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**MOHAMED BOUDIAF UNIVERSITY -M'SILA**

**Faculty of Mathematics and Computer Science**

**Department of Mathematics**

# *Master degree in Mathematics*

**Domain** : Mathematics and Informatics

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# Title:

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*Aggregation Operators on Bounded Trellises*

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**Presented by :**

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**Before the jury composed of:**

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# Contents

## الملخص

تتناول هذه المذكرة دراسة البنيات الرياضية المرتبطة بالمجموعات المرتبة والشبكيات (Lattices) وشبه الشبكيات (Trellises) ، انطلاقًا من العلاقات الثنائية وخصائصها الأساسية وطرق تمثيلها، ثم الانتقال إلى دراسة العلاقات الترتيبية والعناصر الخاصة داخل المجموعات المرتبة، مع التعمق في بنية الشبكيات المحدودة وخصائصها الجبرية. كما تهتم المذكرة بالمجموعات شبه المرتبة والـ Trellises باعتبارها تعميمًا لبعض البنيات المرتبة، حيث تم عرض أهم خصائصها والعمليات الثنائية المعرفة عليها وبعض المجموعات الجزئية المرتبطة بها. وفي الأخير تم التركيز على المؤثرات التجميعية على الشبكيات والـ Trellises المحدودة، مع دراسة أشهر هذه المؤثرات وخصائص عمليتي t-norm و t-conorm وطرق بنائهما، إضافة إلى إبراز أهميتهما في المنطق الضبابي ونظم اتخاذ القرار والتطبيقات الرياضية الحديثة.



**الكلمات المفتاحية** :العلاقات الثنائية، المجموعات المرتبة، المجموعات شبه المرتبة، الشبكيات المحدودة، Trellises ، المؤثرات التجميعية، t-norm ، t-conorm .



## Abstract

This master memory examines some mathematical relational structures related to ordered sets, lattices, and trellises (or weakly associative lattices) as a generalization of lattices. In particular, we focuses on pseudo-ordered sets and trellises, presenting their main properties, binary operations defined on them, and some related subsets. Finally, we study some associative operators on bounded lattices and trellises, examining the most common properties of these operators, we pay particular attention to the notions of t-norm and t-conorm operations, and their construction methods. It also highlights their importance in fuzzy logic, decision-making and modern mathematical applications.



**Keywords:** Ordered sets, Pseudo order sets, Bounded lattices, Bounded trellises, Aggregation operators, t-norms, t-conorms.

# شُكر وإهداء

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿وَقُل رَّبِّ زِدۡنِي عِلۡمٗا﴾

**الحمد لله الذي بنعْمته تتمّ الصالحات، نحمُده سبحانه على البدء والختام، حمدًا كثيرًا طيبًا مباركًا فيه كما يحب ربنا ويرضى .**

**وفاءً وتقديرًا واعترافًا منا بالجميل، أتقدم بخالص الشكر و الإمتنان إلى الأستاذ:لمنور زدام،على جهوده القيمة وتوجيهاته السديدة التي كان لها الأثر الكبير في إنجاح هذا العمل، جزاه الله خير الجزاء، وبارك في عِلمه وعمَله، وجعل ذلك في ميزان حسناته.**

**وإنه من دواعي فخري واعتزازي أنّ تُناقش هذه المذكرة أمام لجنة المناقشة المكونة من الأستاذ سعداوي الخيروالدكتورفراحتية نسيم الذين أتّوجه إليهم بخالص الشكر والامتنان على قبولهم مناقشة هذه المذكرة،وإسهامهم بإرشاداتهم القّيمة وتوجيهاتهم المتميزة، فجزاهم الله عني خير الجزاء.**

## بكُل حبَّ أهدِي ثمرةَ نجاحِي وتخرجِي:

**إلى القلبِ الكبير الذِي منحنِي الأمان، و إلى اليد التي دفعتني دائمًا نحو النجاح، إلى من كان حضُورهُ قوَة، وكلامهُ دعمًا، ودُعاؤُه رفيقًا لكل خطواتي، أهديك تخرجي بكل فخر ومحبة، فأنت السبب بعد الله في وصولي إلى هذا اليوم...أبي الغالي "محمد قدح"**

**إلى من كان دُعاؤُها سرُ نجاحي، وحنانها سر قوتي، وصبرها سببُ وصولي إلى هذا اليوم، إلى من سهرت وضحت دون مقابلاً، إلى من كانت سندي في ضعفي، ونورَ طريقي في ظلام أيامي، أهديك تخرجي هذا، فهو ثمرةُ قلبكِ قبل جهدي، ونجاحُكِ قبل نجاحي...أمي الغالية"حياة بركة"**

**إلى رُوحُ الرُوح...عمقي وقوتي...ملجأي ومعتمدي...وحضوري في غيبتي...إلى إخوتي الغاليين، دمتم لي سندًا وفرحَا.**

**لا يسعنِي إلا أنّ أُعبرَ عن خالص إمتناني وتقديري لكل من ساندني ورافقني في هذه الرحلة، راجيًا من الله أنّ يجزي الجميع على عطائِهم ودعمِهم الصادق.**

# Introduction

Order theory and lattice structures play a fundamental role in many areas of mathematics and its applications, particularly in algebra, logic, and theoretical computer science. Classical lattice theory is based on partially ordered sets in which every pair of elements admits a least upper bound and a greatest lower bound. These structures provide a natural framework for studying order, hierarchy, and algebraic operations [4, 24].

In recent decades, several generalizations of partially ordered sets have been introduced and extend the scope of lattice theory. Among these generalizations, pseudo-ordered sets (psosets) and trellises [8, 26] have attracted considerable attention. Unlike classical posets, pseudo-ordered structures relax some of the standard axioms, particularly transitivity, which allows a richer and more flexible framework for modeling non-classical ordering phenomena.

Trellises, introduced by Fried [8] and Skala [26], extend the notion of lattices by considering structures where every pair of elements admits both a least upper bound and a greatest lower bound, but where the underlying order may fail to be transitive. This relaxation leads to new algebraic and combinatorial properties that are not present in classical lattice theory, making trellises a useful tool in the study of generalized order structures.

On the other hand, aggregation functions constitute an important class of operations defined on ordered structures [12, 20, **?**]. They are widely used in decision theory, fuzzy logic, and information fusion. Recently, aggregation operators have been studied not only on classical lattices but also on more general structures such as bounded posets and trellises, in order to better model uncertainty and non-linear aggregation phenomena.

The main objective of this work is to study algebraic structures related to ordered sets, particularly pseudo-ordered sets and trellises, and to investigate aggregation operators defined on bounded lattices and bounded trellises. We aim to present fundamental definitions, illustrate key properties, and explore the relationship between order-theoretic concepts and aggregation operations.

This manuscript is organized as follows:

- The first chapter recalls basic notions of binary relations, ordered sets, and lattice theory.

- The second chapter introduces pseudo-ordered sets and trellises, together with their main properties and examples.

- The third chapter focuses on aggregation operators on bounded lattices and their extension to trellises.

Finally, we conclude with perspectives and possible directions for further research in this area.

# Chapter 1

# Generalities on ordered sets and lattices

In this chapter, we recall the necessary basic concepts of binary relations, ordered sets, and lattices, We present the main definitions and properties of relations and order structures, together with important examples and representations. These concepts provide the theoretical foundation for the study of bounded lattices .[4, 18, 24, 25]

## 1.1 Binary relations

In this section, we recall the notion of binary relations and their properties.

### 1.1.1 Definitions and examples

**Definition 1.1 (Sets).** *A set $\mathbb{S}$ is any well-defined collection of objects called elements or members of the set . we write $x \in \mathbb{S}$ to denote that $x$ is an element of $\mathbb{S}$.the notion $x \notin \mathbb{S}$ denotes that $x$ is not an element of $\mathbb{S}$ .*

**Remark 1.1.**

- *One way describing a set that has a finite number of elements is by listing the elements of the set between braces. For example:*
  1. *The set $V$ of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.*
  2. *The set of all positive integers that are less than 4 can be written as $\{1, 2, 3\}$.*
- *The order in with the elements of a set are listed is not important for example: (2) can be written $\{1, 3, 2\}$ or $\{3, 2, 1\}$ or $\{3, 1, 2\}$ or $\{2, 1, 3\}$ and $\{2, 3, 1\}$. But the preferred writing is $\{1, 2, 3\}$.*

**Definition 1.2 (Cartesion Product).** *If $A$ and $B$ are two nonempty sets, we define the product between two sets which called the Cartesian product $A \times B$ as the set of all ordered pair $(a, b)$ with*

*$a \in A$ and $b \in B$, i.e,*

$$A \times B = \{(a, b) \mid a \in \ A \ \textit{and} \ b \in B\}$$

**Example 1.1.** *let $A = \{1, 2, 3\}$ and $B = \{x, y\}$, then*

$$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$$

*and*

$$B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$$

**Remark 1.2.** *As a remark, the cartesian product does not necessary commutative. As can seen from Example 1.1 that $A \times B \neq B \times A$ .*

**Theorem 1.1.** *For any two finite nonempty sets $A$ and $B$, $|A \times B| = |A| \times |B|$, such that $|A|$ is the number of the elements of $A$, called cardinal of $A$.*

**Definition 1.3** (**Binary relations**)**.** *Let $A$ and $B$ be a nonempty sets, a binary relation $\mathcal{R}$ from $A$ to $B$ is a subset of the Cartesian product $A \times B$. If $\mathcal{R} \subseteq A \times B$ and $(a, b) \in \mathcal{R}$, we say that $a$ is related to $b$ by $\mathcal{R}$ and we can write $a\mathcal{R}b$. If $a$ is not related to $b$ by $\mathcal{R}$, we write $a \not\mathcal{R} b$. Frequently, if $A$ and $B$ are equal, in this case we often say that $\mathcal{R} \subseteq A \times A$ is a binary relation on $A$, instead of a binary relation from $A$ to $A$.*

**Example 1.2.** *Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$ be two finite sets, so $\mathcal{R} = \{(1, x), (2, y), (3, x)\}$ is a binary relation from $A$ to $B$. The fact that $(1, x) \in \mathcal{R}$, we can write $1\mathcal{R}x$ and we say that 1 is related to $x$ by the binary relation $\mathcal{R}$.*

**Definition 1.4** (**Inverse Relations**)**.** *Let $\mathcal{R}$ be a relation from a set $A$ to a set $B$. The inverse of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$ , is the relation from $B$ to $A$ which consists of those ordered pairs which, when reversed, belong to $\mathcal{R}$; that is,*

$$\mathcal{R}^{-1} = \{(b, a) | (a, b) \in \mathcal{R}\}$$

**Example 1.3.** *Let $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $\mathcal{R} = \{(1, y), (1, z), (3, y)\}$. Then ,*
*$\mathcal{R}^{-1} = \{(y, 1), (y, 3), (z, 1)\}$ is a inverse of $\mathcal{R}$.*
*Clearly, if $\mathcal{R}$ is any relation, then $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$. Also, the domain and range of $\mathcal{R}^{-1}$ are equal, respectively, to the range and domain of $\mathcal{R}$. Moreover, if $\mathcal{R}$ is a relation on $A$ , then $\mathcal{R}^{-1}$ is also a relation on $A$.*

**Definition 1.5** (**Composition of Relations**)**.** *Let $A$, $B$ and $C$ be sets, and let $\mathcal{R}$ be a relation from $A$ to $B$ and let $\mathcal{S}$ be a relation from $B$ to $C$. That is, $\mathcal{R}$ is a subset of $A \times B$ and $\mathcal{S}$ is a subset of $B \times C$. Then $\mathcal{R}$ and $\mathcal{S}$ give rise to a relation from $A$ to $C$ denoted by $\mathcal{R} \circ \mathcal{S}$ and defined by :*

$$a(\mathcal{R} \circ \mathcal{S})c \ \textit{if for some} \ b \in B \ \textit{we have} \ a\mathcal{R}b \ \textit{and} \ b\mathcal{S}c$$

*That is ,*

$$\mathcal{R} \circ \mathcal{S} = \{(a,c) \in A \times C \mid \exists b \in B : (a,b) \in \mathcal{R} \text{ and } (b,c) \in \mathcal{S}\}.$$

$$A \xrightarrow{\mathcal{R}} B \xrightarrow{\mathcal{S}} C$$

$$a \xrightarrow{\mathcal{R}} b \xrightarrow{\mathcal{S}} c$$

$$a \xrightarrow{\mathcal{R}\circ\mathcal{S}} c$$

*The relation $\mathcal{R} \circ \mathcal{S}$ is called the composition of $\mathcal{R}$ and $\mathcal{S}$, it is sometimes denoted simply by $\mathcal{RS}$. Suppose $\mathcal{R}$ is a relation on a set $A$, that is $\mathcal{R}$ is a relation from a set $A$ to itself. Then $\mathcal{R} \circ \mathcal{R}$, the composition of $\mathcal{R}$ with itself, is always defined. Also, $\mathcal{R} \circ \mathcal{R}$ is sometimes denoted by $\mathcal{R}^2$ . Similarly, $\mathcal{R}^3 = \mathcal{R}^2 \circ \mathcal{R} = \mathcal{R} \circ \mathcal{R} \circ \mathcal{R}$, and so on. Thus*

$$\mathcal{R}^n = \underbrace{\mathcal{R} \circ \mathcal{R} \circ \mathcal{R} \circ \cdots \circ \mathcal{R}}_{n \text{ times}}$$

*is defined for all positive $n$.*

**Example 1.4.** *According to the previous definition 1.5, we have the following example, let $A = \{1,2\}$, $B = \{a,b\}$, and $C = \{x,y\}$ then*

$$\mathcal{R} = \{(1,a),(2,b)\}$$

$$\mathcal{S} = \{(a,x),(b,y)\}$$

$$1 \xrightarrow{\mathcal{R}} a \xrightarrow{\mathcal{S}} x$$

$$2 \xrightarrow{\mathcal{R}} b \xrightarrow{\mathcal{S}} y$$

$$1Ra \text{ and } a\mathcal{S}x \Rightarrow 1(\mathcal{R} \circ \mathcal{S})x$$

$$2\mathcal{R}b \text{ and } b\mathcal{S}y \Rightarrow 2(\mathcal{R} \circ \mathcal{S})y$$

*In the following Theorem, the composition of associative.*

**Theorem 1.2.** *Let $A$, $B$, $C$ and $D$ be sets. Suppose $\mathcal{R}$ is a relation from $A$ to $B$, $\mathcal{S}$ is a relation from $B$ to $C$, and $\mathcal{T}$ is a relation from $C$ to $D$. Then*

$$(\mathcal{R} \circ \mathcal{S}) \circ \mathcal{T} = \mathcal{R} \circ (\mathcal{S} \circ \mathcal{T}).$$

### 1.1.2 Properties of binary relatoins

**Definition 1.6.** *Let $E$ be a set and $\mathcal{R}$ be a binary relation on $E$. The relation $\mathcal{R}$ is said to be:*

- *reflexive if $(a,a) \in \mathcal{R}$, for all $a \in E$.*
- *irreflexive if $(a,a) \notin \mathcal{R}$, for all $a \in E$.*
- *symmetric, $(a,b) \in \mathcal{R}$ if and only if $(b,a) \in \mathcal{R}$, for all $a, b \in E$.*
- *antisymmetric if $(a,b) \in \mathcal{R}$ and $(b,a) \in \mathcal{R}$, then $a = b$, for all $a, b \in E$.*
- *transitive if $(a,b) \in \mathcal{R}$ and $(b,c) \in \mathcal{R}$, then $(a,c) \in \mathcal{R}$, for all $a, b, c \in E$.*

**Example 1.5.** *Let $A = \{1,2,3,4\}$ be a finite set. We define on $E$ the following binary relations as follows:*

$$\begin{gathered}
\mathcal{R}_1 = \{(1,1),(1,2),(2,2),(2,3),(3,3),(4,4)\}, \\
\mathcal{R}_2 = \{(1,1),(1,2),(2,1),(2,2),(3,1),(4,4)\}, \\
\mathcal{R}_3 = \{(1,2),(2,1)\}, \\
\mathcal{R}_4 = \{(1,1),(1,2),(2,1),(2,2)\}, \\
\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4)\} \text{ and} \\
\mathcal{R}_6 = \{(2,1),(3,1),(3,2),(4,4)\}.
\end{gathered}$$

*Then it holds that :*

1) *$\mathcal{R}_1$ is reflexive because it contains all pairs of the form $(a,a)$ for every element $a \in E$, i.e., it contains $(1,1),(2,2),(3,3)$ and $(4,4)$. Then it is not irreflexive.*
2) *$\mathcal{R}_2$ is not reflexive because the pair $(3,3) \notin \mathcal{R}_2$, also it is not irreflexive because $(1,1) \in \mathcal{R}_2$.*
3) *$\mathcal{R}_3$ is irreflexive because $(a,a) \notin \mathcal{R}_4$, for any $a \in E$. Then it is not reflexive.*
4) *$\mathcal{R}_4$ is symmetric because for every $(a,b) \in \mathcal{R}_4$, we have $(b,a) \in \mathcal{R}_4$ like $(1,2)$ and $(2,1)$ the both are in $\mathcal{R}_4$.*
5) *$\mathcal{R}_5$ is antisymmetric, then it not symmetric because we have $(1,2) \in \mathcal{R}_5$, but $(2,1) \notin \mathcal{R}_5$.*
6) *$\mathcal{R}_6$ is transitive because $(3,2),(2,1)$ and $(3,1)$ are there in $\mathcal{R}_6$. But $\mathcal{R}_3$ is not transitive, indeed $(1,2) \in \mathcal{R}_3$, $(2,1) \in \mathcal{R}_3$ and $(1,1) \notin \mathcal{R}_3$.*

### 1.1.3 Representations of binary relations

In this subsection, we present some types of representations of a binary relations.

**Representation of a binary relation by a matrix:**

Let $E = \{a_1, a_2, ..., a_m\}$ and $F = \{b_1, b_2, ..., b_n\}$ are finite sets containing $m$ and $n$ elements respectively. Let $\mathcal{R}$ be a binary relation from $E$ to $F$. Then $\mathcal{R}$ can be represented by the matrix $M_{\mathcal{R}} = [m_{ij}]$ of $m \times n$ elements which is defined by :

$$m_{ij} = \begin{cases} 1 & \text{, if } (x_i, x_j) \in \mathcal{R}, \\ 0 & \text{, if } (x_i, x_j) \notin \mathcal{R}. \end{cases}$$

**Example 1.6.** *Let $E = \{1, 2, 3\}$ be a finite set and $\mathcal{R} = \{(a, b) \in A^2 \mid a \text{ devides } b\}$ be a binary relation on $E$. Then :*

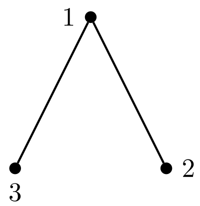
$$\mathcal{R} = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}.$$

*In this case, $\mathcal{R}$ can be represented by the following matrix :*

$$M_{\mathcal{R}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Representation of a binary relation by a directed graph:**

**Definition 1.7.** *There is another way of picturing a binary relation on a finite set using a direct graph. If a binary relation $\mathcal{R}$ is defined on a finite set $E$ then the elements of $E$ are represented by vertices, and the ordered pairs of $\mathcal{R}$ are represented by directed edges. As an example, we take the binary relation given in Example 1.6. Its directed graph is shown in the following figure.*

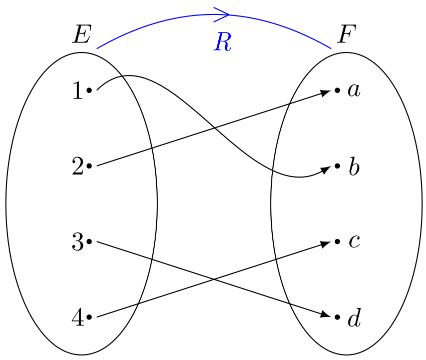



**Representation of a binary relation by Arrow a Diagram:**

**Definition 1.8.** *A binary relation between two finite sets $E$ and $F$ can be represented using an arrow diagram. As an example, let $E = \{1, 2, 3, 4\}$ and $F = \{a, b, c, d\}$ be two finite sets, We define a binary relation $\mathcal{R}$ between $E$ and $F$ as :*

$$\mathcal{R} = \{(1, b), (2, a), (3, d), (4, c)\}.$$

*The above diagram is its arrow diagram defined as follow :*



## 1.2 Ordered sets

In this section, we give the notion of ordered sets and its particular elements.

### 1.2.1 Order relations

**Definition 1.9.** *A binary relation $\mathcal{R}$ on a set $X$ is An order if it satisfies the following three properties:*

1. *Reflexivity: $\forall x \in X, x\mathcal{R}x$.*

2. *Antisymmetry: $\forall x, y \in X, x\mathcal{R}y$ and $y\mathcal{R}x \to x = y$.*

3. *Transitivity: $\forall x, y, z \in X, x\mathcal{R}y$ and $y\mathcal{R}z \to x\mathcal{R}z$.*

**Example 1.7.** *The inclusion relation $\subseteq$ is a partial order on the power set of a set $S$. Indeed, $A \subseteq A$ whenever $A$ is a subset of $S$, then $\subseteq$ is reflexive. It is antisymmetric because if $A \subseteq B$ and $B \subseteq A$ imply that $A = B$.*
*Finally, $\subseteq$ is transitive, because if $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial order on $P(S)$. Consequently, $(P(S), \subseteq)$ is a poset.*

**Example 1.8.** *The divisibility relation ($a\mathcal{R}b$ if and only if $a|b$) is a partial order on $\mathbb{N}^*$.*

**Definition 1.10.** *A set $P$ equipped with an order relation $\leq$ is called a partially ordered set (poset, for short) and denoted by $(P, \leq)$.*

**Example 1.9.** *The two structures $(\mathbb{N}, \leq)$ and $(\mathbb{N}^*, |)$ are partially ordered sets (posets).*

**Definition 1.11.** *Partial ordered sets provide a common frame for many combinatorial configurations. Formally, a partially ordered set (or poset,for short) is a set $P$ together with a binary relation $<$ between its elements which is transitive and antysymmetric :if $x < y$ and $y < z$ then $x < z$, but $x < y$ and $y < x$ connot both hold. We write $x \leqslant y$ if $x < y$ or $x = y$.Elements $x$ and $y$ are comprable if either $x \leqslant y$ or $y \leqslant x$ (or both) hold. A chain in a posetP is a subset $C \subseteq P$ such that any two of its points are comparable. Dually, an antichain is a subset $A \subseteq P$ such that no two of its points are comparable. Observe that $|C \cap A| \leqslant 1$,i.e,every chain $C$ and every antichain $A$ can have at most element in common (for two points in their intersection would be both comparable and incomprable ). Here aare some frequently encountered examples for posets:a family of sets is partially ordered by set inclusion;aset of positive integers is partially ordered by division; a set of vectors in Rn is partially ordered by$(a_1, ..., a_n) < (b_1, ..., b_n)$ if $a_i \leqslant b_i$ for all $i$, and $a_i < b_i$ for at leat one $i$. Small posets may be visualized by drawings, known as hasse diagrams:x is lower in the plane than $y$ whenever $x < y$ and there is ni other poit $z \in P$ for which both $x < z$ and $z < y$.*

**Example 1.10.** *The ordered set $(\mathbb{N}, \leq)$ is a chain .*

**Definition 1.12.** *Let $C$ be a chain of an ordered set $X$. $C$ is said to be maximal if it is not containd (strictly) in any chain of $X$.*

**Proposition 1.1.** *The product of two chain is not necessarily a chain .*

**Example 1.11.** *Let the chain of 5 elements $C_5 = \{0, 1, 2, 3, 4\}$*
*$C_5 \times C_5$={(x,y) $x \in C_5, y \in C_5$}.*
*$(a, b) \leq (c, d) \leftrightarrow a \leq c$ and $b \leq d$.*
*$(0, 4) \leq (4, 1)$ are not comparable.*

**Definition 1.13.** *The Hasse diagram of a finite poset $(P, \leq)$ is a picture of the digraph whose vertexes are the elements of $P$ and which has line segments between of their vertexes. If an element $y \in P$ covers an element $x \in P$, we get the vertex $y$ is higher up than the vertex $x$ and they are connected with a line segment .*

**Example 1.12.** *(a) Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by the relation "$x$ divides $y$". The diagram of $A$ is given in Figure 1.1.*

*(b) Let $B = \{a, b, c, d, e\}$ The diagram in Figure 1.2 defines a partial order on $B$ in the natural way. That is, $d \leq b, d \leq a, e \leq c$ and so on.*

*(c) The diagram of a finite linearly ordered set, i.e., a finite chain, consists simply of one path. For example, Figure1.3 shows the diagram of a chain with five elements.*

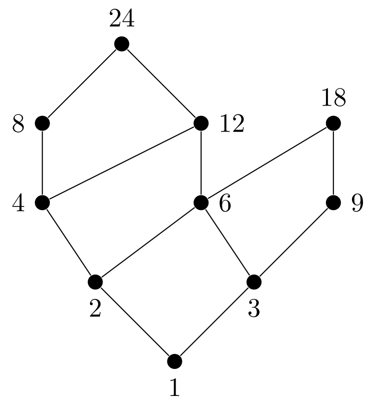



Figure 1.1: fig1.1 .

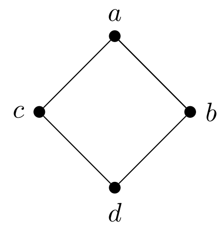



Figure 1.2: fig1.2 .

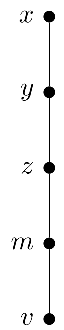



Figure 1.3: fig1.3 .

### 1.2.2 Particular elements of an ordered set

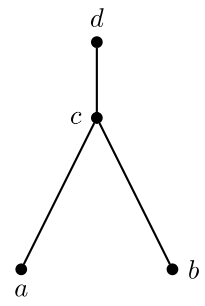
Next, we present some particular elements of an ordered set.

**Definition 1.14.** *Let $(E, \leqslant)$ be a poset and $A$ be a subset of $E$ ($A \subseteq E$). We say that:*

- *an element $x \in E$ is a lower bound of $A$ if $x \leqslant a$, for all $a \in A$. The set of lower bounds of $A$, denoted by $A^L$;*
- *an element $x \in E$ is an upper bound of $A$ if $a \leqslant x$, for all $a \in A$. The set of upper bounds of $A$, denoted by $A^U$;*
- *an element $m \in A$ is the minimum of $A$, if $m \leqslant a$, for all $a \in A$;*
- *an element $M \in A$ is the maximum of $A$, if $a \leqslant M$, for all $a \in A$;*
- *an element $a_1 \in E$ is the least upper bound of $A$ if $a_1$ is the minimum of $A^U$;*
- *an element $a_0 \in E$ is the greatest lower bound of $A$ if $a_0$ is the maximum of $A^L$;*
- *an element $x \in E$ is said to be minimal, if there is not an other element $y \in E$ such that $y < x$;*

- *an element $x \in E$ is said to be maximal, if there is not an other element $y \in E$ such that $x < y$.*

**Example 1.13.** *Let $(E = \{a, b, c, d\}, \leqslant)$ be a finite poset represented by its following Hasse diagram.*



In this poset, we have the following :

- the maximal element of $E$ is $d$;
- the minimal elements of $E$ are $a$ and $b$;
- the greatest element of $E$ is $d$;
- $E$ has not a least element.

### 1.2.3 Morphisms of ordered sets

**Definition 1.15.** *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be posets and $f : P_1 \to P_2$ be a mapping between $P_1$ and $P_2$. The mapping $f$ is called an order-preserving (order-morphism) form $(P_1, \leq_1)$ to $(P_2, \leq_2)$ if, for any $x, y \in P_1$:*

$$x \leq_1 y \to f(x) \leq_2 f(y)$$

*If $P_1 = P_2$, then the mapping $f$ is called an endomorphism.*

**Definition 1.16.** *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be posets and $f : P_1 \to P_2$ be a mapping between $P_1$ and $P_2$. The mapping $f$ is called an order-isomorphism form $(P_1, \leq_1)$ to $(P_2, \leq_2)$ if, it is surjective, for any $x, y \in P_1$:*

$$x \leq_1 y \leftrightarrow f(x) \leq_2 f(y)$$

*If $P_1 = P_2$, then the order-isomorphism f is called an automorphism.*

**Example 1.14.** *Let $(P_1 = \{1, 2, 3, 6\}, \leq_1)$ and $(P_2 = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}, \leq_2\})$ be two posets such that $\leq_1$ is the divisibility order "$|$" and $\leq_2$ is the inclusion order " $\subseteq$ ". Their Hasse diagrams are showen in the Figure 1.1and in the Figure 1.2. Let $f : P_1 \to P_2$ be a mapping defined by the following table:*

| $x$ | 1 | 2 | 3 | 6 |
|---|---|---|---|---|
| $f(x)$ | $\varnothing$ | $\{a\}$ | $\{b\}$ | $\{a,b\}$ |

*Thus, it is not difficult to see that $f$ is an order-isomorphism.*

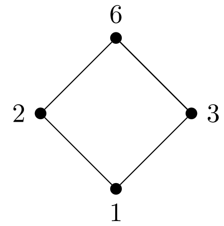



Figure 1.4: Hasse diagram of the psoset $P_1$ .

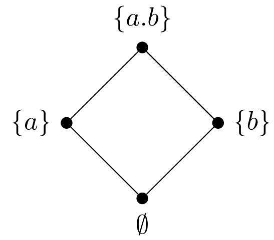



Figure 1.5: Hasse diagram of the psoset $P_2$ .

## 1.3 Bounded lattices

**Definition 1.17.** *A lattice $L$ is said to have a lower bound $0$ if for any element $x$ in $L$ we have $0 \leq x$. Analogously, $L$ is said to have an upper bound $I$ if for any $x$ in $L$ we have $x \leq I$. We say $L$ is bounded if $L$ has both a lower bound $0$ and an upper bound $I$. In such a lattice we have the identities*

$$a \vee I = I, a \wedge I = a, a \vee 0 = a, a \wedge 0 = 0.$$

*for any element $a$ in $L$.*

**Theorem 1.3.** *Every finite lattice $L$ is bounded*

**Example 1.15.**
1. *The lattice $(P(S), \subseteq, \cap, \cup)$ is bounded ($S$ is its greatest element and $\varnothing$ its least element).*
2. *The lattice $(Z, \leq, min, max)$ is not bounded ($Z$ has neither a greatest nor a least element).*

# Chapter 2

# Pseudo-ordered sets and trellises

This chapter introduces pseudo ordered sets and trellises as generalizations of partially ordered sets and lattices obtained by dropping the transitivity property. It presents the main definitions, examples, and fundamental properties related to these structures. Special attention is given to trellises, their algebraic operations, and transitive elements.

## 2.1 Pseudo-ordered sets

This section contains the fundamental definitions and properties of pseudo-ordered sets, along with illustrative examples and several concepts that will be used later. [2, 1, 21, 23]

### 2.1.1 Definitions and basic concepts

**Definition 2.1.** *Let $X$ be a non-empty set, a pseudo order on $X$ is a binary relation $\trianglelefteq$ on $X$ such that, for all $x, y \in X$, the conditions hold :*

1. *$x \trianglelefteq x$ (reflexivity) ;*
2. *$x \trianglelefteq y$ and $y \trianglelefteq x$ implies $x = y$ (antisymmetry).*

*A pair $(X, \trianglelefteq)$ satisfying these properties is called a pseudo-ordered set, or more briefly, a psoset. This notion generalizes partially ordered sets by relaxing the requirement of transitivity.*
*Similarly to the setting of ordered sets, a finite pseudo-ordered sets can be represented by Hasse-type diagram with the convention that: $x$ is below $y$ and is joined to $y$ by a line if $x \trianglelefteq y$. Otherwise, i.e., if $x$ and $y$ are not related, then $x$ and $y$ will be joined by a dashed curve.*

**Example 2.1.** *1. Let $X = \mathbb{R}$ be the set of real numbers. The relation on $X$ defined, for any $x, y \in X$:*

$$x \trianglelefteq y \textit{ if and only if } 0 \leq y - x \leq a \textit{ where } a \in \mathbb{R}^+$$

*is a pseudo order relation on $X$.*

2. *Let $X = \{a, b, c, d, e\}$ and the binary relation defined on $X$ as follows:*
   $\trianglelefteq = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e),(b,c),(b,e),(c,d),(c,e),(d,e)\}$,
   *the couple $(X, \trianglelefteq)$ is a pseudo ordered set. We note that $\trianglelefteq$ is not transitive because $(b,c)$ and $(c,d) \in \trianglelefteq$, while $(b,d) \notin \trianglelefteq$.*
   *In the sequel we will represent finite pseudo ordered sets by Hasse diagram with the convention that if $x$ is below $y$ and is joined to $y$ by a line if $x \trianglelefteq y$. If $x$ is below $y$, and $x$ and $y$ are not related, then $x$ and $y$ will be joined by a dashed curve.*

**Remark 2.1.** *It is easily seen that any ordered set (poset) on a set $A$ is a pseudo-ordered set (psoset) on $A$.*

According to Skala, we have the following definitions :

**Definition 2.2.** *Let $(X, \trianglelefteq)$ be a pseudo-ordered set and $Y \subseteq X$. An element $x \in X$ is called an upper bound of $Y$ if $y \trianglelefteq a$ for any $y \in Y$. If $x$ is an upper bound of $Y$ and satisfies $x \trianglelefteq d$ for every upper bound $d$ of $Y$, then $x$ is called the least upper bound (l.u.b.) or join of $Y$, denoted by $x = \cup Y$. The notions of lower bound and greatest lower bound (g.l.b.) or meet, denoted by $\cap B$, are defined dually.*
*The next definition plays an important role in pseudo-ordered set theory since it is analogous to the definition of a totally-ordered set.*

**Definition 2.3.** *Let $(X, \trianglelefteq)$ be a psoset and $B$ be a subset of $X$. We define a relation $\trianglelefteq_B$ on $B$ as follows : for any $b', y \in B$ there is a finite sequence $(b_1, ..., b_n) \in B$ such that $b \trianglelefteq b_1 \trianglelefteq ... \trianglelefteq b_n \trianglelefteq b'$. $(B, \trianglelefteq_B)$ .*
*is called pseudo chain or (P-chain, for short) if for each pair of elements $b$, $b'$ of $B$ either of the relations $b \trianglelefteq_B b'$ or $b' \trianglelefteq_B b$ holds, ($\trianglelefteq_B$ can be defined dually). A subset $B$ is called a cycle if, for each pair of elements $b$ and $b'$ of $B$ both the relations $b \trianglelefteq_B b'$ and $b' \trianglelefteq_B b$ hold.*

**Example 2.2.** *Let $X = \{0, b, c, d, 1\}$ be a set and a pseudo-order relation on $X$, where $0 \trianglelefteq b \trianglelefteq c \trianglelefteq d \trianglelefteq 1$(i.e.; $0 \trianglelefteq x \trianglelefteq 1$, for any $x \in X$ ). Then $(X, \trianglelefteq)$ is p-chain and the Hasse-type diagram of $X$ is a depicted by the following figure (The hasse-type diagrams are the same):*

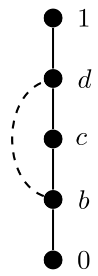


Figure 2.1: Hasse-type diagram of the trellis $(X, \leq)$.

**Remark 2.2.** *Unlike total orders, in a pseudo-ordered set, not every pair of elements is necessarily comparable. This allows for independent elements that do not have any ordering between them.*

**Definition 2.4.** *Let $(X, \trianglelefteq)$ be a psoset. A subset $B \subseteq A$ is called a cycle if every pair of elements $b, b' \in B$ satisfies both $b \subseteq_B b'$ and $b' \subseteq_B b$. A nontrivial cycle is one that contains more than one element. A psoset is said to be acyclic if it does not contain any nontrivial cycle.*

**Remark 2.3.** (i) *Empty set is a cycle;*

(ii) *Any single element set in a psoset is also a cycle;*

(iii) *A nontrivial cycle is a cycle having more than one element and it contains at least three elements.*

**Definition 2.5.** *A preorder relation (preorder, for short) is a binary relation $\lesssim$ over a set $X$ which is reflexive ( $a \lesssim a$ for all $a \in X$), and transitive ($a \lesssim b$ and $b \lesssim c$ implies $a \lesssim c$, for any $a, b, c \in X$) but not necessarily antisymmetric. A set with a preorder relation is called a preorder set.*

*In the following Proposition, we will study properties pseudo ordered sets*

**Proposition 2.1.** *Let $(X, \trianglelefteq)$ be a pseudo-ordered set and $A$ be a subset on $X$. The binary relation $\lesssim$ defined on $A$ as: $x \lesssim y$ if and only if there exists a finite sequence $(a_1, ..., a_n)$ of elements from $X$ such that $x \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq a_n \trianglelefteq y$ is a preorder relation on $A$.*

**Proof**. Let $(X, \trianglelefteq)$ be a pseudo ordered set and $A$ be a non-empty set on $X$.

1. For any $x \in A$, we have $x \trianglelefteq x$ this implies $x \lesssim x$. Hence, $\lesssim$ is reflexive;

2. Let $x, y, z \in A$. Suppose that $x \lesssim y$ and $y \trianglelefteq z$ and we show that $x \trianglelefteq z$.

From the supposition, it follows that there exist two sequences $(a_1, ..., a_n)$ and $(b_1, ..., b_n) \in X$ such that $x \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq a_n \trianglelefteq y$ and $y \trianglelefteq b_1 \trianglelefteq ... \trianglelefteq b_n \trianglelefteq z$, it is easy to see that, there exists another sequence $(a_1, ..., b_n) \in X$ such that $x \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq b_n \trianglelefteq z$. That is, x $\lesssim$ z. Hence, $\lesssim$ is transitive. Therefore, $\lesssim$ is a preorder relation on $A$.

**Remark 2.4.** *If for each pair of elements $x, y$ of $A$ at least one of the relations $x \lesssim y$ or $y \lesssim x$ holds then $A$ will be called a pseudo-chain or (p-chain, for short).*
*A pseudo order on a set $X$ is said to be linear if $X$ itself is a pseudo-chain.*

*It can easily shown that any pseudo order can be extended to a linear which leads the following theorem.*

**Definition 2.6.** *Let $(X, \trianglelefteq)$ be a psoset. We can define the dual psoset of $X$, $(X^d, \trianglelefteq)$ by: $x \trianglelefteq y$ to holds in $X^d$ if and only if $x \trianglelefteq y$ holds in $X$. For $X$ finite, we obtain a diagram for $X^d$ simply by "tuning upside down" a diagram of $X$. Given a statement $\varnothing$ about pseudo ordered sets which is true in all pseudo ordered sets, the dual statement $\varnothing^d$is also true in all pseudo ordered sets. The notion of consistent family and contraction set of a pseudo ordered set $(X, \trianglelefteq)$ was introduced by Skala[27] .*

### 2.1.2 Fundamental properties of pseudo-ordered sets

**Definition 2.7 (Contraction sets over a psoset).** *[27]*
*Let $(X, \trianglelefteq)$ be a pseudo-ordered set and $\mathcal{A}$ be a family of subsets of $X$ (i.e., $\{X_1, X_2, X_3, ...\} \subseteq P(X)$). The family $\mathcal{A}$ is said to be consistent if it satisfies the following condition: For any two distinct sets $R, S \in X$, there do not exist elements $r_1, r_2 \in R$ and $s_1, s_2 \in S$ such that $r_1 \trianglelefteq s_1$ and $s_2 \trianglelefteq r_2$. On such a family $\mathcal{A}$, we define the following pseudo-order $\blacktriangleleft$ as follows:*

$$R \blacktriangleleft S \leftrightarrow \exists r \in R, s \in S \text{ such that } \quad r \trianglelefteq s$$

*is a pseudo-order on $X$. Also, The pseudo-ordered set $(\mathcal{A}, \blacktriangleleft)$ is called the contraction set of $(X, \trianglelefteq)$.*

**Example 2.3.** *On the set of all positive integers set $m|n$ if and only if $m$ is a divisor of $n$. Consider the family of subsets: $\mathcal{A} = \{\{2k-1, 2k\}|k = 1, 2, 3, ..\}$,( i.e., $\{1, 2\}, \{3, 4\}, \{5, 6\}, ...$). Then the family $\mathcal{A}$ is consistent, because for any two different sets in the family, you cannot find elements that satisfy both directions of the relation. For $R = \{10, 11\}$, $S = \{100, 101\}$ and $T = \{201, 202\}$, it holds that:*

- *$R \blacktriangleleft S$ because 10 divides 100,*
- *$S \blacktriangleleft T$ because 101 divides 202,*
- *But $R \ntrianglelefteq T$ because 10 and 11 do not divide 201 or 202 .*

*Hence, the induced pseudo-order is not transitive. Even though $|$ is a partial order, the pseudo order induced is not .*

**Theorem 2.1.** *[27]*
*Any pseudo ordered set is isomorphic with a contraction set of partially ordered set.*

**Theorem 2.2.** *Any pseudo-order $\trianglelefteq$ on a set $A$ can be extended to a linear pseudo-order $\trianglelefteq'$ on $A$ such that any cycle with respect to $\trianglelefteq'$ is also a cycle with respect to $\trianglelefteq$.*

*The following theorem is immediately obtained by using Theorem 2.2*

**Theorem 2.3.** *[27]*
*Any pseudo order is the intersection of all its linear extensions.*

***Proof.*** *Let $X$ be a pseudo-ordered set and $\mathcal{L}$ the set of all its linear extensions. By Theorem 2.2, $\mathcal{L}$ is nonempty. The intersection of $\mathcal{L}$ is clearly an extension of the pseudo-order $\trianglelefteq$ on $X$. In order to prove it equals $\trianglelefteq$ we show that if $a \ntrianglelefteq b$ there exists a linear extension $\trianglelefteq'$ of $\trianglelefteq$ such that $a \ntrianglelefteq b$. Since $a \ntrianglelefteq b$, we can define the pseudo-order $\trianglelefteq^*$ by $b \trianglelefteq^* a$ and for $(a, b) \neq (x, y)$, $x \trianglelefteq^* y$ if and only if $x \trianglelefteq y$. Theorem 2.2 guarantees that there exists a linear extension $\trianglelefteq'$ of $\trianglelefteq^*$. Hence also $b \trianglelefteq' a$ and therefore $a \trianglelefteq' b$.*

**Definition 2.8.** *[27]*
*A psoset is called to have the fixed point property if every order-preserving mapping $f : X \longrightarrow X$ has a fixed point(i.e.,there exists an element $x$ of $X$ such that $f(x) = x$).*

## 2.2 Trellises

The notion of trellis was introduced by Fried [9] and Skala [26][27] as one of the most important algebraic structures in order theory. Which is considered as an extension of the notion of lattice by dropping the property of transitivity. A trellis is defined as a psoset $(X, \trianglelefteq)$ in which pair of elements has a least upper bound and a greatest lower bound. If A is a subset of $X$ and has a greatest lower bound or a least upper bound, then they are unique and will be denoted by $\bigwedge A$ and $\bigvee A$, respectively. If A is a pair $\{a, b\}$, we also write $\bigwedge A = a \wedge b$ and $\bigvee A = a \vee b$.

### 2.2.1 Definitions and properties

**Definition 2.9.** *Let $(X, \trianglelefteq)$ be a non-empty pseudo ordered set.*

1. *The structure $(X, \trianglelefteq)$ is called a $\wedge$-semi-trellis if $x \wedge y$ exists for any $x, y \in X$.*

2. *The structure $(X, \trianglelefteq)$ is called a $\vee$ semi-trellis if $x \vee y$ exists for any $x, y \in X$.*

3. *The structure $(X, \trianglelefteq)$ is called a trellis if it is both a $\wedge$ semi-trellis and a $\vee$ semi-trellis. (i.e., both $x \wedge y$ and $x \vee y$ exist for every pair $x, y \in X$).*

*In other words, a trellis is an algebraic structure $(X, \trianglelefteq, \wedge, \vee)$ where the binary operations $\wedge$ and $\vee$ satisfy the following properties, for any $x, y, z \in X$.*

(i) *Idempotency: $x \wedge x = x \vee x = x$ ;*

(ii) *Commutativity :* $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$ *;*

(iii) *Absorption laws :* $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ *;*

(iv) *Weak associativity :* $x \wedge ((x \vee y) \wedge (x \vee z)) = x = x \vee ((x \wedge y) \wedge (x \wedge z))$ *.*

*For a given trellis* $(X, \trianglelefteq, \wedge, \vee)$ *and* $x, y \in X$*, the following statements are equivalent*

1. $x \trianglelefteq y$ *;*

2. $x \vee y = y$ *;*

3. $x \wedge y = x$*.*

*The notions of sub-trellis, complete trellis, distributive and modular trellis are de- fined analogously to the corresponding notions in partially ordered sets.*

**Example 2.4.** *Let* $X = \{a, b, c, d, e, f\}$ *be a set and let be a pseudo-order relation on* $X$*. Then the structure* $(X, \trianglelefteq)$*, is a trellis. The meet and join operations are defined analogously to those in lattices, as shown in the following table:*

| $\wedge$ \ $\vee$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
| $b$ | $a$ | $b$ | $a$ | $d$ | $e$ | $f$ |
| $c$ | $a$ | $a$ | $c$ | $d$ | $e$ | $f$ |
| $d$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
| $e$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
| $f$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |

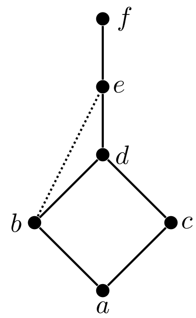



Figure 2.2: Hasse type diagram of the trellis $(X, \leq)$.

**Theorem 2.4.** *[26, 27] A set* $X$ *with two commutative, absorptive, and weak associative binary operations* $\wedge$ *and* $\vee$ *is a trellis if* $a \trianglelefteq b$ *is defined as* $a \wedge b = a$ *and / or* $a \vee b = b$*. These operations are also idempotent.*

**Remark 2.5.** *The main distinction between a lattice and a trellis as algebraic structures lies in the associativity of the binary operations. In a trellis, these operations are generally not associative as opposed to a lattice. This lack of associativity is one of the key characteristics that differentiates trellises from lattices.*

**Remark 2.6.** *It is will be known that every finite lattice is complete, but this property is not true in the finite trellis. Indeed, let us consider the trellis* $(T, \trianglelefteq, \wedge, \vee)$ *given in the following table and figure.*

| $\vee$ \ $\wedge$ | $a$ | $b$ | $c$ | $d$ | $d$ |
|---|---|---|---|---|---|
| $a$ | | $b$ | $c$ | $d$ | $a$ |
| $b$ | $a$ | | $e$ | $e$ | $e$ |
| $c$ | $a$ | $a$ | | $b$ | $e$ |
| $d$ | $a$ | $a$ | $a$ | | $e$ |
| $e$ | $e$ | $b$ | $c$ | $d$ | |

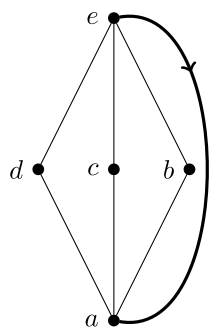



Figure 2.3: Finite trellis not complete.

We can see that $\mathbb{T}$ is not complete, since for example $\vee, b, d$ does not exist.

**Theorem 2.5.** *[27]*
*Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ *be a trellis, then the following statements are equivalent:*

(i) $\trianglelefteq$ *is transitive ;*

(ii) *the operations* $\vee$ *and* $\wedge$ *are associative;*

(iii) *One of the operations* $\wedge$ *or* $\vee$ *is associative.*

(iv) *If* $a \trianglelefteq b$*, then* $a \trianglelefteq b \vee x$ *for each* $x \in X$*.*

(v) *If* $a \trianglelefteq b$*, then* $x \wedge a \trianglelefteq b$ *for each* $x \in X$*.*

**Definition 2.10.** *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis and* $B$ *a proper non-empty subset of* $X$ *(i.e.,* $B \subsetneq A$ *and* $a \neq \varnothing$*). The set* $B$ *is called:*

(i) *a* $\wedge$*-sub-trellis (resp. a* $\vee$*-sub-trellis) of* $X$ *if* $x \wedge y \in B$ *(resp.* $x \vee y \in B$*), for any* $x, y \in B$*, where* $\wedge$ and $\vee$ *are taken in* $X$*;*

(ii) *a sub-trellis of* $X$ *if it is both a* $\wedge$*- and a* $\vee$*-sub-trellis of* $X$*.*

*A bounded trellis is a trellis* $(X, \trianglelefteq, \wedge, \vee)$ *that additionally has a least element denoted by* $0$ *and a greatest element denoted by* $1$ *satisfying* $0 \trianglelefteq x \trianglelefteq 1$*, for any* $x \in X$*. For a bounded trellis the notation* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *is used. Also, a trellis* $(X, \trianglelefteq, \wedge, \vee)$ *is complete if every subset of* $A$ *has a meet and a join.*

**Remark 2.7.** *Bounded sub-trellises play an important role in the study of completeness and compactness conditions in trellises.*

**Remark 2.8.** *It is well-known that every finite lattice is complete. However, this property does not necessarily hold for finite trellises. Indeed, consider the cyclic structure* $A = (\{a, b, c\}, )$*, where the relation is defined by* $a \trianglelefteq b \trianglelefteq c \trianglelefteq a$*. This structure forms a trellis, since every pair of elements admits a least upper bound and a greatest lower bound. Nevertheless, it is not a complete trellis, because the join* $\vee\{a, b, c\}$ *and the meet* $\wedge\{a, b, c\}$ *do not exist in* $A$*. Hence, this example shows that a finite trellis need not be complete.*

**Theorem 2.6.** *[11] A trellis of finite length is complete if and only if every cycle has the meet and the join.*

**Theorem 2.7.** *[27] If every non-empty subset of a pseudo ordered set* $(X, \trianglelefteq)$ *with the smallest element* $0$ *has a least upper bound, then* $\mathbb{T}$ *is a complete trellis.*

**Proposition 2.2.** *[9] Any distributive trellis* $(X, \trianglelefteq, \wedge, \vee)$ *is a lattice .*

**Definition 2.11.** *[27] A trellis* $(X, \trianglelefteq, \wedge, \vee)$ *is called a modular, if* $x \trianglelefteq z$ *implies that* $x \vee (y \wedge z) = (x \vee y) \wedge z$*, for any* $y \in X$*.*

**Remark 2.9.** *In a modular trellis, if* $x \trianglelefteq y$*,* $x \vee z = y \vee z$ *and* $x \wedge z = y \wedge z$ *for some element* $z$*, then* $x = y$*.*
*Though* $x \trianglelefteq y$ *does not necessarily imply* $x \trianglelefteq y \vee z$ *or* $x \wedge z \trianglelefteq y$ *for every element* $z$*, the following special cases do hold.*

**Proposition 2.3.** *[27] Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a modular trellis and* $x, y, z \in X$*. If* $x \trianglelefteq y \trianglelefteq z$*, then* $x \wedge z \trianglelefteq y \trianglelefteq x \vee z$*.*
***Proof.*** *Let* $x, y, z \in X$*. Since* $y \trianglelefteq z$ *and* $y \vee (x \wedge z) = (y \vee x) \wedge z = y \wedge z = y$*, it holds that* $x \wedge z \trianglelefteq y$*. By duality, it follows that* $y \trianglelefteq x \vee z$*.*

**Proposition 2.4.** *[27] Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded modular trellis. The following implications hold for any $x, y, z \in X$:*

*(i) If $x \trianglelefteq y$ and $y \wedge z = 0$, then $x \trianglelefteq y \vee z$;*

*(ii) If $y \trianglelefteq x$ and $y \vee z = 1$, then $y \wedge z \trianglelefteq x$.*

**Definition 2.12.** *[27]*
*A mapping $f$ from a trellis $\mathcal{L} = (X, E)$ into a trellis $(M = X', \trianglelefteq')$ (i.e.,$f : \mathcal{L} \to \mathcal{M}$) is called:*

*(1) a $\wedge$-homomorphism if $f(x \wedge_{\mathcal{L}} y) = f(x) \wedge_{\mathcal{M}} f(y)$, for any $x, y \in X$;*

*(2) a $\vee$-homomorphism if $f(x \vee_{\mathcal{L}} y) = f(x) \vee_{\mathcal{M}} f(y)$, for any $x, y \in X$;*

*(3) a homomorphism if $Q$ is both $\wedge$- and $\vee$-homomorphism;*

*(2) an isomorphism if $Q$ is a one to one homomorphism ;*

*(3) Isomorphism if $f$ is both one to one and onto homomorphism.*

*A homomorphism $f$ of a trellis $X$ into itself is called an endomorphism.*
*Next, we introduce some important special elements of a trellis.*

### 2.2.2 Binary operations on trellises

In this subsection, we present some basic definitions and properties of binary operations on a psoset or trellis. Many of these notions are adapted from their counterparts in the theory of posets and lattices (see, e.g., [24]).
Let $P = (X, \trianglelefteq)$ be a psoset. A binary operation $\mathbf{F} : X \times X \to X$ on $\mathbb{P}$ is called:

(i) associative, if $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in X$;

(ii) conjunctive (resp. disjunctive), if $F(x, y)x \wedge y(resp.x \vee yF(x, y))$, for any $x, y \in X$;

(iii) right-increasing, if $x \trianglelefteq y$ implies $F(z, x) \trianglelefteq F(z, y)$, for any $x, y, z \in X$;

(iv) left-increasing, if $x \trianglelefteq y$ implies $F(x, z) \trianglelefteq F(y, z)$, for any $x, y, z \in X$;

(v) increasing, if $x \trianglelefteq y$ and $z \trianglelefteq t$ implies $F(x, z) \trianglelefteq F(y, t)$, for any $x, y, z, t \in X$.

An element $e \in X$ is called a neutral element of $F$, if $F(e, x) = F(x, e) = x$, for any $x \in X$.

**Example 2.5.** *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. Then the meet $(\wedge)$ (resp. the join $(\vee)$) is con junctive (resp. disjunctive) binary operation.*

**Remark 2.10.** *One easily verifies that any increasing binary operation on a trellis is right and left-increasing. The converse is true, in the presence of the transitivity of $\trianglelefteq$, Indeed, let $(A =$*

$\{0, a, b, c, d, 1\}, \trianglelefteq, \wedge, \vee)$ *be a trellis given by the Hasse-type diagram in Figure 2.4 and $F$ a binary operation on $A$ defined by the following table:*

| $F(x,y)$ | $0$ | $a$ | $b$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ | $c$ | $1$ |
| $a$ | $0$ | $a$ | $b$ | $c$ | $c$ | $1$ |
| $b$ | $0$ | $a$ | $c$ | $d$ | $d$ | $1$ |
| $c$ | $a$ | $a$ | $c$ | $d$ | $d$ | $1$ |
| $d$ | $b$ | $c$ | $c$ | $d$ | $1$ | $1$ |
| $1$ | $c$ | $c$ | $c$ | $1$ | $1$ | $1$ |

*One easily verifies that $F$ is right- and left-increasing, but it is not increasing by the the fact that $a \trianglelefteq b, b \trianglelefteq c$ and $F(a, b) = b \ntrianglelefteq d = F(b, c)$. In the following example, we show that on a given*

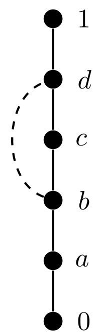



Figure 2.4: Hasse-type diagram of the trellis $(A, \leq)$.

*trellis $(X, \trianglelefteq, \wedge, \vee)$, the binary operations $\wedge$ and $\vee$ do not necessarily be right- and left-increasing.*

**Example 2.6.** *Let $(A = 0, a, b, c, d, 1, \trianglelefteq, \wedge, \vee)$ be a trellis given by the above Hasse-type diagram in Figure 2.4. Since $c \trianglelefteq d$ and $b \wedge c = c \wedge b = b \trianglelefteq a = b \wedge d = d \wedge b$ , it holds that the meet $(\wedge)$ is not right- and not left-increasing. Similarly for the join binary operation $(\vee)$.*

**Proposition 2.5.** *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. Then it holds that the binary operation $\wedge$ (or $\vee$) is increasing if and only if is transitive.*

***Proof.*** *We only give the proof for $(\wedge)$ as for $(\vee)$ is similarly. Suppose that $\wedge$ is increasing and is not transitive (i.e., $x \trianglelefteq y \trianglelefteq z$ and $x \ntrianglelefteq z$, for some $x, y, z \in x$). Since $\wedge$ is increasing and $y \trianglelefteq z$, it follows that $x = x \wedge y \trianglelefteq x \wedge z$. Hence, $x = x \wedge z$. Thus, $x \trianglelefteq z$, a contradiction. Therefore, is transitive. The proof of the converse implication is immediate.* ■

*Combining Theorem 2.5 and Proposition 2.5 leads to the following corollary.*

**Corollary 2.1.** *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. The following statements are equivalent:*

(i) $\wedge$ *(resp.* $\vee$*) is increasing;*

(ii) $\trianglelefteq$ *is transitive;*

(iii) $\wedge$ *(resp.* $\vee$*) is associative.*

### 2.2.3 Specific subsets in trellises

In this subsection, we present some specific elements on a trellis that will be needed to study t-norms and t-conorms on a bounded trellis.

**Definition 2.13.** *[27]*
*Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis. An element $\alpha \in X$ is called:*

(i) *right-transitive, if $\alpha \trianglelefteq x \trianglelefteq y$ implies $\alpha \trianglelefteq y$, for any $x, y \in X$ ;*

(ii) *left-transitive, if $x \trianglelefteq y \trianglelefteq \alpha$ implies $x \trianglelefteq \alpha$, for any $x, y \in X$ ;*

(iii) *middle-transitive, if $x \trianglelefteq \alpha \trianglelefteq y$ implies $x \trianglelefteq y$, for any $x, y \in X$ ;*

(iv) *transitive, if $\alpha$ is right-, left and middle- transitive.*

**Example 2.7.** *In the trellis $\mathbb{T}$ given in Figure 2.1, note that $b \trianglelefteq c \trianglelefteq d$ and $b \ntrianglelefteq d$ then $b$ isn't right transitive. Otherwise, since if $x \trianglelefteq y \trianglelefteq e$ implies $x \trianglelefteq e$, for any $x, y \in T$, then it holds that $e$ is a left-transitive element.*

**Definition 2.14.** *[27]*
*Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis and $(x, y, z)$ a sequence of elements, then :*

(i) *$(x, y, z) \in X$ is called $\wedge$-associative, if $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;*

(ii) *$(x, y, z) \in X$ is $\vee$-associative, if $(x \vee y) \vee z = x \vee (y \vee z)$;*

(iii) *An element $\alpha \in X$ is called $\wedge$-associative (resp. $\vee$-associative), if any sequence of three elements of $X$ including $\alpha$ is $\wedge$-associative (resp. $\vee$-associative);*

(iv) *$\alpha$ is called associative, if it is both $\wedge$- and $\vee$-associative.*
*Notice that for the notion of associative element $\alpha \in X$, the commutativity of the meet and the join operations is sufficient to consider only the sequence $(\alpha, x, y) \in X$.*

**Proposition 2.6.** *[27] Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. Then it holds that any $\wedge$-associative (resp. $\vee$-associative) element is transitive.*

**Remark 2.11.** *The converse of the Theorem 2.4 does not hold in general. Indeed, let $(X, \trianglelefteq, \wedge, \vee)$ the trellis given by the Hasse-type diagram in Figure 2.2. Note that $c$ is transitive but not $\vee$ associative since $(c \vee b) \vee e = e$ but $c \vee (b \vee e) = f$.*
*The following results show two cases that the notion of associative elements and transitive elements are equivalent.*

**Proposition 2.7.** *[27] Let $(X, \trianglelefteq, \wedge, \vee)$ be a pseudo-chain. Then it holds that an element is associative if and only if it is transitive.*

**Proposition 2.8.** *[27] Let $(X, \trianglelefteq, \wedge, \vee)$ be a modular trellis. Then it holds that an element is associative if and only if it is transitive.*

**Theorem 2.8.** *[27] Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. It holds that*

(i) *if $x_1, \ldots, x_k$ are right-transitive, then the join of $\{x_1, \ldots, x_k\}$ exists and equals $x_{i_1} \vee \ldots \vee x_{i_k}$ for any permutation $i_1, \ldots, i_k$ of $1, ..., k$. Moreover,$\vee$ $\{x_1, \ldots, x_k\}$ is right-transitive. Similarly for the left-transitive elements $x_1, \ldots, x_k$ and the meet of $\{x_1, ..., x_k\}$.*

(ii) *if $x_1, \ldots, x_k$ are $\wedge$-associative, then the meet of $\{x_1, \ldots, x_k\}$ exists and equals $x_{i_1} \wedge \ldots \wedge x_{i_k}$ for any permutation $i_1, \ldots, i_k$ of $1, \ldots, k$. Moreover,$\wedge$ $\{x_1, \ldots, x_k\}$ is $\wedge$-associative. Similarly for the $\vee$-associative elements $x_1, \ldots, x_k$ and the join of $\{x_1, \ldots, x_k\}$.*

In the following proposition, a study left, right-transitive.

**Proposition 2.9.** *[27]*
*Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis and $x, a \in X$. It hold that:*

(i) *If $a$ is right-transitive , then*

   *(a)* $a \trianglelefteq x_1 \trianglelefteq ... \trianglelefteq x_k$, *implies* $a \trianglelefteq x_k$ ;

   *(b)* $a \trianglelefteq x$, *implies that* $a \wedge y \trianglelefteq x \wedge y$, *for any* $y \in X$.

(ii) *If $a$ is left-transitive ,then*

   *(a)* $x_1 \trianglelefteq ... \trianglelefteq x_k \trianglelefteq a$, *implies* $x_1 \trianglelefteq a$ ;

   *(b)* $x \trianglelefteq a$, *implies that* $x \vee y \trianglelefteq a \vee y$, *for any* $y \in X$.

(iii) *if $a$ is $\wedge$-associative, then $x \trianglelefteq y$ implies that $a \wedge x \;\; a \wedge y$, for any $x, y \in X$.*

(iv) *if $a$ is $\vee$-associative, then $x \trianglelefteq y$ implies that $a \vee x \;\; a \vee y$, for any $x, y \in X$.*

**Remark 2.12.** *The converse of Proposition 2.6 does not hold in general. Indeed, let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ the trellis given by the Hasse diagram in Figure 2.5. Note that $d$ is transitive but not $\vee$-associative since $(d \vee a) \vee c = c$ but $d \vee (a \vee c) = u$.*

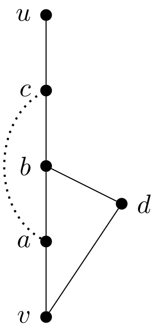



Figure 2.5: Representaition of a trellis. .

# Chapter 3

# Aggregation operators on bounded trellises

In this chapter, we study specific algebraic operations on bounded lattices and trellises .

## 3.1 Aggregation operators on bounded lattices

This section contains the basic definitions and properties of Aggregation operators and some illustrative examples on bounded lattices .

### 3.1.1 Preliminary concepts

**Definition 3.1.** *Let* $\mathbb{L} = (X, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice, and* $n \in X$ *be fixed. A mapping* $A : X^n \to X$ *is called an ( n-ary) aggregation function on* $X$ *whenever it is increasing,* $A(x) \leq A(y)$ *whenever* $x \trianglelefteq y$ *(*i.e., $x_1 \leq y_1, ..., x_n \leq y_n$*) and it satisfies boundary conditions:*

$$A(0, ..., 0) = 0, A(1, ..., 1) = 1.$$

**Example 3.1.** *Let* $\mathbb{L} = (X, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and let* $\mathrm{Q} : X \to X$ *be an unary operator defined for any* $b \in X$ *as follows:*

$$\mathrm{Q}_b(X) = \begin{cases} 1 & x \leqslant b, x \neq 1; \\ 0 & otherwise. \end{cases}$$

*Clearly,* $\mathrm{Q}_b$ *is an unary aggregation operator, for any* $b \in X$*. Moreover, it represents a characteristic function of the principal Ideal* $G(b) = \{x \in X : x \leq b\}$ *generated by* $b$*, provided* $b \neq 1$*.*

*Using the same notion of an n-ary aggregation operator for the case* $n = 2$*, we obtain the following definition.*

**Definition 3.2.** *[20] Let $\mathbb{L} = (X, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice. A binary operation $A : X \times X \to X$ is called an aggregation operator on $X$ if it is increasing and satisfies the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$.*

**Properties 3.1.** *[12]*

- *An aggregation operator $A$ is called idempotent if $A(x, ..., x) = x$ for all $x \in X$.*
- *An aggregation operator $A$ is called 0-positive (resp. 1-positive) if $A(x_1, ..., x_n) = 0$ (resp. $A(x_1, ..., x_n) = 1$) if and only if $x_i = 0$(resp.$x_i = 1$) for all $i \in \{1, 2, ..., n\}$.*
- *The greatest aggregation operator on $X$ is $A$ which takes value 0 if $x = [0]^n$ and 1 otherwise, while the smallest aggregation operator on $X$ is $A_\perp$ which takes value 1 if $x = [1]^n$ and 0 otherwise.*
- *The greatest idempotent aggregation operator on $X$ is $A_\vee$ defined by $A_\vee(X) = \vee_{i=1}^n x_i$ for all $x \in X^n$, and the smallest idempotent aggregation operator on $X$ is $A_\wedge$ defined by $A_\wedge(x) = \wedge_{i=1}^n x_i$ for all $x \in X^n$.*
- *$A_\wedge$ is 1-positive and $A_\vee$ is 0-positive. We call $A_\wedge$ is the $\wedge$-aggregation operator on $X$ and $A_\vee$ is the $\vee$-aggregation operator on $X$.*

*Let $\mathcal{A}(X)$ denote the set of all aggregation operators on $X$, and equip $\mathcal{A}(X)$ with the following order: for all $A, B \in \mathcal{A}(X)$,*

$$A \leq B \text{ whenever } A(x,y) \leq B(x,y), \text{ for any } x, y \in X.$$

**Proposition 3.1.** *Let $\mathbb{L} = (X, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. Then the smallest and the greatest aggregation operators in $\mathbf{A}(X)$ are, respectively, defined by:*

$$A_\perp(X) = \begin{cases} 1 & ,\text{if } x = y = 1; \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad A_\top(X) = \begin{cases} 0 & x = y = 0, ; \\ 1 & \text{otherwise.} \end{cases}$$

*Thus, we have $A_\perp \leqslant A \leqslant A_\top$, for any aggregation operator $A \in \mathcal{A}(X)$.*

### 3.1.2 Famous aggregations operators on bounded lattices

**Definition 3.3 (t-norm).** *Let $\mathbb{L} = (X, \leqslant, \wedge, \vee, 1)$ be a lattice with a greatest element denoted by 1. A binary operation $T : X \times X \to X$ is called a triangular norm (t-norm, for short) if, for all $x, y, z \in X$, the following conditions hold:*

(i) *T(x,y) = T(y,x); (Commutativity)*

(ii) *T(x, T(y,z)) = T(T(x,y),z); (Associativity)*

(iii) *$T(x, y) \leq T(x, z)$, whenever $y \leq z$ ; (Monotonicity)*

(iv) $T(x,1) = x$. *(Neutral element)*

**Definition 3.4 (t-conorm).** *Let* $\mathbb{T} = (X, \leq, \wedge, \vee, 0)$ *be a trellis with least element* 0. *A binary operation* $S : X \times X \to X$ *: is called a triangular conorm (t-conorm, for short) if, for all* $x, y, z \in X$, *it is commutative, associative, monotone, and has 0 as a neutral element,* i.e., $S(x, 0) = x$, *for all* $x \in X$.

**Definition 3.5 (Uninorms).** *Let* $\mathbb{T} = (X, \leq, \wedge, \vee, 0, 1)$ *be a bounded trellis and* $e \in X$. *A binary operation* $U : X \times X \to X$ *is called a uninorm if it is commutative, associative, monotone and has* $e$ *as a neutral element,* i.e., $U(x, e) = x$, *for all* $x \in X$.

If $e = 1$, then $U$ is a t-norm. If $e = 0$, then $U$ is a t-conorm.

**Example 3.2.** *Let* $(L, \leqslant, 0, 1)$ *be a bounded lattice. Then:*

(ii) *The meet operation* $\wedge$ *(resp, the join* $\vee$*) is a t-norm (resp, a t-conorm) on* $X$.

(ii) *The operations* $T_D$ *and* $S_D$ *on* $X$, *defined by:*

$$T_D(x,y) = \begin{cases} x \wedge y & \textit{if} \quad x = 1 \quad \textit{or} \quad y = 1; \\ 0 & \textit{otherwise.} \end{cases}$$

*and*

$$S_D(x,y) = \begin{cases} x \vee y & \textit{if} \quad x = 0 \quad \textit{or} \quad y = 0; \\ 1 & \textit{otherwise.} \end{cases}$$

are a t-norm and a t-conorm, respectively.

**Theorem 3.1.** *For any t-norm* $T$ *on a bounded lattice* $X$, *we have*

$$T_D \leqslant T \leqslant \wedge,$$

*where* $T_D$ *is the drastic t-norm, and* $\wedge$ *is the greatest t-norm on* $X$.

**Theorem 3.2.** *For any t-conorm* $S$ *on a bounded lattice* $X$, *we have*

$$\vee \leqslant S \leqslant S_D,$$

*where* $S_D$ *is the drastic t-conorm, and* $\vee$ *is the smallest t-conorm on* $X$.

**Theorem 3.3.** *Let* $\mathbb{L} = (X, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* $T$ *a t-norm on* $X$. *Then* $T = \wedge$ *if and only if* $T(x, x) = x$, *for any* $x \in X$.

### 3.1.3 Construction of t-norms and t-conorms on bounded lattices

Consider a bounded lattice $(X, \leqslant, 0, 1)$ , an element $a \in X \setminus \{0, 1\}$, a t-norm $V : X \times X \to X$ and a t-conorm $W : X \times X \to X$. An ordinal sum extension $T$ of $V$ to $X$ and $S$ of $W$ to $X$ is given by:

$$T(x,y) = \begin{cases} V(x,y) & , \text{ if } (x,y) \in X, \\ x \wedge y & , \text{ otherwise.} \end{cases}$$

and

$$S(x,y)=\begin{cases} W(x,y) & \text{, if } (x,y)\in X,\\ x\vee y & \text{, otherwise.}\end{cases}$$

However, the above-defined mapping $T$ need not be a t-norm, in general. Similarly, $S$ need not be a t-conorm, in general.

**Example 3.3.** *Given the lattice $L=\{0,t,s,a,k,1\}$ with order given in Figure 1 and consider the t-norm,$V:[a,1]^2\to[a,1]$:*

$$V(x,y)=\begin{cases} x\wedge y & ,1\in\{x,y\},\\ a & \text{, } otherwise.\end{cases}$$

*for all $x,y\in[a,1]$. Then the operation $T$ is constructed as Table 1 by using the formula (1), but $T$ is not a t-norm on $L$.*

| $T$ | 0 | $t$ | $s$ | $a$ | $k$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $t$ | 0 | $t$ | 0 | $t$ | $t$ | $t$ |
| $s$ | 0 | 0 | $s$ | 0 | $s$ | $s$ |
| $a$ | 0 | $t$ | 0 | $a$ | $a$ | $a$ |
| $k$ | 0 | $t$ | $s$ | $a$ | $a$ | $k$ |
| 1 | 0 | $t$ | $s$ | $a$ | $k$ | 1 |

If we take elements $s,k\in L$, then $s\leqslant k$. But we have that $T(s,k)=s||a=T(k,k)$. Hence, the operation $T$ does not satisfy monotonicity. Moreover, $T(T(k,k),s)=T(a,s)=0$ and $T(k,T(k,s))=T(k,s)=s$ for elements $s,k\in L$. Hence, the operation $T$ does not satisfy associativity. So, we obtain that $T$ is not a t-norm on $L$.

**Example 3.4.** *Given the lattice $L=\{0,t,s,a,k,1\}$ with order given in Figure 2 and consider the t-cnorm,$W:[0,a]^2\to[0,a]$:*

$$W(x,y)=\begin{cases} x\vee y & \text{, } 0\in\{x,y\},\\ a & \text{, } otherwise.\end{cases}$$

*for all $x,y\in[0,a]$. Then the operation $S$ is constructed as Table 2 by using the formula (2), but $S$ is not a t-cnorm on $L$.*

| $T$ | $0$ | $t$ | $s$ | $a$ | $k$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $k$ | $a$ | $s$ | $t$ | $1$ |
| $t$ | $t$ | $a$ | $a$ | $a$ | $a$ | $1$ |
| $s$ | $s$ | $a$ | $a$ | $1$ | $t$ | $1$ |
| $a$ | $a$ | $a$ | $1$ | $s$ | $1$ | $1$ |
| $k$ | $k$ | $t$ | $t$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

If we take elements $k, s \in L$, then $k \leqslant s$. But we have that $S(k,k) = a||s = S(s,k)$. Hence, the operation $S$ does not satisfy monotonicity. Moreover, $S(S(k,k),s) = S(a,s) = 1$ and $S(k,S(k,s)) = S(k,s) = s$ for elements $K, s \in L$. Hence, the operation $S$ does not satisfy associativity. So, we obtain that $S$ is not a t-conorm on $L$.

**Theorem 3.4.** *Let $(X, \leqslant, 0, 1)$ be a bounded lattice and $a \in X \setminus 0, 1$. If $V$ is a t-norm on $[a,1]$ and $W$ is a t-conorm on $[0,a]$, then the functions $T : X \times X \to X$ and $S : X \times X \to X$ are, respectively, a t-norm and a t-conorm on $X$, where:*

$$T(x,y) = \begin{cases} V(x,y) & , \textit{if} \quad (x,y) \in [a,1]^2, \\ x \wedge y & , \textit{if} \quad 1 \in \{x,y\}, \\ 0 & , \textit{otherwise}. \end{cases}$$

$$S(x,y) = \begin{cases} W(x,y) & , \textit{if} \quad (x,y) \in [0,a]^2, \\ x \vee y & , \textit{if} \quad 0 \in \{x,y\}, \\ 1 & , \textit{otherwise}. \end{cases}$$

## 3.2 Aggregations operators on bounded trellises

### 3.2.1 Specific subsets of a trellis

In this subection, we introduce our notations for some interesting subsets of a trellis.

**Notation 3.1.** *[29] Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis. We denote by:*

(i) $X^{rtr}$*: the set of right-transitive elements of $X$;*

(ii) $X^{ltr}$*: the set of left-transitive elements of $X$;*

(iii) $X^{tr}$*: the set of transitive elements of $X$;*

(iv) $X^{\wedge-ass}$*: the set of $\wedge$-associative elements of $X$;*

(v) $X^{\vee-ass}$*: the set of $\vee$-associative elements of $X$;*

(vi) $X^{ass}$*: the set of associative elements of* $X$*;*

(vii) $X^{dis}$*: the set of distributive elements of* $X$*.*

*It is easy to see that if* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ *is a bounded trellis, then* $(X^{\alpha}, \trianglelefteq, 0, 1)$ *is a bounded psoset, for any* $\alpha \in \{dis, ass, \wedge - ass, \vee - ass, tr, ltr, rtr\}$*.*

**Proposition 3.2.** *[29] Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ *be a trellis. The following chains of inclusions hold:*

(i) $X^{dis} \subseteq X^{ass} \subseteq X^{\wedge - ass} \subseteq X^{tr} \subseteq X^{rtr}$*;*

(ii) $X^{dis} \subseteq X^{ass} \subseteq X^{\vee - ass} \subseteq X^{tr} \subseteq X^{ltr}$*.*

*Recall that* $X^{rtr}$ *and* $X^{ltr}$ *do not include any non-trivial cycle.*

*The following corollary is an immediate consequence of Notation 3.1 and proposition 2.8.*

**Corollary 3.1.** *[29] Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ *be a trellis. If* $\mathbb{T}$ *is modular or a pseudo-chain, then the following chain of equalities holds:*

$X^{ass} = X^{\wedge - ass} = X^{\vee - ass} = X^{tr}$ *.*

*The following proposition recalls some results about the lattice structure of some of the above subsets.*

**Proposition 3.3.** *[29] Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ *be a trellis. It holds that:*

(i) *The subset* $X^{dis}$ *is a distributive sub-lattice;*

(ii) *If* $\mathbb{T}$ *is modular or a pseudo-chain, then the subsets* $X^{tr} = X^{ass}$ *are sub-lattices.*

*The following proposition provides some results about the trellis structure of some of the above subsets. The proof is an immediate application of Proposition 2.9.*

**Proposition 3.4.** *[29] Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ *be a trellis. It holds that:*

(i) $(X^{ltr}, \trianglelefteq, \wedge)$ *is a* $\wedge$*-sub-trellis of* $\mathbb{T}$*;*

(ii) $(X^{rtr} \trianglelefteq, \vee)$ *is a* $\vee$*-sub-trellis of* $\mathbb{T}$*;*

(iii) $(X^{\wedge - ass}, \trianglelefteq, \wedge)$ *is a* $\wedge$*-sub-trellis of* $\mathbb{T}$*;*

(iv) $(X^{\vee - ass}, \trianglelefteq, \vee)$ *is a* $\vee$*-sub-trellis of* $\mathbb{T}$*.*

**Proposition 3.5.** *[29] Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ *be a trellis and* $A$ *a subset of* $X$*. If* $\mathbb{T}$ *is a pseudo-chain and* $A \subseteq X^{rtr}$ *or* $A \subseteq X^{ltr}$*, then* $A$ *is a sub-trellis of* $\mathbb{T}$*.*

**proof** Let $x, y \in A$. There exists a finite sequence (a1,...,an) of elements in X such that $x \trianglelefteq a_1 ... a_n \trianglelefteq y$ or $y \trianglelefteq a_1 ... a_n \trianglelefteq x$. The fact that $A \subseteq X^{rtr}$ or $A \subseteq X^{ltr}$ implies that $x \trianglelefteq y$ or $y \trianglelefteq x$. Thus, $x \wedge y = x \in A$ or $x \wedge y = y \in A$. In a similar way, we obtain that $x \vee y = y \in A$ or $x \vee y = x \in A$. Thus, $A$ is a sub-trellis of $\mathbb{T}$.

**Proposition 3.6.** *[29] Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis and $A$ a subset of $X$. The following equalities hold:*

(i) *If $A \subseteq X^{rtr}$, then $x \vee (y \vee z) = (x \vee y) \vee z$, for any $x, y, z \in A$;*

(ii) *If $A \subseteq X^{ltr}$, then $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, for any $x, y, z \in A$.*

**Proof.** We only give the proof of $(i)$, the proof of $(ii)$ being dual. Suppose that $A \subseteq X^{rtr}$, and let $x, y, z \in A$. We have $x \trianglelefteq x \vee y (x \vee y) \vee z$, $y \trianglelefteq x \vee y (x \vee y) \vee z$ and $z \trianglelefteq (x \vee y) \vee z$. Since $A \subseteq X^{rtr}$, it follows that $x (x \vee y) \vee z$, $y \trianglelefteq (x \vee y) \vee z$ and $z \trianglelefteq (x \vee y) \vee z$. Moreover, $y \vee z (x \vee y) \vee z$. Hence, $x \vee (y \vee z) (x \vee y) \vee z$. In a similar way, we obtain that $(x \vee y) \vee z \trianglelefteq x \vee (y \vee z)$. Thus, $x \vee (y \vee z) = (x \vee y) \vee z$.

In view of Proposition 3.6, one can deduce the following corollary.

**Corollary 3.2.** *[29] Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis and $A$ a subset of $X^{rtr}$ or $X^{ltr}$.*

(i) *If $A$ is a $\wedge$-sub-trellis of $\mathbb{T}$, then it is a $\wedge$-sub-lattice of $\mathbb{T}$;*

(ii) *If $A$ is a $\vee$-sub-trellis of $\mathbb{T}$, then it is a $\vee$-sub-lattice of $\mathbb{T}$;*

(iii) *If $A$ is a sub-trellis of $\mathbb{T}$, then it is a sub-lattice of $\mathbb{T}$.*

**Corollary 3.3.** *[29] Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis and $\alpha \in \{ass, \wedge - ass, \vee - ass, tr, ltr, rtr\}$. If $X^{\alpha}$ is a sub-trellis of $\mathbb{T}$, then it is a sub-lattice of $\mathbb{T}$.*

*Combining Proposition 3.6 leads to the following result.*

**Proposition 3.7.** *[29] Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis. The following statements are equivalent:*

(i) *$\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ is a lattice;*

(ii) *$X^{rtr} = X$;*

(iii) *$X^{ltr} = X$;*

(iv) *$X^{\wedge - ass} = X$;*

(v) *$X^{\vee - ass} = X$.*

### 3.2.2 Triangular norm and conorm on bounded trellises

**Definition 3.6 (t-norm ).** *Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 1)$ be a trellis with a greatest element denoted by 1. A binary operation $T : X \times X \to X$ is called a triangular norm (t-norm) if, for all $x, y, z \in X$, the following conditions hold:*

(i) *T(x,y) = T(y,x ); (Commutativity)*

(ii) *T(x, T(y,z)) = T(T(x,y),z); (Associativity)*

(iii) $T(x,y) \trianglelefteq T(x,z)$ ; *whenever* $y \trianglelefteq z$ *(Monotonicity)*

(iv) $T(x,1) = x$. *(Neutral element)*

**Remark 3.1.** • *The meet operation* $\wedge$ *on a proper bounded trellis* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ *is not a t-norm on* $T$ *as it is not necessarily increasing or associative.*

• *Any t-norm on a bounded trellis is conjunctive.*

**Proposition 3.8.** *Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ *be a bounded trellis. If there exists an idempotent t-norm* $T$ *on* $T$*,then* $T = \wedge$ *and* $T$ *is a bounded lattice.*

**Definition 3.7 (t-conorm).** *Let* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 0)$ *be a trellis with least element* $0$. *A binary operation* $S : X \times X \to X$ *is called a t-conorm if, for all* $x, y, z \in X$, *it is commutative, associative, monotone (i.e.,*$y \trianglelefteq z \to S(x,y) \trianglelefteq S(x,z)$*), and has 0 as a neutral element, i.e.,* $S(x,0) = x$.

**Remark 3.2.** • *The join operation* $\vee$ *on a proper bounded trellis* $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ *is not a t-conorm on* $T$ *as it is not necessarily increasing or associative.*

• *Any t-conorm on a bounded trellis is disjunctive.*

**Example 3.5.** *Let* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *be a bounded trellis.*

*(1) The binary operation* $T_D$ *on* $X$, *called drastic t-norm, defined by:*

$$T_D(x,y) = \begin{cases} x & , \text{ if } y = 1, \\ y & , \text{ if } x = 1, \\ 0 & , \text{ otherwise.} \end{cases}$$

*(2) The binary operation* $S_D$ *on* $X$, *called drastic t-conorm, defined by:*

$$S_D(x,y) = \begin{cases} x & , \text{ if } y = 0, \\ y & , \text{ if } x = 0, \\ 1 & , \text{ otherwise.} \end{cases}$$

# Conclusion

In this work, we have studied several algebraic structures related to order theory, especially partially ordered sets, pseudo-ordered sets, and trellises.

We have investigated some aggregation operators on bounded lattices and bounded trellises, together with some important classes such as t-norms and t-conorms. The study showed that trellises provide a flexible framework for extending aggregation theory beyond classical lattices, even when transitivity does not hold.

Finally, this work opens the door to further research on aggregation operators and the algebraic properties of generalized ordered structures, particularly in fuzzy logic, decision-making, and aggregation function theory.

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