



PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC
RESEARCH



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Master MEMORY

Field : Mathematics and computer sciences

Branch : Mathematics

Option : Functional analysis

Theme

On some ϕ -summing operators

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University year 2019/2020

Acknowledgments

First of all, Alhamd lillah for helping me to accomplish this work.

I would like to express my sincere gratitude to my supervisor the Doctor BELAALA Maatougui for helping and guiding me during my research and writing.

Besides my supervisor, I would like to thank the president of the jury the Doctor YAHY Rachid and the examiner BELAADA Abdelaziz for devoting time and effort to read and examine my work.

Finally, I would like to thank my dear family who were the real support for me.

DEDICATION

This work is dedicated

*To my beloved parents Alaoua and Halima,
and to my dear sisters and brothers;
Nedjwa, Faris, Nawri, Dalila, Malika,
Naima, Farid and Nawal for encouraging
me. Also, to my best friends Hedda,
Khalissa, Ezzahra and Zahra, for their
endless support.*

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Introduction

The notion of p -summing linear operators introduced by Grothendieck in 1956 and Pietsch in 1968. after that a lot of authors have interested to generalize this notion to multilinear and polynomial, sublinear case.

In 1987 Roshdi Khalil and Wassim Deeb have given in [7] the notion of ϕ -summing operators. A linear operator $T \in \mathcal{L}(E, F)$ is called ϕ -summing if there exists a constant $C \geq 0$ such that for all $(x_i)_{i=1}^n$ in E

$$\sum_{i=1}^n \phi \|T(x_i)\| \leq C \sup_{x^* \in B_{E^*}} \sum_{i=1}^n \phi |\langle x^*, x_i \rangle|. \quad (1)$$

where ϕ is a modulus function. The smallest constant $C \geq 0$ such that the inequality (1) hold is called the metric of T , and denoted by $\pi_\phi(T)$. This class of mappings is denoted by $\Pi_\phi(E, F)$.

In 2017, M. Belaala [8] considered the Lipschitz case to give a new version in this category. He has proven Pietsch domination theorem.

In the first chapter, we will give a general overview on metric spaces, Lipschitz spaces and Lipschitz operator and their properties.

In the second chapter, we based on the paper [7] of R. Khalil and W. Deeb . We will introduce ϕ -summing operators for modulus functions ϕ and study this class of operators and their properties.

In the third Pietsch, we based on the paper [8] of Maatougui Belaala which is titled by “Lipschitz ϕ -umming Operators”. We will introduce the notion of Lipchitz ϕ -summing operator between metric space, we prove non linear Pietsch domination theorem for such operators and it is shown that class is Lipchitz ideal.

In the fourth chapter, we based on the article of Verónica Dimant which is titled by “Strongly p -summing multilinear operators”. We will give a new notion of multilinear ϕ -summing operators.

Notations

- \mathbb{K} The field of real or complex numbers;
- $L(X, Y)$ The set of all linear operators;
- $\mathcal{L}(X, Y)$ The set of all bounded linear operators T from X into Y ;
- $L(X_1, \dots, X_m; Y)$ The space of all m -linear applications;
- $Lip(X, E)$ The space of all bounded Lipschitz maps f from X into E ;
- $Lip_0(X, E)$ The space of all Lipschitz maps f from X into E for which $f(e) = 0$;
- $\mathcal{M}(X)$ The linear space of all molecules on X ;
- $\mathcal{F}(X)$ The Lipschitz-free space;
- $\mathbb{A}(X)$ *Arens-Eells space*;
- $\Pi_p^L(X, Y)$ The set of all Lipschitz P -summing operators from X to Y ;
- $L^\phi(E, F)$ The space of all metrically bounded operator from $l^\phi(E)$ into $l^\phi(F)$;
- $\Pi^\phi(E, F)$ The set of all ϕ -summing operators from E to F ;
- $S(M)$ The unit sphere of M ;
- E^* The topological dual of E ;
- B_{E^*} The closed unit ball of E^* ;
- $B_{E^\#}$ The closed unit ball of $E^\#$;
- $\Pi_\phi^L(X, Y)$ The set of all Lipschitz ϕ -summing operators from E to F ;
- $\Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$ The set of all multilinear ϕ -summing operators;
- $\mathcal{L}_S(mX; Y)$ The set of all symmetric multilinear mappings from mX into Y ;
- $\mathcal{P}(mX; Y)$ The set of all m -homogeneous polynomials from X into Y .

Chapter 1

Preliminaries

This chapter is dedicated to recalling some previous concepts that accompany us during the completion of this work to exploit and use them as a means of clarifying the new notions.

1.1 Distance

Definition 1.1.1 *Let X be a non empty set. We say that d is a distance on X if d is an application from X^2 into \mathbb{R}^+ such that for all x, y, z in X , we have*

1. $d(x, y) = 0 \Leftrightarrow x = y$ (separation),
2. $d(x, y) = d(y, x)$ (symmetry),
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality).

The space X equipped with d is called metric space and denoted by (X, d) .

1.2 Normed space

Definition 1.2.1 *Let X be a vector space on field \mathbb{K} . The mapping $N : X \times X \longrightarrow \mathbb{R}^+$ is called norm on X if:*

- (1) $N(x) \geq 0$ for all $x \in X$,

- (2) $N(x) = 0 \iff x = 0$ for all $x \in X$,
- (3) $N(\lambda x) = |\lambda| N(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$,
- (4) $N(x + y) \leq N(x) + N(y)$ for all $x, y \in X$.

The space X equipped with N is called normed space and denoted by (X, N) .

Remark 1.2.1 If N only satisfies (1) , (2) and (4), N is called a semi-norm on X .

1.3 Bounded linear operator

Definition 1.3.1 Let X and Y be two vector spaces on the same field \mathbb{K} ($\mathbb{K} = \mathbb{R} \vee \mathbb{C}$). A map $T : X \longrightarrow Y$ is a linear operator (usually called a linear transformation in linear algebra) if:

1. For all $x_1, x_2 \in X, T(x_1 + x_2) = T(x_1) + T(x_2)$ (T is additive),
2. For all $x \in X, \lambda \in \mathbb{K}, T(\lambda x) = \lambda T(x)$ (T is homogeneous).

In other word, for all $x_1, x_2 \in X, \alpha, \beta \in \mathbb{K}$

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2).$$

We denoted by $L(X, Y)$ the space of linear operators T from X into Y .

Definition 1.3.2 Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two normed vector spaces (both over \mathbb{R} or over \mathbb{C}). A linear operator $T : X \longrightarrow Y$ is bounded if there is a constant $C \geq 0$ such that

$$\|T(x)\|_Y \leq C \|x\|_X \quad \text{for all } x \in X.$$

We denoted by $\mathcal{L}(X, Y)$ the space of bounded linear operators T from X into Y .

Proposition 1.3.1 *Let X, Y be normed vector spaces and let $T : X \longrightarrow Y$ be a linear operator. The following statements are equivalent.*

- (1) T is a bounded linear operator,
- (2) T is continuous everywhere in X ,
- (3) T is continuous at 0 in X .

Proof. (1) \implies (2) Let C as in the definition of bounded linear operator and for all $x, x' \in X$. By linearity of T we have

$$\|T(x) - T(x')\|_Y = \|T(x - x')\|_Y \leq C \|x - x'\|_X .$$

which implies (2).

(2) \implies (3) is trivial.

(3) \implies (1) If T is continuous at 0 there exists $\delta > 0$ such that for all $x' \in X$ with $\|x'\| < \delta$ we have $\|T(x')\| < 1$. Now let $x \in X$ and $x \neq 0$ ($x' = \frac{\delta}{2} \frac{x}{\|x\|}$). Then

$$\left\| \delta \frac{x}{2 \|x\|_X} \right\|_X = \frac{\delta}{2} \text{ and thus } \left\| T \left(\delta \frac{x}{2 \|x\|_X} \right) \right\|_Y < 1$$

But by the linearity of T and the homogeneity of the norm we get

$$\left\| T \left(\delta \frac{x}{2 \|x\|_X} \right) \right\|_Y = \left\| \delta \frac{T(x)}{2 \|x\|_X} \right\|_Y = \frac{\delta}{2 \|x\|_X} \|T(x)\|_Y$$

and therefore $\|T(x)\|_Y \leq C \|x\|_X$ with $C = \frac{2}{\delta}$. ■

1.4 Lipschitz operator

Definition 1.4.1 *A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ between two metric spaces is Lipschitz of constant $C \geq 0$ (C -Lipschitz) such that*

$$\forall x, y \in X, d_Y(f(x), f(y)) \leq C d_X(x, y) \tag{1.4.1}$$

if $C = 1$, the map is called nonexpansive (and contraction if $C < 1$).

For a Lipschitz map f , we define its Lipschitz constant by

$$\|f\|_{Lip} = Lip(f) = \sup_{x \neq y} \frac{d_y(f(x), f(y))}{d_x(x, y)} = \inf\{C : C \text{ verifying (1.4.1)}\}.$$

Let $(X, e_X, d_X), (Y, e_Y, d_Y)$ be pointed metric spaces. We say a map $f : (X, e_X, d_X) \rightarrow (Y, e_Y, d_Y)$ preserves distinguished point if $f(e_X) = e_Y$.

Remark 1.4.1 *if $C_1 \leq C_2$ and T is C_1 -Lipschitz then T is C_2 -Lipschitz .*

Proposition 1.4.1 *Let f, g be Lipschitz maps from X into \mathbb{R} and $\lambda \in \mathbb{R}$, Then*

(1) $Lip(f + g) \leq Lip(f) + Lip(g)$.

(2) $Lip(\lambda f) = |\lambda| Lip(f)$.

Proof.

(1) Let f, g be Lipschitz maps from X into \mathbb{R} . For all $x, y \in X$ we have

$$\begin{aligned} |(f + g)(x) - (f + g)(y)| &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq (Lip(f) + Lip(g)) d_X(x, y). \end{aligned}$$

So $f + g$ is Lipschitz and

$$Lip(f + g) \leq Lip(f) + Lip(g).$$

(2) Let $\lambda \in \mathbb{R}$ and f is Lipschitz map, For all $x, y \in X$ we have

$$\begin{aligned} Lip(\lambda f) &= \sup_{x \neq y} \frac{|\lambda f(x) - \lambda f(y)|}{d_x(x, y)} \\ &= \sup_{x \neq y} \frac{|\lambda| |f(x) - f(y)|}{d_x(x, y)} \\ &= |\lambda| Lip(f). \end{aligned}$$

So λf is Lipschitz and

$$Lip(\lambda f) = |\lambda| Lip(f).$$

■

Proposition 1.4.2 *Let X, Y and Z be metric spaces and let $f : (X, d_X) \rightarrow (Y, d_Y)$, $g : (Y, d_Y) \rightarrow (Z, d_Z)$ be Lipschitz maps. Then $g \circ f : (X, d_X) \rightarrow (Z, d_Z)$ is Lipschitz and*

$$Lip(g \circ f) \leq Lip(g) Lip(f).$$

Proof. For x, y in X , we have

$$\begin{aligned} d_Z(g \circ f(x), g \circ f(y)) &\leq \text{Lip}(g) d_Y(f(x), f(y)) \\ &\leq \text{Lip}(g) \text{Lip}(f) d_X(x, y) \end{aligned}$$

and this shows the proposition. ■

1.5 Lipschitz spaces

Definition 1.5.1 Let X be a metric space and E Banach space. We denote by $\text{Lip}(X, E)$ the space of all bounded Lipschitz maps f from X into E with norm

$$\|f\| = \max(\text{Lip}(f), \|f\|_\infty).$$

In other word

$$\text{Lip}(X, E) = \{f : X \longrightarrow E \text{ is bounded Lipschitz}\}.$$

Remark 1.5.1 The space $\text{Lip}(X, E)$ is a vector space.

If $E = \mathbb{R}$ then

$$\text{Lip}(X, \mathbb{R}) = \text{Lip}(X).$$

$X^\#$ is called Lipschitz dual and $X^* \subset X^\#$.

Definition 1.5.2 Let (X, e, d) be a pointed metric space and E Banach space. We denote by $\text{Lip}_0(X, E)$ the space of all Lipschitz maps f from X into E for which $f(e) = 0$, with norm

$$\text{Lip}(f) = \sup_{x \neq y} \frac{d_E(f(x), f(y))}{d_X(x, y)}.$$

In other word

$$\text{Lip}_0(X, E) = \{f : X \longrightarrow E \text{ is Lipschitz and } f(e) = 0\}.$$

Remark 1.5.2 The space $\text{Lip}_0(X, E)$ is a Banach space.

If $E = \mathbb{R}$ then

$$\begin{aligned} \text{Lip}_0(X, \mathbb{R}) &= \text{Lip}_0(X) \\ &= X^\#. \end{aligned}$$

1.6 Lipschitz-free space

1.6.1 Adjoint of Lipschitz mapping

Definition 1.6.1 Let X, Y be pointed metric spaces and $T : X \longrightarrow Y$ a Lipschitz map which preserves base point. Defined the Lipschitz adjoint (or dual) of T is defined by

$$\begin{aligned} T^\# : \text{Lip}_0(Y) &\longrightarrow \text{Lip}_0(X) \\ g &\longmapsto T^\#(g) = g \circ T. \end{aligned}$$

Proposition 1.6.1 Let X, Y be pointed metric spaces and $T : X \longrightarrow Y$ a Lipschitz map which preserves base point. Then $T^\#$ is a bounded linear map (continuous) and

$$\|T^\#\| = \text{Lip}(T).$$

1.6.2 Arens Eells space

Definition 1.6.2 Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies

$$\sum_{x \in X} m(x) = 0.$$

We denote by $\mathcal{M}(X)$ the linear space of all molecules on X .

For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A . For $m \in \mathcal{M}(X)$ we can write

$$m = \sum_{j=1}^n \lambda_j m_{x_j x'_j}$$

for some suitable scalars λ_j , and we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^n |\lambda_j| d(x_j, x'_j), m = \sum_{j=1}^n \lambda_j m_{x_j x'_j} \right\},$$

where the infimum is taken over all representations of the molecule m . Denote by $\mathbb{A}(X)$ the completion of the normed space $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [3] in 1956. The terminology of *Arens-Eells space* $\mathbb{A}(X)$ is due to Weaver

[14]. A different notations were used in [6] by Godefroy and Kalton . It is the Lipschitz-free space denoted by $\mathcal{F}(X)$.

Proposition 1.6.2 *Let (X, e, d) be a pointed metric space.the space $(\mathcal{A}(X))^*$ is isometrically isomorphic to $Lip_0(X)$.*

Proof. Define

$$\begin{aligned} S : (\mathcal{A}(X))^* &\longrightarrow Lip_0(X) \\ \phi &\longmapsto S(\phi) \end{aligned}$$

by

$$(S\phi)(x) = \phi(m_{xe}).$$

Since $\|\chi_{\{x\}} - \chi_{\{x'\}}\|_{\mathcal{A}(X)} = d(x, x')$ for all $x, x' \in X$, we have

$$\begin{aligned} |(S\phi)(x) - (S\phi)(x')| &= |\phi(m_{xe}) - \phi(m_{x'e})| \\ &= |\phi(m_{xx'})| \\ &\leq \|\phi\| d(x, x'). \end{aligned}$$

Then

$$Lip(S\phi) \leq S\phi.$$

Also $(S\phi)(e) = \phi(e)$ so indeed $S\phi \in Lip_0(X)$. It follows that S is a nonexpansive linear mapping from $\mathcal{A}(X)$ to $Lip_0(X)$.

Define now

$$\begin{aligned} R : Lip_0(X) &\longrightarrow (\mathcal{A}(X))^* \\ f &\longmapsto R(f). \end{aligned}$$

By

$$R(f)(m) = \sum_{x \in X} m(x)f(x).$$

For $f \in Lip_0(X)$ and m a molecule .If $m = \sum_{j=1}^n \lambda_j m_{x_j x'_j}$, we have

$$\begin{aligned} |R(f)(m)| &= \left| \sum_{x \in X} m(x) f(x) \right| \\ &\leq \sum_{j=1}^n |\lambda_j| |f(x_j) - f(x'_j)| \\ &\leq Lip(f) \sum_{j=1}^n |\lambda_j| d(x_j, x'_j). \end{aligned}$$

Hence $|R(f)(m)| \leq Lip(f) \|m\|_{\mathcal{M}(X)}$, which uniquely extends to a continuous linear functional on the completion $\mathcal{A}(X)$ of $\mathcal{M}(X)$, denoted by the same symbol Rf . Thus $Rf \in (\mathcal{A}(X))^*$ and $\|Rf\| \leq Lip(f)$. Straightforward calculations show that R and S are inverses. Indeed, for all $x \in X$

$$\begin{aligned} (SoR)(f)(x) &= S(R(f))(x) \\ &= R(f)(m_{xe}) \\ &= f(x) \end{aligned}$$

and for all $m \in \mathcal{M}(X)$

$$\begin{aligned} (RoS)(\phi)(m) &= R(S(\phi))(m) \\ &= \sum_{x \in X} m(x) S(\phi)(x) \\ &= \sum_{j=1}^n \lambda_j (S(\phi)(x_j) - S(\phi)(x'_j)) \\ &= \sum_{j=1}^n \lambda_j \phi(m_{x_j x'_j}) \\ &= \phi(m). \end{aligned}$$

The operators R, S are nonexpansive and $RoS = SoR = Id$, so S is isometric ($\|x\| = \|(RoS)(x)\| \leq \|R\| \|S(x)\| \leq \|S(x)\|$) and hence $Lip_0(X)$ is isometrically isomorphic to $(\mathcal{A}(X))^*$. ■

Proposition 1.6.3 *The application*

$$\begin{aligned} i_X : X &\longrightarrow \mathcal{A}(X) \\ x &\longmapsto i_X(x) = m_{xe} \end{aligned}$$

is an isometric embedding of X into $\mathcal{A}(X)$.

Proposition 1.6.4 *Let (X, e, d) be a pointed metric space.*

1. For any molecule m we have

$$\|m\|_{\mathcal{E}(X)} = \sup \left\{ |\langle m, f \rangle| = \left| \sum_{x \in X} m(x) f(x) \right| : f \in B_{X^\#} \right\}.$$

2. $\|\cdot\|_{\mathcal{E}(X)}$ is the largest seminorm on $\mathcal{M}(X)$ which satisfies for all x, y in X ,

$$\|\chi_{\{x\}} - \chi_{\{y\}}\|_{\mathcal{E}(X)} = d(x, y).$$

Theorem 1.6.1 *Let (X, e, d) be a pointed metric space. Let E be a Banach space and let $T : X \rightarrow E$ be a Lipschitz map which preserves base point (i.e., $T(e) = 0$). Then there is a unique bounded linear operator $u : \mathcal{E}(X) \rightarrow E$ such that $T = u \circ i_X$ and $\|u\| = \text{Lip}(T)$ ($i_X : X \rightarrow \mathcal{E}(X)$).*

$$\begin{array}{ccc} & \mathcal{E}(X) & \\ & i_X \uparrow & \searrow u \\ X & \xrightarrow{T} & E \end{array}$$

1.6.3 Banach free space

Definition 1.6.3 *The Lipschitz free space on X is*

$$\mathcal{F}(X) = \overline{\{\text{span} \delta_x : x \in X\}}^{\text{Lip}_0(X)^*}.$$

We say that $\gamma \in \mathcal{F}(X)$ is finitely supported if $\gamma \in \{\text{span} \delta_x : x \in X\}$. Then, the support of such a γ (denoted $\text{supp } \gamma$) is the smallest subset F of X which contains e and such that $\gamma \in \{\text{span} \delta_x : x \in F\}$.

Proposition 1.6.5 *For any metric space X , $\mathcal{F}(X)^*$ is isometrically isomorphic to $\text{Lip}_0(X)$.*

Theorem 1.6.2 *Let (X, e, d) be a pointed metric space and let E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz map such that $T(e) = 0$. Then, there is a unique linear map u (noted T_L): $\mathcal{F}(X) \rightarrow E$ with $\|T_L\| = \text{Lip}(T)$ and such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \delta_x \downarrow & \nearrow T_L & \\ \mathcal{F}(X) & & \end{array}$$

Moreover, the linear isometry $\phi : \text{Lip}_0(X) \rightarrow B(\mathcal{F}(X))$ such that $\phi(T) = T_L$ is onto.

1.7 Lipschitz operator ideals

Definition 1.7.1 [1] A Lipschitz operator ideal \mathcal{I}_{Lip} is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components

$$\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip}$$

satisfy:

- (i) $\mathcal{I}_{Lip}(X, E)$ is a linear subspace of $Lip_0(X, E)$,
- (ii) $v \circ g \in \mathcal{I}_{Lip}(X, E)$ for $v \in E$ and $g \in X^\#$,
- (iii) The ideal property: if $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $W \in \mathcal{L}(E, F)$, then the composition $W \circ T \circ S$ is in $\mathcal{I}_{Lip}(Y, F)$.

A Lipschitz operator ideal \mathcal{I}_{Lip} is a normed (Banach) Lipschitz operator ideal if there is $\|\cdot\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \rightarrow [0, +\infty[$ that satisfies:

- (i') For every pointed metric space X and every Banach space E , the pair $(\mathcal{I}_{Lip}(X, E), \|\cdot\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X, E)$,
- (ii') $\|Id_{\mathbb{K}} : \mathbb{K} \rightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$,
- (iii') If $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $W \in \mathcal{L}(E, F)$, then

$$\|W \circ T \circ S\|_{\mathcal{I}_{Lip}} \leq Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|W\|.$$

Chapter 2

Linear ϕ -summing operators

In 1987 R. Khalil and W. Deeb introduced the notion of linear ϕ -summing operators. They used in their definition the modulus function. In this chapter, we will study this class of operators and their properties.

2.1 Modulus function

Definition 2.1.1 A function $\phi: [0, \infty[\rightarrow [0, \infty[$ is called a modulus function if

1. ϕ is a continuous function on $[0, \infty[$,
2. $\phi(0) = 0$,
3. $\phi(x + y) \leq \phi(x) + \phi(y), \forall x, y \in [0, \infty[$ (subadditive),
4. ϕ is strictly increasing.

Example 2.1.1 these functions $\phi(x) = x^p$, and $\phi(x) = \ln(1 + x)$ for $x \geq 0$ when $0 < p \leq 1$ are modulus functions.

Definition 2.1.2 [7] Let E and F be two Banach spaces and ϕ be a modulus function on $[0, \infty)$. Consider the following two spaces:

(i) $l^\phi \langle E \rangle = \{ (x_n) : \sup_{\|x^*\| \leq 1} \sum_n \phi |\langle x_n, x^* \rangle| < \infty, x_n \in E \}$,

(ii) $l^\phi(F) = \{(x_n) : \sum_n \phi \|x_n\| < \infty, x_n \in F\}$.

For $x = (x_n) \in l^\phi(E)$, we define

$$\|x\|_\varepsilon = \sup_{\|x^*\| \leq 1} \sum_n \phi |\langle x_n, x^* \rangle|,$$

and for $y = (y_n) \in l^\phi(F)$, we define

$$\|y\|_\pi = \sum_n \phi \|y_n\|.$$

Theorem 2.1.1 [7, Theorem 1.1] *The spaces $(l^\phi(E), \|\cdot\|_\varepsilon)$ and $(l^\phi(F), \|\cdot\|_\pi)$ are complete metric linear spaces.*

Remark 2.1.1 *The spaces $l^\phi(E)$ and $l^\phi(F)$ are generalizations of the spaces $l^p(E)$ and $l^p(F)$ for $0 < p < 1$.*

Definition 2.1.3 *A linear operator $T : l^\phi(E) \rightarrow l^\phi(F)$ will be called metrically bounded if for all $x = (x_n) \in l^\phi(E)$ there is a $\lambda > 0$ such that*

$$\|T(x)\|_\varepsilon \leq \lambda \|x\|_\varepsilon.$$

We let $L^\phi(E, F)$ denote the space of all metrically bounded operator from $l^\phi(E)$ into $l^\phi(F)$ with norm

$$\|T\|_\phi = \inf \{ \lambda : \|T(x)\|_\varepsilon \leq \lambda \|x\|_\varepsilon, x \in l^\phi(E) \}.$$

Remark 2.1.2 *for all $x = (x_n) \in l^\phi(E)$. Clearly every metrically bounded operator is continuous.*

2.2 Linear p -summing operators

Definition 2.2.1 *Let E and F be two Banach spaces, a bounded linear operator $T : E \rightarrow F$ is called p -summing, $0 < p < \infty$, if for all sequences $\{x_i\}_{1 \leq i \leq n} \in E$ there exists $\lambda > 0$ such that*

$$\sum_{i=1}^n \|T(x_i)\|^p \leq \lambda \sup_{x^* \in B_{X^*}} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p$$

We denote by $\Pi^p(E, F)$ the space of all p -summing operators from E to F .

2.3 Linear ϕ -summing operators

Definition 2.3.1 For Banach spaces E and F , a bounded linear operator $T: E \rightarrow F$ is called ϕ -summing if for all finite set $(x_i)_{i=1}^n$ in E there exists a constant $C \geq 0$ such that

$$\sum_{i=1}^n \phi(\|T(x_i)\|) \leq C \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|\langle x_i, x^* \rangle|). \quad (2.3.1)$$

In other word

$$\sum_{i=1}^n \phi(\|T(x_i)\|) \leq C \sup_{\xi \in B_{\mathcal{L}(X)}} \sum_{i=1}^n \phi(|\xi(x_i)|).$$

The definition is a generalization of the definition of p -summing operators for $0 < p \leq 1$.

Let $\Pi^\phi(E, F)$ be the set of all ϕ -summing operators from E to F . Every $T \in \Pi^\phi(E, F)$ defines an element $\tilde{T} \in \Pi^\phi(E, F)$ via:

$$\begin{aligned} \tilde{T}: l^\phi(E) &\longrightarrow l^\phi(F) \\ (x_n) &\longmapsto \tilde{T}((x_n)) = (T(x_n)). \end{aligned}$$

For $T \in \Pi^\phi(E, F)$ we define the ϕ -summing metric of T as $\|T\|_\phi = \left\| \tilde{T} \right\|_\phi$: Hence

$$\|T\|_\phi = \inf \{C : C \text{ verifying (2.3.1)}\}.$$

Theorem 2.3.1 $[\gamma](\Pi^\phi(E, F), \|\cdot\|_\phi)$ is a complete metric linear space.

Theorem 2.3.2 $[\gamma]$ Let $T \in \Pi^\phi(E, F)$, $f \in L(G, E)$, and $g \in L(F, H)$ Then $T \circ f \in \Pi^\phi(G, E)$ and $g \circ T \in \Pi^\phi(E, H)$. Further

$$\|T \circ f\|_\phi \leq (\|f\| + 1) \|T\|_\phi$$

and

$$\|g \circ T\|_\phi \leq (\|g\| + 1) \|T\|_\phi.$$

Proof. The proof follows from the fact that for all $a > 0$ and $\phi(at) \leq (1 + a)\phi(t)$ which is a consequence of the monotonicity and subadditivity of ϕ . ■

Proposition 2.3.1 (ideal property)

Let $S \in \mathcal{L}(Y, X)$, $T \in \Pi_\phi(X, E)$, and $w \in \mathcal{L}(E, F)$, then $w \circ T \circ S$ is linear ϕ -summing and $\pi_\phi(w \circ T \circ S) \leq (1 + \|w\|)\pi_\phi(T)(1 + Lip(S))$.

2.4 Pietsch domination theorem

Theorem 2.4.1 *Let $T \in L(E, F)$. The followings are equivalent:*

(i) $T \in \Pi^\phi(E, F)$.

(ii) *There exists $\lambda > 0$ and $\nu \in S(M)$ (the unit sphere of M) such that*

$$\phi \|T(x)\| \leq \lambda \int_{B_{E^*}} \phi |\langle x, x^* \rangle| d\nu(x^*).$$

Proof. (ii) \implies (i) This is evident.

(i) \implies (ii) Let $T \in \Pi^\phi(E, F)$ and $\lambda = \|T\|_\phi$. For every finite sequence $(x_i)_{i=1}^n$ in E , define the map:

$$\begin{aligned} Q : S(M) &\longrightarrow \mathbb{C} \\ \mu &\longmapsto Q(\mu) = \sum_{i=1}^n \phi \|T(x_i)\| - \lambda \sum_{i=1}^n \int_{B_{E^*}} \phi |\langle x_i, x^* \rangle| d\mu. \end{aligned} \quad (2.4.1)$$

Clearly, the function Q is convex. Further, there is a point $\mu_0 \in S(M)$ such that . Indeed choose μ_0 the dirac measure at x_0^* , where

$$\sum_{i=1}^n \phi |\langle x_i, x_0^* \rangle| = \sup_{x^* \in B_{E^*}} \sum_{i=1}^n \phi |\langle x_i, x^* \rangle|.$$

Further, if $\{Q_1, \dots, Q_r\}$ is a collection of such functions defined by (2.4.1), then for any $(a_1, \dots, a_r), \sum_{i=1}^r a_k = 1$, there is Q defined in a similar way, such that $\sum_{i=1}^r a_k Q_k(u)$ for all $\mu \in S(M)$. Hence the collection of functions in $S(M)$ defined by (2.4.1) satisfies Fan's lemma. Consequently there is a measure ν in $S(M)$ such that $Q(\nu) \leq 0$ for all Q defined by (2.4.1). In particular if Q is defined by (2.4.1) with associated sequence $\{x\}, x \in E$, we get

$$\phi \|T(x)\| \leq \lambda \int_{B_{E^*}} \phi |\langle x, x^* \rangle| d\nu.$$

■

Chapter 3

Lipschitz ϕ -summing operators

In this chapter we contribute to the theory of absolutely p -summing Lipschitz mappings by studying the class of Lipschitz ϕ -summing mappings. we have shown that our class of operators is a Lipschitz ideal in the sense of [1] M18. After giving the domination Theorem of this class, we have studied the relationships between a Lipschitz operator T and its linearization \hat{T} .

Our purpose in this chapter is to extend this concept to a more general setting introducing the notion of Lipschitz ϕ -summing operator between pointed metric spaces. Our notion generalizes that of Lipschitz p -summing operator given by M. A. S. Saleh in [2] for $0 < p < 1$. The study of Lipschitz p -summing operators with $p \geq 1$ between metric spaces was initiated by J. D. Farmer and W. B. Johnson in [5]. They stated a nonlinear version for such operators of the celebrated Pietsch factorization-domination theorem of p -summing linear operators [10]. In [4], G. Botelho, D. Pellegrino and P. Rueda proved an abstract version of the Pietsch domination theorem (improved in [11] by D. Pellegrino and J. Santos) which gives a unified proof for the Pietsch domination theorem, for the result of J. D. Farmer and W. B. Johnson and for other known Pietsch domination theorems in classes of mappings that generalize the ideal of p -summing linear operators. We study in this chapter the basic properties of Lipschitz ϕ -summing operators and establish a nonlinear version of Pietsch domination theorem for such operators which is not covered by the general Pietsch domination theorem stated in [11].

3.1 Lipschitz p -summing operators

Definition 3.1.1 A Lipschitz map $T : X \longrightarrow Y$ is called Lipschitz p -summing ($1 \leq p < \infty$), if there is a positive constant C such that for all $\{x_i\}_{1 \leq i \leq n}, \{y_i\}_{1 \leq i \leq n}$ in X and all $\{a_i\}_{1 \leq i \leq n} \subset \mathbb{R}^+$, we have

$$\sum_{i=1}^n a_i d(T(x_i), T(y_i))^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i |f(x_i) - f(y_i)|^p. \quad (3.1.1)$$

We denote the Lipschitz p -summing ($1 \leq p < \infty$) norm by

$$\pi_p^L = \{\inf C : C \text{ verifying (3.1.1)}\}.$$

We denote by $\Pi_p^L(X, Y)$ the set of all Lipschitz p -summing operators from X to Y .

Proposition 3.1.1 (Ideal property). Let X, Y, E, F be metric spaces. Let $v : E \longrightarrow X$, $w : Y \longrightarrow F$ be Lipschitz functions and $T : X \longrightarrow Y$ be a Lipschitz p -summing operator. Then $w \circ T \circ v$ is Lipschitz p -summing operator and

$$\pi_p^L(w \circ T \circ v) \leq \text{Lip}(w) \pi_p^L(T) \text{Lip}(v).$$

Theorem 3.1.1 [5] (Pietsch domination theorem). Let $1 \leq p < \infty$. The following properties are equivalent for a mapping $T \in \text{Lip}_0(X, Y)$ and a positive constant C .

1. $\pi_p^L(T) \leq C$.
2. There is a probability μ on $B_{X^\#}$ such that

$$d(T(x), T(y)) \leq C \left(\int_{B_{X^\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}}.$$

Theorem 3.1.2 If $1 \leq p \leq q < \infty$, then $\Pi_p^L(X, Y) \subset \Pi_q^L(X, Y)$. Moreover, $\pi_q^L(T) \leq \pi_p^L(T)$.

3.2 Lipschitz ϕ -summing operators between metric spaces

Definition 3.2.1 [8] Let ϕ be a modulus function and let X, Y be pointed metric spaces. A Lipschitz map $T \in \text{Lip}(X, Y)$ is called Lipschitz ϕ -summing if there exists a constant $C \geq 0$ such that regardless of the choice of points $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and the choice of numbers $(a_i)_{i=1}^n$ in \mathbb{R}^+ , we have the inequality

$$\sum_{i=1}^n a_i \phi(d(T(x_i), T(x'_i))) \leq C \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i \phi(|f(x_i) - f(x'_i)|). \quad (3.2.1)$$

The infimum of such constants C is denoted by $\pi_\phi^L(T)$, in other word

$$\pi_\phi^L(T) = \{\inf C : C \text{ verifying (3.2.1)}\}.$$

And the set of such maps T by $\Pi_\phi^L(X, Y)$.

Remark 3.2.1 Definition 3.2.1 is the same if we restrict to $a_i = 1$ (see [5]).

Taking the modulus function $\phi(t) = t^p$ ($0 < p \leq 1$) for all $t \geq 0$, Definition 3.2.1 covers the notions of Lipschitz p -summing operators with $p \in (0, 1)$ introduced by M. A. S. Saleh [2], and of Lipschitz 1-summing operators given by J. D. Farmer and W. B. Johnson [5].

If (X, d) is a pointed metric space, we can consider a new metric on X by putting $d_\phi = \phi \circ d$. We then define $\text{Lip}_\phi(X, Y) = \text{Lip}((X, \phi \circ d), (Y, \phi \circ d))$ and

$$\text{Lip}_\phi(T) = \sup \left\{ \frac{\phi(d(T(x), T(y)))}{\phi(d(x, y))} : x, y \in X, x \neq y \right\}$$

for all $T \in \text{Lip}_\phi(X, Y)$. The following assertions can be checked easily.

Proposition 3.2.1 [8] Let ϕ be a modulus function and X, Y are pointed metric spaces.

- (1) $\text{Lip}(X, Y) \subset \text{Lip}_\phi(X, Y)$ and $\text{Lip}_\phi(T) \leq 1 + \text{Lip}(T)$ for all $T \in \text{Lip}(X, Y)$.
- (2) $\Pi_\phi^L(X, Y) \subset \text{Lip}_\phi(X, Y)$ and $\text{Lip}_\phi(T) \leq \pi_\phi^L(T)$ for all $T \in \Pi_\phi^L(X, Y)$.

Proof.

(1) If $T \in \text{Lip}(X, Y)$, then

$$d(T(x), T(y)) \leq \text{Lip}(T)d(x, y),$$

for all $x, y \in X$, and thus

$$\begin{aligned} \phi(d(T(x), T(y))) &\leq \phi(\text{Lip}(T)d(x, y)) \\ &\leq (1 + \text{Lip}(T))\phi(d(x, y)), \end{aligned}$$

so for all $x, y \in X$, and this proves (1).

(2) If $T \in \Pi_\phi^L(X, Y)$, we have

$$\sum_{i=1}^n \phi(d(T(x_i), T(x'_i))) \leq \pi_\phi^L(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|),$$

for all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. In particular ($n = 1$),

$$\begin{aligned} \phi(d(T(x), T(y))) &\leq \pi_\phi^L(T) \sup_{f \in B_{X^\#}} \phi(|f(x) - f(y)|) \\ &= \pi_\phi^L(T) \phi(d(x, y)), \end{aligned}$$

for all $x, y \in X$, and thus we obtain (2). For the last equality above, notice that

$$\phi(d(x, y)) = \phi(|h_y(x) - h_y(y)|) \leq \sup_{f \in B_{X^\#}} \phi(|f(x) - f(y)|) \leq \phi(d(x, y)),$$

where h_Y is the function in $B_{X^\#}$ defined by

$$h_y(z) = d(z, y) - d(0, y),$$

for all $z \in X$ and we have

$$h_y(x) - h_y(y) = d(x, y).$$

■

We now study the relation between a Lipschitz ϕ -summing operator and its linear version.

Proposition 3.2.2 [8] *Let ϕ be a modulus function, X, Y are Banach spaces and $T: X \rightarrow Y$ be a ϕ -summing bounded linear operator. Then T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(T)$.*

Proof. Let $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. We have

$$\begin{aligned} \sum_{i=1}^n \phi(\|T(x_i) - T(x'_i)\|) &= \sum_{i=1}^n \phi(\|T(x_i - x'_i)\|) \\ &\leq \pi_\phi(T) \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|x^*(x_i - x'_i)|) \\ &= \pi_\phi(T) \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|x^*(x_i) - x^*(x'_i)|) \\ &\leq \pi_\phi(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|), \end{aligned}$$

and the proposition follows. ■

We next show that $\Pi_\phi^L(X, Y)$ has the ideal property.

Proposition 3.2.3 *Let ϕ be a modulus function, let X, Y, Z, W be pointed metric spaces, let $B \in \text{Lip}(X, Y)$, $T \in \Pi_\phi^L(Y, Z)$ and $A \in \text{Lip}_\phi(Z, W)$. Then $A \circ T \circ B \in \Pi_\phi^L(X, W)$ and*

$$\pi_\phi^L(A \circ T \circ B) \leq \text{Lip}_\phi(A) \pi_\phi^L(T) (1 + \text{Lip}(B)).$$

Proof. For all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X , we have

$$\begin{aligned} \sum_{i=1}^n \phi(d(A \circ T \circ B(x_i), A \circ T \circ B(x'_i))) &\leq \text{Lip}_\phi(A) \sum_{i=1}^n \phi(d(T \circ B(x_i), T \circ B(x'_i))) \\ &\leq \text{Lip}_\phi(A) \pi_\phi^L(T) \sup_{g \in B_{Y^\#}} \sum_{i=1}^n \phi(|g(B(x_i)) - g(B(x'_i))|), \end{aligned}$$

we use the inequality $\phi(at) \leq (1+a)\phi(t)$, we find

$$\begin{aligned} \phi(|g(B(x_i)) - g(B(x'_i))|) &= \phi(\text{Lip}(B) \left| \frac{g \circ B}{\text{Lip}(B)}(x_i) - \frac{g \circ B}{\text{Lip}(B)}(x'_i) \right|) \\ &\leq (1 + \text{Lip}(B)) \phi\left(\left| \frac{g \circ B}{\text{Lip}(B)}(x_i) - \frac{g \circ B}{\text{Lip}(B)}(x'_i) \right| \right). \end{aligned}$$

Then, the above inequality can be estimated as

$$\sum_{i=1}^n \phi(d(A \circ T \circ B(x_i), A \circ T \circ B(x'_i))) \leq \text{Lip}_\phi(A) \pi_\phi^L(T) (1 + \text{Lip}(B)) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|).$$

■

It is clear that $\Pi_\phi^L(X, Y)$ also enjoys the injectivity property. More specifically, we have.

Proposition 3.2.4 *Let ϕ be a modulus function and let X, Y, Z be pointed metric spaces. If $\iota: Y \rightarrow Z$ is an isometry, then $T \in \Pi_\phi^L(X, Y)$ if and only if $\iota \circ T \in \Pi_\phi^L(X, Z)$. In this case,*

$$\pi_\phi^L(\iota \circ T) = \pi_\phi^L(T).$$

Proof. It suffices to observe that

$$\sum_{i=1}^n \phi(d(\iota \circ T(x_i), \iota T(x'_i))) = \sum_{i=1}^n \phi(d(T(x_i), T(x'_i)))$$

for all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. ■

Proposition 3.2.5 *Let X be a pointed metric space and E be a Banach space. The space $\Pi_\phi^L(X, E)$ is a Lipschitz operator ideal.*

Proof. (i) Let $T \in \Pi_\phi^L(X, E)$, $\alpha \in \mathbb{K}$. Let ϕ be a modulus function. Then given $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \in X$ we obtain

$$\begin{aligned} \sum_{i=1}^n \phi(\|\alpha T(x_i) - \alpha T(x'_i)\|) &= \sum_{i=1}^n \phi(|\alpha| \|T(x_i) - T(x'_i)\|) \\ &\leq (1 + |\alpha|) \sum_{i=1}^n \phi(\|T(x_i) - T(x'_i)\|) \\ &\leq (1 + |\alpha|) C \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi |f(x_i) - f(x'_i)|. \end{aligned}$$

So, αT is Lipschitz ϕ -summing and $\pi_\phi^L(\alpha T) \leq (1 + |\alpha|)\pi_\phi^L(T)$.

Let $T_1, T_2 \in \Pi_\phi^L(X, E)$, and ϕ be a modulus function. Then given $x_1, \dots, x_n \in X; x'_1, \dots, x'_n \in X$ we obtain

$$\begin{aligned}
 & \sum_{i=1}^n \phi(\|(T_1 + T_2)(x_i) - (T_1 + T_2)(x'_i)\|) \\
 = & \sum_{i=1}^n \phi(\|(T_1(x_i) - T_1(x'_i)) + (T_2(x_i) - T_2(x'_i))\|) \\
 \leq & \sum_{i=1}^n \phi(\|T_1(x_i) - T_1(x'_i)\|) + \sum_{i=1}^n \phi(\|T_2(x_i) - T_2(x'_i)\|) \\
 \leq & \pi_\phi^L(T_1) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi|f(x_i) - f(x'_i)| + \pi_\phi^L(T_2) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi|f(x_i) - f(x'_i)|. \\
 \leq & (\pi_\phi^L(T_1) + \pi_\phi^L(T_2)) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi|f(x_i) - f(x'_i)|.
 \end{aligned}$$

So, $T_1 + T_2$ is Lipschitz ϕ -summing and $\pi_\phi^L(T_1 + T_2) \leq \pi_\phi^L(T_1) + \pi_\phi^L(T_2)$.

(ii) Now let $0 \neq g \in X^\#$, and $v \in E$. For all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X we have

$$\begin{aligned}
 \sum_{i=1}^n \phi(\|vg(x_i) - vg(x'_i)\|) &= \sum_{i=1}^n \phi(\|v\| |g(x_i) - g(x'_i)|) \\
 &\leq (1 + \|v\|) Lip(g) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|).
 \end{aligned}$$

Then vg is Lipschitz ϕ -summing and $\pi_\phi^L(vg) \leq (1 + \|v\|) Lip(g)$.

(iii) Now we prove Π_ϕ^L having the ideal property. Let $S \in Lip_0(Y, X)$, $T \in \Pi_\phi^L(X, E)$, and $W \in \mathcal{L}(E, F)$. For all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X we have

$$\begin{aligned}
 \sum_{i=1}^n \phi(\|W \circ T \circ S(x_i) - W \circ T \circ S(x'_i)\|) &\leq (1 + \|W\|) \sum_{i=1}^n \phi(\|W \circ T \circ S(x_i) - W \circ T \circ S(x'_i)\|) \\
 &\leq (1 + \|W\|) \pi_\phi^L(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(\|f \circ S(x_i) - f \circ S(x'_i)\|) \\
 &= (1 + \|W\|) \pi_\phi^L(T) (1 + Lip(S)) \sup_{g \in B_{Y^\#}} \sum_{i=1}^n \phi(|g(x_i) - g(x'_i)|).
 \end{aligned}$$

Which means that $W \circ T \circ S$ is Lipschitz ϕ -summing and

$$\pi_\phi^L(W \circ T \circ S) \leq (1 + \|W\|) \pi_\phi^L(T) (1 + Lip(S)).$$

■

We next state a version of Pietsch domination theorem for Lipschitz ϕ -summing operators, which is inspired by [5, Theorem 1].

3.3 Pietsch domination theorem

Theorem 3.3.1 *Let ϕ be a modulus function, let X, Y be pointed metric spaces and let $T \in \text{Lip}(X, Y)$. Then T is Lipschitz ϕ -summing if and only if there exist a constant C and a regular Borel probability measure μ on $B_{X^\#}$ such that*

$$\phi(d(T(x), T(x'))) \leq C \int_{B_{X^\#}} \phi(|f(x) - f(x')|) d\mu(f) \quad (3.3.1)$$

for all $x, x' \in X$. In such a case, $\pi_\phi^L(T)$ is the minimum of all the constants C for which such a measure exists. in other word

$$\pi_\phi^L(T) = \{\inf C : C \text{ verifying (3.3.1)}\}.$$

Proof. Assume that $T \in \Pi_\phi^L(X, Y)$. For any $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and $(\lambda_i)_{i=1}^n$ in \mathbb{R}^+ , define the function $\varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n} : B_{X^\#} \rightarrow \mathbb{R}$ by

$$\varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n}(f) = \sum_{i=1}^n \lambda_i \phi(d(T(x_i), T(x'_i))) - \pi_\phi^L(T) \sum_{i=1}^n \lambda_i \phi(|f(x_i) - f(x'_i)|).$$

Let A be the set consisting of all such functions.

A is a convex subset of the vector space $C(B_{X^\#})$ indeed let $\varphi_{(x_i, y_i, \alpha_i)_{1 \leq i \leq n}}$ and $\varphi_{(x'_j, y'_j, \beta_j)_{1 \leq j \leq m}} \in A$ such that $\alpha_i > 0, x_i, x'_i \in X$ ($1 \leq i \leq n$) $\beta_j > 0, y_j, y'_j \in X$ ($1 \leq j \leq m$). We have

$$\begin{aligned} \varphi_{(x_i, x'_i, \alpha_i)_{1 \leq i \leq n}}(f) &= \sum_{i=1}^n \alpha_i C \phi(|f(x_i) - f(x'_i)|) - \sum_{i=1}^n \alpha_i \phi(\|T(x_i) - T(x'_i)\|). \\ \varphi_{(y_j, y'_j, \beta_j)_{1 \leq j \leq m}}(f) &= \sum_{j=1}^m \beta_j C \phi(|f(y_j) - f(y'_j)|) - \sum_{j=1}^m \beta_j \phi(\|T(y_j) - T(y'_j)\|). \end{aligned}$$

So,

$$\begin{aligned} &\varphi_{(x_i, x'_i, \alpha_i)_{1 \leq i \leq n}}(f) + \varphi_{(y_j, y'_j, \beta_j)_{1 \leq j \leq m}}(f) = \\ &\sum_{k=1}^{n+m} \gamma_k C \phi(|f(z_k) - f(z'_k)|) - \sum_{k=1}^{n+m} \gamma_k \phi(\|T(z_k) - T(z'_k)\|) \in A, \gamma_k > 0, z_k, z'_k \in X, \end{aligned}$$

which

$$T(z_k) = \begin{cases} T(x_k), & 1 \leq k \leq n \\ T(y_{k-n}), & 1+n \leq k \leq n+m \end{cases},$$

$$T(z'_k) = \begin{cases} T(x'_k), & 1 \leq k \leq n \\ T(y'_{k-n}), & 1+n \leq k \leq n+m \end{cases},$$

$$\gamma_k = \begin{cases} \alpha_k, & 1 \leq k \leq n \\ \beta_{k-n}, & 1+n \leq k \leq n+m \end{cases}.$$

Let $\lambda > 0$, $\lambda\varphi_{(x_i, x'_i, \alpha_i)}_{1 \leq i \leq n}(f) = \sum_{i=1}^n \lambda a_i C\phi(|f(x_i) - f(x'_i)|) - \sum_{i=1}^n \lambda a_i \phi(\|T(x_i) - T(x'_i)\|) \in A$. Consider now the set

$$B = \{\varphi \in C(B_{X\#}) : \varphi(f) > 0, \forall f \in B_{X\#}\}.$$

Clearly, B is a convex open subset of the Banach space $C(B_{X\#})$. Moreover, A and B are disjoint. Indeed, for otherwise, we could find $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and $(\lambda_i)_{i=1}^n$ in \mathbb{R}^+ such that $\varphi_{(\lambda_i, x_i, x'_i)}_{i=1}^n(f) > 0$ for all $f \in B_{X\#}$, that is,

$$\pi_\phi^L(T) \sum_{i=1}^n \lambda_i \phi(|f(x_i) - f(x'_i)|) < \sum_{i=1}^n \lambda_i \phi(d(T(x_i), T(x'_i)))$$

for all $f \in B_{X\#}$. But since $B_{X\#}$ is weak* compact and $f \mapsto \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|)$ is a continuous function from $(B_{X\#}, w^*)$ to \mathbb{R} , we would have

$$\pi_\phi^L(T) \sup_{f \in B_{X\#}} \sum_{i=1}^n \lambda_i \phi(|f(x_i) - f(x'_i)|) < \sum_{i=1}^n \lambda_i \phi(d(T(x_i), T(x'_i))),$$

and this is impossible because T is in $\Pi_\phi^L(X, Y)$.

Then, by the Hahn–Banach separation theorem and the Riesz representation theorem, there exist a regular Borel probability measure μ on $B_{X\#}$ and a scalar $c \in \mathbb{R}$ such that

$$\int_{B_{X\#}} \varphi_{(\lambda_i, x_i, x'_i)}_{i=1}^n d\mu \leq c < \int_{B_{X\#}} \varphi d\mu$$

for all $\varphi_{(\lambda_i, x_i, x'_i)}_{i=1}^n \in A$ and all $\varphi \in B$. Observe that $c = 0$. Indeed, $0 \leq c$ since the zero function $\varphi_{(\lambda_i, x_i, x'_i)}_{i=1}^n$ is in A ; and $c \leq 0$ since all positive constant functions are in B . It follows that

$$\int_{B_{X\#}} \varphi_{(1, x, x')}(f) d\mu(f) \leq 0$$

for all $x, x' \in X$, and since μ is a probability measure, we conclude that

$$\phi(d(T(x), T(x'))) \leq \pi_\phi^L(T) \int_{B_{X\#}} \phi(|f(x) - f(x')|) d\mu(f).$$

for all $x, x' \in X$.

Conversely, assume that the second statement of the theorem holds and let $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. We have

$$\phi(d(T(x_i), T(x'_i))) \leq C \int_{B_{X^\#}} \phi(|f(x_i) - f(x'_i)|) d\mu(f)$$

for $i = 1, \dots, n$. It follows that

$$\begin{aligned} \sum_{i=1}^n \phi(d(T(x_i), T(x'_i))) &\leq C \int_{B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|) d\mu(f) \\ &\leq C \int_{B_{X^\#}} \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|) d\mu(f) \\ &= C \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|). \end{aligned}$$

Hence $T \in \Pi_\phi^L(X, Y)$ and $\pi_\phi^L(T) \leq C$.

Finally, it is clear from the proof that $\pi_\phi^L(T)$ is the minimum value of C . ■

Remark 3.3.1 *Theorem 3.3.1 extends a nonlinear version of Pietsch domination theorem given in [2, Corollary 6.14] for Lipschitz p -summing operators with $0 < p < 1$.*

The Lipschitz-free Banach space $\mathcal{F}(X)$ over a pointed metric space X is the closed linear span in $(X^\#)^*$ of the evaluation functionals $\delta_x: X^\# \rightarrow \mathbb{R}$ with $x \in X$, where $\delta_x(f) = f(x)$ for all $f \in X^\#$. It is known that the Dirac map $\delta_X: X \rightarrow \mathcal{F}(X)$, given by $\delta_X(x) = \delta_x$, is a (nonlinear) isometry, and $\mathcal{F}(X)^*$ is isometrically isomorphic to $X^\#$ via the evaluation map $Q_X: X^\# \rightarrow \mathcal{F}(X)^*$ given by $Q_X(f)(\gamma) = \gamma(f)$ for all $f \in X^\#$ and $\gamma \in \mathcal{F}(X)$.

Let us recall (see [?, Lemma 3.1]) that if X and Y are pointed metric spaces and $T \in \text{Lip}(X, Y)$, then there exists a unique operator $\widehat{T} \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $\widehat{T}\delta_X = \delta_Y T$. Furthermore, $\|\widehat{T}\| = \text{Lip}(T)$.

We can now study the relationship of a Lipschitz ϕ -summing operator T in $\text{Lip}(X, Y)$ and its linearization \widehat{T} in $\mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$.

Proposition 3.3.1 *Let ϕ be a modulus function, let X, Y be pointed metric spaces and let $T \in \text{Lip}(X, Y)$. If \widehat{T} is ϕ -summing, then T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(\widehat{T})$.*

Proof. Notice first that

$$d(T(x), T(y)) = \|\delta_{T(x)} - \delta_{T(y)}\| = \|\widehat{T}(\delta_x) - \widehat{T}(\delta_y)\| = \|\widehat{T}(\delta_x - \delta_y)\|$$

for all $x, y \in X$. If \widehat{T} is ϕ -summing, we have

$$\sum_{i=1}^n \phi(\|\widehat{T}(\gamma_i)\|) \leq \pi_\phi(\widehat{T}) \sup_{F \in B_{\mathcal{F}(X)^*}} \sum_{i=1}^n \phi(|F(\gamma_i)|)$$

for any set $(\gamma_i)_{i=1}^n \subset \mathcal{F}(X)$. Then

$$\begin{aligned} \sum_{i=1}^n \phi(d(T(x_i), T(y_i))) &\leq \pi_\phi(\widehat{T}) \sup_{F \in B_{\mathcal{F}(X)^*}} \sum_{i=1}^n \phi(|F(\delta_{x_i} - \delta_{y_i})|) \\ &= \pi_\phi(\widehat{T}) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|Q_X(f)(\delta_{x_i} - \delta_{y_i})|) \\ &= \pi_\phi(\widehat{T}) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(y_i)|). \end{aligned}$$

Consequently, T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(\widehat{T})$. ■

Chapter 4

Multilinear ϕ -summing operators

This work is based on an article of Verónica Dimant[12] titled Strongly p -summing multilinear operators. 21 September 2001.

4.1 Bounded multilinear operator

Definition 4.1.1 Let $m \in \mathbb{N}$ and $X_j (1 \leq j \leq m)$ be a Banach spaces over $\mathbb{K} (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})$. An operator $T : X_1, \dots, X_m \longrightarrow Y$ is called multilinear (or m -linear) if the mappings

$$\begin{aligned} T_j : X_j &\longrightarrow Y \\ x^j &\longmapsto T(x^1, \dots, x^j, \dots, x^m) \end{aligned}$$

are linear for each $x^k \in X_k, k \neq j$, in other words for all $\alpha, \beta \in \mathbb{K}$ and $x^i, y^j \in X_j (1 \leq j \leq m)$

$$T(x^1, \dots, \alpha x^j + \beta y^j, \dots, x^m) = \alpha T(x^1, \dots, x^j, \dots, x^m) + \beta T(x^1, \dots, y^j, \dots, x^m).$$

We denoted by $L(X_1, \dots, X_m; Y)$ the space of all m -linear applications T from X_1, \dots, X_m into Y .

Remark 4.1.1 In particular, we say that T is bilinear if $m = 2$, and $X_1 = X_2 = X$, trilinear if $m = 3$ and $X_1 = X_2 = X_3 = X$ and multilinear form if $X_1 = X_2 = \dots = X_m = X$ and $Y = \mathbb{K}$.

Example 4.1.1 *An operator*

$$\begin{aligned} T : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto T(x, y) = xy \end{aligned}$$

is bilinear because for all $x, x', y, y', \alpha, \alpha', \beta, \beta' \in \mathbb{R}$

$$1. T(\alpha x + \beta x', y) = (\alpha x + \beta x')y = \alpha xy + \beta x'y = \alpha T(x, y) + \beta T(x', y).$$

$$2. T(x, \alpha' y + \beta' y') = x(\alpha' y + \beta' y') = \alpha' xy + \beta' xy' = \alpha' T(x, y) + \beta' T(x, y').$$

Definition 4.1.2 *The multilinear application $T : X_1, \dots, X_m \longrightarrow Y$ is continuous if it is continuous as a function between two normed spaces. As a consequence of this definition, and the following equality*

$$T(x^1, \dots, x^m) - T(y^1, \dots, y^m) = T(x^1 - y^1, \dots, x^m) + T(x^1, x^2 - y^2, \dots, x^m) + \dots + T(x^1, \dots, x^m - y^m),$$

we have a result that gives a characterization of continuous m -linear mapping.

Proposition 4.1.1 *Let X_1, \dots, X_m, Y be normed spaces. For all $T \in L(X_1, \dots, X_m; Y)$, the following statements are equivalent*

- (1) T is continuous,
- (2) T is continuous in $(0, \dots, 0)$,
- (3) There exists a constant $C > 0$ such that

$$\|T(x^1, \dots, x^m)\| \leq C \|x^1\| \dots \|x^m\| \text{ for all } x_j \in X_j, (\forall 1 \leq j \leq m).$$

4.2 Bounded multilinear ϕ -summing operators

Definition 4.2.1 *Let ϕ be a modulus function and let X_1, X_2, \dots, X_m, Y are Banach spaces. The bounded multilinear operator $T : X_1 \times X_2 \times \dots \times X_m \longrightarrow Y$ is ϕ -summing if there exists a constant $C \geq 0$ such that*

$$\sum_{i=1}^n \phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|) \leq C \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|). \quad (4.2.1)$$

The set of all multilinear ϕ -summing operators is denoted by $\Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$.

For $T \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$. We define

$$\|T\|_m^\phi = \inf \{ C : C \text{ verifying (4.2.1)} \}.$$

Remark 4.2.1 1. For $m = 1$, we find

$$\begin{aligned} (4.2.1) \implies \sum_{i=1}^n \phi(\|T(x_i)\|) &\leq C \sup_{\xi \in B_{\mathcal{L}(X)}} \sum_{i=1}^n \phi(|\xi(x_i)|). \\ \implies \sum_{i=1}^n \phi(\|T(x_i)\|) &\leq C \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|\langle x_i, x^* \rangle|). \end{aligned}$$

2. If $\phi(x) = x^p$, $0 < p \leq 1$ (modulus function) we have

$$(4.2.1) \implies \sum_{i=1}^n \|T(x_i^1, x_i^2, \dots, x_i^m)\|^p \leq C \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n |\xi(x_i^1, x_i^2, \dots, x_i^m)|^p.$$

4.3 Consequence

Proposition 4.3.1 The space $\Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$ is sublinear space from the space $\mathcal{L}(X_1, X_2, \dots, X_m; Y)$

Proof.

1) Let $\lambda \in \mathbb{R}$ and $T \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$ and let ϕ a modulus function, we have

$$\|\lambda T(x_i^1, x_i^2, \dots, x_i^m)\| = |\lambda| \|T(x_i^1, x_i^2, \dots, x_i^m)\|.$$

Then

$$\begin{aligned} \phi(\|\lambda T(x_i^1, x_i^2, \dots, x_i^m)\|) &= \phi(|\lambda| \|T(x_i^1, x_i^2, \dots, x_i^m)\|) \\ &\leq (1 + |\lambda|) \phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|). \end{aligned}$$

So

$$\begin{aligned} \sum_{i=1}^n \phi(\|\lambda T(x_i^1, x_i^2, \dots, x_i^m)\|) &\leq (1 + |\lambda|) \sum_{i=1}^n \phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|) \\ &\leq (1 + |\lambda|) C \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) \\ &= K \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|). \end{aligned}$$

With $C > 0$ and $K = ((1 + |\lambda|)C) > 0$.

Then

$$\begin{cases} \lambda T \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Y) \\ \|\lambda T\|_m^\phi \leq (1 + |\lambda|) \|T\|_m^\phi. \end{cases} \quad (*)$$

2) Let $T_1, T_2 \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$, then

$$\|(T_1 + T_2)(x_i^1, x_i^2, \dots, x_i^m)\| \leq \|T_1(x_i^1, x_i^2, \dots, x_i^m)\| + \|T_2(x_i^1, x_i^2, \dots, x_i^m)\|.$$

Then

$$\begin{aligned} \phi(\|(T_1 + T_2)(x_i^1, x_i^2, \dots, x_i^m)\|) &\leq \phi(\|T_1(x_i^1, x_i^2, \dots, x_i^m)\|) + \phi(\|T_2(x_i^1, x_i^2, \dots, x_i^m)\|) \\ &\leq \|T_1\|_m^\phi \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) \\ &\quad + \|T_2\|_m^\phi \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) \\ &= (\|T_1\|_m^\phi + \|T_2\|_m^\phi) \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|). \end{aligned}$$

Then

$$\begin{cases} (T_1 + T_2) \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Y) \\ (\|T_1 + T_2\|_m^\phi \leq \|T_1\|_m^\phi + \|T_2\|_m^\phi) \end{cases} \quad (**)$$

From (*) and (**) we deduce the $\Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$ is subnormed space of $\mathcal{L}(X_1, X_2, \dots, X_m; Y)$.

■

Proposition 4.3.2 (ideal property). If $T \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$ and $u_j : E_j \longrightarrow X_j$, ($1 \leq j \leq m$), $u_j \in \mathcal{L}(E_j, X_j)$ and $w : Y \longrightarrow Z$, $w \in \mathcal{L}(Y, Z)$.

Then

$$w \circ T \circ (u_1, u_2, \dots, u_m) \in \Pi_m^\phi(E_1, E_2, \dots, E_m; Z)$$

and

$$\|w \circ T \circ (u_1, u_2, \dots, u_m)\| \leq (1 + \|w\|) \|T\|_m^\phi.$$

Proof.

1) we prove $w \circ T \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Z)$

$$\|w \circ T(x_i^1, x_i^2, \dots, x_i^m)\| \leq \|w\| \|T(x_i^1, x_i^2, \dots, x_i^m)\|.$$

Then

$$\begin{aligned} \phi(\|w \circ T(x_i^1, x_i^2, \dots, x_i^m)\|) &\leq \phi(\|w\| \|T(x_i^1, x_i^2, \dots, x_i^m)\|) \\ &\leq (1 + \|w\|) \phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|). \end{aligned}$$

So

$$\begin{aligned} \sum_{i=1}^n \phi(\|w \circ T(x_i^1, x_i^2, \dots, x_i^m)\|) &\leq (1 + \|w\|) \sum_{i=1}^n \phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|) \\ &\leq (1 + \|w\|) \|T\|_m^\phi \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(\|\xi(x_i^1, x_i^2, \dots, x_i^m)\|). \end{aligned}$$

Then

$$\begin{cases} w \circ T \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Z) \\ \|w \circ T\|_m^\phi \leq (1 + \|w\|) \|T\|_m^\phi \end{cases} \quad (\#)$$

2) For $T \in \Pi_m^\phi(X_1, X_2, \dots, X_m; Y)$, $u_j \in \mathcal{L}(E_j, X_j)$, $(1 \leq j \leq m)$ and $e^j \in E_j$, we prove $T \circ (u_1, \dots, u_m) \in \Pi_m^\phi(E_1, E_2, \dots, E_m; Y)$.

$$T \circ (u_1, u_2, \dots, u_m) (e_i^1, e_i^2, \dots, e_i^m) = T \circ (u_1(e_i^1), u_2(e_i^2), \dots, u_m(e_i^m)).$$

Then

$$\|T \circ (u_1, u_2, \dots, u_m) (e_i^1, e_i^2, \dots, e_i^m)\| = \|T \circ (u_1(e_i^1), u_2(e_i^2), \dots, u_m(e_i^m))\|.$$

So

$$\sum_{i=1}^n \phi \|T \circ (u_1, u_2, \dots, u_m) (e_i^1, e_i^2, \dots, e_i^m)\|$$

$$\begin{aligned}
 &= \sum_{i=1}^n \phi \|T \circ (u_1(e_i^1), u_2(e_i^2), \dots, u_m(e_i^m))\| \\
 &\leq \|T\|_m^\phi \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi (|\xi(u_1(e_i^1), u_2(e_i^2), \dots, u_m(e_i^m))|) \\
 &= \|T\|_m^\phi \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi (|\langle (e_i^1, e_i^2, \dots, e_i^m), (u_1^*, u_2^*, \dots, u_m^*)(\xi) \rangle|) \\
 &= \|T\|_m^\phi \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi \|(u_1^*, u_2^*, \dots, u_m^*)\| \left(\left| \left\langle x_i^j, \frac{(u_1^*, u_2^*, \dots, u_m^*)}{\|(u_1^*, u_2^*, \dots, u_m^*)\|}(\xi) \right\rangle \right| \right) \\
 &\leq \|T\|_m^\phi \sup_{\gamma \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n (1 + \|(u_1^*, u_2^*, \dots, u_m^*)\|) \phi (|\langle (e_i^1, e_i^2, \dots, e_i^m), \gamma \rangle|) \\
 &= \|T\|_m^\phi (1 + \|(u_1^*, u_2^*, \dots, u_m^*)\|) \sup_{\gamma \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi (|\langle (e_i^1, e_i^2, \dots, e_i^m), \gamma \rangle|) \\
 &= \|T\|_m^\phi (1 + \|(u_1^*, u_2^*, \dots, u_m^*)\|) \sup_{\gamma \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi (|\gamma((e_i^1, e_i^2, \dots, e_i^m))|).
 \end{aligned}$$

Fainly we find

$$\begin{cases} T \circ (u_1, u_2, \dots, u_m) \in \Pi_m^\phi(E_1, E_2, \dots, E_m; Y) \\ \|T \circ (u_1, u_2, \dots, u_m)\| \leq \|T\|_m^\phi (1 + \|(u_1^*, u_2^*, \dots, u_m^*)\|). \end{cases} \quad (\#\#)$$

From (#) and (\#\#) we de duce the ideal property of this class.

$$\begin{array}{ccccccc}
 X_1 & \times & X_2 & \times & \dots & \times & X_m & \xrightarrow{T} & Y & \xrightarrow{w} & Z \\
 u_1 \uparrow & & u_2 \uparrow & & & & u_m \uparrow & & \swarrow & & w \circ T \circ (u_1, u_2, \dots, u_m) \\
 E_1 & \times & E_2 & \times & \dots & \times & E_m & & & &
 \end{array}$$

Remark 4.3.1 *Some symbols have been shortened because typing does not appear.*

■

4.4 Pietsch domination theorem

Theorem 4.4.1 *The followings are equivalent for a multilinear operator $T \in \mathcal{L}(X_1, X_2, \dots, X_m; Y)$ and a positive constant C .*

(i) *The operator $T : X_1 \times X_2 \times \dots \times X_m \longrightarrow Y$ is ϕ -summing.*

(ii) There is a probability μ on $B_{\mathcal{L}(X_1, X_2, \dots, X_m)}$ such that

$$\phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|) \leq C \int_{B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) d\mu(\xi).$$

Proof. (ii) \implies (i)

$$\phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|) \leq C \int_{B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) d\mu(\xi).$$

Then

$$\begin{aligned} \sum_{i=1}^n \phi(\|T(x_i^1, x_i^2, \dots, x_i^m)\|) &\leq C \sum_{i=1}^n \int_{B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) d\mu(\xi) \\ &= C \int_{B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) d\mu(\xi) \\ &\leq C \int_{B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|) d\mu(\xi) \\ &\leq C \sup_{\xi \in B_{\mathcal{L}(X_1, X_2, \dots, X_m)}} \sum_{i=1}^n \phi(|\xi(x_i^1, x_i^2, \dots, x_i^m)|). \end{aligned}$$

So $T : X_1 \times X_2 \times \dots \times X_m \longrightarrow Y$ is ϕ -summing. \blacksquare

4.5 Polynomials ϕ -summing

4.5.1 symmetric multilinear operators

For all $m \in \mathbb{N}^*$, we denote by \mathcal{S}_m the set of all permutations of $\{1, \dots, m\}$, in other words the set of all bijections

$$\sigma : \{1, \dots, m\} \longrightarrow \{1, \dots, m\}.$$

Definition 4.5.1 Let $m \in \mathbb{N}^*$ and X, Y a vector spaces over a field \mathbb{K} . A multilinear operator $T : X, X, \dots, X \longrightarrow Y$, is symmetric if

$$T(x^1, x^2, \dots, x^m) = T(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)}),$$

for every $x_1, x_2, \dots, x_m \in X$ and every $\sigma \in \mathcal{S}_m$.

$\mathcal{L}_S^m(X; Y)$ will denote the subspace of $\mathcal{L}^m(X; Y)$ consisting of all the symmetric multilinear mappings from ${}^m X$ into Y .

Proposition 4.5.1 For all $T \in \mathcal{L}(^m X; Y)$. Let T_S be the associate symmetric operator of T . Then, the following are checked

- (1) $T_S \in \mathcal{L}_S(^m X; Y)$,
- (2) $T_S = T$ if, and only if $T \in \mathcal{L}_S(^m X; Y)$,
- (3) $(T_S)_S = T_S$,
- (4) If $x \in X$, then $T(x, x, \dots, x) = T_S(x, x, \dots, x)$.

4.5.2 homogeneous polynomial

Definition 4.5.2 Let $m \in \mathbb{N}^*$. A map $P : X \rightarrow Y$ is called m -homogeneous polynomial, if there exists a symmetric m -linear mapping $T_P (T_P \in \mathcal{L}_S(^m X; Y))$, such that $P(x) = T_P(x, \dots, x)$ for all $x \in X$.

We denote by $\mathcal{P}(^m X; Y)$, the Banach space of all continuous m -homogeneous polynomials from X into Y endowed with the norm

$$\|P\| = \sup \{ \|P(x)\| : \|x\| \leq 1 \} = \inf \{ C : \|P(x)\| \leq C \cdot \|x\|^m \}.$$

If $Y = \mathbb{K}$, then $\mathcal{P}(^m X; Y) = \mathcal{P}(^m X)$.

Definition 4.5.3 Let T a symmetric m -linear operator ($T \in \mathcal{L}_S(^m X; Y)$) and $P : X \rightarrow Y$ is m -homogeneous polynomials P is said ϕ -summing if there exists a constant $C \geq 0$ such that

$$\sum_{i=1}^n \phi(\|T(x_i, x_i, \dots, x_i)\|) \leq C \sup_{\xi \in B_{\mathcal{L}_S(^m X)}} \sum_{i=1}^n \phi(|\xi(x_i, x_i, \dots, x_i)|).$$

In other word

$$\sum_{i=1}^n \phi(\|P(x_i)\|) \leq C \sup_{Q \in B_{\mathcal{P}(^m X)}} \sum_{i=1}^n \phi(|Q(x_i)|).$$

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المُلخَص : يندرج محتوى هذه المذكرة في إطار التحليل الدالي، و بشكل أكثر دقة في الهندسة غير الخطية ، استلهمت أفكار هذا العمل من مقال J.D. Farmer et W.B. Johnson المعنون بـ "Lipschitz p-summing operators" و مقال رشدي خليل و وسيم الديب المعنون بـ " ϕ -summing operators in Banach spaces" و كذلك مقال بلعلي المعتوق المعنون بـ "Lipschitz ϕ -summing operators" سنقوم في هذا العمل بدراسة مؤثر لبشترز ϕ -جمعي. أدخلنا مفهوم مؤثر خطي ϕ -جمعي و مفهوم مؤثر لبشترز ϕ -جمعي (غير خطي) بين فضاءين مترين مع إعطاء صيغة غير خطية لنظرية الهيمنة لـ (Pietsch) الخاصة بهذه المؤثرات و بعض الخواص المتعلقة بهذا الصنف. و في نهاية هذه المذكرة سنقدم مفهوم مؤثر متعدد خطي ϕ -جمعي مع إعطاء صيغة متعددة الخطية لنظرية الهيمنة لـ (Pietsch).

الكلمات المفتاحية : مؤثر خطي p جمعي، مؤثر لبشترز p جمعي، مؤثر خطي ϕ -جمعي، مؤثر لبشترز ϕ -جمعي.

Résumé: Le contenu de cette thèse s'inscrit dans le cadre de l'analyse fonctionnelle et plus précisément dans la géométrie non linéaire. Les notions de ce travail nous ont été inspirées de l'article de J.D. Farmer et W.B. Johnson intitulé "les opérateurs Lipschitz p-sommants" et de l'article de Roshdi Khalil et W. Deeb intitulé "les opérateurs ϕ -sommants dans les espaces de Banach", ainsi du papier de M. Belaala intitulé "Les opérateurs Lipschitz ϕ -sommants ". Dans ce mémoire, on va étudier les opérateurs Lipschitz ϕ -sommants. on introduit la notion d'opérateur linéaire ϕ -sommant and Lipschitz ϕ -sommant entre espaces métriques en donnant une version non linéaire du théorème de domination de Pietsch pour ces opérateurs et quelques propriétés concernant cette classe. Et à la fin de ce travail, nous avons présenté la notion d'opérateur multilinéaire ϕ -sommant et en donnant une version multilinéaire du théorème de domination de Pietsch.

Mots-clés: opérateur lipschitzien p-sommant, opérateur linéaire p-sommant, opérateur linéaire ϕ -sommant, opérateur lipschitzien ϕ -sommant, théorème de domination de Pietsch.

Abstract: The context of this memory falls within the framework of functional analysis and more precisely in nonlinear geometry. The notions of this work were inspired from the article of J.D. Farmer and W.B. Johnson entitled "Lipschitz p-summing operators" and from the paper of R. Khalil and W. Deeb entitled " ϕ -summing operators in Banach spaces", also that of M. Belaala entitled "Lipschitz ϕ -summing operators". In this memory, we will study the Lipschitz ϕ -summing operators. we have introduced the notion of linear ϕ -summing operator and Lipschitz ϕ -summing operator between metric spaces by giving a nonlinear version of Pietsch domination theorem for these operators and some properties concerning this class, and the end of this work, we presented the notion of multilinear ϕ -summing operators and giving a multilinear version of Pietsch domination theorem.

Key-words : linear p-summing operator, Lipschitz p-summing operator, linear ϕ -summing operator, Lipschitz ϕ -summing operator, Pietsch Domination theorem.