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MOHAMED BOUDIAF UNIVERSITY -M'SILA
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Transitive Closure via Composition of Relations

Presented by :
Manel Mebdoua

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Before the jury composed of:

Chair :	M^r *ZEDAM* Lemnaouar	Prof, Med Boudiaf University-M'Sila
Supervisor :	M^{me} *BAKRI* Norelhouda	M.C.B, Med Boudiaf University-M'Sila
Examiner :	M^r *SAADAOUI* Kheir	M.C.A, Med Boudiaf University-M'Sila

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Dedication

To myself. . .
After God, you were my strength in moments of weakness, my hope when the light dimmed. Today, I honor you with this achievement one that mirrors your perseverance alone.

To my mother. . .
You, my answered prayer, my steadfast support when days grew difficult. I dedicate my success to you the fruit of your patience, your prayers, and your contentment.

To my dear brothers and sisters...
You were always my help and my shield, the laughter during times of exhaustion. Your presence beside me was a blessing, your words propelled me forward. Thank you for being part of my journey.

Contents

Introduction

Relations form a foundational construct in mathematics, capturing associations between elements either within a single set or across multiple sets. Relations come in many flavours, such as binary or ternary, crisp or fuzzy, et cetera. Among these, binary relations (associating pairs of elements) are pivotal in discrete mathematics and underpin core areas such as computer science, logic, database theory, and graph theory. Binary relations can display various interesting properties, such as reflexivity, symmetry, and notably, transitivity [3, 12, 18, 30, 35].

Transitivity is by far the most interesting property of (crisp and fuzzy) binary relations within finite domains [18], or in infinite domains [14, 29, 38]. Transitivity is essential in formal deduction, reachability in graphs, and logical inference chains. In fuzzy frameworks, transitivity is often defined via T-transitivity, using triangular norms for generalized conjunction [33]. Crucially, composing relations provides a natural and compact way to express transitivity as relational inclusion [32].

Composition has deep historical roots in relational calculus and has been developed substantially in the fuzzy domain. Bandler and Kohout introduced fuzzy relational products based on quantifier substitution [3, 4]; Goguen expanded this approach via lattice-theoretic fuzzy relations [26]; and De Baets and Kerre deepened the theoretical foundations of fuzzy relational compositions [13]. More recent work by Štěpnička and Holčapek extends the framework to generalized fuzzy quantifiers [39], while interval-valued fuzzy relations and their transitive closures have been further explored by González-del-Campo et al. [9, 43]. Beyond fuzzy systems, relational compositions appear in fuzzy relational equations [27, 11], formal concept analysis [25, 35], temporal and spatial reasoning within relation algebras [21], inference systems [23, 24], medical diagnosis [12], and relational databases [37]. Computational studies include algorithms tailored for symmetric matrices [10], interval-valued fuzzy closures [9], and dynamic graphs [23].

In practice, binary relations often fail to meet the transitivity condition. More broadly, if a binary relation R lacks a desired property P, one may ask whether it is possible to determine the smallest binary relation that contains R and satisfies P, this is known as the P-closure. This concept of closure are fundamental in various mathematical fields like geometry [41] and logic [31], as well as computational disciplines such as database theory [17] and data analysis [22]. They are also relevant in practical domains like medical diagnosis [2, 5, 6] and handwriting recognition [28]. Bandler and Kohout [7] have provided general findings that apply to both crisp and fuzzy binary relations, while more focused studies on transitivity specifically can be found in [16].

Despite this rich literature, the use of relation compositions to derive transitive closures (especially through matrix-based representations) remains underrepresented in educational materials and computational frameworks. Standard methods often rely on graph algorithms or on matrix algorithms such as Warshall's or Floyd–Warshall's [12, 30, 21]. However, the composition perspective offers both theoretical elegance and practical alignment, especially when translating relational operations into matrix computations. This thesis addresses this gap by delivering a complete exploration of transitive closure via relation composition, combining set-theoretic, graphical, and matrix-based approaches. The main aim of the present work is to clarify how relational composition constructs transitive closures, formalizing the process using matrix operations to facilitate algorithmic implementation, investigating Warshall's matrix-based algorithm for computing transitive closures, and comparing these methods with respect to their theoretical foundations. By pursuing these objectives, this work strengthens relation theory's foundational principles and supports practical and pedagogical applications in discrete mathematics and theoretical computer science. It also lays groundwork for extending these approaches to fuzzy relations [8, 34, 43] and optimizing transitive closure methods in large-scale and dynamic systems [23].

This work is structured as follows:

- **Chapter 1** introduces binary relations, their properties, and representations.
- **Chapter 2** explores relation composition and its role in generating transitive closures.
- **Chapter 3** focuses on computational techniques including Warshall's algorithm, matrix implementations, and comparative analyses.
- Finally, **the conclusion** and **future work** are drawn.

Chapter 1

Preliminaries

In this chapter, we provide some basic notions and definitions concerning binary relations and their properties, as well as specific classes of binary relations. For more details, we refer to [7, 30, 34]

1.1 Binary relations

1.1.1 Product sets

Definition 1.1. [34] Let X and Y be two non-empty sets. The product of these two sets is the Cartesian product and is denoted by $X \times Y$, defined as:

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

One frequently writes X^2 instead of $X \times X$.

Example 1.1. Let X and Y be two finite sets such that $X = \{1, 2\}$ and $Y = \{a, b, c\}$. Then:

(i) $X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

(ii) $Y \times X = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

(iii) $X \times X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

One can easily see that $X \times Y \neq Y \times X$.

1.1.2 Binary relations

Definition 1.2. [30] Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be two finite sets. A subset $R \subseteq X \times Y$ is called a **binary relation** (or simply, a relation) between X and Y, if $(x, y) \in R$ (i.e., x is related to y), and we write:

$$xRy.$$

If $X = Y$, then $R \subseteq X \times X$, and we say that R is a relation on X.

The **domain** of a relation R is the set of all first elements of the ordered pairs which belong to R, and **the range** is the set of second elements.

Example 1.2. Let $X = \{1, 2, 3\}$ and $Y = \{x, y, z\}$, and let the relation $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from X to Y since $R \subseteq X \times Y$. With respect to this relation:

$$1Ry, \quad 1Rz, \quad 3Ry,$$

but the following pairs are not in R:

$$1Rx, \quad 2Rx, \quad 2Ry, \quad 2Rz, \quad 3Rx, \quad 3Rz$$

The **domain** of R is:

$$\text{dom}(R) = \{1, 3\}$$

The **range** of R is:

$$\text{ran}(R) = \{y, z\}$$

Three special binary relations on X are:

(i) the null relation $O_{X^2} = \emptyset$,

(ii) the binary identity relation $\Delta_X = \{(x, x) | x \in X\}$,

(iii) the universal binary relation X^2.

Two elements x and y of a set X equipped with a binary relation R are called comparable elements if it holds that $(x, y) \in R$ or $(y, x) \in R$, otherwise, they are called incomparable elements.

1.1.3 Operations on binary relations

Let X be a non-empty set and R_1 and R_2 be two binary relations on X,then [1]:

Inclusion:
A binary relation R_1 on a set X is said to be included in a binary relation R_2 on the same set X, denoted by $R_1 \subseteq R_2$ if, for any $(x, y) \in X$, $(x, y) \in R_1$ implies that $(x, y) \in R_2$.

Intersection:
The intersection of relations R_1 and R_2 on X is the binary relation $R_1 \cap R_2$ on X defined as

$$R_1 \cap R_2 = \{(x, y) \in X^2 | (x, y) \in R_1 \wedge (x, y) \in R_2\}.$$

If $R_1 \cap R_2 = \emptyset$, then R_1 and R_2 are called disjoint relations.

Union:
The union of two binary relations R_1 and R_2 on X is the binary relation $R_1 \cup R_2$ on X defined as

$$R_1 \cup R_2 = \{(x, y) \in X^2 | (x, y) \in R_1 \vee (x, y) \in R_2\}.$$

Transpose:
For a given binary relation R on a set X, we denote the transpose (converse) of R by R^t, i.e., for any $x, y \in X$, $(x, y) \in R^t$ means that $(y, x) \in R$.

Complement:
We denote the complement of R by R^c, i.e., for any $x, y \in X$, $(x, y) \in R^c$ means that $(x, y) \notin R$.

The dual:
We denote the dual of R by R^d, i.e., for any $x, y \in X$, $(x, y) \in R^d$ means that $(y, x) \notin R$.
We will discuss the composition in the next chapter.

1.2 Properties of binary relation

A binary relation can have various properties, depending on how the elements are related [30].

1.2.1 Reflexive relations

A relation R on set X is **reflexive** if for every element $x \in X$, the pair $(x, x) \in R$. That is, a relation is reflexive if it contains all pairs (x, x) for any element x in X. If even one such pair is missing, the relation is **not reflexive**.

Example 1.3. Consider the set X= {5,6,7,8} and the following five relations on X:

- $R_1 = (5, 5), (5, 6), (6, 7), (5, 7), (8, 8)$;
- $R_2 = (5, 5), (6, 6), (7, 7), (8, 8), (5, 6), (6, 5)$;
- $R_3 = (5, 7), (6, 5)$;
- $R_4 = \emptyset$ (the empty relation);
- $R_5 = X \times X$ (the universal relation).

Since the set X contains four elements 5,6,7,8, a relation on X is ***reflexive*** if it includes the pair (5,5),(6,6),(7,7) and (8,8).

- R_2 and R_5 are reflexive since they contain all the necessary pairs.
- $R_1, R_2,$ and R_4 are not reflexive because at least one of these pair is missing (for example, (6,6) is missing in R_1, and R_4 contains no pairs at all).

1.2.2 Symmetric and antisymmetric relations

A relation R on a set X is **symmetric** if for any pair $(x, y) \in R$, the pair (y, x) also is in R. In other words, if x is related to y,then y must also be related to x.
If there exists at least one pair $(x, y) \in R$ such that $(y, x) \notin R$, then R is **not symmetric**.

Example 1.4. In Example 1.3, R_1 is not symmetric since $(5, 6) \in R_1$ but $(6, 5) \notin R_1$. R_3 is not symmetric since $(5, 6) \in R_3$ but $(6, 5) \notin R_3$. The other relations are symmetric.

A relation R on a set X is antisymmetric if whenever aRb and bRa then $x = y$, that is, if $x \neq y$ and xRy then $y \not R x$. Thus, R is not antisymmetric if there exist distinct elements x and y in X such that xRy and yRx.

Example 1.5. In Example 1.3, R_2 is not antisymmetric since (5,6) and (6,5) belong to R_2, but $5 \neq 6$. Similarly, the universal relation R_5 is not antisymmetric. All other relations are antisymmetric.

1.2.3 Transitive relations

A relation R on set X is considered **transitive** if, whenever x is related to y, (xRy) and y is related to z (yRz), then x must also be related to z (xRz). In other words, if both $(x, y) \in R$ and $(y, z) \in R$, then it must follow that $(x, z) \in R$.
If there exist elements x,y and z in X such that $(x, y) \in R$ and $(y, z) \in R$, but $(x, z) \notin R$, then the relation R is **not transitive**.

Example 1.6. In Example 1.3, the relation R_3 is not transitive since $(6, 5), (5, 7) \in R_3$ but $(6, 7) \notin R_3$. All other relations are transitive.

A preorder is binary relation over a set X which is reflexive and transitive. A partial order (order for short) is a binary relation on a set X that is reflexive, antisymmetric and transitive, a set with an order relation is called an ordered set (also called a poset). A tolerance is a binary relation on a set X that is reflexive and symmetric. A binary relation on a set X is said to be an equivalence relation if is it reflexive, symmetric and transitive.

There are several ways to represent a relation between finite sets. As demonstrated before, one approach is to list the relation's ordered pairs. Another method, involves using a table. In the following, we explore two additional techniques for representing relations. One employs visual representations known as directed graphs, while the other involves the use of zero-one matrices. Typically, matrices are well-suited for implementing relations in computer programs. In contrast, directed graphs are often preferred for visually analyzing and understanding the properties of these relations.

1.3 Graph represntation of relations

Graphs serve as a visual tool for representing relationships. Different types of graphs are used for this purpose, including directed graphs (digraphs) and undirected graphs. Here is a detailed look at these representations [30].

1.3.1 Directed graphs

Definition 1.3. A directed graph, or digraph, consists of a set X of vertices (or nodes) together with a set Y of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (x, y), and the vertex b is called the terminal vertex of this edge.

Properties:

1. A relation R is reflexive if every node in a directed graph has a loop.
2. A relation R is irreflexive if no node in a directed graph has a loop.
3. A relation R is symmetric if, for every edge between two distinct nodes, there is always an edge in the reverse direction.
4. A relation R is asymmetric if for any two distinct nodes, there are no edges in both directions between them.
5. A relation R is transitive if, whenever there is an edge from x to y and an edge from y to z, there must also be an edge from x to z.

Example 1.7. The directed graph of the relation

$$R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$$

, on the set $\{1, 2, 3, 4\}$ is shown in Figure 1.1

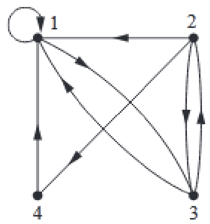

Figure 1.1: The Directed Graph of the Relation R.

Example 1.8. The ordered pairs in the relation R represented by the directed graph shown in Figure 1.2 are:

$$R = \{(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,3),(4,1),(4,3)\}$$

. Each of these pairs corresponds to an edge of the directed graph, with (2,2) and (3,3) corresponding to loops.

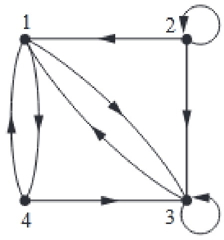

Figure 1.2: The Directed Graph of the Relation R.

Remark 1.1. Note that a symmetric relation can be represented by an undirected graph, which is a graph where edges do not have directions.

1.3.2 Undirected graphs

In an undirected graph, the edges do not have a specific direction. This type of representation is useful when the relationship is symmetric. In other words, if node a is connected to node b, then node b is also connected to node a.

Example 1.9. Consider a relation R on the set $X = \{1, 2, 3\}$, where $R = \{(1, 2)(2, 3)\}$. The corresponding undirected graph consists of three vertices: 1, 2 and 3, with edges connecting 1 and 2, as well as 2 and 3.

1.4 Matrix representation of relations

A relation can be represented in different ways, and one common method is using a matrix. A matrix is simply a rectangular arrangement of numbers organized into rows and columns. When a matrix has n rows and m columns, it is referred to as an $n \times m$ matrix. Each element in the matrix is located at the intersection of a specific row and column, and the element in the ith row and jth column is denoted by M_{ij}.

Example 1.10. Consider the following 2×3 matrix M:

$$M = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

From this matrix, we can determine that : $M_{13} = 0$, $M_{22} = 2$.

1.4.1 Matrix addition and multiplication

Before using matrices to solve problems, it is important to understand the basic operations that can be performed on them.

Definition 1.4. Let X and Y be two matrices of the same size. The sum of these matrices, denoted as $X + Y$, results in another matrix where each element is computed as:

$$X + Y = [x_{ij} + y_{ij}]$$

Example 1.11. Adding two matrices:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 \\ 3 & 0 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

This shows that matrix addition is performed by adding corresponding elements from both matrices.

Definition 1.5. Let X be an $m \times k$ matrix with entries x_{ij}, and Y be a $k \times n$ matrix with entries y_{ij}. The product $X \times Y$ is an $m \times n$ matrix $Z = [z_{ij}]$, where each entry z_{ij} is computed as:

$$z_{ij} = \sum_{l=1}^{k} x_{il} y_{lj} = x_{i1} y_{1j} + x_{i2} y_{2j} + \cdots + x_{ik} y_{kj}.$$

Example 1.12. Given:

$$X = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

The product $X \times Y$ is calculated as:

$$X \times Y = \begin{bmatrix} (1 \cdot 2 + 0 \cdot 1 + 4 \cdot 3) & (1 \cdot 4 + 0 \cdot 1 + 4 \cdot 0) \\ (2 \cdot 2 + 1 \cdot 1 + 1 \cdot 3) & (2 \cdot 4 + 1 \cdot 1 + 1 \cdot 0) \\ (3 \cdot 2 + 1 \cdot 1 + 0 \cdot 3) & (3 \cdot 4 + 1 \cdot 1 + 0 \cdot 0) \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \end{bmatrix}.$$

1.4.2 Matrix representation of relations

One useful way to represent relations between elements of finite sets is by using matrices, which provide a clear and structured format.

Definition 1.6. [12] Let R be a relation from set $X = \{x_1, x_2, \ldots, x_m\}$ to set $Y = \{y_1, y_2, \ldots, y_n\}$. The relation R can be represented by an $m \times n$ matrix $M_R = [m_{ij}]$, where:

$$m_{ij} = \begin{cases} 1 & \text{if } (x_i, y_j) \in R, \\ 0 & \text{if } (x_i, y_j) \notin R. \end{cases}$$

Example 1.13. Let $X = \{1, 3, 5\}$, $Y = \{1, 2\}$, and $R = \{(1, 1), (3, 2), (5, 1)\}$. The matrix representation of R is:

$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

1.4.3 Useful characteristics

The 0–1 matrix representation of a relation provides a practical and efficient way to verify several important properties, such as reflexivity, symmetry, and antisymmetry.

Reflexivity

A relation R on a set is said to be *reflexive* if every element is related to itself; that is, $\forall x\, ((x, x) \in R)$. In terms of the 0–1 matrix $M_R = [m_{i,j}]$ representing the relation, this property requires that all diagonal entries satisfy $m_{i,i} = 1$ for $i = 1, 2, \ldots, n$. Thus, to verify reflexivity, one simply needs to check whether all diagonal elements of the matrix are equal to 1:

$$\text{If } m_{i,i} = 1 \text{ for all } i, \text{ then } R \text{ is reflexive; otherwise, it is not.}$$

Symmetry

A relation R is *symmetric* if for all elements x and y, whenever xRy, then yRx; that is, $\forall x, y\, ((x, y) \in R \Rightarrow (y, x) \in R)$. In matrix terms, this condition is satisfied when $m_{i,j} = m_{j,i}$ for all $i, j = 1, 2, \ldots, n$. Equivalently, the matrix of the relation must be equal to its transpose:

$$\mathbf{M}_R = (\mathbf{M}_R)^T.$$

Antisymmetry

A relation R is *antisymmetric* if, for all $x \neq y$, whenever $(x, y) \in R$, it must be that $(y, x) \notin R$. Translated to matrix terms, this means that for all $i \neq j$, if $m_{i,j} = 1$, then $m_{j,i} = 0$. This condition can also be written using logical disjunction:

$$\forall i \neq j, \quad (m_{i,j} = 0) \vee (m_{j,i} = 0),$$

or, using logical negation of conjunction:

$$\forall i \neq j, \quad \neg(m_{i,j} \wedge m_{j,i}).$$

Example 1.14. Consider the following relation matrix:

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

1. **Not reflexive**: The entry $m_{2,2} = 0$, while reflexivity requires all diagonal entries to be 1.
2. **Not symmetric**: For instance, $m_{1,2} = 1$ but $m_{2,1} = 0$. Since $m_{i,j} \neq m_{j,i}$, the relation is not symmetric.
3. **Antisymmetric**: For all $i \neq j$, if $m_{i,j} = 1$, then $m_{j,i} = 0$. This condition holds for all such pairs in M_R, so the relation is antisymmetric.

1.4.4 Conversion to/from matrix representation

In general, converting a relation to and from its matrix representation is straightforward:

1. To obtain its matrix representation, assign 1 to each M_{ij} where $(x_i, y_j) \in R$, and set all other entries to 0.

2. To retrieve the relation from its matrix representation, start with an empty set R and add (x_i, y_j) for every $M_{ij} = 1$.

Chapter 2

Composition of relations and its role in transitive closure

In this chapter, we study the problem of closing a binary relation with respect to various relational properties, with a focus on the transitivity property. Transitivity is by far the most interesting property of binary relations. Also, the fact that the transitivity property can conveniently be expressed as a relational inclusion using the notion of composition, leads us to study the composition of binary relations and their role in defining and constructing transitive closures. We begin by introducing the fundamental notions of binary relational composition and review its basic properties. Then, we focus on the closures of a binary relation defined via the iterative composition of binary relations.

2.1 Composition of binary relations

In the theory of binary relations, a major role is played by the composition of relations, as it is the most important operation that allows combining relations. In this section, an explanation of the usual notion of composition of two binary relations is given. One easily observes that there does not exist a single type of composition of two binary relations. Let X, Y and Z be sets, and let R be a relation from X to Y and let S be a relation from Y to Z. That is, R is a subset of $X \times Y$ and S is a subset of $Y \times Z$. In the following, all the possible types of Composition of two binary relations R and S are given as follows [1]:

$$R \circ_1 S = \{(x, z) \in X \times Z \mid (\exists y \in Y)((x, y) \in R \wedge (y, z) \in S)\};$$

$$R \circ_2 S = \{(x, z) \in X \times Z \mid (\exists y \in Y)((x, y) \in R \wedge (z, y) \in S)\};$$

$$R \circ_3 S = \{(x, z) \in X \times Z \mid (\exists y \in Y)((y, x) \in R \wedge (y, z) \in S)\};$$

$$R \circ_4 S = \{(x, z) \in X \times Z \mid (\exists y \in Y)((y, x) \in R \wedge (z, y) \in S)\};$$

$$R \circ_5 S = \{(x, z) \in X \times Z \mid (\exists y \in Y)((y, z) \in R \wedge (x, y) \in S)\};$$

$$R \circ_6 S = \{(x, z) \in X \times Z \mid (\exists y \in Y)((z, y) \in R \wedge (x, y) \in S)\};$$

$$R \circ_7 S = \{(x, z) \in X \times Z \mid (\exists y \in Y)((y, z) \in R \wedge (y, x) \in S)\};$$

$$R \circ_8 S = \{(x,z) \in X \times Z \mid (\exists y \in Y)((z,y) \in R \wedge (y,x) \in S)\}.$$

Only the two types of composition ($\circ_i$, $i \in \{1,2\}$) of binary relations are considered, as the other types can be defined by using the same formulas based on the commutativity of the conjunction and the notion of transpose. i.e.,

$$R \circ_i S = S \circ_{i-4} R, \text{ for any } i \in \{5,\ldots,8\},$$

$$R \circ_i S = R^t \circ_{5-i} S^t, \text{ for any } i \in \{3,4\}.$$

It is clear that associativity is the most important property of the compositions of relations.

Proposition 2.1. Among the two compositions ($\circ_i, i \in \{1,2\}$) of binary relations, only the first composition $\circ_1$ is associative; i.e., for any binary relations R, S, and T, it holds that:

$$(R \circ_1 S) \circ_1 T = R \circ_1 (S \circ_1 T).$$

Proof.

$$\begin{aligned}
(R \circ_1 S) \circ_1 T &= \{(x,z) \in X \times Z \mid (\exists y \in Y)((x,y) \in R \circ_1 S \ \wedge \ (y,z) \in T)\} \\
&= \{(x,z) \in X \times Z \mid (\exists y,t \in Y)((x,t) \in R \ \wedge \ (t,y) \in S \ \wedge \ (y,z) \in T)\} \\
&= \{(x,z) \in X \times Z \mid (\exists t \in Y)((x,t) \in R \ \wedge \ (t,z) \in S \circ_1 T)\} \\
&= \{(x,z) \in X \times Z \mid (x,z) \in R \circ_1 (S \circ_1 T)\} \\
&= R \circ_1 (S \circ_1 T).
\end{aligned}$$

□

Remark 2.1. One can easily provide counterexamples showing that the remaining seven compositions are not associative. For instance, the following example shows that $\circ_2$ is not associative. Indeed, let R, S and T be the binary relations on $X = \{x_1, x_2, x_3, x_4\}$ given by:

$$\begin{aligned}
R &= \{(x_1,x_2)\}, \\
S &= \{(x_2,x_3),(x_1,x_2)\}, \\
T &= \{(x_3,x_1),(x_2,x_2)\}.
\end{aligned}$$

One easily verifies that

$$\begin{aligned}
(R \circ_2 S) \circ_2 T &= \{(x_1,x_3)\}, \\
R \circ_2 (S \circ_2 T) &= \{(x_1,x_1)\},
\end{aligned}$$

It is clear that

$$(R \circ_2 S) \circ_2 T \neq R \circ_2 (S \circ_2 T).$$

In literature, the composition $\circ_1$ is the usual type of composition of two binary relations R and S and it is denoted by $\circ$, i.e.,

$$R \circ S = \{(x,z) \in X \times Z \mid (\exists y \in Y)((x,y) \in R \wedge (y,z) \in S)\};$$

Example 2.1. Let $A = \{1,2,3,4\}$, $B = \{a,b,c,d\}$, $C = \{x,y,z\}$, and let

$$R = \{(1,a),(2,d),(3,a),(3,b),(3,d)\} \quad \text{and} \quad S = \{(b,x),(b,z),(c,y),(d,z)\}.$$

Consider the arrow diagrams of R and S as in Figure 2.1. Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus:

$$2(R \circ S)z \quad \text{since} \quad 2Rd \quad \text{and} \quad dSz.$$

Similarly, there is a path from 3 to x and a path from 3 to z. Hence,

$$3(R \circ S)x \quad \text{and} \quad 3(R \circ S)z.$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}.$$

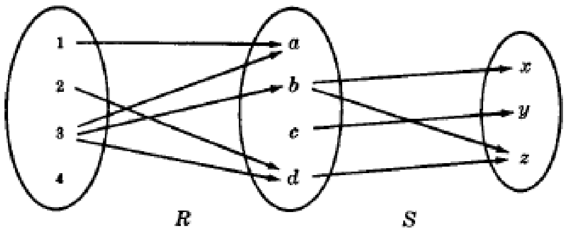

Figure 2.1: Diagrams of R and S

Suppose R is a relation on a set X, that is, R is a relation from a set X to itself. Then $R \circ R$, the composition of R with itself, is always defined. The powers of a relation R can be recursively defined from the definition of a composite of two relations.

Definition 2.1. Let $\mathbb{N}^*$ be the set of non-zero natural numbers. The n-th power R^n of a binary relation R on X is recursively defined as

$$R^1 = R \quad \text{and} \quad R^n = R^{n-1} \circ R,$$

for any $n \in \mathbb{N}^*$ and $n > 1$.

Example 2.2. To better understand the concept, let's consider the following example. Let R and S be two binary relations on $X = \{1, 2, 3, 4\}$ given by:

$$R = \{(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)\}, \quad S = \{(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)\}.$$

We are required to find the composition $R \circ S$. Hence, we observe that $(1, 2) \in R$ and $(2, 3) \in S$, which implies $(1, 3) \in S \circ R$. Let's identify all such possible pairs:

- $(1, 1) \in R$ and $(1, 4) \in S \Rightarrow (1, 4) \in R \circ S$,
- $(1, 1) \in R$ and $(1, 3) \in S \Rightarrow (1, 3) \in R \circ S$,
- $(1, 3) \in R$ and $(3, 1) \in S \Rightarrow (1, 1) \in R \circ S$,
- $(2, 4) \in R$ and $(4, 1) \in S \Rightarrow (2, 1) \in R \circ S$,
- $(3, 2) \in R$ and $(2, 3) \in S \Rightarrow (3, 3) \in R \circ S$.

Thus,

$$R \circ S = \{(1, 3), (1, 4), (1, 1), (2, 1), (3, 3)\}.$$

2.1.1 Matrix of composite relations

We now turn our attention to determining the matrix for the composite of relations [12].

Definition 2.2. Let X, Y, Z be three sets with $|X| = m$, $|Y| = p$, and $|Z| = n$. Let R be a relation from X to Y, represented by the $m \times p$ matrix $M_R = [x_{ij}]$. Let S be a relation from Y to Z, represented by the $p \times n$ matrix $M_S = [y_{ij}]$. Let $R \circ S$ denote the composite relation from X to Z. Then, $R \circ S$ is represented by the $m \times n$ matrix:

$$M_{R \circ S} = M_R \odot M_S = [c_{ij}],$$

where each entry z_{ij} is defined as:

$$z_{ij} = \bigvee_{k=1}^{p} (x_{ik} \wedge y_{kj}) = (x_{i1} \wedge y_{1j}) \vee (x_{i2} \wedge y_{2j}) \vee \cdots \vee (x_{ip} \wedge y_{pj}).$$

Remark 2.2. The matrix operator $\odot$ resembles standard matrix multiplication but with the following modifications:

- The addition operator $(+)$ is replaced by logical disjunction $(\vee)$.
- The multiplication operator $(\times)$ is replaced by logical conjunction $(\wedge)$.

Example 2.3. Let $X = Y = Z = \{x, y, z\}$.
The relations are given by:
R = {(x, x), (x, z), (y, x), (y, y)}
S = {(x, y), (y, z), (z, x), (z, z)}
Now, we compute the composition, which results in the relation:
$R \circ S = \{(x, x), (x, y), (x, z), (y, y), (y, z)\}$
The corresponding adjacency matrices are:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The matrix representation of $R \circ S$ is obtained using the Boolean product:

$$M_{R \circ S} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark 2.3. Consider a relation R on a set X, where R is represented by a matrix M_R. For any integer $k \geq 1$, the matrix representation of R^k (the k-fold composition of R) is given by:

$$M_{R^k} = \underbrace{M_R \odot M_R \odot \cdots \odot M_R}_{k \text{ times}}$$

This represents the composition of R with itself k times.

One can easliy prove the associativity property using the matrix of composite relations.

Remark 2.4. Let X, Y, Z, and U be sets, with R a relation from X to Y, S a relation from Y to Z, and T a relation from Z to U. Then, the composition of relations satisfies the associative property:

$$T \circ (S \circ R) = (T \circ S) \circ R$$

Let the Boolean matrices representing the relations R,S, and T be M_R, M_S, and M_T, respectively. As demonstrated before, the Boolean matrix product corresponds to the composition of relations, i.e., $M_{R\circ S} = M_R \odot M_S$.
Calculating both sides:

$$M_{(R\circ S)\circ T} = M_{R\circ S} \odot M_T = (M_R \odot M_S) \odot M_T,$$

$$M_{R\circ(S\circ T)} = M_R \odot M_{S\circ T} = M_R \odot (M_S \odot M_T).$$

Since Boolean matrix multiplication is associative:

$$(M_R \odot M_S) \odot M_T = M_R \odot (M_S \odot M_T).$$

Hence,

$$M_{(R\circ S)\circ T} = M_{R\circ(S\circ T)}.$$

Because their Boolean matrices are identical, we conclude:

$$(R \circ S) \circ T = R \circ (S \circ T).$$

2.1.2 Interaction of the composition with basic set operations and transpose

Theorem 2.1. [1] Let R_1, R_2, S_1, S_2, and S be binary relations on a set X. The following statements hold:

(i) If $R_1 \subseteq R_2$ and $S_1 \subseteq S_2$, then $R_1 \circ S_1 \subseteq R_2 \circ S_2$;

(ii) $(R_1 \cap R_2) \circ S = (R_1 \circ S) \cap (R_2 \circ S)$ and $S \circ (R_1 \cap R_2) = (S \circ R_1) \cap (S \circ R_2)$;

(iii) $(R_1 \cup R_2) \circ S = (R_1 \circ S) \cup (R_2 \circ S)$ and $S \circ (R_1 \cup R_2) = (S \circ R_1) \cup (S \circ R_2)$.

***Proof*.** We only give the proof for property (i), as the other results can be proved similarly. Suppose that $R_1 \subseteq R_2$ and $S_1 \subseteq S_2$. Let $(x, z) \in R_1 \circ S_1$. Then there exists $y \in X$ such that $(x, y) \in R_1$ and $(y, z) \in S_1$. Since $R_1 \subseteq R_2$ and $S_1 \subseteq S_2$, it follows that $(x, z) \in R_2 \circ S_2$. Thus, $R_1 \circ S_1 \subseteq R_2 \circ S_2$. □

Remark 2.5. The composition of relations is generally not commutative. For example, let $X = \{x, y\}$, $R = \{(x, x), (y, x), (y, y)\}$, and $S = \{(x, y), (y, x), (y, y)\}$.
Here,

$$S \circ R = \{(x, y), (y, x), (y, y)\},$$

whereas,

$$R \circ S = \{(x, x), (x, y), (y, x), (y, y)\}.$$

Clearly,

$$S \circ R \neq R \circ S.$$

Thus, the composition of relations is not commutative.

Theorem 2.2. Let X, Y, and Z be sets, R a relation from X to Y, and S a relation from Y to Z. Then, the transpose (inverse) of the composition satisfies:

$$(R \circ S)^t = S^t \circ R^t$$

Proof. Let $z \in Z$ and $x \in X$. By definition of inverse relations:

$$(z, x) \in (R \circ S)^t \iff (x, z) \in R \circ S.$$

This holds if and only if there exists $y \in Y$ such that $(x, y) \in R$ and $(y, z) \in S$. Equivalently:

$$(z, y) \in S^t \text{ and } (y, x) \in R^t \implies (z, x) \in S^t \circ R^t.$$

Thus, $(R \circ S)^t \subseteq S^t \circ R^t$.
Conversely, if $(z, x) \in S^t \circ R^t$, there exists $y \in Y$ such that $(z, y) \in S^t$ and $(y, x) \in R^t$. This implies $(x, y) \in R$ and $(y, z) \in S$, so $(x, z) \in R \circ S$. Therefore:

$$(z, x) \in (R \circ S)^t \implies S^t \circ R^t \subseteq (R \circ S)^t.$$

Since both inclusions hold, we conclude:

$$(R \circ S)^t = S^t \circ R^t.$$

This completes the proof. □

Corollary 2.1. Let R be a binary relation on a set X. It holds that:

$$(R^n)^t = \left(R^t\right)^n.$$

2.1.3 Properties of the composition of relations

In the following, we express the properties of the composition of binary relations [19].

Theorem 2.3. Let R be a binary relation on a set X. Then,

1. If R is reflexive, then R^t is reflexive.
2. $R \circ R^t$ is reflexive if and only if the domain of R is X.
3. R is antisymmetric if and only if $R \cap R^t \subset \Delta_X$.

Proof.

1. If R is reflexive, then $\forall x \in X, (x, x) \in R$, so $(x, x) \in R^t$, and thus R^t is reflexive.
2. For the direct implication, assume that $R \circ R^t$ is reflexive and show that the domain of R is X.

$$\begin{aligned} R \circ R^t \text{ is reflexive } &\Rightarrow \forall x \in X, x(R \circ R^t)x, \\ &\Rightarrow \forall x \in X, \exists y \in X \text{ such that } (xRy) \wedge (yR^t x), \\ &\Rightarrow \forall x \in X, \exists y \in X : xRy \wedge xRy, \\ &\Rightarrow \forall x \in X, \exists y \in X : xRy, \\ &\Rightarrow Dom(R) = X. \end{aligned}$$

   For the converse implication, assume $Dom(R) = X$ (i.e., $\forall x \in X$, $\exists y \in X : xRy$) and show that $R \circ R^t$ is reflexive:

   $\Rightarrow \forall x \in X, \exists y \in X : (xRy) \wedge (yR^t x)$.

   $\Rightarrow \forall x \in X : xR \circ R^t x$. Thus, $R \circ R^t$ is reflexive.

3. R is antisymmetric $\Rightarrow R \cap R^t \subset \Delta_X$.
   Assume R is antisymmetric and let $(x, y) \in R \cap R^t$. This is equivalent to: $(x, y) \in R$ and $(x, y) \in R^t \Leftrightarrow (x, y) \in R$ and $(y, x) \in R$. By antisymmetry, $x = y$, so $(x, y) = (x, x) \in \Delta_X$, proving $R \cap R^t \subset \Delta_X$.

   Conversely, assume $R \cap R^t \subset \Delta_X$ and show R is antisymmetric. Let $x, y \in X$ such that: $\begin{cases} xRy \\ \text{and} \\ yRx \end{cases} \begin{cases} xRy \\ \text{and} \\ xR^ty \end{cases} \Rightarrow (x, y) \in R \cap R^t$. Since $R \cap R^t \subset \Delta_X$, $x = y$, confirming R is antisymmetric.

□

2.2 Transitivity of binary relations in terms of compositions

In this subsection, we discuss the transitivity property from the point of view of the composition of binary relations [35].

Theorem 2.4. Let R be a binary relation on a set X, R is transitive if and only if $R \circ R \subset R$.

Proof. First , we show that R being transitive implies $R \circ R \subset R$.
Let $(x, z) \in R \circ R$, then $\exists y \in X : (xRy) \wedge (yRz)$. Since R is transitive, xRz (i.e., $(x, z) \in R$), hence $R \circ R \subset R$.
Now, we show that if $R \circ R \subset R$, then R is transitive.
Let $x, y, z \in X$ such that: $\begin{cases} xRy \\ \wedge \\ yRz \end{cases} \Rightarrow x(R \circ R)z \Rightarrow (x, z) \in R \circ R$. Since $R \circ R \subset R$, $(x, z) \in R$, proving R is transitive. □

Theorem 2.5. The relation R on a set X is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof. We first prove the "if" part of the theorem. Suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$ In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(x, y) \in R$ and $(y, z) \in R$, then by the definition of composition of relations, $(x, z) \in R^2$. Since $R^2 \subseteq R$, it follows that $(x, z) \in R$. Hence, R is transitive.
Now we prove the "only if" part of the theorem using mathematical induction.
Note that this part of the theorem is trivially true for $n = 1$):
Now, assume that $R^n \subseteq R$ for some positive integer n. This is the inductive hypothesis. We must show that $R^{n+1} \subseteq R$.
Let $(x, y) \in R^{n+1}$. Since $R^{n+1} = R \circ R^n$, there exists an element $t \in X$ such that $(x, t) \in R$ and $(t, y) \in R^n$.
By the inductive hypothesis, $(t, y) \in R$, and since $(x, t) \in R$ as well, and R is transitive, it follows that $(x, y) \in R$.
Therefore, $R^{n+1} \subseteq R$. By the principle of mathematical induction, $R^n \subseteq R$ for all $n \geq 1$. □

2.3 Closures of a binary relation

Where P is any property which a binary relation R on a set X may have or fail to have, the P-closure of R is defined to be the smallest relation S containing R and possessing P. Bandler and Kohout have discussed the concept of closure of a given binary relation with respect to a property P as follows [1].

2.3.1 General results

Definition 2.3. If P is a property which a binary relation R on a set X may have or fail to have, then a binary relation S is the P-closure of R, written $S = P^{\text{cl}}(R)$, if and only if S satisfies all of:

(i) S has property P;

(ii) $R \subseteq S$;

(iii) If $R \subseteq T$ and T has property P, then $S \subseteq T$.

It is clear that a P-closure, if it exists, must be unique.

Corollary 2.2. A binary relation R on a set X possesses property P if and only if $R = P^{\text{cl}}(R)$.

For many properties P, a P-closure exists for some binary relations but not for others. Thus, all R that already possess P have a P-closure trivially (by the above corollary), but this guarantees nothing for other R. The interesting closures are those for properties where every R has a P-closure, because in these cases, and of course only in these, there is a P-closure operator on the entire set of binary relations on X. The following theorem states the conditions for this to occur.

Theorem 2.6. A P-closure exists for all binary relations R on a set X if and only if:

(i) The universal relation X^2 possesses P;

(ii) The intersection of every (non-empty) family of binary relations, each of which possesses P, also possesses P.

Theorem 2.7. [30] Let C be a family of relations R defined on a set X, and let T denote the intersection of all relations R in C, that is:

$$T = \bigcap \{R \mid R \in C\}.$$

(i) If every relation R in C is symmetric, then T is symmetric.

(ii) If every relation R in C is transitive, then T is transitive.

Proof. (i) We show if every relation R in C is symmetric, then T is symmetric.
Assume $(x, y) \in T$. By the definition of intersection, (x, y) must belong to every relation $R \in C$. Since each R is symmetric, $(y, x) \in R$ for all $R \in C$. Therefore, $(y, x) \in T$, proving that T is symmetric.
(ii) We show if every relation R in C is transitive, then T is transitive.
Suppose $(x, y) \in T$ and $(y, z) \in T$. By the definition of intersection, both (x, y) and (y, z) belong to every relation $R \in C$. Since each R is transitive, $(x, z) \in R$ for all $R \in C$. Consequently, $(x, z) \in T$, which establishes the transitivity of T.

□

Theorem 2.8. [30]

(i) Symmetry and transitivity are R-closable for any relation R.

(ii) The P-closure of R, denoted $P^{\text{cl}}(R)$, is defined as:

$$P^{\text{cl}}(R) = \bigcap \{S \mid S \text{ is a } P\text{-relation and } R \subseteq S\}\,.$$

Proof. (i) Symmetry:
The universal relation $X \times X$ is symmetric and contains R, satisfying condition (i) from Theorem 2.6. By theorem 2.7(i), the intersection of symmetric relations is symmetric, satisfying condition (ii) from Theorem 2.6.
Transitivity:
The universal relation $X \times X$ is transitive and contains R, satisfying condition (i) from Theorem 2.6. By theorem 2.7(ii), the intersection of transitive relations is transitive, satisfying condition (ii) from Theorem 2.6.
Thus, symmetry and transitivity are R-closable for any relation R.

(ii) Define T as the intersection of all P-relations containing R, expressed as

$$T = \bigcap \big\{S \mid S \text{ is a } P\text{-relation and } R \subseteq S\big\}.$$

Since P is R-closable, the set of such P-relations is non-empty (by property (i)), and their intersection T is itself a P-relation (by property (ii)). Furthermore, because each S in this collection contains R, it follows that $R \subseteq T$. Consequently, T is a P-relation containing R. By definition of $P^{\text{cl}}(R)$ as the *smallest* P-relation containing R, we have $P^{\text{cl}}(R) \subseteq T$. Conversely, $P^{\text{cl}}(R)$ itself is one of the P-relations in the set over which the intersection is taken (since it contains R and satisfies P), implying $T \subseteq P^{\text{cl}}(R)$. This double inclusion ($P^{\text{cl}}(R) \subseteq T$ and $T \subseteq P^{\text{cl}}(R)$) establishes that $P^{\text{cl}}(R) = T$.

□

Theorem 2.9. If P and P' are properties for which closures exist (satisfying the conditions of Theorem 2.6), and if P^{cl} and P'^{cl} commute with each other, then $(P \wedge P')$ also satisfies the conditions of Theorem 2.6 , and has a closure given by:

$$(P \wedge P')^{\text{cl}} = P^{\text{cl}}(P'^{\text{cl}}) = P'^{\text{cl}}(P^{\text{cl}}).$$

2.3.2 Closures for some special properties of binary relations

Bandler and Kohout [7] studied the closures for some special properties of binary relations, such as the symmetric closure, transitive closure, etc. In the following, we discuss the closures of a binary relation for the reflexivity, symmetry and transitivity properties. Since these properties are all preserved under intersection and that the universal relation X^2 possesses all of them. Hence, the closures for these properties always exist.

Theorem 2.10. Let R be a binary relation on a set X. It holds that:

(i) The reflexive closure of R is $r(R) = R \cup \Delta_X$;

(ii) The symmetric closure of R is $s(R) = R \cup R^t$;

(iii) The transitive closure of R is $t(R) = \bigcup_{n \geq 1} R^n$;

Proof. (i) We show that $R \cup \Delta_X = r(R)$ is the reflexive closure of R.

- It is clear that $R \subset R \cup \Delta_X$.
- $R \cup \Delta_X$ is reflexive because $\Delta_X \subset R \cup \Delta_X$.

- Let μ be a reflexive relation containing R. We show that $R \cup \Delta_X \subset \mu$. We have $R \subset \mu$, and since μ is reflexive, $\Delta_X \subset \mu$. Therefore, $R \cup \Delta_X \subset \mu$.

Finally, $r(R) = R \cup \Delta_X$ is the reflexive closure of R.

(ii) We show that $R \cup R^t = s(R)$ is the symmetric closure of R.

- It is clear that $R \subset s(R) = R \cup R^t$.
- $(x, y) \in s(R) = R \cup R^t$. Then $(x, y) \in R$ or $(x, y) \in R^t$, which implies $(y, x) \in R^t$ or $(y, x) \in R$. Therefore, $(y, x) \in s(R) = R \cup R^t$, proving that $s(R)$ is symmetric.
- μ be a symmetric relation containing R We show that $s(R)) = R \cup R^t \subset \mu$, Since $R \subset \mu$ and μ is symmetric, $R^t \subset \mu$ Thus $s(R)) = R \cup R^t \subset \mu$.

Finally, $s(R)) = R \cup R^t$ is the symmetric closure of R.

(iii) We show that $t(R) = \cup_{i \geq 1} R^i$ is the transitive closure of R.

- For $i = 1$, $R \subset t(R) = \cup_{i \geq 1} R^i$.
- To prove that $t(R)$ is transitive, let $x, y, z \in X$ such that:
$$\begin{cases} x\,(\cup_{i\geq 1}R^i)\,y \\ \text{and} \quad y\,(\cup_{i\geq 1}R^i)\,z \end{cases} \Rightarrow x\,(\cup_{i\geq 1}R^i)\,z.$$
$(x, y) \in \cup_{i\geq 1}R^i \Leftrightarrow \exists s \in \mathbb{N}^*$ such that $(x, y) \in R^s = \underbrace{R \circ R \circ \ldots \circ R}_{s \text{ times}} \Leftrightarrow$ $\exists (x_1, x_2, \ldots, x_{s-1}) \in X^{s-1}$ such that: $(xRx_1) \wedge (x_1Rx_2) \wedge \ldots \wedge (x_{s-1}Ry)$ ....(1).
$(y, z) \in \cup_{i\geq 1}R^i \Leftrightarrow \exists k \in \mathbb{N}^*$ such that $(y, z) \in R^k = \underbrace{R \circ R \circ \ldots \circ R}_{k \text{ times}} \Leftrightarrow$ $\exists (y_1, y_2, \ldots, y_{k-1}) \in X^{k-1}$ such that:$(yRy_1) \wedge (y_1Ry_2) \wedge \ldots \wedge (y_{k-1}Rz)$....(2).
From (1) and (2), we have: $((xRx_1) \wedge (x_1Rx_2) \wedge \ldots \wedge (x_{s-1}Ry) \wedge (yRy_1) \wedge (y_1Ry_2) \wedge \ldots \wedge (y_{k-1}Rz)) \Rightarrow (x, z) \in R^{k+s}$,Since $k + s \in \mathbb{N}^*$so $(x, z) \in \cup_{i\geq 1}R^i$.
- μ be a transitive relation containing R. We show that $t(R) = \cup_{i\geq 1}R^i \subset \mu$. Let $(x, y) \in \cup_{i\geq 1}R^i \Leftrightarrow \exists b \in \mathbb{N}^*$ such that $(x, y) \in R^b \Leftrightarrow \exists (x_1, x_2, \ldots, x_{b-1}) \in X^{b-1}$ such that: $(xRx_1) \wedge (x_1Rx_2) \wedge \ldots \wedge (x_{b-1}Ry)$. Since $R \subset \mu$, we have $(x\mu x_1) \wedge (x_1\mu x_2) \wedge \ldots \wedge (x_{b-1}\mu y)$. By transitivity of μ, we conclude $(x\mu y)$, so $t(R) = \cup_{i\geq 1}R^i \subset \mu$.

Finally, $t(R) = \cup_{i\geq 1}R^i \subset \mu$ is the transitive closure of R.

□

Remark 2.6. Let R be a binary relation defined on a set X. Then:

(i) R is reflexive if and only if $R = r(R)$.

(ii) R is symmetric if and only if $R = s(R)$.

(iii) R is transitive if and only if $R = t(R)$.

(iv) The transitive closure is idempotent, meaning that applying the transitive closure operation twice gives the same result: $t(t(R)) = t(R)$.

We denote by R^+ the transitive closure of R.

Example 2.4. Let $X = \{x, y, z, t, u\}$, and define the relation $R = \{(a, b), (b, c), (b, d), (d, e)\}$ on X. Then:

$$r(R) = R \cup \Delta_X = \{(x, y), (y, z), (y, t), (t, u), (x, x), (y, y), (z, z), (t, t), (u, u)\}.$$
$$s(R) = R \cup R^t = \{(x, y), (y, x), (y, z), (z, x), (y, t), (t, y), (t, u), (u, t)\}.$$

Example 2.5. Let $X = \{1, 2, 3, 4\}$, and let $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$. we will compute the transitive closure of R via composition of relations

- Set $X = \{1, 2, 3, 4\}$
- Relation $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$

1- Compute R^1

$$R^1 = R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$$

2- Compute $R^2 = R^1 \circ R$
Find all pairs (x, z) such that $\exists y$ where $(x, y) \in R$ and $(y, z) \in R$:

- $(1, 2) \in R$ and $(2, 3) \in R \Rightarrow (1, 3)$ added
- $(1, 2) \in R$ and $(2, 1) \in R \Rightarrow (1, 1)$ added
- $(2, 3) \in R$ and $(3, 4) \in R \Rightarrow (2, 4)$ added
- $(2, 1) \in R$ and $(1, 2) \in R \Rightarrow (2, 2)$ added

$$R^2 = \{(1, 3), (1, 1), (2, 4), (2, 2)\}$$

3- Compute $R^3 = R^2 \circ R$
Check for new pairs by composing R^2 with R:

- $(1, 3) \in R^2$ and $(3, 4) \in R \Rightarrow (1, 4)$ added

$$R^3 = \{(1, 4)\}$$

4- Compute $R^4 = R^3 \circ R$
No new pairs can be composed:

$$R^4 = \emptyset$$

5- The Transitive Closure $R^+$is:

$$R^+ = R^1 \cup R^2 \cup R^3 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$$

Chapter 3

Transitive closures and their computation

This chapter explores the concept of transitive closure, a fundamental operation in the study of relations, particularly within the context of directed graphs. We begin by examining the theoretical foundations of transitive closure through graph-based representations, emphasizing the role of paths in determining connectivity. This is followed by a discussion of matrix-based methods, which provide an algebraic approach to computing transitive closures. A key focus is placed on Warshall's algorithm, a widely used and efficient technique for computing transitive closures in finite directed graphs. Finally, the chapter concludes with a comparative analysis of the various methods, highlighting their advantages, limitations, and appropriate use cases depending on the structure and size of the data involved.

3.1 Theory of transitive closure via digraph representation

3.1.1 Paths in Directed Graphs

For any binary relation R defined on a finite set X, the transitive closure R^+ can be determined by analyzing all possible paths in the directed graph (digraph) representation of R. We now show some terminology that we will use for this purpose [35]. A path in a directed graph is obtained by traversing along edges (in the same direction as indicated by the arrow on the edge).

Definition 3.1. A *path* from a to b in the directed graph G is a sequence of edges

$$(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$$

in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$; that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex of the next edge in the path. This path is denoted by

$$x_0, x_1, x_2, \ldots, x_{n-1}, x_n$$

and has length n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

Remark 3.1. A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

The term *path* also applies to relations. Carrying over the definition from directed graphs to relations, there is a path from a to b in R if there is a sequence of elements

$$a, x_1, x_2, \ldots, x_{n-1}, b$$

with $(a, x_1) \in R$, $(x_1, x_2) \in R$, ..., and $(x_{n-1}, b) \in R$. Theorem 3.1 can be obtained from the definition of a path in a relation.

Theorem 3.1. Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Proof. We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, so the theorem holds for $n = 1$. Now, assume that the theorem is true for a positive integer n; that is, there is a path of length n from a to b if and only if $(a, b) \in R^n$. This is the inductive hypothesis.
There is a path of length $n + 1$ from a to b if and only if there is an element $c \in A$ such that:

- There is a path of length one from a to c, so $(a, c) \in R$, and
- There is a path of length n from c to b, that is, $(c, b) \in R^n$.

By the inductive hypothesis, this implies there is a path of length $n+1$ from a to b if and only if there exists $c \in A$ with $(a, c) \in R$ and $(c, b) \in R^n$.
But this condition is equivalent to $(a, b) \in R^{n+1}$. Therefore, the statement holds for $n + 1$.
By the principle of mathematical induction, the theorem holds for all positive integers n. □

3.1.2 Transitive closure via digraph representation

We now show that finding the transitive closure of a relation is equivalent to determining which pairs of vertices in the associated directed graph are connected by a path [35]. With this in mind, we define a new relation.

Definition 3.2. Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.
Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b, it follows that R^* is the union of all the sets R^n. In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

The connectivity relation is useful in many models.

The following theorem show the transitive closure and connectivity relation.

Theorem 3.2. The transitive closure of a relation R equals the connectivity relation R^*.

Proof. Note that R^* contains R by definition. To show that R^* is the transitive closure of R, we must also show that R^* is transitive and that $R^* \subseteq S$ whenever S is a transitive relation that contains R.

1. Transitivity of R^*: If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. We obtain a path from a to c by following the path from a to b and then the path from b to c. Hence, $(a, c) \in R^*$, and R^* is transitive.

2. Minimality of R^*: Suppose S is a transitive relation containing R. Because S is transitive, $S^n \subseteq S$ for all $n \in \mathbb{N}$ (by Theorem 2.5). Furthermore,

$$S^* = \bigcup_{k=1}^{\infty} S^k \subseteq S.$$

Now, since $R \subseteq S$, any path in R is also a path in S. Hence, $R^* \subseteq S^* \subseteq S$. Thus, any transitive relation containing R must also contain R^*. Therefore, R^* is the transitive closure of R. □

Now that we know that the transitive closure equals the connectivity relation, we turn our attention to the problem of computing this relation. We do not need to examine arbitrarily long paths to determine whether there is a path between two vertices in a finite directed graph. As the following lemma shows, it is sufficient to examine paths containing no more than n edges, where n is the number of elements in the set.

Lemma 3.1. Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.

Proof . Suppose there is a path from a to b in R. Let m be the length of the shortest such path. Suppose that

$$x_0, x_1, x_2, \ldots, x_{m-1}, x_m,$$

where $x_0 = a$ and $x_m = b$, is such a path.
Suppose that $a = b$ and that $m > n$, so that $m \geq n + 1$. By the pigeonhole principle, because there are n vertices in A, among the m vertices

$$x_0, x_1, \ldots, x_{m-1},$$

at least two must be equal. Suppose that $x_i = x_j$ with $0 \leq i < j \leq m - 1$. Then the path contains a circuit from x_i to itself. This circuit can be removed from the path from a to b, yielding a shorter path, namely,

$$x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_{m-1}, x_m,$$

from a to b.
Hence, the path of shortest length must have length less than or equal to n.

□

From Lemma 3.1, we see that the transitive closure of R on a finite set is the union of R, R^2, R^3, ..., and R^n. This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i, for some positive integer i with $i \leq n$. Therefore,

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n.$$

Example 3.1. Let $X = \{1, 2, 3, 4\}$, and let $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$. Find the transitive closure of R.

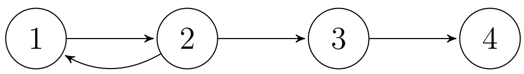

Figure 3.1: Digraph of R

We can determine R^+ by geometrically computing all paths from the digraph. From vertex 1 we have paths to vertices 1, 2, 3 and 4. So the ordered pairs $(1, 1)$, $(1, 2)$, $(1, 3)$ and $(1, 4) \in R^+$. From vertex 2 we have paths to vertices 1, 2, 3 and 4. This gives us the ordered pairs $(2, 1)$, $(2, 2)$, $(2, 3)$ and $(2, 4)$. From vertex 3 we have only one path to vertex 4. This gives us the ordered pair $(3, 4)$. From vertex 4 we do not have any paths. So we have

$$R^+ = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$$

3.2 Matrix representation of transitive closure

For any relation R on a finite set with n elements, if M_R denotes its matrix representation, then the matrix M_{R^+} of the transitive closure R^+ can be obtained through the following Boolean matrix operation [19]:

$$M_{R^+} = M_R \vee (M_R)^2_{\odot} \vee (M_R)^3_{\odot} \vee \cdots \vee (M_R)^n_{\odot}$$

where:

- $M_R^{[n]}$ represents the k-th Boolean power of M_R
- $\vee$ denotes the element-wise logical or operation

This matrix formulation provides a direct computational method for determining the transitive closure of R.

Example 3.2. solve the previous example (Example 3.1) using a Matrix
Writing down the Boolean matrix we get

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we compute the higher powers of M_R we get:

$$(M_R)^2_{\odot} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)^3_{\odot} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)^4_{\odot} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we observe that $(M_R)^n_{\odot}$ equals $(M_R)^2_{\odot}$ if n is even, and equals $(M_R)^3_{\odot}$ if n is odd and greater than 1. Hence we get,

$$M_{R^+} = M_R \vee (M_R)^2_{\odot} \vee (M_R)^3_{\odot}$$

Thus,

$$M_{R^+} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is nothing but the matrix representation of the relation we obtained earlier by the digraph method (1). Thus we see that we need not consider all the powers of R^n to obtain R^+ when the set X is finite. This result always holds good.

3.3 Warshall's algorithm

The methods used to solve the previous examples have certain drawbacks. The graphical method is unsystematic and impractical for large sets. The matrix method is better than the graphical method and can be implemented with the help of a program, but it tends to become costly in terms of time and space requirement in the case of large matrices. So this method is inefficient too. The Warshall's algorithm as described next helps to overcome these drawbacks. [19]

3.3.1 Algorithm overview

Consider a set $X = \{x_1, x_2, \ldots, x_n\}$ and a relation R defined on X. For a path $c_1, c_2, \ldots, c_m$ in R, the vertices other than the endpoints c_1 and c_m are termed interior vertices. We define a Boolean matrix W_k (where $1 \leq k \leq n$) such that $W_k[i][j] = 1$ if and only if there exists a path from x_i to x_j in R with all interior vertices (if any) belonging to $\{x_1, x_2, \ldots, x_k\}$.
When $k = n$, all vertices are included in $\{x_1, x_2, \ldots, x_n\}$, so $W_n[i][j] = 1$ indicates a path from x_i to x_j in R. Thus, W_n represents the transitive closure R^*. Starting with $W_0 = M_R$ (the matrix representation of R), the sequence $W_0, W_1, \ldots, W_n$ is computed, where $W_n = M_{R^+}$. Warshall's Algorithm efficiently computes each W_k from W_{k-1}, avoiding the complexity of matrix exponentiation.
Computing W_k from W_{k-1}
Let $W_k = [p_{ij}]$ and $W_{k-1} = [s_{ij}]$. If $p_{ij} = 1$, there is a path from x_i to x_j with interior vertices from $\{x_1, \ldots, x_k\}$. Two cases arise:
1. If x_k is not an interior vertex, all interior vertices are from $\{x_1, \ldots, x_{k-1}\}$, so $s_{ij} = 1$.
2. If x_k is an interior vertex, the path splits into subpaths: x_i to x_k and x_k to x_j, with all interior vertices from $\{x_1, \ldots, x_{k-1}\}$. Thus, $s_{ik} = 1$ and $s_{kj} = 1$.
Therefore, $p_{ij} = 1$ if either:
- $s_{ij} = 1$ (Case 1), or
- $s_{ik} = 1$ and $s_{kj} = 1$ (Case 2).
Procedure for Updating W_k
1. Copy all entries from W_{k-1} to W_k.
2. Identify positions $(y_1, y_2, \ldots)$ in column k of W_{k-1} where the entry is 1, and positions $(z_1, z_2, \ldots)$ in row k of W_{k-1} where the entry is 1.
3. For each pair (y_i, z_j), set $W_k[y_i][z_j] = 1$.

Example 3.3. We determine the transitive closure of the relation described in Example 3.1.

We begin with:

$$W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we have $n = 4$.
To find W_1, $k = 1$. We can see that W_0 has 1's in column 1 at location 2, and in row 1 at location 2. Thus W_1 has a new 1 at position (2, 2).

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_2, $k = 2$. W_1 has 1's in column 2 at locations 1 and 2, and in row 2 at locations 1, 2 and 3. So the new 1's would go to positions (1,1), (1,2), (1,3), (2,1), (2,2), and (2,3) (if not already there).

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_3, $k = 3$. W_2 has 1's in column 3 at locations 1 and 2, and in row 3 at location 4. So the new 1's would come at positions (1,4) and (2,4) (if not already there).

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_4, $k = 4$. W_3 has 1's in column 4 at locations 1, 2 and 3 but no 1's in row 4. So no new 1's are added. Hence $W_4 = W_3$.
This gives the matrix representation which matches the one obtained in Example 3.1

3.3.2 The algorithm

To determine the matrix **CLOSURE** representing the transitive closure of a relation R with an $n \times n$ matrix representation **MAT**, we use **Warshall's Algorithm**:

1. **CLOSURE** $\leftarrow$ **MAT**
2. **for** $k = 1$ **to** n
3.   **for** $i = 1$ **to** n
4.   **for** $j = 1$ **to** n
5.   **CLOSURE**$(i, j) \leftarrow$ **CLOSURE**$(i, j) \vee ($**CLOSURE**$(i, k) \wedge$ **CLOSURE**$(k, j))$

This algorithm consists of three nested loops, each iterating from 1 to n, resulting in a time complexity of $O(n^3)$. In contrast, the matrix multiplication method for finding the transitive closure has a higher time complexity of $O(n^4)$. This is because each matrix multiplication requires $O(n^3)$ operations, and the process is repeated $n - 1$ times to compute the matrices $(M_R)^2_\circ, (M_R)^3_\circ, \ldots, (M_R)^n_\circ$. The total steps amount to $n^3(n-1)$, leading to $O(n^4)$. Thus, Warshall's algorithm is both simpler and more efficient than the matrix multiplication approach.

To get direct results, for instance, use the following Python code implements Warshall's algorithm to compute the transitive closure of the relation represented as an adjacency matrix in Example 3.1:

```
def warshall(mat):
    n = len(mat)
    closure = [row[:] for row in mat]  # Make a copy to avoid modifying
        the original matrix

    for k in range(n):
        for i in range(n):
```

```
            for j in range(n):
                closure[i][j] = closure[i][j] or (closure[i][k] and
                    closure[k][j])

    return closure

# Example: Relation R represented as a 4x4 adjacency matrix
R = [
    [0, 1, 0, 0],
    [1, 0, 1, 0],
    [0, 0, 0, 1],
    [0, 0, 0, 0]
]

closure = warshall(R)

# Display the result
for row in closure:
    print(row)
# Output:
[
    [1, 1, 1, 1],
    [1, 1, 1, 1],
    [0, 0, 0, 1],
    [0, 0, 0, 0]
]
```

3.3.3 An application of warshall's algorithm

Theorem 3.3. [19] If R and S are equivalence relations on a set X, then the smallest equivalence relation containing both R and S is $(R \cup S)^*$.

Proof. A relation is reflexive if it contains the identity relation Δ. Since R and S are reflexive, $\Delta \subseteq R$ and $\Delta \subseteq S$, which implies $\Delta \subseteq R \cup S \subseteq (R \cup S)^*$. Therefore, $(R \cup S)^*$ is reflexive.
For symmetry, consider $(x, y) \in R$. Since R is symmetric, $(y, x) \in R$. As $R \subseteq (R \cup S) \subseteq (R \cup S)^*$, both (x, y) and (y, x) belong to $(R \cup S)^*$. The same logic applies if $(x, y) \in S$, proving $(R \cup S)^*$ is symmetric.
The transitive property of $(R \cup S)^*$ follows directly from its definition as the smallest transitive relation containing $R \cup S$. Hence, $(R \cup S)^*$ is the smallest equivalence relation encompassing both R and S.
Furthermore, since $(R \cup S)^*$ is an equivalence relation and it contains $(R \cup S)$, it inherently includes both R and S. This confirms that $(R \cup S)^*$ is indeed the smallest equivalence relation encompassing both R and S. □

Example 3.4. Consider the set $X = \{1, 2, 3, 4, 5\}$ with the equivalence relations R and S defined as follows:

- $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$
- $S = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

The partition of X corresponding to R is $X/R = \{\{1, 2\}, \{3, 4\}, \{5\}\}$, and the partition corresponding to S is $X/S = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$. The goal is to determine the smallest equivalence relation containing both R and S and to find the resulting partition of X.

Hence, the matrix representations of R and S are:

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The union of R and S is represented by the matrix:

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

To compute the transitive closure $M_{(R\cup S)^*}$, we apply Warshall's algorithm:

1. Initialize $W_0 = M_{R\cup S}$.
2. For $k = 1$:
   - Column 1 has 1's at positions 1 and 2, and row 1 has 1's at positions 1 and 2.
   - No new 1's are added, so $W_1 = W_0$.
3. For $k = 2$:
   - Column 2 has 1's at positions 1 and 2, and row 2 has 1's at positions 1 and 2.
   - No new 1's are added, so $W_2 = W_1$.
4. For $k = 3$:
   - Column 3 has 1's at positions 3 and 4, and row 3 has 1's at positions 3 and 4.
   - No new 1's are added, so $W_3 = W_2$.

The process continues similarly for $k = 4$ and $k = 5$, but no additional 1's are introduced, confirming that the transitive closure remains unchanged from $M_{R\cup S}$.
We now proceed to compute W_4 for $k = 4$. The matrix W_3 has 1's at positions 3, 4, and 5 in column 4, and at positions 3, 4, and 5 in row 4. This requires adding new 1's at positions $(3, 5)$ and $(5, 3)$ in W_3 to obtain W_4. The resulting matrix is:

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Next, for $k = 5$, we observe that W_4 already has 1's at positions 3, 4, and 5 in column 5, as well as in row 5. Therefore, no additional 1's are needed, and $W_5 = W_4$.
The transitive closure $(R \cup S)^*$ is given by the set of pairs:

$$(R \cup S)^* = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$$

The corresponding partition of the set X is:

$$\{\{1, 2\}, \{3, 4, 5\}\}$$

3.4 A comparison of transitive closure computation methods

The techniques used to determine the transitive closure of a relation differ in terms of efficiency, clarity, and ease of execution. One common approach is the repeated composition method, which involves continuously composing the relation with itself until no further changes occur. Though conceptually straightforward, this method becomes inefficient for large datasets due to the high number of operations and extended processing time it requires.

Alternatively, the matrix method converts the relation into a binary matrix and uses matrix multiplication to determine successive powers of the relation. This technique is more practical for computational use and typically faster than manual composition, though it requires a solid understanding of mathematical operations and can become complex with extensive data.

Graph-based methods offer intuitive visual representations, depicting elements as nodes connected by edges, which makes it easier to interpret the structure and possible paths within the relation. However, they are not well-suited for precise calculations or handling large relations.

Among the most effective techniques is Warshall's algorithm, which systematically updates the matrix of a relation to include all reachable pairs. It is known for its speed, accuracy, and suitability for programming environments, though it does not provide the visual insight offered by graph representations.

Ultimately, the best method for computing a transitive closure depends on factors such as the size of the dataset, the complexity of the relation, and whether the goal is conceptual understanding or efficient implementation.

Conclusion and Future Work

This work explored the fundamental aspects of binary relations, specifically transitive closure and how it was computed through the composition of relations. Our starting point was to construct a strong foundation in the definitions, properties, and representations of relations, both visually and numerically. The composition of relations was analyzed in detail to demonstrate its role in constructing transitive closures. The matrix-based approach was emphasized, leading to a formal presentation of Warshall's algorithm. Finally, a comparative analysis of computation methods was provided to highlight the strengths and limitations of each approach. The study strengthened key theoretical concepts and presented a structured method for understanding transitive closure in the context of discrete mathematics and theoretical computer science.

Extending these methods to fuzzy relations and analyzing the optimization of transitive closure computation in large-scale data structures and dynamic systems is a promising direction for future research.

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