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Theme

On the Vibrations of Traveling Strings and Membranes

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بسم الله الرحمن الرحيم

Dedication

I dedicate this modest work :

- To my parents,
- To my brothers and sisters,
- To my aunts,
- To all my family,
- To all friends and all my department family,
- To all my adorable ones that I have known during all my life ...

Acknowledgement

Praise be to Allah who has enabled me to finish this work.

First of all, I want to thank my supervisor Dr. Sengouga for every effort give me, and I apologize to him for any shortcomings, you have all my respect and appreciation, thank you.

If I say thank you, my thanks will not fulfill your rights, this the words for you, mom and dad and my sisters, brothers and all my family and friends and my fiance. I'm really grateful for your help, you have been my main supporter in life, I will never forget you favor for the rest of my life.

I am also want to say for the jury thanks for considering this and on reading this work.

Contents

Introduction

Wave motion is a fundamental phenomenon in science and engineering, occurring in contexts such as acoustics, electromagnetism, elasticity, and fluid dynamics. Mathematically, waves are often described using partial differential equations (PDEs) that capture how disturbances propagate through a medium over time.

This dissertation investigates a particular class of wave equations: those describing vibrating strings and membranes that are *moving* through space with constant velocity. These are known as *travelling strings* and *travelling membranes*. In contrast to classical problems defined on fixed spatial domains, the moving boundary introduces significant mathematical challenges, especially when damping effects are present.

We begin by revisiting the classical wave equation in one spatial dimension, deriving it via Hamilton's principle (a variational approach rooted in mechanics) and solving it using the method of separation of variables. We then generalize the model to account for uniform translation of the string, analyzing both the undamped and damped scenarios.

In the subsequent chapter, we extend the discussion to two-dimensional domains, focusing on membranes (thin elastic surfaces) undergoing translational motion. As in the one-dimensional case, we consider both undamped and damped vibrations. To simplify analysis, coordinate transformations are introduced that convert the moving domain into a fixed reference frame.

Several works have explored similar themes. For instance, W. L. Miranker [3] examined travelling strings and proposed the model:

$$\begin{cases} u_{tt} - 2vu_{xt} + (v^2 - c^2)u_{xx} = 0, & x \in (0,\ell),\ t > 0 \\ u(0,t) = u(\ell,t) = 0, & t > 0 \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & x \in (0,\ell). \end{cases}$$

Similarly, V. P. Boldin and M. V. Trubin [1] studied travelling membranes using the model:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x \in (0,\ell),\ y \in (0,b),\ t > 0 \\ u(vt,y,t) = u(vt+\ell,y,t) = 0, & y \in (0,b),\ t > 0 \\ u(x,0,t) = u(x,b,t) = 0, & x \in (0,\ell),\ t > 0, \\ u(x,y,0) = u^0(x,y), \quad u_t(x,y,0) = u^1(x,y), & x \in (0,\ell),\ y \in (0,b). \end{cases}$$

This dissertation is organized as follows:

- **Chapter 1** introduces mathematical tools used throughout the work, including functional spaces (such as L^p and Sobolev spaces), Fourier series, and elements of the calculus of variations, including Gâteaux derivatives and critical points. We also review the classical wave equation.

- **Chapter 2** addresses the problem of travelling strings. We derive the governing equations using Hamilton's principle, establish well-posedness, and solve them via variable transformation and separation of variables, considering both damped and undamped settings.

- **Chapter 3** generalizes the study to travelling membranes. We formulate the wave equation in two spatial dimensions with moving boundaries and incorporate damping. Analytical techniques similar to those used for strings are employed, with added complexity due to the higher dimensionality.

- Some references, used in this work, are given at the end of the manuscript.

Chapter 1

Preliminaries

1.1 Functional Spaces

In this section, we recall some functional spaces and theorems that will be used throughout this work. These spaces provide the mathematical framework for analyzing the behavior of solutions to partial differential equations.

1.1.1 L^p Spaces

Let Ω be an open subset of $\mathbb{R}^n$ equipped with the Lebesgue measure dx.

Definition 1.1 *Let $p \in \mathbb{R}$ with $1 < p < \infty$. The L^p space is defined by:*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ is measurable and } |f|^p \in L^1(\Omega) \right\},$$

with the norm

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

Definition 1.2 *We define the space*

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \exists C > 0 \text{ such that } |f(x)| \leq C \text{ a.e. on } \Omega \},$$

with the essential supremum norm

$$\|f\|_{L^\infty} = \inf \{ C > 0 \mid |f(x)| \leq C \text{ a.e. in } \Omega \}.$$

Theorem 1.3 (Dominated Convergence Theorem) *Let (f_n) be a sequence in $L^p(\Omega)$ satisfying:*

- *$f_n(x) \to f(x)$ a.e. on Ω,*
- *$\exists g \in L^p(\Omega)$ such that $|f_n(x)| \leq g(x)$ a.e. for all n.*

Then $f \in L^p(\Omega)$ and $\|f_n - f\|_{L^p(\Omega)} \to 0$ as $n \to \infty$.

1.1.2 H^1 Spaces

Definition 1.4 *A function $v \in L^2(\Omega)$ belongs to $H^1(\Omega)$ if there exist functions $g_1, g_2, \ldots, g_n \in L^2(\Omega)$ such that:*

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_\Omega v \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega g_i \varphi \, dx.$$

Then, g_i is called the distributional derivative $\frac{\partial v}{\partial x_i}$.

Definition 1.5 *The Sobolev space $H^1(\Omega)$ is defined as:*

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \;\middle|\; \frac{\partial v}{\partial x_i} \in L^2(\Omega), \ i = 1, \ldots, n \right\},$$

where $\frac{\partial v}{\partial x_i}$ denotes the distributional derivative.

The space $H^1(\Omega)$ is equipped with the scalar product:

$$\langle u, v \rangle = \int_\Omega uv \, dx + \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx,$$

with the associated norm:

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega v^2 \, dx + \sum_{i=1}^{n} \int_\Omega \left(\frac{\partial v}{\partial x_i} \right)^2 dx \right)^{1/2}.$$

1.2 Fourier Series

1.2.1 Fourier Expansion

Definition 1.6 *Let f be a piecewise continuous and $2T$-periodic function. The Fourier series of f is given by:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega x) + b_n \sin(n\omega x) \right), \quad \textit{where } \omega = \frac{\pi}{T}.$$

Alternatively, it can be written in complex form as:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}.$$

1.2.2 Fourier Coefficients

Definition 1.7 *Let u be a $2T$-periodic function such that it admits a Fourier series expansion:*

$$u(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega x) + b_k \sin(k\omega x) \right), \quad \omega = \frac{\pi}{T}. \tag{1.1}$$

To compute the coefficients a_k and b_k, we use the orthogonality relations:

$$\int_{-T}^{T} \cos(k\omega x)\cos(m\omega x)\,dx = \int_{-T}^{T} \sin(k\omega x)\sin(m\omega x)\,dx = 0, \quad \text{if } k \neq m,$$
$$\int_{-T}^{T} \cos(k\omega x)\sin(m\omega x)\,dx = 0, \quad \text{for all } k, m \geq 0,$$
$$\int_{-T}^{T} \cos^2(k\omega x)\,dx = \int_{-T}^{T} \sin^2(k\omega x)\,dx = T.$$

Assuming uniform convergence of (1.1), multiplication by $\cos(n\omega x)$ and integration over $[-T, T]$ yields:

$$a_n = \frac{1}{T}\int_{-T}^{T} u(x)\cos(n\omega x)\,dx, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

Similarly,

$$b_n = \frac{1}{T}\int_{-T}^{T} u(x)\sin(n\omega x)\,dx. \tag{1.3}$$

Fourier Coefficients of the Derivative. If $u \in C^1(\mathbb{R})$ and is $2T$-periodic, then for $k \geq 1$:

$$a_k' = \frac{1}{T}\int_{-T}^{T} u'(x)\cos(k\omega x)\,dx = k\omega b_k,$$
$$b_k' = \frac{1}{T}\int_{-T}^{T} u'(x)\sin(k\omega x)\,dx = -k\omega a_k.$$

Thus,

$$a_k' = k\omega b_k, \quad b_k' = -k\omega a_k. \tag{1.4}$$

Complex Form. Using Euler's formula

$$e^{\pm ik\omega x} = \cos(k\omega x) \pm i\sin(k\omega x),$$

the series (1.1) can be written as:

$$u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega x},$$

with complex coefficients:

$$c_k = \frac{1}{2T}\int_{-T}^{T} u(x)e^{-ik\omega x}\,dx.$$

The relations with the real coefficients are:

$$c_0 = \frac{1}{2}a_0, \quad c_k = \frac{1}{2}(a_k - ib_k), \quad c_{-k} = \overline{c_k}, \quad \text{for } k > 0.$$

Convergence in L^2. Let $S_N(x)$ be the N-th partial sum:

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos(k\omega x) + b_k \sin(k\omega x) \right).$$

Theorem 1.8 *If $u \in L^2(-T,T)$, then $S_N \to u$ in the L^2 norm:*

$$\lim_{N\to\infty} \int_{-T}^{T} |S_N(x) - u(x)|^2 \, dx = 0.$$

Moreover, Parseval's identity holds:

$$\int_{-T}^{T} |u(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

As a consequence, the Fourier coefficients decay to zero:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0.$$

1.3 Differentiability of Functionals and Critical Points

In this section, we introduce the notion of differentiability for functionals defined on Banach spaces and define the concept of critical points.

1.3.1 Gâteaux Differentiability

Definition 1.9 *Let E be a Banach space and $\Omega \subset E$ an open set. Let $I : \Omega \to \mathbb{R}$ be a functional. We say that I is* Gâteaux differentiable *(or G-differentiable) at a point $u \in \Omega$ if there exists a continuous linear functional $A \in E'$ such that, for all $v \in E$,*

$$\lim_{t\to 0} \frac{I(u+tv) - I(u)}{t} = \langle A, v \rangle. \tag{1.5}$$

This functional A, when it exists, is unique and is called the Gâteaux differential *of I at u, denoted by $I'_G(u)$.*

Definition 1.10 *If I is Gâteaux differentiable at every point u in an open set $U \subset E$, we say that I is* Gâteaux differentiable on *U. The map*

$$\begin{aligned} I'_G &: U \to E', \\ u &\mapsto I'_G(u), \end{aligned}$$

is called the Gâteaux derivative *of I.*

1.3.2 Critical Points

Definition 1.11 *Let Ω be an open subset of a Banach space E, and suppose that $I \in C^1(\Omega, \mathbb{R})$. A point $u \in \Omega$ is called a* critical point *of I if*

$$I'(u) = 0.$$

A point $u \in \Omega$ such that $I'(u) \neq 0$ is called a regular point *of I.*

Given $c \in \mathbb{R}$, we say that c is a critical value *of I if there exists $u \in \Omega$ such that*

$$I(u) = c \quad \textit{and} \quad I'(u) = 0.$$

Otherwise, c is called a regular value *of I.*

1.4 The One-Dimensional Wave Equation

In this section, we study the transverse vibrations of a stretched string of length ℓ and uniform linear mass density ρ. The string is fixed at both ends and allowed to move freely in the transverse direction. The tension in the string is denoted by $\mathbf{T}$. We derive the wave equation governing the motion of the string using a principle known in mechanics.

1.4.1 Derivation of the Equation of Motion

Let $u(x,t)$ denote the transverse displacement of the string at position $x \in [0, \ell]$ and time t. The velocity of a point on the string is given by the partial derivative with respect to time:

$$\frac{d}{dt} u(x,t) = u_t(x,t).$$

The kinetic energy K of the string between $x = 0$ and $x = \ell$ at time t is given by:

$$K(t) = \frac{1}{2} \int_0^{\ell} \rho u_t^2(x,t)\, dx,$$

where ρ is the linear mass density.

The potential energy V due to the stretching of the string under tension $\mathbf{T}$ is:

$$V(t) = \frac{1}{2} \int_0^{\ell} \mathbf{T} u_x^2(x,t)\, dx,$$

where $u_x = \partial u / \partial x$ is the spatial derivative.

The Lagrangian is then defined as the difference between the kinetic and potential energies:

$$L(u) = K(t) - V(t) = \frac{1}{2} \int_0^{\ell} \rho u_t^2 - \mathbf{T} u_x^2\, dx.$$

To derive the equation of motion, we apply Hamilton's principle: *the actual motion of the string corresponds to a critical point of the functional*

$$I(u) = \int_{t_1}^{t_2} L(u)\,dt = \int_{t_1}^{t_2} \int_0^\ell \rho u_t^2 - \mathbf{T} u_x^2 \,dx\,dt.$$

Let w be a smooth (admissible) function satisfying:

$$w(x,t_1) = w(x,t_2) = 0, \quad \text{for all } x \in (0,\ell),$$
$$w(0,t) = w(\ell,t) = 0, \quad \text{for all } t \in (t_1,t_2).$$

Then,

$$\lim_{\sigma\to 0} \int_{t_1}^{t_2} \int_0^\ell \frac{1}{\sigma}(\rho\sigma^2 w_t^2 + 2\rho\sigma u_t w_t - \mathbf{T}\sigma^2 w_x^2 - 2\mathbf{T}\sigma u_x w_x)dxdt.$$

By using the Lebesgue dominate convergence theorem, one has:

$$\int_{t_1}^{t_2} \int_0^\ell \lim_{\sigma\to 0} \frac{1}{\sigma}(\rho\sigma^2 w_t^2 + 2\rho\sigma u_t w_t - \mathbf{T}\sigma^2 w_x^2 - 2\mathbf{T}\sigma u_x w_x)dxdt.$$
$$= \int_{t_1}^{t_2} \int_0^\ell \lim_{\sigma\to 0} \rho\sigma w_t^2 + 2\rho u_t w_t - \mathbf{T}\sigma w_x^2 - 2\mathbf{T} u_x w_x dxdt.$$
$$= \int_{t_1}^{t_2} \int_0^\ell 2\rho u_t w_t - 2\mathbf{T} u_x w_x dxdt.$$

Integrating by parts, we infer that

$$\begin{aligned} \lim_{\sigma\to 0} \frac{I\left(u+\sigma w\right) - I\left(u\right)}{\sigma} &= 2\left[\int_0^\ell \rho u_t\left(x,t\right) w\left(x,t\right) dx\right]_{t=t_1}^{t=t_2} - 2\int_{t_1}^{t_2} \int_0^\ell \rho u_{tt}\left(x,t\right) w\left(x,t\right) dxdt \\ &\quad -2\left[\int_{t_1}^{t_2} \mathbf{T} u_x\left(x,t\right) w\left(x,t\right) \partial t\right]_{x=0}^{x=\ell} + 2\int_{t_1}^{t_2} \int_0^\ell \mathbf{T} u_{xx}\left(x,t\right) w\left(x,t\right) dxdt, \end{aligned}$$

hence

$$\lim_{\sigma\to 0} \frac{I\left(u+\sigma w\right) - I\left(u\right)}{\sigma} = -2\int_{t_1}^{t_2} \int_0^\ell (\rho u_{tt} - \mathbf{T} u_{xx}) w dxdt = 0.$$

Since this integral vanishes for all admissible w, we must have:

$$\rho u_{tt} - \mathbf{T} u_{xx} = 0 \quad \text{in } (0,\ell) \times (t_1,t_2).$$

Defining $c^2 = \mathbf{T}/\rho$, we obtain the classical wave equation:

$$\boxed{u_{tt} - c^2 u_{xx} = 0, \quad x \in (0,\ell),\ t > 0.} \tag{1.6}$$

In fact, $c = \sqrt{\mathbf{T}/\rho}$ is the propagation speed of the wave in the string.

To complete the model, we impose the boundary and initial conditions:

$$\begin{aligned} &u(0,t) = 0, \quad u(\ell,t) = 0, && t > 0, \\ &u(x,0) = u^0(x), && x \in (0,\ell), \\ &u_t(x,0) = u^1(x), && x \in (0,\ell). \end{aligned}$$

This defines the initial-boundary value problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in (0,\ell),\ t > 0, \\ u(0,t) = u(\ell,t) = 0, & t > 0, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in (0,\ell). \end{cases} \tag{1.7}$$

1.4.2 Separation of Variables Method

It is well known that the wave equation (2.1) admits a unique solution u, which can be obtained by several classical methods, such as the method of separation of variables or the d'Alembert formula. For simplicity, we consider the case where $c = 1$.

• The method of separation of variables assumes that the solution can be written in the form:

$$u(x,t) = X(x)T(t).$$

Substituting into the wave equation yields:

$$u_{tt} = X(x)T''(t), \quad u_{xx} = X''(x)T(t).$$

Plugging into the PDE:

$$X(x)T''(t) = X''(x)T(t). \tag{1.8}$$

Dividing both sides by $X(x)T(t)$ (assuming neither is zero), we get:

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

where $-\lambda^2$ is the separation constant. The minus sign ensures that we get non-trivial oscillatory solutions.

• Applying the boundary conditions $u(0,t) = u(\ell,t) = 0$ implies that $X(0) = X(\ell) = 0$. Thus, X must satisfy the Sturm-Liouville problem:

$$X'' + \lambda^2 X(x) = 0, \quad X(0) = X(\ell) = 0.$$

The general solution of this ODE is:

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x).$$

Applying $X(0) = 0$ yields $B = 0$. Then $X(\ell) = 0$ gives:

$$\sin(\lambda \ell) = 0 \Rightarrow \lambda = \lambda_n := \frac{n\pi}{\ell}, \quad n = 1, 2, 3, \ldots$$

Thus, the eigenfunctions are:

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right).$$

- The temporal equation becomes:

$$T''(t) + \lambda_n^2 T(t) = 0,$$

whose general solution is:

$$T_n(t) = a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t).$$

Hence, the corresponding solution u_n is:

$$u_n(x,t) = [a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)] \sin(\lambda_n x).$$

Using the principle of superposition, the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)] \sin(\lambda_n x).$$

To determine the coefficients a_n and b_n, we use the initial conditions:

$$u(x,0) = u^0(x) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x), \qquad u_t(x,0) = u^1(x) = \sum_{n=1}^{\infty} \lambda_n b_n \sin(\lambda_n x).$$

Thus,

$$a_n = \frac{2}{\ell} \int_0^\ell u^0(x) \sin(\lambda_n x)\, dx, \qquad b_n = \frac{2}{\ell \lambda_n} \int_0^\ell u^1(x) \sin(\lambda_n x)\, dx.$$

Theorem 1.12 *The solution of the initial-boundary value problem* (2.1) *with* $c = 1$ *is given by:*

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{\ell} t\right) + b_n \sin\left(\frac{n\pi}{\ell} t\right) \right] \sin\left(\frac{n\pi x}{\ell}\right),$$

where the coefficients a_n *and* b_n *depend on the initial displacement and velocity.*

Chapter 2

The Wave Equation for Travelling Strings

This chapter addresses the transverse vibration of a string, of length ℓ, that is also traveling at a constant speed $\mathbf{v}$ between two pulleys, see Figure 2.1.

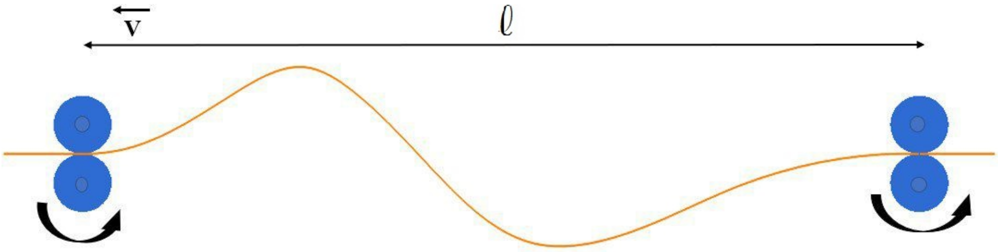

Figure 2.1: A vibrating string that travels at constant speed $\mathbf{v}$.

2.1 Undamped Travelling Strings

2.1.1 Derivation of the Equation of Motion

In this part, we use the same principle used in the first chapter, but this time the string is also moving horizontally at a constant speed $\mathbf{v}$. Thus, the total derivative of u is given by

$$\frac{d}{dt}u(x,t) = u_t + x_t u_x = u_t - \mathbf{v}u_x.$$

The kinetic energy of the string is

$$K = \frac{1}{2}\int_0^\ell \rho\left[\mathbf{v}^2 + (u_t - \mathbf{v}u_x)^2\right]dx,$$

where ρ is the linear mass density of the string.

The potential energy is

$$V = \frac{1}{2}\int_0^\ell \mathbf{T} u_x^2 \, dx,$$

where $\mathbf{T}$ is the tension.

Let us consider the functional

$$I(u) = \int_{t_1}^{t_2}\int_0^\ell \left(\rho\left[\mathbf{v}^2 + (u_t - \mathbf{v}u_x)^2\right] - \mathbf{T}u_x^2\right) dx\, dt,$$

and let w be an admissible function satisfying

$$\begin{aligned} w(x,t_1) &= w(x,t_2) = 0, \quad \text{for } 0 < x < \ell, \\ w(\cdot,t) &\in H_0^1(0,\ell), \quad \text{for } t_1 < t < t_2. \end{aligned}$$

Then, $I(u+\sigma w) - I(u)$ equals:

$$\int_{t_1}^{t_2}\int_0^\ell \rho\sigma^2 w_t^2 + 2\rho\sigma u_t w_t + \rho\mathbf{v}^2\sigma^2 w_x^2 + 2\rho\sigma\mathbf{v}^2 u_x w_x \\ - 2\rho\sigma\mathbf{v}u_t w_x - 2\rho\sigma\mathbf{v}w_t u_x - 2\rho\sigma^2\mathbf{v}w_t w_x - \mathbf{T}\sigma^2 w_x^2 - 2\mathbf{T}\sigma u_x w_x \, dx\, dt.$$

By Hamilton's principle, we have

$$\delta I = \lim_{\sigma\to 0}\frac{I(u+\sigma w) - I(u)}{\sigma} = 0,$$

which gives:

$$\int_{t_1}^{t_2}\int_0^\ell 2\rho u_t w_t - 2\rho\mathbf{v}^2 u_x w_x - 2\rho\mathbf{v}u_t w_x - 2\rho\mathbf{v}w_t u_x - 2\mathbf{T}u_x w_x \, dx\, dt = 0.$$

Integrating by parts and using the vanishing of w on the boundary and at t_1, t_2, we arrive at:

$$\int_{t_1}^{t_2}\int_0^\ell \left(2\rho u_{tt} - 2\rho\mathbf{v}^2 u_{xx} - 4\rho\mathbf{v}u_{tx} - 2\mathbf{T}u_{xx}\right) w \, dx\, dt = 0.$$

Since w can be arbitrary chosen, we obtain the PDE:

$$2\rho u_{tt} + 2\rho\mathbf{v}^2 u_{xx} - 4\rho\mathbf{v}u_{tx} - 2\mathbf{T}u_{xx} = 0,$$

or, simplified:

$$u_{tt} - 2\mathbf{v}u_{tx} + (\mathbf{v}^2 - \frac{\mathbf{T}}{\rho})u_{xx} = 0.$$

Finally, we get the wave equation for the traveling string:

$$\begin{cases} u_{tt} - 2\mathbf{v}u_{tx} - (c^2 - \mathbf{v}^2)u_{xx} = 0, & x \in (\mathbf{v}t, \mathbf{v}t + \ell),\ t > 0, \\ u(0,t) = u(\ell,t) = 0, & t > 0, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in (0,\ell), \end{cases} \tag{2.1}$$

where $c = \sqrt{\mathbf{T}/\rho}$ is the wave propagation speed, and u^0, u^1 are the initial displacement and velocity of the string.

2.1.2 Energy Conservation and Uniqueness of the Solution

To show the uniqueness of the solution to Problem (2.1), we define the energy functional:

$$E(t) = \frac{1}{2}\int_0^\ell u_t^2(x,t) + (c^2 - \mathbf{v}^2)u_x^2(x,t)\,dx.$$

Computing its time derivative yields:

$$\frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt}\int_0^\ell u_t^2 + (c^2 - \mathbf{v}^2)u_x^2\,dx = \int_0^\ell u_{tt}u_t + (c^2 - \mathbf{v}^2)u_{xt}u_x\,dx.$$

Integrating the second term by parts in x:

$$\int_0^\ell u_{xt}u_x\,dx = [u_t u_x]_{x=0}^{x=\ell} - \int_0^\ell u_t u_{xx}\,dx.$$

Using the boundary conditions $u(0,t) = u(\ell,t) = 0$, we obtain:

$$\frac{d}{dt}E(t) = \int_0^\ell u_{tt}u_t - (c^2 - \mathbf{v}^2)u_{xx}u_t\,dx.$$

Now, we consider the additional term:

$$2\mathbf{v}\int_0^\ell u_{tx}u_t\,dx = \mathbf{v}\int_0^\ell \frac{\partial}{\partial x}(u_t^2)\,dx = \mathbf{v}\left[u_t^2\right]_{x=0}^{x=\ell} = 0,$$

again by the boundary conditions.

Thus, we define a modified energy derivative:

$$\frac{d}{dt}E(t) = \int_0^\ell \left(u_{tt} - (c^2 - \mathbf{v}^2)u_{xx} - 2\mathbf{v}u_{tx}\right)u_t\,dx = 0,$$

using the fact that u satisfies the wave equation (2.1). Hence, $E(t)$ is constant:

$$E(t) = E(0) = \frac{1}{2}\int_0^\ell u_t^2(x,0) + (c^2 - \mathbf{v}^2)u_x^2(x,0)\,dx, \quad \forall t \geq 0. \tag{2.2}$$

As a consequence, we have

Theorem 2.1 *The solution to Problem (2.1) is unique if $c > \mathbf{v}$.*

Proof. Assume that there exist two solutions u_1 and u_2. Let $w = u_1 - u_2$, which satisfies:

$$\begin{cases} w_{tt} - 2\mathbf{v}w_{tx} - (c^2 - \mathbf{v}^2)w_{xx} = 0, & x \in (\mathbf{v}t, \mathbf{v}t + \ell),\ t > 0, \\ w(x,0) = w_t(x,0) = 0, & x \in (\mathbf{v}t, \mathbf{v}t + \ell), \\ w(0,t) = w(\ell,t) = 0, & t > 0. \end{cases}$$

Applying the energy identity (2.2) to w, and using the initial conditions $w(x,0) = w_t(x,0) = 0$, we obtain:

$$E(t) = 0, \quad \forall t \geq 0.$$

That is,

$$\frac{1}{2}\int_0^\ell w_t^2(x,t) + (c^2 - \mathbf{v}^2)w_x^2(x,t)\,dx = 0.$$

Since $c > \mathbf{v}$, this implies that both $w_t(x,t) = 0$ and $w_x(x,t) = 0$, hence $w(x,t) = 0$. Therefore, $u_1 = u_2$, and the solution is unique. ■

2.1.3 Change of Coordinates

We consider the transformation of variables:

$$x + \mathbf{v}t = \xi, \quad t = \tau.$$

Then, using the chain rule, we compute the partial derivatives:

$$u_x = u_\xi, \qquad u_t = \mathbf{v}u_\xi + u_\tau.$$

It follows that:

$$\begin{aligned} u_{xx} &= u_{\xi\xi}, \\ u_{xt} &= (u_x)_t = (u_\xi)_t = \mathbf{v}u_{\xi\xi} + u_{\xi\tau}, \\ u_{tt} &= (\mathbf{v}u_\xi + u_\tau)_t = \mathbf{v}(\mathbf{v}u_\xi + u_\tau)_\xi + (\mathbf{v}u_\xi + u_\tau)_\tau = \mathbf{v}^2 u_{\xi\xi} + 2\mathbf{v}u_{\xi\tau} + u_{\tau\tau}. \end{aligned}$$

Substituting these expressions into the wave equation (2.1), we obtain:

$$\begin{aligned} u_{tt} - 2\mathbf{v}u_{tx} - (c^2 - \mathbf{v}^2)u_{xx} &= (\mathbf{v}^2 u_{\xi\xi} + 2\mathbf{v}u_{\xi\tau} + u_{\tau\tau}) - 2\mathbf{v}(\mathbf{v}u_{\xi\xi} + u_{\xi\tau}) - (c^2 - \mathbf{v}^2)u_{\xi\xi} \\ &= u_{\tau\tau} - c^2 u_{\xi\xi}. \end{aligned}$$

Therefore, the equation (2.1) becomes:

$$\begin{cases} u_{\tau\tau} - c^2 u_{\xi\xi} = 0, & \xi \in (\mathbf{v}\tau, \ell + \mathbf{v}\tau), \ \tau > 0, \\ u(\mathbf{v}\tau, \tau) = u(\ell + \mathbf{v}\tau, \tau) = 0, & \tau > 0, \\ u(\xi, 0) = u^0(\xi), \quad u_\tau(\xi, 0) = u^1(\xi), & \xi \in (\mathbf{v}\tau, \ell + \mathbf{v}\tau). \end{cases} \tag{2.3}$$

Note that the new spatial variable ξ varies over the moving interval $(\mathbf{v}\tau, \ell + \mathbf{v}\tau)$ for each time $\tau > 0$.

2.1.4 The Exact Solution

In this section, we show how to write the exact solution using a series representation. We work in the transformed coordinates (ξ, τ), where the wave equation simplifies to the standard D'Alembert form.

D'Alembert's general solution for the transformed problem (2.3) is:

$$u(\xi, \tau) = f(\xi + c\tau) + g(\xi - c\tau). \tag{2.4}$$

We apply the first boundary condition at $\xi = \mathbf{v}\tau$:

$$u(\mathbf{v}\tau, \tau) = f((\mathbf{v} + c)\tau) + g((\mathbf{v} - c)\tau) = 0.$$

We define:

$$a = \mathbf{v} + c, \qquad b = \mathbf{v} - c,$$

and deduce:

$$g(\tau) = -f\left(\frac{a\tau}{b}\right).$$

Substituting this into (2.4), we get:

$$u(\xi, \tau) = f(\xi + c\tau) - f\left(\frac{a}{b}(\xi - c\tau)\right). \tag{2.5}$$

Applying the second boundary condition $u(\ell + \mathbf{v}\tau, \tau) = 0$ and substituting into (2.5) yields:

$$f(a\tau + \ell) = f\left(\frac{a}{b}(b\tau + \ell)\right),$$

which implies that f is periodic with period $\alpha\ell$, where:

$$\alpha = \frac{2c}{c - \mathbf{v}}.$$

We expand $f(\xi)$ in its Fourier series over the interval $(0, \alpha\ell)$:

$$f(\xi) = \sum_{n=-\infty}^{+\infty} f_n \exp\left(\frac{2n\pi i\xi}{\alpha\ell}\right). \tag{2.6}$$

Using (2.5), we find:

$$u(\xi, \tau) = \sum_{n=-\infty}^{+\infty} f_n \left[\exp\left(\frac{2n\pi i}{\alpha\ell}(\xi + c\tau)\right) - \exp\left(\frac{2n\pi i a}{b\alpha\ell}(\xi - c\tau)\right)\right].$$

Summarizing, we have the following

Theorem 2.2 *The solution to Problem (2.7) is given by the series:*

$$u(\xi, \tau) = \sum_{n=-\infty}^{+\infty} f_n \left[\exp\left(\frac{2n\pi i}{\alpha\ell}(\xi + c\tau)\right) - \exp\left(\frac{2n\pi i a}{b\alpha\ell}(\xi - c\tau)\right)\right],$$

where $\alpha = \frac{2c}{c-\mathbf{v}}$ *and the coefficients* f_n *depend on the initial conditions.*

To express the solution using trigonometric functions, we decompose the complex coefficients f_n as follows:

$$f_n = \sigma_n + i\tau_n, \qquad s_n = \sigma_n + \sigma_{-n}, \qquad t_n = \tau_n - \tau_{-n}.$$

We split the series into positive and negative indices and use Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

After some algebra, we find:

$$u(\xi,\tau) = \sum_{n=1}^{+\infty} \left[s_n \left(\cos\left(\frac{2n\pi}{\alpha\ell}(\xi + c\tau) \right) - \cos\left(\frac{2n\pi a}{b\alpha\ell}(\xi - c\tau) \right) \right) \right.$$
$$\left. + t_n \left(\sin\left(\frac{2n\pi}{\alpha\ell}(\xi + c\tau) \right) - \sin\left(\frac{2n\pi a}{b\alpha\ell}(\xi - c\tau) \right) \right) \right].$$

Using the trigonometric identities:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B, \quad \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$

we further decompose $u(\xi,\tau)$ as:

$$\begin{aligned} u(\xi,\tau) = \sum_{n=1}^{+\infty} s_n &\left[\cos\left(\frac{2n\pi}{\alpha\ell}\xi \right) \cos\left(\frac{2n\pi}{\alpha\ell}c\tau \right) - \cos\left(\frac{2n\pi a}{b\alpha\ell}\xi \right) \cos\left(\frac{2n\pi a}{b\alpha\ell}c\tau \right) \right. \\ &\left. - \sin\left(\frac{2n\pi}{\alpha\ell}\xi \right) \sin\left(\frac{2n\pi}{\alpha\ell}c\tau \right) + \sin\left(\frac{2n\pi a}{b\alpha\ell}\xi \right) \sin\left(\frac{2n\pi a}{b\alpha\ell}c\tau \right) \right] \\ + t_n &\left[\sin\left(\frac{2n\pi}{\alpha\ell}\xi \right) \cos\left(\frac{2n\pi}{\alpha\ell}c\tau \right) - \sin\left(\frac{2n\pi a}{b\alpha\ell}\xi \right) \cos\left(\frac{2n\pi a}{b\alpha\ell}c\tau \right) \right. \\ &\left. + \cos\left(\frac{2n\pi}{\alpha\ell}\xi \right) \sin\left(\frac{2n\pi}{\alpha\ell}c\tau \right) + \cos\left(\frac{2n\pi a}{b\alpha\ell}\xi \right) \sin\left(\frac{2n\pi a}{b\alpha\ell}c\tau \right) \right]. \end{aligned}$$

We return to the original variables $x = \xi - \mathbf{v}t$, $\tau = t$. Using $a = \mathbf{v} + c$, this gives:

$$\begin{aligned} &u(x,t) = \\ &\sum_{n=1}^{+\infty} \left[s_n \left(\cos\left(\frac{2n\pi}{\alpha\ell}x \right) - \cos\left(\frac{2n\pi a}{b\alpha\ell}x \right) \right) + t_n \left(\sin\left(\frac{2n\pi}{\alpha\ell}x \right) - \sin\left(\frac{2n\pi a}{b\alpha\ell}x \right) \right) \right] \cos\left(\frac{2n\pi}{\alpha\ell}at \right) \\ &+ \left[-s_n \left(\sin\left(\frac{2n\pi}{\alpha\ell}x \right) - \sin\left(\frac{2n\pi a}{b\alpha\ell}x \right) \right) + t_n \left(\cos\left(\frac{2n\pi}{\alpha\ell}x \right) + \cos\left(\frac{2n\pi a}{b\alpha\ell}x \right) \right) \right] \sin\left(\frac{2n\pi}{\alpha\ell}at \right). \end{aligned}$$

Thus, the function $u(x,t)$ is periodic in t with period:

$$T = \frac{\alpha\ell}{a} = \frac{2c\ell}{c^2 - \mathbf{v}^2}.$$

2.2 Damped Travelling Strings

In this section, we present a derivation of the solution to a damped wave equation defined on a moving interval. For simplicity, we take $c = 1$ and consider the wave equation with linear damping:

$$\begin{cases} u_{tt} - u_{xx} + 2\delta u_t = 0, & x \in (\mathbf{v}t, \ell + \mathbf{v}t),\ t > 0, \\ u(\mathbf{v}t, t) = u(\ell + \mathbf{v}t, t) = 0, & t > 0, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in (0,\ell). \end{cases} \tag{2.7}$$

To simplify the PDE, we use a change of variables and an integrating factor. Define:

$$\phi(\xi,\tau) = u(x,t)e^{\delta t}, \qquad u(x,t) = e^{-\delta t}\phi(\xi,\tau), \tag{2.8}$$

where his time ξ and τ denotes

$$\xi = x - \mathbf{v}t, \qquad \tau = t - \mathbf{v}x.$$

We first compute the transformed derivatives. Since

$$x = \frac{1}{1-\mathbf{v}^2}(\xi + \mathbf{v}\tau) \quad \text{and} \quad t = \frac{1}{1-\mathbf{v}^2}(\tau + \mathbf{v}\xi),$$

we compute partial derivatives:

First derivatives:

$$u_t = \left(\left(e^{-\delta t}\phi(\xi,\tau)\right)\right)_t = -\delta e^{-\delta t}\phi + e^{-\delta t}(\phi_\xi \xi_t + \phi_\tau \tau_t),$$

where $\xi_t = -\mathbf{v}$ and $\tau_t = 1$. Hence,

$$u_t = e^{-\delta t}(-\delta\phi - \mathbf{v}\phi_\xi + \phi_\tau).$$

Second time derivative:

$$u_{tt} = \left(e^{-\delta t}(-\delta\phi - \mathbf{v}\phi_\xi + \phi_\tau)\right)_t = e^{-\delta t}\left(\delta^2\phi + 2\delta\mathbf{v}\phi_\xi - 2\delta\phi_\tau + \mathbf{v}^2\phi_{\xi\xi} - 2\mathbf{v}\phi_{\xi\tau} + \phi_{\tau\tau}\right).$$

First space derivative:

$$u_x = (e^{-\delta t}\phi(\xi,\tau))_x = e^{-\delta t}(\phi_\xi \xi_x + \phi_\tau \tau_x),$$

where $\xi_x = 1$ and $\tau_x = -\mathbf{v}$. Thus,

$$u_x = e^{-\delta t}(\phi_\xi - \mathbf{v}\phi_\tau).$$

Second space derivative:

$$u_{xx} = e^{-\delta t}(\phi_\xi - \mathbf{v}\phi_\tau)_x = e^{-\delta t}(\phi_{\xi\xi} - 2\mathbf{v}\phi_{\xi\tau} + \mathbf{v}^2\phi_{\tau\tau}).$$

Substitute into the PDE:

$$\begin{aligned} u_{tt} - u_{xx} + 2\delta u_t = e^{-\delta t}\big[&\delta^2\phi + 2\delta\mathbf{v}\phi_\xi - 2\delta\phi_\tau + \mathbf{v}^2\phi_{\xi\xi} - 2\mathbf{v}\phi_{\xi\tau} + \phi_{\tau\tau} \\ &-(\phi_{\xi\xi} - 2\mathbf{v}\phi_{\xi\tau} + \mathbf{v}^2\phi_{\tau\tau}) + 2\delta(-\delta\phi - \mathbf{v}\phi_\xi + \phi_\tau)\big] = 0. \end{aligned}$$

Collect like terms:

$$u_{tt} - u_{xx} + 2\delta u_t = e^{-\delta t}\left[\phi_{\tau\tau}(1-\mathbf{v}^2) - \phi_{\xi\xi}(1-\mathbf{v}^2) - \delta^2\phi\right] = 0.$$

This eventually simplifies to the equation:

$$\phi_{\tau\tau} - \phi_{\xi\xi} + \beta\phi = 0, \quad \text{with } \beta = -\frac{\delta^2}{1-\mathbf{v}^2}. \tag{2.9}$$

We solve it using separation of variables:

$$\phi(\xi, \tau) = f(\xi)g(\tau).$$

Plaguing this in the equation (2.9), we get

$$\frac{g''(\tau)}{g(\tau)} + \beta - \frac{f''(\xi)}{f(\xi)} = 0.$$

Then:

$$\frac{f''(\xi)}{f(\xi)} = -\lambda^2 \quad \text{and} \quad \frac{g''(\tau)}{g(\tau)} = -\left(\lambda^2 + \beta\right),$$

On one hand

$$f'' + \lambda^2 f = 0$$

Taking into account the boundary conditions at $\xi = 0$ and $\xi = \ell$, we get

$$f(\xi) = C\sin(\lambda\xi), \quad \lambda_n := \frac{n\pi}{\ell}.$$

On the other hand, g is given by

$$g(\tau) = A\cos(\omega_n\tau) + B\sin(\omega_n\tau), \quad \omega^2 = \lambda_n^2 + \beta.$$

So the solution is:

$$\phi_n(\xi, \tau) = \sin(\lambda_n\xi)\left(a_n\cos(\omega_n\tau) + b_n\sin(\omega_n\tau)\right), \quad \omega_n^2 = \lambda_n^2 + \beta.$$

Thus, the full solution is:

$$u(x,t) = e^{-\delta t}\sum_{n=1}^{\infty}\sin(\lambda_n(x - \mathbf{v}t))\left(a_n\cos(\omega_n(t - \mathbf{v}x)) + b_n\sin(\omega_n(t - \mathbf{v}x))\right).$$

To summarize, we have

Theorem 2.3 *The solution of Problem (2.7) is given by the series:*

$$u(x,t) = e^{-\delta t}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{\ell}(x - \mathbf{v}t)\right)\left(a_n\cos\left(\omega_n(t - \mathbf{v}x)\right) + b_n\sin\left(\omega_n(t - \mathbf{v}x)\right)\right),$$

where:

$$\omega_n = \sqrt{\lambda_n^2 + \beta} = \sqrt{\lambda_n^2 - \frac{\delta^2}{1-\mathbf{v}^2}},$$

and a_n, b_n depend on the initial data.

Chapter 3

The wave equation for travelling membranes

3.1 Undamped travelling membranes

In this chapter, we study the vibrations of a membrane travelling horizontally at a constant speed **v** between two cylindrical pulleys, as shown in Figure 3.1.

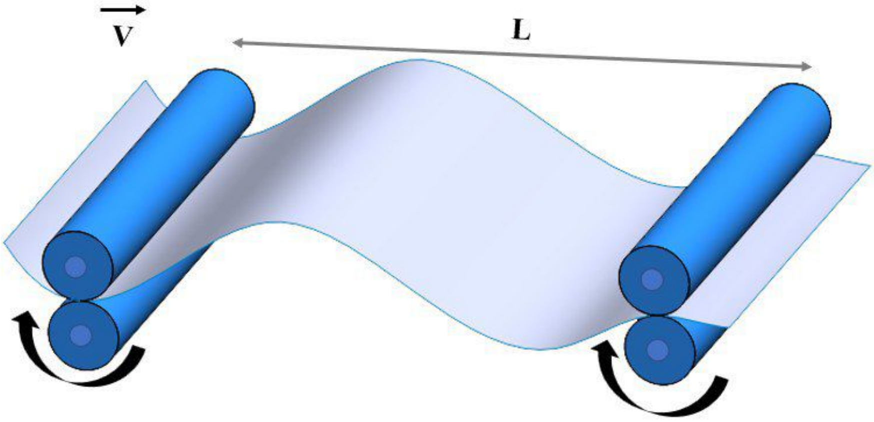

Figure 3.1: A travelling membrane

3.1.1 Derivation of the Equation of Motion

Let ℓ be the distance between the pulleys and b the width of the membrane. The reference domain is defined as

$$\Omega := (0, \ell) \times (0, b).$$

To derive the equation of motion, we apply Hamilton's principle. The kinetic and potential energies of the membrane are, respectively:

$$\begin{aligned} K &= \frac{1}{2}\int_\Omega \rho\left[\mathbf{v}^2 + (u_t - \mathbf{v}u_x)^2\right]dx\,dy,\\ V &= \frac{1}{2}\int_\Omega \mathbf{T}(u_x^2 + u_y^2)dx\,dy, \end{aligned}$$

where ρ is the density and $\mathbf{T}$ the tension.

The action functional is given by

$$I(u) = \int_{t_1}^{t_2}\int_\Omega \left[\rho\left(\mathbf{v}^2 + (u_t - \mathbf{v}u_x)^2\right) - \mathbf{T}(u_x^2 + u_y^2)\right]dx\,dy\,dt.$$

Expanding the integrand, we get:

$$I(u) = \int_{t_1}^{t_2}\int_\Omega \left[\rho\mathbf{v}^2 + \rho u_t^2 + \rho\mathbf{v}^2u_x^2 - 2\rho\mathbf{v}u_tu_x - \mathbf{T}u_x^2 - \mathbf{T}u_y^2\right]dx\,dy\,dt.$$

Let w be a test function such that

$$w(x,y,t_1) = w(x,y,t_2) = 0, \quad \text{and } w(\cdot,\cdot,t) \in V(\Omega),$$

where $V(\Omega)$ is defined by

$$V(\Omega) = \{\varphi \in H^1(\Omega) : \varphi(0,y) = \varphi(\ell,y) = 0 \text{ for a.e. } y \in (0,b)\}.$$

Note that $H_0^1(\Omega) \subset V(\Omega) \subset H^1(\Omega)$.

We compute the variation of the action functional:

$$\begin{aligned} I(u+\sigma w) - I(u) = \int_{t_1}^{t_2}\int_\Omega \Big[&\rho\left((u_t + \sigma w_t - \mathbf{v}(u_x + \sigma w_x))^2 - (u_t - \mathbf{v}u_x)^2\right)\\ &- \mathbf{T}\left((u_x + \sigma w_x)^2 + (u_y + \sigma w_y)^2 - u_x^2 - u_y^2\right)\Big]dx\,dy\,dt. \end{aligned}$$

Expanding each square and simplifying:

$$\begin{aligned} I(u+\sigma w) - I(u) = \int_{t_1}^{t_2}\int_\Omega \Big[&2\sigma\rho u_tw_t - 2\sigma\rho\mathbf{v}u_tw_x - 2\sigma\rho\mathbf{v}u_xw_t + 2\sigma\rho\mathbf{v}^2u_xw_x\\ &- 2\sigma\mathbf{T}u_xw_x - 2\sigma\mathbf{T}u_yw_y\Big]dx\,dy\,dt + O(\sigma^2). \end{aligned}$$

Taking the limit, as $\sigma \to 0$, in

$$\begin{aligned} \delta I(u;w) &= \lim_{\sigma\to 0}\frac{I(u+\sigma w) - I(u)}{\sigma}\\ &= \int_{t_1}^{t_2}\int_\Omega 2\rho u_tw_t - 2\rho\mathbf{v}u_tw_x - 2\rho\mathbf{v}u_xw_t + 2\rho\mathbf{v}^2u_xw_x - 2\mathbf{T}u_xw_x - 2\mathbf{T}u_yw_y\;dx\,dy\,dt. \end{aligned}$$

Integrating by parts and using the boundary and initial conditions (so that all boundary terms vanish), we obtain:

$$\delta I(u;w) = \int_{t_1}^{t_2}\int_\Omega \Big(-2\rho u_{tt} + 4\rho \mathbf{v} u_{tx} - 2\rho \mathbf{v}^2 u_{xx} + 2\mathbf{T} u_{xx} + 2\mathbf{T} u_{yy}\Big) w \; dx\, dy\, dt.$$

Since this must vanish for all admissible functions w, the integrand must be zero:

$$\rho u_{tt} - 2\rho \mathbf{v} u_{tx} + \rho \mathbf{v}^2 u_{xx} - \mathbf{T} u_{xx} - \mathbf{T} u_{yy} = 0.$$

Dividing through by ρ and setting $c^2 = \mathbf{T}/\rho$, we arrive at the final equation of motion:

$$u_{tt} - 2\mathbf{v} u_{tx} + (\mathbf{v}^2 - c^2) u_{xx} - c^2 u_{yy} = 0. \tag{3.1}$$

This is subject to the boundary and initial conditions:

$$\begin{cases} u(0,y,t) = u(\ell,y,t) = 0, & \text{for } y \in (0,b), t > 0, \\ u(x,0,t) = u(x,b,t) = 0, & \text{for } x \in (0,\ell), t > 0, \\ u(x,y,0) = u^0(x,y), \quad u_t(x,y,0) = u^1(x,y), & \text{for } (x,y) \in \Omega. \end{cases} \tag{3.2}$$

Thus, we have derived the equation governing transverse vibrations of a horizontally travelling, undamped membrane.

3.1.2 Energy Conservation and Uniqueness of Solution

The uniqueness of the solution to the initial-boundary value problem defined by equations (3.1)–(3.2) can be established using energy methods.

Define the energy of a solution $u(x,y,t)$ at time t as:

$$E(t) = \frac{1}{2}\int_\Omega \left[u_t^2 + (c^2 - \mathbf{v}^2) u_x^2 + c^2 u_y^2\right] dx\, dy.$$

To prove that $E(t)$ is conserved in time, we compute its derivative:

$$\frac{d}{dt}E(t) = \int_\Omega u_t u_{tt} + (c^2 - \mathbf{v}^2) u_x u_{xt} + c^2 u_y u_{yt} \; dx\, dy.$$

We integrate the second and third terms by parts (in x and y respectively), using the boundary conditions:

$$\int_\Omega u_x u_{xt} dx = [u_t u_x]_{x=0}^{x=\ell} - \int_\Omega u_t u_{xx} dx,$$
$$\int_\Omega u_y u_{yt} dy = [u_t u_y]_{y=0}^{y=b} - \int_\Omega u_t u_{yy} dy.$$

The boundary terms vanish due to the boundary conditions, so we obtain:

$$\frac{d}{dt}E(t) = \int_\Omega u_t u_{tt} - (c^2 - \mathbf{v}^2) u_t u_{xx} - c^2 u_t u_{yy} \; dx\, dy.$$

We now recall from the wave equation (3.1) that:

$$u_{tt} = 2\mathbf{v}u_{tx} - (\mathbf{v}^2 - c^2)u_{xx} + c^2 u_{yy}.$$

Substitute into the energy derivative:

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_\Omega u_t \left[2\mathbf{v}u_{tx} - (\mathbf{v}^2 - c^2)u_{xx} + c^2 u_{yy}\right] - (c^2 - \mathbf{v}^2)u_t u_{xx} - c^2 u_t u_{yy}\, dx\, dy \\ &= 2\mathbf{v}\int_\Omega u_t u_{tx}\, dx\, dy - (\mathbf{v}^2 - c^2)\int_\Omega u_t u_{xx}\, dx\, dy + c^2 \int_\Omega u_t u_{yy}\, dx\, dy \\ &\quad - (c^2 - \mathbf{v}^2)\int_\Omega u_t u_{xx}\, dx\, dy - c^2 \int_\Omega u_t u_{yy}\, dx\, dy.\end{aligned}$$

The terms involving u_{xx} and u_{yy} cancel, and we are left with:

$$\frac{d}{dt}E(t) = 2\mathbf{v}\int_\Omega u_t u_{tx} dx\, dy.$$

Integrating this term by parts in x:

$$\int_\Omega u_t u_{tx} dx\, dy = \frac{1}{2}\int_\Omega \partial_x(u_t^2) dx\, dy = \frac{1}{2}\int_0^b \left[u_t^2(\ell, y, t) - u_t^2(0, y, t)\right] dy = 0,$$

again due to the fixed boundary conditions at $x = 0$ and $x = \ell$. Hence, the energy is conserved and

$$E(t) = E(0), \quad \text{for } t > 0.$$

As a consequence, we have

Theorem 3.1 *The solution of Problem (3.1)–(3.2) is unique if $c > \mathbf{v}$.*

Proof. Let u_1 and u_2 be two solutions with the same initial and boundary data. Then $w = u_1 - u_2$ satisfies the homogeneous problem:

$$\begin{cases} w_{tt} - 2\mathbf{v}w_{tx} + (\mathbf{v}^2 - c^2)w_{xx} - c^2 w_{yy} = 0, & \text{in } \Omega, t > 0, \\ w = 0, & \text{on } \partial\Omega, \\ w(x, y, 0) = w_t(x, y, 0) = 0, & \text{in } \Omega. \end{cases}$$

The energy associated to w is zero at $t = 0$, and remains zero for all t by conservation. Since the energy is a sum of positive terms and $c > \mathbf{v}$, it follows that $w = 0$, i.e., $u_1 = u_2$. ■

3.1.3 Change of Coordinates

Let us perform a change of variables:

$$\xi = x + \mathbf{v}t, \quad \tau = t, \quad y = y.$$

Under this transformation:

$$\begin{aligned} u_x &= u_\xi, & u_t &= \mathbf{v}u_\xi + u_\tau, \\ u_{xx} &= u_{\xi\xi}, & u_{xt} &= \mathbf{v}u_{\xi\xi} + u_{\xi\tau}, \\ u_{tt} &= \mathbf{v}^2 u_{\xi\xi} + 2\mathbf{v}u_{\xi\tau} + u_{\tau\tau}. \end{aligned}$$

Substituting into the equation of motion (3.1):

$$\begin{aligned} u_{tt} - 2\mathbf{v}u_{tx} + &(\mathbf{v}^2 - c^2)u_{xx} - c^2 u_{yy} \\ &= \mathbf{v}^2 u_{\xi\xi} + 2\mathbf{v}u_{\xi\tau} + u_{\tau\tau} - 2\mathbf{v}(\mathbf{v}u_{\xi\xi} + u_{\xi\tau}) + (\mathbf{v}^2 - c^2)u_{\xi\xi} - c^2 u_{yy} \\ &= u_{\tau\tau} - c^2(u_{\xi\xi} + u_{yy}). \end{aligned}$$

Therefore, in the new coordinates, the PDE becomes:

$$u_{\tau\tau} - c^2(u_{\xi\xi} + u_{yy}) = 0.$$

The new domain is time-dependent since $\xi \in (\mathbf{v}\tau, \ell + \mathbf{v}\tau)$. Thus, the transformed initial-boundary value problem reads:

$$\begin{cases} u_{\tau\tau} - c^2(u_{\xi\xi} + u_{yy}) = 0, & \xi \in (\mathbf{v}\tau, \ell + \mathbf{v}\tau), y \in (0, b), \tau > 0, \\ u(\mathbf{v}\tau, y, \tau) = u(\ell + \mathbf{v}\tau, y, \tau) = 0, & y \in (0, b), \tau > 0, \\ u_y(\xi, 0, \tau) = u_y(\xi, b, \tau) = 0, & \xi \in (\mathbf{v}\tau, \ell + \mathbf{v}\tau), \tau > 0, \\ u(\xi, y, 0) = u^0(\xi), \quad u_\tau(\xi, y, 0) = u^1(\xi), & \xi \in (0, \ell), y \in (0, b). \end{cases}$$

This represents the wave equation in a moving domain with homogeneous mixed boundary conditions.

3.1.4 Exact Solution

For simplicity, we fix $c^2 = 1$ and consider the transverse vibrations $u(x, y, t)$ of a homogeneous membrane in a moving rectangle with boundaries $0 < y < b$ and $x \in (\mathbf{v}t, \mathbf{v}t + \ell)$. The governing equation is:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0, & x \in (\mathbf{v}t, \mathbf{v}t + \ell), \ y \in (0, b), \ t > 0, \\ u(\mathbf{v}t, y, t) = u(\mathbf{v}t + \ell, y, t) = 0, & t > 0, \\ u_y(x, 0, t) = u_y(x, b, t) = 0, & x \in (\mathbf{v}t, \mathbf{v}t + \ell), \ t > 0. \end{cases} \tag{3.3}$$

with initial condition u^0 and u^1.

We seek a solution by separation of variables in the form:

$$u(x, y, t) = \varphi(x, t)Y(y).$$

Substituting into the PDE yields:

$$\varphi_{tt}Y - \varphi_{xx}Y - \varphi Y'' = 0.$$

Dividing both sides by φY (assuming $\varphi \neq 0$ and $Y \neq 0$):

$$\frac{\varphi_{tt}}{\varphi} - \frac{\varphi_{xx}}{\varphi} = \frac{Y''}{Y} = -k^2.$$

This separation yields two ordinary differential equations:

- The y-dependent part:

  $$Y'' + k^2 Y = 0, \quad Y'(0) = Y'(b) = 0.$$

  The general solution is

  $$Y(y) = R\cos(ky) + Z\sin(ky).$$

  The boundary condition $Y'(0) = 0$ implies $Z = 0$, so:

  $$Y(y) = R\cos(ky).$$

  The second condition $Y'(b) = 0$ implies $\sin(kb) = 0$, so:

  $$k = k_m := \frac{m\pi}{b}, \quad m \in \mathbb{N}^*.$$

  Therefore:

  $$Y(y) = R\cos\left(\frac{m\pi y}{b}\right).$$

- The equation for $\varphi(x,t)$ becomes:

  $$\varphi_{tt} - \varphi_{xx} + k_m^2\varphi = 0, \quad \varphi(\mathbf{v}t, t) = \varphi(\mathbf{v}t + \ell, t) = 0.$$

To handle the moving domain, we switch to the moving frame:

$$\xi = x - \mathbf{v}t, \quad \tau = t.$$

Let $s(\xi,\tau) = \varphi(x,t)$, so that derivatives transform as:

$$\varphi_t = -\mathbf{v}s_\xi + s_\tau, \qquad \varphi_{tt} = \mathbf{v}^2 s_{\xi\xi} - 2\mathbf{v}s_{\xi\tau} + s_{\tau\tau},$$

$$\varphi_x = s_\xi, \qquad \varphi_{xx} = s_{\xi\xi}.$$

Substituting into the PDE:

$$\begin{aligned}\varphi_{tt} - \varphi_{xx} + k_m^2\varphi &= (\mathbf{v}^2 s_{\xi\xi} - 2\mathbf{v}s_{\xi\tau} + s_{\tau\tau}) - s_{\xi\xi} + k_m^2 s \\ &= (1-\mathbf{v}^2)s_{\tau\tau} - (1-\mathbf{v}^2)s_{\xi\xi} + k_m^2 s.\end{aligned}$$

This simplifies to:

$$\begin{cases} s_{\tau\tau} - s_{\xi\xi} + \dfrac{k_m^2}{1-\mathbf{v}^2} s = 0, \quad \xi \in (0,\ell), \ \tau > 0, \\ s(0,\tau) = s(\ell,\tau) = 0. \end{cases} \tag{3.4}$$

We solve this by separation of variables

$$s(\xi,\tau) = h(\xi)g(\tau),$$

leading to:

$$\frac{g''}{g} = -\left(\lambda^2 + \frac{k_m^2}{1-\mathbf{v}^2}\right), \quad \text{and} \quad \frac{h''}{h} = -\lambda^2.$$

So we solve:

$$h'' + \lambda^2 h = 0,$$

Due to the Boundary conditions $h(0) = h(\ell) = 0$, the solution is

$$h(\xi) = \sin(\lambda_n \xi), \quad \text{where} \ \ \lambda_n := \frac{n\pi}{\ell},$$

The function g satisfies

$$g'' + \omega_{mn}^2 g = 0, \quad \text{where} \ \ \omega_{mn}^2 := \lambda_n^2 + \frac{k_m^2}{1-\mathbf{v}^2}$$

Solutions are:

$$g(\tau) = a_{mn}\cos(\omega_{mn}\tau) + b_{mn}\sin(\omega_{mn}\tau).$$

Returning to original variables $x = \xi + \mathbf{v}t$, $t = \tau$, we find:

$$u_{mn}(x,y,t) = \sin(\lambda_n(x-\mathbf{v}t))\left[a_{mn}\cos(\omega_{mn}t) + b_{mn}\sin(\omega_{mn}t)\right]\cos(k_m y).$$

By superposition:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn}\cos(\omega_{mn}t) + b_{mn}\sin(\omega_{mn}t)\right]\sin(\lambda_n(x-\mathbf{v}t))\cos(k_m y),$$

with

$$\lambda_n = \frac{n\pi}{\ell}, \quad k_m = \frac{m\pi}{b}, \quad \omega_{mn} = \sqrt{\lambda_n^2 + \frac{k_m^2}{1-\mathbf{v}^2}}.$$

Summarizing, we have

Theorem 3.2 *The solution of the wave equation* (3.3) *in the moving domain is given by:*

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn}\cos(\omega_{mn}t) + b_{mn}\sin(\omega_{mn}t)\right]\sin\left(\lambda_n(x-\mathbf{v}t)\right)\cos(k_m y),$$

where a_{mn}, b_{mn} *are constants determined by the initial conditions.*

3.1.5 Damped Travelling Membranes

We now consider the damped version of the wave equation in a moving domain. Let $u(x, y, t)$ denote the displacement in a rectangle of length ℓ and height b moving with speed $\mathbf{v}$. We assume the presence of interior damping with coefficient $2\delta > 0$:

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} + 2\delta u_t = 0, & x \in (\mathbf{v}t, \mathbf{v}t + \ell),\ y \in (0, b),\ t > 0, \\ u(\mathbf{v}t, y, t) = u(\mathbf{v}t + \ell, y, t) = 0, & t > 0, \\ u_y(x, 0, t) = u_y(x, b, t) = 0, & x \in (\mathbf{v}t, \mathbf{v}t + \ell),\ t > 0. \end{cases} \tag{3.5}$$

We separate variables in the form

$$u(x, y, t) = \varphi(x, t)Y(y)$$

and substitute into the PDE:

$$\varphi_{tt}Y - \varphi_{xx}Y - \varphi Y'' + 2\delta\varphi_t Y = 0.$$

Dividing by φY (assuming nonzero functions), we get:

$$\frac{\varphi_{tt}}{\varphi} - \frac{\varphi_{xx}}{\varphi} + 2\delta\frac{\varphi_t}{\varphi} = \frac{Y''}{Y} = -k^2.$$

This yields two equations:

- The y-equation:

  $$Y''^2Y = 0, \quad Y'(0) = Y'(b) = 0.$$

  Its solution is:

  $$Y(y) = R\cos(ky), \quad \text{with } k = \frac{m\pi}{b},\ m \in \mathbb{N}^*.$$

- The x, t equation:

  $$\varphi_{tt} - \varphi_{xx} + 2\delta\varphi_t + k_m^2\varphi = 0, \quad \varphi(\mathbf{v}t, t) = \varphi(\mathbf{v}t + \ell, t) = 0. \tag{3.6}$$

We switch to the moving frame:

$$\xi = x - \mathbf{v}t, \quad \tau = t.$$

Let $s(\xi, \tau) = \varphi(x, t)e^{\delta t}$, so that $\varphi(x, t) = e^{-\delta t}s(\xi, \tau)$. Then:

$$\begin{aligned} \varphi_t &= -\delta e^{-\delta t}s + e^{-\delta t}s_\tau - \mathbf{v}e^{-\delta t}s_\xi, \\ \varphi_{tt} &= e^{-\delta t}\left(s_{\tau\tau} - 2\mathbf{v}s_{\xi\tau} + \mathbf{v}^2 s_{\xi\xi} - 2\delta s_\tau + 2\delta\mathbf{v}s_\xi + \delta^2 s\right), \\ \varphi_x &= e^{-\delta t}s_\xi, \quad \varphi_{xx} = e^{-\delta t}s_{\xi\xi}. \end{aligned}$$

Substitute into (3.6):

$$\varphi_{tt}-\varphi_{xx}+2\delta\varphi_t+k_m^2\varphi=e^{-\delta t}\left(s_{\tau\tau}-2\mathbf{v}s_{\xi\tau}+\mathbf{v}^2s_{\xi\xi}-s_{\xi\xi}+2\delta s_\tau-2\delta\mathbf{v}s_\xi+k_m^2s-\delta^2s\right).$$

Group and simplify:

$$s_{\tau\tau}-2\mathbf{v}s_{\xi\tau}+(\mathbf{v}^2-1)s_{\xi\xi}+2\delta s_\tau-2\delta\mathbf{v}s_\xi+(k_m^2-\delta^2)s=0.$$

This can be reduced to a standard wave equation using the form

$$s(\xi,\tau)=h(\xi)g(\tau),$$

but we first note that the cross term $s_{\xi\tau}$ complicates separation. However, *assuming v is small*, we can neglect these terms. Otherwise, we proceed by assuming that s still satisfies a wave-like equation with effective potential:

$$s_{\tau\tau}-s_{\xi\xi}+\gamma s=0,\quad \text{where } \gamma:=\frac{k_m^2-\delta^2}{1-\mathbf{v}^2}.$$

Now, we solve by separation of variables:

$$s(\xi,\tau)=h(\xi)g(\tau).$$

Substitute into the equation, we get

$$\frac{g''}{g}-\gamma-\frac{h''}{h}=0.$$

Thus, h satisfies

$$h''+\lambda^2h=0.$$

Due to the boundary conditions $h(0)=h(\ell)=0$, we get

$$h(\xi)=\sin\left(\frac{n\pi}{\ell}\xi\right),\quad \text{where } \lambda=\lambda_n:=\frac{n\pi}{\ell},$$

For the function g, we have

$$g''-\left(\gamma+\lambda^2\right)g=0$$

hence

$$g(\tau)=a_{mn}\cos(\omega_{mn}\tau)+b_{mn}\sin(\omega_{mn}\tau),$$

where

$$\omega_{mn}^2:=\lambda_n^2+\gamma=\frac{n^2\pi^2}{\ell^2}+\frac{k_m^2-\delta^2}{1-\mathbf{v}^2}.$$

Therefore, the solution is:

$$u(x,y,t)=e^{-\delta t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn}\cos(\omega_{mn}t)+b_{mn}\sin(\omega_{mn}t)\right]\sin\left(\lambda_n(x-\mathbf{v}t)\right)\cos(k_my).$$

Finally, we can state these result in the following

Theorem 3.3 *The solution of the damped wave equation* (3.5) *is given by:*

$$u(x,y,t) = e^{-\delta t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[a_{mn} \cos(\omega_{mn} t) + b_{mn} \sin(\omega_{mn} t)\right] \sin\left(\lambda_n (x - \mathbf{v} t)\right) \cos(k_m y),$$

where

$$\lambda_n = \frac{n\pi}{\ell}, \quad k_m = \frac{m\pi}{b}, \quad \omega_{mn} = \sqrt{\frac{n^2\pi^2}{\ell^2} + \frac{k_m^2 - \delta^2}{1 - \mathbf{v}^2}}.$$

Bibliography

[1] V. P. Boldin, and, .M.V Trubin, (1991). Wave effects in a stretched band. *Soviet applied mechanics*, 27(9):900–904.

[2] H. Brezis, (1983) *Analyse fonctionnelle, theorie et applications*, Masson, Paris.

[3] W. L. Miranker, (1960), The wave equation in a medium in motion. *IBM J Res Dev*, 4(1):36–42.

[4] S. Salsa, (2016)*Partial Differential Equations in Action*,from modelling to theory, volume (99). Springer.

[5] W.A Strauss, 2007). *Partial differential equations: An introduction.* John Wiley & Sons.

الملخص:

في هذه الدراسة، نفحص معادلات الموجة أحادية الأبعاد وثنائية الأبعاد في مجالات تتغير مع الزمن. نقوم باشتقاق معادلة الموجة باستخدام مبدأ هاميلتون. ثم نستخدم طريقة فصل المتغيرات لحل المعادلة. كما ندرس أيضًا الحالة التي تتضمن تخميدًا خطيًا داخليًا.

الكلمات المفتاحية: معادلة الموجة، ميادين تتغير مع الزمن، فصل المتغيرات، التخميد.

Abstract

In this study, we examine the one-dimensional and two-dimensional wave equations in time-dependent domains. We derive the wave equation using Hamilton's principle. Then, we employ the method of separation of variables to solve the equation. We also consider the case with interior linear damping.

Keywords: Wave equation, time-dependent domains, separation of variables method, damping.

Résumé

Dans cette étude, nous examinons les équations d'onde unidimensionnelles et bidimensionnelles dans des domaines dépendant du temps. Nous dérivons l'équation d'onde en utilisant le principe de Hamilton. Ensuite, nous employons la méthode de séparation des variables pour résoudre l'équation. Nous considérons également le cas avec un amortissement linéaire intérieur.

Mots clés: Équation d'onde, domaines dépendant du temps, séparation des variables, amortissement.