\,(decomposition\ theorem).$$

Definition 1.1.7. *Given a fuzzy set A defined on X and any number $\alpha \in [0, 1]$, the strong α-cut A^+_α is the crisp set that contains all the element of the universal set X whose membership grades in A are strictly greater than the specified value of α:*

$$A^+_\alpha = \{x \in X : \mu_A(x) > \alpha\}.$$

Example 1.1.4. *Let* $X = \{0, 1, 2, ..., 9\}$ *and* $A = \{\langle 1, 0.9\rangle, \langle 2, 0.3\rangle, \langle 3, 0.5\rangle, \langle 4, 1.0\rangle, \langle 5, 0.8\rangle\}$ *we have for any level* α *in* $[0, 1]$:
$A_1 = \{x \in X | \mu_A(x) \geq 1\} = \{4\}$.
$A_{0.9} = \{x \in X | \mu_A(x) \geq 0.9\} = \{1, 4\}$.
$A_{0.8} = \{x \in X | \mu_A(x) \geq 0.8\} = \{1, 4, 5\}$.
$A_{0.5} = \{x \in X | \mu_A(x) \geq 0.5\} = \{1, 3, 4, 5\}$.
$A_{0.3} = \{x \in X | \mu_A(x) \geq 0.3\} = \{1, 2, 3, 4, 5\}$.

Proposition 1.1.1. *Let* A, B *two fuzzy sets, for all* $\alpha, \beta \in [0, 1]$

1. $A_\alpha^+ \subseteq A_\alpha$.
2. $\alpha \leq \beta$ *implies* $A_\beta \subseteq A_\alpha$.
3. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
4. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.

Proof 1.1.1.

(1) *To proof* $A_\alpha^+ \subseteq A_\alpha$

$$
\begin{aligned}
\textit{If } x \in A_\alpha^+ &\Rightarrow x \in A_\alpha \\
\textit{Let } x \in A_\alpha^+ &\Rightarrow A(x) > \alpha \\
&\Rightarrow x \in A_\alpha \\
&\Rightarrow A_\alpha^+ \subseteq A_\alpha.
\end{aligned}
$$

(2) *To proof if* $\alpha \leq \beta$ *implies* $A_\beta \subseteq A_\alpha$.

$$
\begin{aligned}
\textit{Let } x \in A_\beta &\Rightarrow A(x) \geq \beta \geq \alpha \\
&\Rightarrow A(x) \geq \alpha \\
&\Rightarrow x \in A_\alpha.
\end{aligned}
$$

(3) *To proof if* $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$, *this means*
(1) $(A \cap B)_\alpha \subseteq A_\alpha \cap B_\alpha$.
(2) $(A \cap B)_\alpha \supseteq A_\alpha \cap B_\alpha$.

1 − *Let* $x \in (A \cap B)_\alpha$

$$
\begin{aligned}
&\Rightarrow (A \cap B)(x) \geq \alpha \\
&\Rightarrow \min[A(x), B(x)] \geq \alpha \\
&\Rightarrow A(x) \geq \alpha \wedge B(x) \geq \alpha \\
&\Rightarrow x \in A_\alpha \wedge x \in B_\alpha \\
&\Rightarrow x \in (A_\alpha \cap B_\alpha) \\
&\Rightarrow (A \cap B)_\alpha \subseteq A_\alpha \cap B_\alpha.
\end{aligned}
$$

2 − *Let* $x \in A_\alpha \cap B_\alpha$

$\Rightarrow \quad x \in A_\alpha \wedge x \in B_\alpha$
$\Rightarrow \quad A(x) \geq \alpha \wedge B(x) \geq \alpha$
$\Rightarrow \quad \min[A(x), B(x)] \geq \alpha$
$\Rightarrow \quad (A \cap B)(x) \geq \alpha$
$\Rightarrow \quad x \in (A \cap B)_\alpha$
$\Rightarrow \quad A_\alpha \cap B_\alpha \subseteq (A \cap B)_\alpha.$

From (1) *and* (2) *we have* $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.

(4) *To proof if* $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$, *this means*
(1) $(A \cup B)_\alpha \subseteq A_\alpha \cup B_\alpha$.
(2) $(A \cup B)_\alpha \supseteq A_\alpha \cup B_\alpha$.

$1-$ *Let* $x \in (A \cup B)_\alpha$
$\Rightarrow \quad (A \cup B)(x) \geq \alpha$
$\Rightarrow \quad \max[A(x), B(x)] \geq \alpha$
$\Rightarrow \quad A(x) \geq \alpha \vee B(x) \geq \alpha$
$\Rightarrow \quad x \in A_\alpha \vee x \in B_\alpha$
$\Rightarrow \quad x \in A_\alpha \cup B_\alpha$
$\Rightarrow \quad (A \cup B)_\alpha \subseteq A_\alpha \cup B_\alpha.$

$2-$ *Let* $x \in A_\alpha \cup B_\alpha$
$\Rightarrow \quad x \in A_\alpha \vee x \in B_\alpha$
$\Rightarrow \quad A(x) \geq \alpha \vee B(x) \geq \alpha$
$\Rightarrow \quad \max[A(x), B(x)] \geq \alpha$
$\Rightarrow \quad (A \cup B)(x) \geq \alpha$
$\Rightarrow \quad x \in (A \cup B)_\alpha$
$\Rightarrow \quad A_\alpha \cup B_\alpha \subseteq (A \cup B)_\alpha.$

From (1) *and* (2) *we have* $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.

1.2 Fuzzy relations

Definition 1.2.1. [14] *Let* X *and* Y *be two nonempty sets. A binary fuzzy relation from* X *to* Y *could be a fuzzy subset of* $X \times Y$ *characterized by a membership function* μ_R *which associates with each pair* (x, y) *its grade of membership* $R(x, y)$ *within the interval* $[0, 1]$.

$$R = \{((x, y), R(x, y)) \mid (x, y) \in X \times Y\}.$$

Definition 1.2.2. [8, 14] *Let* R *be a fuzzy relation (fuzzy relation on* X, *for short).*

(i) Reflexive: If $R(x, x) = 1$, *for any* $x \in X$.

(ii) Symmetry: If $R(x, y) = R(y, x)$, *for all* $x, y \in X$.

(iii) Antisymmetry: If $x \neq y$ $R(x,y) = 0 \vee R(y,x) = 0$, for all $x, y \in X$.

(iv) Transitive: If $R(x,z) \geq \max\{\min\{R(x,y), R(y,z)\}\}$, for all $x, y, z \in X$.

Example 1.2.1. *Let $X = \{x_1, x_2, x_3, x_4\}$ and R be a fuzzy relation on $X \times X$ defined by*

$$R = \begin{bmatrix} 1 & 0.2 & 1 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 \\ 1 & 0.2 & 1 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 \end{bmatrix}$$

We have $R(x_1,x_1) = R(x_2,x_2) = R(x_3,x_3) = R(x_4,x_4) = 1$.
Then R is reflexive.
We have $R(x_i,x_j) = R(x_j,x_i)$ for all $i, j \in \{1,2,3,4\}$.
Hence R is symmetric

1.2.1 Fuzzy order

Definition 1.2.3. *Let X be a nonempty crisp set and R be a fuzzy relation on X. If R is reflexive, transitive, and antisymmetric, it is called fuzzy order or partial fuzzy order.*
A nonempty set X with a fuzzy order R is termed a fuzzy ordered set and is denoted by (X, R).
It easily follows that every partially ordered set $(X, \leqslant)$ and every fuzzy ordered set (X, R) are often viewed as fuzzy ordered sets.

1.2.2 Fuzzy lattices

Definition 1.2.4. [1] *Let A be a non-empty subset of X and let X be a fuzzy poset, An element $u \in X$ is alleged to be an upper bound of A provided that $R(a,u) > 0$ for all $a \in A$. An upper bound u_0 of A is that the least bound of A if and given that $R(u_0,u) > 0$, for each boundary u of A. a componet $l \in X$ is allged to be a lower bound of A if and providing $R(l,a) > 0$ for all $a \in A$. A bound l_0 of A is that the greatest bound of A if and on condition that $R(l,l_0) > 0$, for every boundary l of A. the smallest amount bound and there for the greatest boundary of a set $\{x,y\}$ denoted $x \vee y$ and $x \wedge y$ respectively.*

Definition 1.2.5. [1] *Let (X, R) be a fuzzy poset. (X, R) is a fuzzy lattice if and only if $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.*

Example 1.2.2. [1] *Let $X = \{x, y, z\}$ and let $R : X \times X \longrightarrow [0,1]$ be a fuzzy relation such that $R(x,x) = R(y,y) = R(z,z) = 1$, $R(x,y) = R(x,z) = R(y,z) = 0$, $R(y,x) = 0.5$, $R(z,x) = 0.3$, and $R(z,y) = 0.2$. then it is easily cheked that R is a fuzzy partial order relation. Also $x \vee y = x$, $x \vee z = x$, $y \vee z = y$, $x \wedge y = y$, $x \wedge z = z$, and $y \wedge z = z$, thus (X, R) is a fuzzy lattice.*

Remark 1.2.1. *Since R is antisymmetric, it follows that the least upper (greatest lower) bound if it exists, is unique.*

Proposition 1.2.1. *Let (X, R) be a fuzzy poset. if (X, R_α) is a lattice for all $\alpha \in]0, 1]$, then (X, R) is a fuzzy lattice.*

Proposition 1.2.2. *Let $(X, R, \wedge, \vee)$ be a fuzzy lattice, and let $x, y, z \in X$. Then it hold that:*

1. $R(x, x \vee y) > 0, R(y, x \vee y) > 0, R(x \wedge y, x) > 0$ *and* $R(x \wedge y, y) > 0$.
2. $R(x, z) > 0$ *and* $R(y, z) > 0$, *then* $R(x \vee y, z) > 0$.
3. $R(z, x) > 0$ *and* $R(z, y) > 0$, *then* $R(z, x \wedge y) > 0$.
4. $R(x, y) > 0$ *if and only if* $x \vee y = y$.
5. $R(x, y) > 0$ *if and only if* $x \wedge y = x$.
6. *If* $R(y, z) > 0$ *then* $R(x \wedge y, x \wedge z) > 0$ *and* $R(x \vee y, x \vee z) > 0$.

Proof 1.2.1. (4) *Suppose* $R(x, y) > 0$. *since* $R(y, y) = 1 > 0$, $R(x \vee y, y) > 0$. *by* (2).*since* $R(y, x \vee y) > 0$. *by* (1), $x \vee y = y$ *by the antisymmetry of* R. *conversely, suppose* $x \vee y = y$. *then* $R(x, y) = R(x, x \vee y) > 0$ *by* (1).

(6) *Suppose* $R(y, z) > 0$. *then*

$$R(x \wedge y, z) \geq \sup_{p \in X} \min[R(x \wedge y, p), R(p, z)].$$

$$\geq \min[R(x \wedge y, y), R(y, z)] > 0.$$

Since $R(x \wedge y, x) > 0$ *by* (1), $x \wedge y$ *is a lower bound of* $\{x, z\}$. *since* $x \wedge z$ *is the greatest lower bound of* $\{x, z\}$, $R(x \wedge y, x \wedge z) > 0$.

$$R(y, x \vee z) \geq \sup_{p \in X} \min[R(y, p), R(p, x \vee z)].$$

$$\geq \min[R(y, z), R(z, x \vee z)] > 0.$$

Since $R(x, x \vee z) > 0$ *by* (1), $R(x \vee y, x \vee z) > 0$ *by* (2).

Proposition 1.2.3. *Let $(X, R, \wedge, \vee)$ be a fuzzy lattice, and let $x, y, z \in X$. Then it hold that:*

1. $x \vee x = x, x \wedge x = x$.
2. $x \vee y = y \vee x, x \wedge y = y \wedge x$.
3. $(x \vee y) \vee z = x \vee (y \vee z), (x \wedge y) \wedge z = x \wedge (y \wedge z)$.
4. $(x \vee y) \wedge x = x, (x \wedge y) \vee x = x$.

Proof 1.2.2. (3) *Since* $R(x, x \vee (y \vee z)) > 0$ *and*

$$R(y, x \vee (y \vee z)) \geq \sup_{k \in X} \min[R(y,k), R(k, x \vee (y \vee z))].$$

$$\geq \min[R(y, y \vee z), R(y \vee z, x \vee (y \vee z))] > 0.$$

Suppose $R(x \vee y, x \vee (y \vee z))] > 0$ *since*

$$R(z, x \vee (y \vee z)) \geq \sup_{k \in X} \min[R(z,k), R(k, x \vee (y \vee z))].$$
$$\geq \min[R(z, y \vee z), R(y \vee z, x \vee (y \vee z))] > 0.$$

$R((x \vee y) \vee z, x \vee (y \vee z)) > 0$ *by* (2). *Similarly we may show* $R(x \vee (y \vee z), (x \vee y) \vee z) > 0$ *by the antisymmetry of* R, $(x \vee y) \vee z = x \vee (y \vee z)$. *Similarly we may show* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (4) *Let* $B = \{x \vee y, x\}$. *Since* $R(x, x \vee y) > 0$ *and* $R(x,x) = 1 > 0$, x *is a lower bound of* B. *If* z *is a lower bound of* B, *then* $R(z, x) > 0$. *Thus* x *is the greatest lower bound of* B. *Hence* $(x \vee y) \wedge x = x$. *Similarly we may show* $(x \wedge y) \vee x = x$.

Remark 1.2.2. *A lattice* L *is said to be complete if every subset* S *of* L *has a least upper bound and every subset of* L *has a greatest lower bound.*

Chapter 2

Intuitionistic fuzzy sets

2.1 Notations and definitions

Definition 2.1.1. [11] *An intuitionistic fuzzy set (IFS) A on universe X is on object of the form*

$$A = \{(x, \mu_A(x), \nu_A(x))\, | x \in X\}\,.$$

where $\mu_A(x) \in [0,1]$ *is called the degree of membership of* x *in* A,
$\nu_A(x) \in [0,1]$ *is called the degree of non membership of* x *in* A, *and* $\mu_A(x)$ *and* $\nu_A(x)$ *satisfy the following condition:*

$$\mu_A(x) + \nu_A(x) \leqslant 1 \text{ for all } x \in X.$$

Example 2.1.1. [2] *Let* $X = \{a, b, c, d\}$ *and let be* A *an intuitionistic fuzzy set in* X *where:*

$$A = \{\langle a, 0.4, 0.3\rangle\,, \langle b, 0.2, 0.3\rangle\,, \langle c, 1.0, 0.0\rangle\,, \langle d, 0.9, 0\rangle\}\,.$$

Definition 2.1.2. [7] *The value of*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

named as the intuitionistic fuzzy set index or hesitation margin of x.
$\pi_A(x)$ *is that the degree of indeterminacy of* $x \in X$ *to the IFS* A *and* $\pi_A(x) \in [0,1]$.

Definition 2.1.3. *(Similar IFS) Two IFS A and B are said to be similar or cognate if*

$$\exists \mu_A(x) = \mu_B(x) \text{ or } \nu_A(x) = \nu_B(x).$$

Definition 2.1.4. *two IFS* A *and* B *are said to be equal if*

$$\mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x).$$

Definition 2.1.5. *(Equivalent IFS) Two IFS* A *and* B *are said to be equivalent to each other, i.e,* A *is equivalent to* B *denoted by* $A \sim B$ *if* $\exists$ *function* $f : \mu_A(x) \longrightarrow \mu_B(x)$ *and* $f : \nu_A(x) \longrightarrow \nu_B(x)$ *which are both injection and surjection (i.e, bijection). Then the function define one-to-one correspondence between* A *and* B.

Definition 2.1.6. *(Proper Subset IFS) A is a proper subset of B, i.e, $A \subset B$ if $A \subseteq B$ and $A \neq B$. It means $\mu_A(x) \leqslant \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ but $\mu_A(x) \neq \mu_B(x)$ and $\nu_A(x) \neq \nu_B(x)$ for $x \in X$.*

Definition 2.1.7. *(The support)* [12] *The support of an intuitionistic fuzzy set $A = (X, \mu_A(x), \nu_A(x))$ is defined as*

$$Supp(A) = \{x \in X : \mu_A(x) \neq 0 \text{ and } \nu_A(x) \neq 1\}.$$

Definition 2.1.8. *(The core)* [12] *The core of an intuitionistic fyzzy set $A = (X, \mu_A(x), \nu_A(x))$ is defined as*

$$Core(A) = \{x \in X : \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}.$$

Definition 2.1.9. *(The height)* [12] *The height of an intuitionistic fyzzy set $A = (X, \mu_A(x), \nu_A(x))$ is defined as*

$$h(A) = \left\{ \sup_{x \in X} (\mu_A(x)), \min_{x \in X} (\nu_A(x)) \right\}.$$

2.2 Basic operations on intuitionistic fuzzy sets [7]

- Inclusion

$$A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \forall x \in X.$$

- Intersection

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}.$$

- Union

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}.$$

- Complement

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}.$$

- Sum $(A \oplus B)$

$$A \oplus B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle | x \in X\}.$$

- Product $(A \otimes B)$

$$A \otimes B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle | x \in X\}.$$

Example 2.2.1. [7] *Let* $X = \{a, b, c, d, e\}$, *let the IFS* A *and* B *have the forms:*
$A = \{\langle a, 0.3, 0.5\rangle, \langle b, 0.6, 0.4\rangle, \langle c, 0.9, 0.0\rangle\}$.
$B = \{\langle a, 0.7, 0.1\rangle, \langle b, 0.8, 0.1\rangle, \langle c, 1.0, 0.0\rangle\}$.
Then:
$A^c = \{\langle a, 0.5, 0.3\rangle, \langle b, 0.4, 0.6\rangle, \langle c, 0.0, 0.9\rangle\}$.
$B^c = \{\langle a, 0.1, 0.7\rangle, \langle b, 0.1, 0.8\rangle, \langle c, 0.0, 1.0\rangle\}$.
$A \cap B = \{\langle a, 0.3, 0.5\rangle, \langle b, 0.6, 0.4\rangle, \langle c, 0.9, 0.0\rangle\}$.
$A \cup B = \{\langle a, 0.7, 0.1\rangle, \langle b, 0.8, 0.1\rangle, \langle c, 1.0, 0.0\rangle\}$.

2.3 Some properties of intuitionistic fuzzy sets

1. Commutativity

$$A \cap B = B \cap A \; , \; A \cup B = B \cup A.$$

2. Associativity

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$(A \cup B) \cup C = A \cup (B \cup C).$$

3. Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. De Morgan " Laws "

$$(A \cup B)^c = A^c \cap B^c.$$

$$(A \cap B)^c = A^c \cup B^c.$$

Proof 2.3.1.
1- Proof that $A \cap B = B \cap A$ *and* $A \cup B = B \cup A$.
$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$.
$= \{\langle x, \min(\mu_B(x), \mu_A(x)), \max(\nu_B(x), \nu_A(x))\rangle : x \in X\}$.
$= B \cap A$.

$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$.
$= \{\langle x, \max(\mu_B(x), \mu_A(x)), \min(\nu_B(x), \nu_A(x))\rangle : x \in X\}$.
$= B \cup A$.

2-
Let $(A \cap B) \cap C = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\} \cap C$.
$= \{\langle x, \min(\min(\mu_A(x), \mu_B(x)), \mu_C(x)), \max(\max(\nu_A(x), \nu_B(x)), \nu_C(x))\rangle : x \in X\}$.

$= \{\langle x, \min(\mu_A(x), \mu_B(x), \mu_C(x)), \max(\nu_A(x), \nu_B(x), \nu_C(x))\rangle : x \in X\}.$
$= \{\langle x, \min(\mu_A(x), \min(\mu_B(x), \mu_C(x))), \max(\nu_A(x), \max(\nu_B(x), \nu_C(x)))\rangle : x \in X\}.$
$= A \cap \{\langle x, \min(\mu_B(x), \mu_C(x)), \max(\nu_B(x), \nu_C(x))\rangle : x \in X\}.$
$= A \cap (B \cap C).$
Let $(A \cup B) \cup C = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\} \cup C.$
$= \{\langle x, \max(\max(\mu_A(x), \mu_B(x)), \mu_C(x)), \min(\min(\nu_A(x), \nu_B(x)), \nu_C(x))\rangle : x \in X\}.$
$= \{\langle x, \max(\mu_A(x), \mu_B(x), \mu_C(x)), \min(\nu_A(x), \nu_B(x), \nu_C(x))\rangle : x \in X\}.$
$= \{\langle x, \max(\mu_A(x), \max(\mu_B(x), \mu_C(x))), \min(\nu_A(x), \min(\nu_B(x), \nu_C(x)))\rangle : x \in X\}.$
$= A \cup \{\langle x, \max(\mu_B(x), \mu_C(x)), \min(\nu_B(x), \nu_C(x))\rangle : x \in X\}.$
$= A \cup (B \cup C).$

3-
Proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Let $A \cap (B \cup C) = A \cap \{\langle x, \max(\mu_B(x), \mu_C(x)), \min(\nu_B(x), \nu_C(x))\rangle : x \in X\}.$
$= \{\langle x, \min(\mu_A(x), \max(\mu_B(x), \mu_C(x))), \max(\nu_A(x), \min(\nu_B(x), \nu_C(x)))\rangle : x \in X\}.$
$= \{\langle x, \max(\min(\mu_A(x), \mu_B(x)), \min(\mu_A(x), \mu_C(x))), \min(\max(\nu_A(x), \nu_B(x)), \max(\nu_A(x), \nu_C(x)))\rangle\}.$
$= (A \cap B) \cup (A \cap C).$

4-
Proof that $(A \cap B)^c = A^c \cup B^c$.
Let $(A \cap B)^c = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}^c.$
$= \{\langle x, \max(\nu_A(x), \nu_B(x)), \min(\mu_A(x), \mu_B(x))\rangle : x \in X\}.$
$= \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\} \cup \{\langle x, \nu_B(x), \mu_B(x)\rangle : x \in X\}.$
$= A^c \cup B^c.$

2.4 (α, β)-cut of intuitionistic fuzzy set

[7] Let A be an intuitionistic fuzzy set, and let $(\alpha, \beta) \in]0,1] \times [0,1[$, $\alpha + \beta \leq 1$. An (α, β)-cut of A is the crisp subset $C_{\{\alpha,\beta\}}(A)$ of X defined by

$$C_{\{\alpha,\beta\}}(A) = \{x \in X | \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}.$$

2.5 Intuitionistic fuzzy relations

Burillo and bustince introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

Definition 2.5.1. *An intuitionistic fuzzy binary relation from a universe X to a universe Y is an intuitionistic fuzzy subset of $X \times Y$, i.e, is an expression R given by:*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle | (x, y) \in X \times Y\}.$$

Where:

$$\mu_R : X \times Y \longrightarrow [0,1], \; and \, \nu_R \longrightarrow [0,1]$$

Satisfy the condition

$$0 \leqslant \mu_R(x,y) + \nu_R(x,y) \leqslant 1.$$

for any $(x,y) \in X \times Y$, *the value* $\mu_R(x,y)$ *is termed the degree of membership of* (x,y) *in* R *and* $\nu_R(x,y)$ *is named the degree of non-membership of* (x,y) *in* R.
Let R *and* P *be two intuitionistic fuzzy relations from a universe* X *to a universe* Y. R *is said to be contained in* P *or we are saying that* P *contains* R, *denoted by* $R \subseteq P$, *if for all* $(x,y) \in X \times Y$ *it holds that* $\mu_R(x,y) \leqslant \mu_P(x,y)$ *and* $\nu_R(x,y) \geq \nu_P(x,y)$. *The transpose (or the inverse)* R^t *of* R *is that the intuitionistic fuzzy relation from the universe* Y *to the universe* X *defined by* $R^t = \{\langle (x,y), \mu_{R^t}(x,y), \nu_{R^t}(x,y) \rangle \,|(x,y) \in X \times Y\}$ *where* $\mu_{R^t}(x,y) = \mu_R(y,x)$ *and* $\nu_{R^t}(x,y) = \nu_R(y,x)$ *for any* $(x,y) \in X \times Y$ *the intersection of two intuitionistic fuzzy relation* R *and* P *from a universe* X *to a universe* Y *is defined as:*

$$R \cap P = \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y)) \rangle \,|(x,y) \in X \times Y\}.$$

the union of two intuitionistic fuzzy relation R *and* P *from a universe* X *to a universe* Y *is defined as:*

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y)) \rangle \,|(x,y) \in X \times Y\}.$$

Let R *be an intuitionistic fuzzy relation from a universe* X *into itself (intuitionistic fuzzy relation on* X, *for short).*

(i) Reflexivity: $\mu_R(x,x) = 1$, *for any* $x \in X$. *in this case we note that* $\nu_R(x,x) = 0$ *for any* $x \in X$.

(ii) Antisymmetry: for any $x, y \in X, x \neq y$ *then*

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x). \\ \nu_R(x,y) \neq \nu_R(y,x). \\ \pi_R(x,y) = \pi_R(y,x). \end{cases}$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) Perfect Antisymmetry: for any $x, y \in X, x \neq y$ *and*

$$\begin{cases} \mu_R(x,y) > 0. \\ \quad or \\ \mu_R(x,y) = 0 \; and \; \nu_R(x,y) < 1. \end{cases}$$

Then:

$$\begin{cases} \mu_R(y,x) = 0. \\ and \\ \nu_R(y,x) = 1. \end{cases}$$

(iv) Transitivity: $R \supseteq R\ o^{\alpha,\beta}_{\lambda,\rho}\ R$. $(\alpha, \beta, \lambda, \rho)$ *ara T-norms or T-conorms triungelaire*

$$\begin{cases} \mu_R(x,y) \wedge \mu_R(y,z) \leq \mu_R(x,z). \\ \nu_R(x,y) \vee \nu_R(y,z) \geq \nu_R(x,z). \end{cases}$$

2.6 Intuitionistic fuzzy lattices

For an intuitionistic fuzzy ordered set (X, μ_R, ν_R) and $x \in X$, the intuitionistic fuzzy sets $R_{\geq[x]}$ and $R_{\leq[x]}$ defined in X by

$$R_{\geq[x]} = \left\{\left\langle y, \mu_{R_{\geq[x]}}(y), \nu_{R_{\geq[x]}}(y)\right\rangle | y \in X\right\}.$$

Where $\mu_{R_{\geq[x]}}(y) = \mu_R(x,y)$ and $\nu_{R_{\geq[x]}}(y) = \nu_R(x,y)$.

$$R_{\leq[x]} = \left\{\left\langle y, \mu_{R_{\leq[x]}}(y), \nu_{R_{\leq[x]}}(y)\right\rangle | y \in X\right\}.$$

Where $\mu_{R_{\leq[x]}}(y) = \mu_R(y,x)$ and $\nu_{R_{\leq[x]}}(y) = \nu_R(y,x)$.

$R_{\geq[x]}$ and $R_{\leq[x]}$ are called the dominating class of x and the class dominated by X respectively.

Definition 2.6.1. [15] *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X.*

(i) The set of upper bounds of A with respect to R is that the intuitionistic fuzzy subset of X defined by:

$$U(R,A)(y) = \bigcap_{x \in A} R_{\geq[x]}(y) \quad ,\textit{for any } \ y \in X.$$

(ii) The set of lower bounds of A with relevance R is that the intuitionistic fuzzy subset of X defined by:

$$L(R,A)(y) = \bigcap_{x \in A} R_{\leq[x]}(y) \quad ,\textit{for any } \ y \in X.$$

Definition 2.6.2. [15] *Let (X, μ_R, ν_R) be an intuitionistic fuzzy ordered set and A be a subset of X An element $x \in X$ is termed the smallest amount bound (or a supremum) of A with relation to R if:*

(i) $x \in Supp(U(R,A))$.

(ii) for all other $y \in Supp(U(R,A))$, $\mu_R(x,y) > 0$ *or* $(\mu_R(x,y) = 0$ *and* $\nu_R(x,y) < 1)$.

An element $x \in X$ is called the greatest lower bound (or an infimum) of A with respect to R if

(i) $x \in Supp(L(R,A))$.

(ii) for all other $y \in Supp\left(L\left(R, A\right)\right), \mu_R\left(y, x\right) > 0$ *or* $\left(\mu_R\left(y, x\right) = 0 \text{ and } \nu_R\left(y, x\right) < 1\right)$.

Definition 2.6.3. [15] *An intuitionistic fuzzy ordered set* (X, μ_R, ν_R) *is called an intuitionistic fuzzy lattice with respect to the intuitionistic fuzzy order* R *(or simply, intuitionistic fuzzy lattice) if each pair of elements* $\{x, y\}$ *of* X *has a supremum and infimum.*

Chapter 3

Picture fuzzy sets

In this section, we define some order on the set D^* in order to use them for studing picture fuzzy sets, and operations about picture fuzzy sets and picture fuzzy relations.

3.1 On the set D^*

Definition 3.1.1. [5] *The set D^*, is a subset of $[0,1]^3$, it is the set of truth value of any picture fuzzy set :*

$$D^* = \left\{x = (x_1, x_2, x_3)\,|x \in [0,1]^3, x_1 + x_2 + x_3 \leq 1\right\}.$$

From now on, we we'll assume that if $x \in D^*$, then the first, the second and also the third components of x, are denoted by x_1, x_2 and x_3, respectively, i.e, $x = (x_1, x_2, x_3)$.

3.1.1 Some orders on D^*

lemma 3.1.1. *Let $x, y \in D^*$, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.*
The relation $\leq_1$ on D^*, defined by
$x \leq_1 y$ if and only if $(x_1 \leq y_1) \wedge (x_2 \leq y_2) \wedge (x_3 \geq y_3)$.
for all $x \in D^*$, is an order relation.

Proof 3.1.1. (1) *Reflexive:*
$x \leq_1 x$, indeed $(x_1 \leq x_1) \wedge (x_2 \leq x_2)$ and $(x_3 \geq x_3)$.
(2) *Antisymmetry:*
If $x \leq_1 y$ and $y \leq_1 x$, this means
$x_1 \leq_1 y_1$ and $y_1 \leq_1 x_1$,
and $x_2 \leq_1 y_2$ and $y_2 \leq_1 x_2$,
and $x_3 \geq_1 y_3$ and $y_3 \geq_1 x_3$.
So, $x_1 = y_1$ and $y_2 = x_2$ and $y_3 = x_3$.

(3) *Transitive:*
If $x \leq_1 y$, *and* $y \leq_1 z \Rightarrow x \leq_1 z$ *this means*
$x_1 \leq_1 y_1$, *and* $y_1 \leq_1 z_1$,
$x_2 \leq_1 y_2$, *and* $y_2 \leq_1 z_2$.
$x_3 \geq_1 y_3$, *and* $y_3 \geq_1 z_3$.
So, $x_1 \leq_1 z_1$, *and* $x_2 \leq_1 z_2$, *and* $x_3 \geq_1 z_3$.
Hence $\leq_1$ *is a picture fuzzy ordering*
It is easy to do see that $(D^*, \leq_1)$ *is not lattice.*
Indeed take $X = (0.7, 0.2, 0.1)$ *and* $Y = (0.1, 0.6, 0.2)$, *then*
$(0.7 \vee 0.1) = 0.7$,
$(0.2 \vee 0.6) = 0.6$,
$(0.1 \wedge 0.2) = 0.1$,
Hence $x \vee y = (0.7, 0.6, 0.1) \notin D^*$

Definition 3.1.2. *Let* $x, y \in D^*$, $x = (x_1, x_2, x_3)$ *and* $y = (y_1, y_2, y_3)$. *The second relation* $\leq_2$ *on* D^* *defined by*
$x \leq_2 y$ *if and only if* $((x_1 < y_1) \wedge (x_3 \geq y_3)) \vee ((x_1 = y_1) \wedge (x_3 > y_3)) \vee ((x_1 = y_1) \wedge (x_3 = y_3) \wedge (x_2 \leq y_2))$.

lemma 3.1.2. *The relation* $\leq_2$ *is an order on* D^*.

Proof 3.1.2. (1) *Reflexive:*
It easy to see that$(x_1 = x_1) \wedge (x_3 = x_3) \wedge (x_2 \leq x_2)$.
Hence $x \leq_2 x$.
(2) *Antisymmetry: If* $x \leq_2 y$ *and* $y \leq_2 x$ *implies* $x = y$
$x \leq_2 y$ *this means:*
(a) $x_1 < y_1 \wedge x_3 \geq y_3$, *or*
(b) $x_1 = y_1 \wedge x_3 > y_3$, *or*
(c) $x_1 = y_1 \wedge x_3 = y_3 \wedge x_2 \leq y_2$.
$y \leq_2 x$ *this means:*
(a') $y_1 < x_1 \wedge y_3 \geq x_3$, *or*
(b') $y_1 = x_1 \wedge y_3 > x_3$, *or*
(c') $y_1 = x_1 \wedge y_3 = x_3 \wedge y_2 \leq x_2$.
There are nine cases to discuss:
case 1: a and a' implies directly that $x = y$.
case 2: a and b' implies directly that $x = y$.
In a same way, we show that the cases
case 3: a and c' implies directly that $x = y$. *case 4: b and a' implies directly that* $x = y$.
case 5: b and b' implies directly that $x = y$.
case 6: b and c' implies directly that $x = y$.
case 7: c and a' implies directly that $x = y$.
case 8: c and b' implies directly that $x = y$.
finaly, the case 9: c and c' give $x_1 = y_1 \wedge x_2 = y_2$ *and* $x_3 = y_3$.
Hence $x = y$.

And $(\leq_2)$ *is antisymmetric.*

(3) Transitive:

If $x \leq_2 y$ *and* $y \leq_2 z$ *implies* $x \leq_2 z$

$x \leq_2 y$ *this means:*

(a) $x_1 < y_1 \wedge x_3 \geq y_3$, *or*

(b) $x_1 = y_1 \wedge x_3 > y_3$, *or*

(c) $x_1 = y_1 \wedge x_3 = y_3 \wedge x_2 \leq y_2$.

$y \leq_2 z$ *this means:*

(a') $y_1 < z_1 \wedge y_3 \geq z_3$, *or*

(b') $y_1 = z_1 \wedge y_3 > z_3$, *or*

(c') $y_1 = z_1 \wedge y_3 = z_3 \wedge y_2 \leq z_2$.

There nine cases to discuss.

It easy to see that the cases

1 (a and a'), 2 (a and b') and 3 (a and c') give directly the transitivty.

In a same way, we can conclude that the cases 4 (b and a'), 5 (b and b') and 6 (b and c') give directly the transitivty.

Finaly, the cases 7 (c and a'), 8 (c and b') and 9 (c and c') give also the transitivty.

Hence $(\leq_2)$ *is Transitive. Consequently,* $(\leq_2)$ *is picture fuzzy ordering.*

It remains to show that $(D^*, \leq_2)$ is a picture fuzzy lattice.

Let $x, y \in D^*$, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.

We can verify that $x \wedge y = (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3)$.

$x \vee y = (x_1 \vee y_1, 0, x_3 \wedge y_3)$. With $1_{D^*} = (1, 0, 0)$ and $0_{D^*} = (0, 0, 1)$

Hence $(D^*, \leq_2, \vee, \wedge, 0_{D^*}, 1_{D^*})$ is a bounded lattice.

3.2 Picture fuzzy negation operator

Picture fuzzy negations are an extension of the fuzzy negations and the intuitionistic fuzzy negations.

Definition 3.2.1. [5] *A picture fuzzy negator is a non increasing mapping* $N : D^* \longrightarrow D^*$ *satisfying*

$$N(0_{D^*}) = 1_{D^*} \text{ and } N(1_{D^*}) = 0_{D^*}.$$

If $N(N(x)) = x$, *for all* $x \in D^*$ *then* N *is called an involutive picture fuzzy negator.*

The mapping N_0 *defined by*

$$N_0(x) = (x_3, 0, x_1).$$

For all $x \in D^*$, *is a picture fuzzy negator.*

Proposition 3.2.1. *N_0 is called the simple picture negation.*

Proof 3.2.1. *Let $1_{D^*} = (1,0,0) \in D^*$ then $N_0(1_{D^*}) = N_0(1,0,0) = (0,0,1) = 0_{D^*}$.*
and $N_0(0_{D^*}) = N_0(0,0,1) = (1,0,0) = 1_{D^*}$.
Let $x, y \in D^*$ and $N_0(x) = (x_3, 0, x_1)$ and $N_0(y) = (y_3, 0, y_1)$, $x \leq_2 y$ consider that
$((x_1 < y_1) \wedge (x_3 \geq y_3)) \vee ((x_1 = y_1) \wedge (x_3 > y_3)) \vee ((x_1 = y_1) \wedge (x_3 = y_3) \wedge (x_2 \leq y_2))$,
So
If $x_1 < y_1 \wedge x_3 = y_3$ then $N_0(y) \leq_2 N_0(x)$.
If $x_1 < y_1 \wedge x_3 > y_3$ then $N_0(y) \leq_2 N_0(x)$.
If $x_1 = y_1 \wedge x_3 > y_3$ then $N_0(y) \leq_2 N_0(x)$.
If $x_1 = y_1 \wedge x_3 = y_3$ then $N_0(y) \leq_2 N_0(x)$.
It shows that the mapping $N_0(x)$ is non-increasing and the operator N_0 is a picture negation operator.

Definition 3.2.2. *Let $x = (x_1, x_2, x_3) \in D^*$. Then let us define $x_4 = 1 - (x_1 + x_2 + x_3)$. The mapping N_s is by $N_s(x) = (x_3, x_4, x_1)$.*

Proposition 3.2.2. *N_s is an involutive picture negation operator and it is called the picture standard negation operator.*

3.3 Picture fuzzy set operations

Definition 3.3.1. *Let X be a non-empty set and let A, B be two (PFS) the inclusion, the union and the intersection operations according to the order relation $\leq_1$ on D^* are defined by*

- *$A \subseteq_1 B$ if and only if $A(x) \leq_1 B(x)$.*
  $A(x) \leq_1 B(x)$ it is means: $(\mu_A(x), \eta_A(x), \nu_A(x)) \leq_1 (\mu_B(x), \eta_B(x), \nu_B(x)) \Longleftrightarrow$
  $\mu_A(x) \leq \mu_B(x)$ *and* $\eta_A(x) \leq \eta_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$.

- *$A \cup_1 B$ is definie par $(A \cup_1 B)(x) = A(x) \vee_1 B(x)$.*
  $A(x) \vee_1 B(x)$ it is means: $(\mu_A(x), \eta_A(x), \nu_A(x)) \vee_1 (\mu_B(x), \eta_B(x), \nu_B(x)) =$
  $(\mu_A(x) \vee \mu_B(x), \eta_A(x) \wedge \eta_B(x), \nu_A(x) \wedge \nu_B(x))$.

- *$A \cap_1 B$ is definie par $(A \cap_1 B)(x) = A(x) \wedge_1 B(x)$.*
  $A(x) \wedge_1 B(x)$ it is means: $(\mu_A(x), \eta_A(x), \nu_A(x)) \wedge_1 (\mu_B(x), \eta_B(x), \nu_B(x)) =$
  $(\mu_A(x) \wedge \mu_B(x), \eta_A(x) \wedge \eta_B(x), \nu_A(x) \vee \nu_B(x))$.

Definition 3.3.2. [7] *A picture fuzzy set A on a universe X is an object within the form of:*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \,|x \in X\}.$$

where $\mu_A(x) \in [0,1]$ is named the degree of positive membership of x in A and $\eta_A(x) \in [0,1]$ is called the degree of neutral membership of x in A.
$\nu_A(x) \in [0,1]$ is called the degree of negative membership of x in A and $\mu_A(x)$ and $\eta_A(x)$ and $\nu_A(x)$ satisfy the subsequent condition:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leqslant 1 \ \textit{for all}\, x \in X.$$

Now $\pi(x) = (1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)))$ might be called the degree of refusal membership of x in A.
Let PFS(x) denoted the set of all the picture fuzzy sets on a universe X.

Definition 3.3.3. *(The height)* [6] *Let A be a PFS of X, then the height is defined as:*

$$h(A) = \left\{ \sup_{x\in X} (\mu_A(x)), \min_{x\in X} (\eta_A(x)), \min_{x\in X} (\nu_A(x)) \right\}.$$

Example 3.3.1. *Let X be a universal set and A be a PFS in X have the forms:*
$X = \{x_1, x_2, x_3\}.$
$A = \{\langle x_1, 0.4, 0.2, 0.3\rangle, \langle x_2, 0.7, 0.1, 0.1\rangle, \langle x_3, 0.6, 0.2, 0.2\rangle\}.$
$h(A) = \{0.7, 0.1, 0.1\}.$

3.4 Basic operations on picture fuzzy sets

- Inclusion [4] We say that A is included in B if:

$$\forall x \in X, \mu_A(x) \leqslant \mu_B(x) \text{ and } \eta_A(x) \leq \eta_B(x) \text{ and } \nu_A(x) \geq \nu_B(x).$$

- Equality [4] We say that $A = B$, if and only if

$$A \subseteq B \text{ and } B \subseteq A.$$

- Union [4]

$$\{x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x)) \,|x \in X\}.$$

- Intersection [4]

$$\{x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x)) \,|x \in X\}.$$

- Complement [4]

$$Co(A) = \{\nu_A(x), \eta_A(x), \mu_A(x), |x \in X\}.$$

Example 3.4.1. *Let X be a universal set and A, B be two PFS in X have the forms:* $X = \{x_1, x_2, x_3\}$ *and:*

$$A = \{\langle x_1, 0.3, 0.3, 0.2\rangle, \langle x_2, 0.7, 0.1, 0.1\rangle, \langle x_3, 0.4, 0.3, 0.2\rangle\}.$$

$$B = \{\langle x_1, 0.3, 0.4, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.1\rangle, \langle x_3, 0.4, 0.3, 0.1\rangle\}.$$

Then:

$$A^c = \{\langle x_1, 0.2, 0.3, 0.3\rangle, \langle x_2, 0.1, 0.1, 0.7\rangle, \langle x_3, 0.2, 0.3, 0.4\rangle\}.$$

$$B^c = \{\langle x_1, 0.1, 0.4, 0.3\rangle, \langle x_2, 0.1, 0.2, 0.6\rangle, \langle x_3, 0.1, 0.3, 0.4\rangle\}.$$

$$A \cup B = \{\langle x_1, 0.3, 0.3, 0.1\rangle, \langle x_2, 0.7, 0.1, 0.1\rangle, \langle x_3, 0.4, 0.3, 0.1\rangle\}.$$

$$A \cap B = \{\langle x_1, 0.3, 0.3, 0.2\rangle, \langle x_2, 0.6, 0.1, 0.1\rangle, \langle x_3, 0.4, 0.3, 0.2\rangle\}.$$

In the same way, we define picture fuzzy sets operation by mean of $\leq_2$.

Definition 3.4.1. *Let X be a non-empty set and let A, B be two (PFS). The inclusion, union and intersection operations according to the order relation $\leq_2$ on D^*, are defined by*

- *$A \subseteq_2 B$ if and only if $A(x) \leq_2 B(x)$.*
  $A(x) \leq_2 B(x)$ it is means: $(\mu_A(x), \eta_A(x), \nu_A(x)) \leq_2 (\mu_B(x), \eta_B(x), \nu_B(x)) \Longleftrightarrow$ $(\mu_A(x) < \mu_B(x), \nu_A(x) \geq \nu_B(x))$ *or* $(\mu_A(x) = \mu_B(x), \nu_A(x) > \nu_B(x))$ *or* $\mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x)$ *and* $\eta_A(x) \leq \eta_B(x)$.
- *$A \cup_2 B$ is definie par $(A \cup_2 B)(x) = A(x) \vee_2 B(x)$.*
  $(\mu_A(x) \vee \mu_B(x), 0, \nu_A(x) \wedge \nu_B(x))$.
- *$A \cap_2 B$ is definie par $(A \cap_2 B)(x) = A(x) \wedge_2 B(x)$.*
  $(\mu_A(x) \wedge \mu_B(x), 1 - (\mu_A(x) \wedge \mu_B(x)) - (\nu_A(x) \vee \nu_B(x)), \nu_A(x) \vee \nu_B(x))$.

Definition 3.4.2. *Let $x, y \in D^*$.*

$$\inf(x, y) = \begin{cases} \min(x, y), & \textit{If } x \leq_2 y \textit{ or } y \leq_2 x, \\ (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3), & \textit{otherwise.} \end{cases}$$

$$\sup(x, y) = \begin{cases} \max(x, y), & \textit{If } x \leq_2 y \textit{ or } y \leq_2 x, \\ (x_1 \vee y_1, 0, x_3 \wedge y_3), & \textit{otherwise.} \end{cases}$$

3.5 Cuts of picture fuzzy sets

Definition 3.5.1. *Let X be a set, $A \in PFS(X)$. The (α, δ, β)-cut of A is the crisp set $C_{\alpha,\delta,\beta}(A)$ defined by:*

$$C_{(\alpha,\delta,\beta)}(A) = \{x \in X \mid \mu_A(x) \geq \alpha, \eta_A(x) \geq \delta \text{ and } \nu_A(x) \leq \beta\}.$$

Where $(\alpha, \delta, \beta) \in [0,1]^3$ with $\alpha + \delta + \beta \leq 1$.

Definition 3.5.2. *(The strong (α, δ, β)-cut)*
Let X be a set, $A \in PFS(X)$. The strong (α, δ, β)-cut of A is the crisp set $C^+_{\alpha,\delta,\beta}(A)$ defined by:

$$C^+_{(\alpha,\delta,\beta)}(A) = \{x \in X \mid \mu_A(x) > \alpha, \eta_A(x) > \delta \text{ and } \nu_A(x) < \beta\}.$$

Where $(\alpha, \delta, \beta) \in [0,1]^3$ with $\alpha + \delta + \beta \leq 1$.

Proposition 3.5.1. *If A and B are two picture fuzzy sets on the universe set X, then:*

1. $A \subseteq B \Rightarrow A_{(\alpha,\delta,\beta)} \subseteq B_{(\alpha,\delta,\beta)}$.
2. $(A \cap B)_{(\alpha,\delta,\beta)} = A_{(\alpha,\delta,\beta)} \cap B_{(\alpha,\delta,\beta)}$.
3. $(A \cup B)_{(\alpha,\delta,\beta)} \supseteq A_{(\alpha,\delta,\beta)} \cup B_{(\alpha,\delta,\beta)}$.

Proof 3.5.1. (1) *Let $x \in A_{(\alpha,\delta,\beta).}$*
$A \subseteq B$ this means $A(x) \leq_1 B(x)$.
$A(x) \geq_1 (\alpha, \delta, \beta)$.
Since $A \subseteq B$, this implies that
$(\alpha, \delta, \beta) \leq_1 A(x) \leq_1 B(x)$.
Hence $x \in B_{(\alpha,\delta,\beta)}$.

(2) *We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so $(A \cap B)_{(\alpha,\delta,\beta)} \subseteq A_{(\alpha,\delta,\beta)}$ and $(A \cap B)_{(\alpha,\delta,\beta)} \subseteq B_{(\alpha,\delta,\beta)} \Rightarrow (A \cap B)_{(\alpha,\delta,\beta)} \subseteq A_{(\alpha,\delta,\beta)} \cap B_{(\alpha,\delta,\beta)}$.*
Conversely, let $x \in A_{(\alpha,\delta,\beta)} \cap B_{(\alpha,\delta,\beta)}$
$\Rightarrow \mu_A(x) \geq \alpha, \mu_B(x) \geq \alpha \Rightarrow \mu_A(x) \wedge \mu_B(x) \geq \alpha \Rightarrow \mu_{A\cap B}(x) \geq \alpha$
$\Rightarrow \eta_A(x) \geq \delta, \eta_B(x) \geq \delta \Rightarrow \eta_A(x) \wedge \eta_B(x) \geq \delta \Rightarrow \eta_{A\cap B}(x) \geq \delta$
$\Rightarrow \nu_A(x) \leq \beta, \nu_B(x) \leq \beta \Rightarrow \nu_A(x) \wedge \nu_B(x) \leq \beta \Rightarrow \nu_{A\cap B}(x) \leq \beta$
$\Rightarrow x \in (A \cap B)_{(\alpha,\beta,\delta)}$
$\Rightarrow A_{(\alpha,\delta,\beta)} \cap B_{(\alpha,\delta,\beta)} \subseteq (A \cap B)_{(\alpha,\delta,\beta)}$.
Hence $(A \cap B)_{(\alpha,\delta,\beta)} = A_{(\alpha,\delta,\beta)} \cap B_{(\alpha,\delta,\beta)}$.

(3) *We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so*
$A_{(\alpha,\delta,\beta)} \subseteq (A \cup B)_{(\alpha,\delta,\beta)}$ *and* $B_{(\alpha,\delta,\beta)} \subseteq (A \cup B)_{(\alpha,\delta,\beta)}$
So $(A \cup B)_{(\alpha,\delta,\beta)} \supseteq A_{(\alpha,\delta,\beta)} \cup B_{(\alpha,\delta,\beta)}$.

3.6 Picture fuzzy relations

Definition 3.6.1. [4] *Let X and Y be ordinary non-empty sets. A picture fuzzy relation R is a picture fuzzy subset of $X \times Y$, i.e.,*

$$R = \{((x, y), \mu_R(x, y), \eta_R(x, y), \nu_R(x, y)) \,|x \in X, y \in Y\}.$$

Where $\mu_R : X \times Y \longrightarrow [0, 1]$, $\eta_R : X \times Y \longrightarrow [0, 1]$,

$\nu_R : X \times Y \longrightarrow [0, 1]$, *with*

$\mu_R(x, y) + \eta_R(x, y) + \nu_R(x, y) \leq 1$, *for each* $(x, y) \in (X \times Y)$.

Definition 3.6.2. [4] *Let $R \in PFR(X \times Y)$. We define the inverse relation R^{-1} between Y and X by*

$\mu_{R^{-1}}(y, x) = \mu_R(x, y), \quad \eta_{R^{-1}}(y, x) = \eta_R(x, y).$

$\nu_{R^{-1}}(y, x) = \nu_R(x, y) \;, \; \forall (x, y) \in (X \times Y).$

Proposition 3.6.1. [4] *Let $R, P, Q \in PFR(X \times Y)$, then*

1. $R \subseteq P$ *implies* $R^{-1} \subseteq P^{-1}$.
2. $(R \cup P)^{-1} = R^{-1} \cup P^{-1}$.
3. $(R^{-1})^{-1} = R$.
4. $(R \cap P)^{-1} = R^{-1} \cap P^{-1}$.
5. $R \cap P \subseteq R$ *and* $\qquad R \cap P \subseteq P$.
6. *If* $(R \supseteq P)$ *and* $(R \supseteq Q)$, *then* $R \supseteq P \cap Q$.

Proof 3.6.1.

1. *If* $R \subseteq P$, *then* $R(x, y) = R^{-1}(y, x)$.
   $R \subseteq P = R(x, y) \leq_1 P(x, y)$ *or* $R(x, y) = R^{-1}(y, x) \leq_1 P(x, y) = P^{-1}(y, x)$
   Hence $R \subseteq P \quad$ *implies* $\quad R^{-1} \subseteq P^{-1}$.

2. *We have* $(R\cup P)^{-1}(x,y)=(R\cup P)(y,x)=R(y,x)\vee P(y,x)=R^{-1}(x,y)\vee P^{-1}(x,y)$
   Hence
   $(R\cup P)^{-1}=R^{-1}\cup P^{-1}$.

3. $(R^{-1})^{-1}(x,y)=R^{-1}(y,x)=R(x,y)$
   Hence $(R^{-1})^{-1}=R$

4. $(R\cap P)^{-1}(x,y)=(R\cap P)(y,x)$
   $=R(y,x)\wedge P(y,x)$
   $=R^{-1}(x,y)\wedge P^{-1}(x,y)$
   $=(R^{-1}\cap P^{-1})(x,y)$
   Hence $(R\cap P)^{-1}=R^{-1}\cap P^{-1}$.

5. $(R\cap P)(x,y)=R(x,y)\wedge P(x,y)$.
   We have $R(x,y)\wedge P(x,y)\leq_1 R(x,y)$ *and* $R(x,y)\wedge P(x,y)\leq_1 P(x,y)$
   Hence $R\cap P\subseteq R$ *and* $R\cap P\subseteq P$.

6. $R\supseteq P\Rightarrow R(x,y)\geq_1 P(x,y)$.
   $R\supseteq Q\Rightarrow R(x,y)\geq_1 Q(x,y)$.
   This implies $R(x,y)\wedge R(x,y)\geq_1 P(x,y)\wedge Q(x,y)$.
   Hence $R(x,y)\geq_1 P(x,y)\wedge Q(x,y)$.
   So $R\supseteq P\cap Q$.

Definition 3.6.3. [13] *The picture fuzzy relation R on X is referred to as:*

- *Reflexive: if for all* $x\in X$, $\mu_R(x,x)=1$.

- *Symmetric: if for all* $x,y\in X$, $\mu_R(x,y)=\mu_R(y,x)$ *and* $\eta_R(x,y)=\eta_R(y,x)$.

- *Transitive: if* $R^2\subset R$, *where* $R^2=R\circ R$.

- *Picture tolerance: if R is reflexive and symmetric.*

- *Picture preorder: if R is reflexive and transitive.*

- *Picture similarity (picture fuzzy equivalence): if R is reflexive and symmetric and transitive.*

Example 3.6.1. [13] *Let* $X = \{x_1, x_2, x_3, x_4\}$ *be a universe set. We consider a relation* R *on* X *as follows (Table 1):*

Table 1. The picture fuzzy relation R *:*

R	x_1	x_2	x_3	x_4
x_1	(1 , 0 , 0)	(0.3 , 0.4 , 0.2)	(0.4 , 0.5 , 0.1)	(0.3 , 0.4 , 0.2)
x_2	(0.3 , 0.4 , 0.2)	(1 , 0 , 0)	(0.7 , 0.2 , 0.05)	(0.4 , 0.5 , 0.1)
x_3	(0.4 , 0.5 , 0.1)	(0.7 , 0.2 , 0.05)	(1 , 0 , 0)	(0.3 , 0.4 , 0.2)
x_4	(0.3 , 0.4 , 0.2)	(0.4 , 0.5 , 0.1)	(0.3 , 0.4 , 0.2)	(1 , 0 , 0)

We have $R(x_1, x_1) = R(x_2, x_2) = R(x_3, x_3) = R(x_4, x_4) = 1$.
Then R *is reflexive.*
We have $R(x_i, x_j) = R(x_j, x_i)$ *for all* $i, j \in \{1, 2, 3, 4\}$.
Hence R *is symmetric*
It is easy that R *is reflexive and symmetric. but it is not transitive, because* $R^2 \nsubseteq R$.

The relation R^2 *is computed in Table 2. Here, we see that*
$(\mu_{R\circ R}(x_1, x_2), \eta_{R\circ R}(x_1, x_2), \nu_{R\circ R}(x_1, x_2)) = (0.4, 0, 0.1) > (\mu_R(x_1, x_2), \eta_R(x_1, x_2), \nu_R(x_1, x_2)) = (0.3, 0.4, 0.2)$

Table 2 . The picture fuzzy relation R^2

:

R^2	x_1	x_2	x_3	x_4
x_1	(1 , 0 , 0)	(0.4 , 0 , 0.1)	(0.4 , 0 , 0.1)	(0.3 , 0 , 0.2)
x_2	(0.3 , 0 , 0.1)	(1 , 0 , 0)	(0.7 , 0 , 0.05)	(0.4 , 0 , 0.2)
x_3	(0.4 , 0 , 0.1)	(0.7 , 0 , 0.05)	(1 , 0 , 0)	(0.7 , 0 , 0.1)
x_4	(0.4 , 0 , 0.1)	(0.4 , 0 , 0.1)	(0.4 , 0 , 0.1)	(1 , 0 , 0)

Conclusion

In this master memory, we have studied the picture fuzzy sets as a generalization of the notion of fuzzy sets. More specically, we have presented the most important defnitions of the set D^* and the orders, examples, operations and properties.

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ملخص

الهدف من هذه المذكرة هو استيعاب فكرة مجموعات الصورة الضبابية (PFS) بالإضافة إلى بعض الخصائص والعمليات المتعلقة بهذه الفكرة.

Abstract

The objective of this memory is to assimilate the notion of picture fuzzy sets (PFS) as well as some properties and operations related to this notion.

Résumé

L'objectif de ce mémoire est d'assimiler la notion des ensembles flous image (PFS) ainsi que quelques propriétés et opérations reliées à cette notion.