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Theme

Some remarks to anisotropic variable exponents Sobolev spaces

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Dedication

Nothing compares to the joy of graduation, as it is one of the most beautiful moments in our lives. The tiredness of the years, the sleepless nights and the prayers of the parents have been harvested, and the psychological pressures we have gone through have disappeared and we have forgotten them as soon as we feel the joy of graduation.

First, I have mercy on my dear father, then I dedicate my graduation to everyone who supported me and stood with me from my childhood to my old age, so my mother, who supported me with all my requirements and my brothers and sisters, and I say to them thank you for this beautiful situation

I wish success to all my colleagues.

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Abstract

The theory of function spaces with variable exponents has rapidly made progress in the past 20 years, so in this memory our goal is to study some basic properties of anisotropic variable exponent Sobolev spaces.

keywords: Anisotropic, variable exponent, Sobolev space, embedding.

Résumé

La théorie des espaces fonctionnels à exposant variable fait des progrès impressionnants tout au cours des vingt dernières années, donc dans cette mémoire, notre objectif est étudier quelques propriétés des espaces de Sobolev anisotrope à exposant variable.

mots-clés: Anisotrope, exposant variable, espace de Sobolev, injections.

ملخص

حققت نظرية فضاءات الوظائف ذات الأسس المتغيرة تقدمًا سريعًا خلال ٢٠ سنة الماضية ، لذلك في هذه المذكرة ، هدفنا هو دراسة بعض الخصائص الأساسية لفضاءات Sobolev ، حالة تباين الأسس كتوابع متغيرة .

الكلمات الرئيسية: تباين الأسس ، متغير الأس ، فضاء Sobolev ، التباين.

List of Symbols

In what follows, we will use the following notations.

$\mathbb{R}^n$	Euclidean, n-dimensional space,
x	Vecteur de $\mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $x_i \in \mathbb{R}$, $1 \leq i \leq n$
$d\mu$	or dx Lebesgue measure N-dimensional
$\|E\|$	or $\mathrm{mes}\ (E)$ Measure of Lebesgue of a set E
$\|\Omega\|$	Measure of the set Ω
$\nearrow$	Limit of an increasing sequence,
χ_E	Characteristic function of the set E,
Ω	Open set in $\mathbb{R}^N$
$\overline{\Omega}$	The closure set of in $\mathbb{R}^N$
a.e.	almost everywhere
$\partial\Omega$	The border of Ω
B	Open ball,
$B(x, r)$	Open ball with center x and radius $r > 0$,
B_E	the closed unit ball of E ,
B_E	$= \{x \in E \mid \|x\| = 1\}$
$W^{k,p(\cdot)}$	Sobolev space,
$W_0^{k,p(\cdot)}$	Sobolev space with zero boundary values,
$W^{-k,p(\cdot)}$	dual space of $W^{-k,p(\cdot)}$,
$W^{1,\overrightarrow{p}}$	anisotropic Sobolev space,
$\mathcal{P}$	set of variable exponents,
$\mathcal{P}^{\log}$	variable exponents p such that $\frac{1}{p}$ is log-Hölder continuous,
$D_i u = \frac{\partial u}{\partial x_i}$	The partial derivative of u with respect to x_i,
$\mathcal{D}(\Omega)$	space of indefinitely differentiable functions on Ω,
p'	dual exponent,
p^*	Sobolev conjugate
p^+, p^-	essential supremum and infimum of p,

C^∞	Smooth functions,
C_0^∞	Smooth functions with compact support,
L	Usually the constant from equivalence of Φ-functions,
Δu	the gradient of u,
∇u	the Laplacian of u,
$\simeq$	is similar or equal to,
X^*	The dual space of X
p^-	$= \min_{x\in\overline{\Omega}}(p(x))$,
p^+	$= \max_{x\in\overline{\Omega}}(p(x))$,
$W^{E,\vec{p}} : E$	= $\left\{\alpha \in \mathbb{N}^n; \lvert\alpha\rvert \leqq 1\right\}$

Introduction

The Sobolev space (see [4],[12]) is a vector space of functions that have weak derivatives. Motivation for studying these spaces is that solutions of partial differential equations, when they exist, belong naturally to Sobolev spaces. The reason Sobolev spaces are so effective for PDEs is that Sobolev spaces are Banach spaces, and thus the powerful tools of functional analysis can be brought to bear. In particular, the existence of weak solutions to many elliptic PDE follows directly from the Lax-Milgram theorem.
In addition variable exponent Lebesgue-Sobolev spaces have been intensively studied during the last years (see [3], [5], [6], [8], [10], [11],[16], [17] and [18] for more details). These spaces of functions provide a useful tool for the study of both isotropic and anisotropic elliptic and parabolic equations with variable exponent (see [1],[2], [7], [15])

In this memory, we try to study an interesting topic like some basic properties of anisotropic variable exponent Sobolev spaces, as well as completeness, separability, uniform convexity, reflexivity, density of smooth functions, duality, convergence, continuous-compact embedding (injection), Poincaré-type inequality, anisotropic Sobolev inequalities, anisotropic embedding theorems and Hölder's inequality, Furthermore, the log-Holder continuous conditions...ect.

The memory consists of three chapters. The first chapter focuses on isotropic case, the second on isotropic case with variable exponents and the third on anisotropic case with variable exponents.

In the preliminaries (Chapter 1) we establish the notation of the memory. We introduce some important results on some definitions, examples of Sobolev spaces and some results about the extension theorems, the boundary trace theorems and the embedding theorems. Also, we recall the most important results about completeness, separability, uniform convexity, reflexivity, density of smooth functions in the case constant exponents.

In Chapter 2 of this memory we introduce the study of some definitions and basic properties of the variable exponent Lebesgue-Sobolev spaces as generalization of some results as in Chapter 1.

In the last chapter (Chapter 3) we give the necessary notations and some properties of anisotropic constant and variable exponent Sobolev spaces and some important results as in In Chapter 2.

CHAPTER 1

SOBOLEV SPACES

1.1 Motivation and definition of Sobolev Spaces

let $C_c^\infty(\Omega)$ be the set of infinitely differentiable functions with compact support $\varphi : \Omega \to \mathbb{R}$. function $\varphi \in C_c^\infty(\Omega)$ will often be called a test function.For each $u \in C^1(\Omega)$ and taking into account that every $\varphi \in C^{\infty_c}(\Omega)$ has compact support, Greens identity yields

$$\int_\Omega u\varphi_{x_i} = -\int_\Omega u_{x_i}\varphi \quad \forall \varphi \in C_c^\infty(\Omega), \forall i = 1, ..., n$$

This motivates the definition of Sobolev spaces as follows:

Definition 1.1.1. [12] For each $1 \leq p \leq \infty$ the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\omega) \middle| \begin{smallmatrix} \exists g^1,...,g_N \in L^P(\Omega) & \text{such that} \\ \int_\Omega u\varphi_{x_i} = -\int_\Omega g_i\varphi & \forall \varphi \in C_c^\infty(\Omega) \end{smallmatrix} \right\}$$

We denote by $u_{x_i} = g_i$ the weak derivative of $u \in W^{1,p}(\Omega)$, which are unique as we will see now. Also, the gradient of u is defined by $\nabla u = (u_{x_1}, ..., u_{x_N})$ if $p = 2$ we write $H^1(\Omega) = W^{1,2}(\Omega)$

In order to prove that the weak derivative is unique, we shall first state the following well known lemma, about L^p spaces:

Lemma 1.1.2. *[12] Let $\Omega \in \mathbb{R}^N$ be an open set and let $u \in l^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

Then,$u = 0$ a.e. on Ω

Now we can proof the uniqueness of weak derivative, as follows:

Proposition 1.1.3. *[12][13] let $u \in L^p(\Omega)$be a function that has a weak derivative u_{x_i}. Then, this weak derivative is unique except in a set of zero measure, that is, if $g, h \in L^p(\Omega)$ are two functions such that $\int_\Omega u\varphi_{x_i} = -\int_\Omega g\varphi = -\int_\Omega h\varphi$ for every $\forall\varphi \in C_c^\infty(\Omega)$,then $g = h$ a.e .*

Proof. Suppose that there exist,$g, h \in L^p(\Omega)$ such that $\int_\Omega u\varphi_{x_i} = -\int_\Omega g\varphi = -\int_\Omega h\varphi, \forall\varphi \in C_c^\infty(\Omega)$ .Let $\psi = g - h$.We shall see that $\Psi = 0$ a.e we have that

$$\int_\Omega \psi\varphi = 0 \quad \forall\varphi \in C_c^\infty(\Omega)$$

Since $\psi \in C_c^\infty(\Omega)$, in particular $\psi \in L^p_{loc} \subset L^1_{loc}$ Therefore, using the previous lemma, $\psi = 0$ a.e., and therefore the weak derivative is well dfined (up to a set of zero measure) □

There is an alternative way to dfined the Sobolev Spaces. Given a function $f \in L^p(\mathbb{R}^N)$ and e_i the i-th vector of the canonical basis of $\mathbb{R}^N$ we say that the i-th partial derivative of f exists in the L^p sense and equals f_{x_i} , if $\epsilon^{-1}(\tau_{\epsilon e_i} f - f) \to -f_{x_i}$ in $L^p(\mathbb{R}^N)$,when $\epsilon \to 0$. The function $\tau_{\epsilon e_i}$ defined by $(\tau_{\epsilon e_i} f)(x) = f(x + \epsilon e_i)$.

With these definitions , we can alternatively define the sobolev space as

$W^{1,p}(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N) | f_{x_i}$ exists in the L^p sense for every $i = 1, ..., N\}$

Both definitions of the partial derivative f_{x_i} can be proved to be equal. In what follows, the first definition will be used, instead of the alternative definition.

For each $u \in W^{1,p}(\Omega)$, we define the norm of u by

$$\|u\|_{W^{1,p}} = \|u\|_p + \sum_1^N \|u_{x_i}\|_p \tag{1.1}$$

Moreover, $H^1(\Omega)$is a Hilbert space equipped with the following scalar product:

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^N (u_{x_i}, v_{x_i})_{L^2} \tag{1.2}$$

Since $H^1(\Omega)$ is a Hilbert space, (1.2) induces a norm in $H^1(\Omega)$,

$$\|u\|'_{H^1} = \left(\|u\|_2^2 + \sum_{i=1}^N \|u_{x_i}\|_2^2\right)^{1/2}$$

This norm is equivalent to1.1 for $p = 2$.

1.2 First properties of Sobolev Spaces

Proposition 1.2.1. *[12][4] Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, the following statemente hold :*

(i) *For each $1 \leq p \leq \infty, W^{1,p}(\Omega)$ is a Banach space.*

(ii) *For each $1 < p < \infty, W^{1,p}(\Omega)$ is refelexive.*

(iii) *For each $1 \leq p < \infty, W^{1,p}(\Omega)$ is a separable.*

Proof. [1]

(i) Let $\{u_n\}_{n\in N}$ be a Cauchy sequence in $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, Then, from 1.1 it follows that $\{u_n\}_{n\in N}$ and $\left\{(u_n)_{x_i}\right\}_{n\in N}$, with $1 \leq i \leq N$, are Cauchy sequences in L^p . Thus, since L^p is a Banach space , it follows that $u_n \to n$ and $(u_n)_{x_i} \to g_i$ in L^p with $u, g_i \in L^p$ Therefore, since

$$\int u_n \varphi_{x_i} = -\int_\Omega (u_n)_{x_i} \varphi \quad \forall \varphi \in C_c^\infty(\Omega)$$

Letting $n \to +\infty$

$$\int u \varphi_{x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega)$$

Therefore, we obtain that $u \in W^{1,p}$, $u_{xi} = g_i$ and thus $\|u_n - u\|_{W^{1,p}} = \|u_n - u\|_p + \sum_{i=1}^N \|U_n - g_i\|_p \to 0,$ as desired.

(ii) Consider the space $E = L^p(\Omega) \times L^p(\Omega)^N$ which is reflexive since it is the product of reflexive spaces. Set the operator $T : W^{1,p}(\Omega) \to E$ defined by $Tu = (u, \nabla u)$ Then, T is an isometry, and since $W^{1,p}(\Omega)$ is a Banach space,$M = T\left(W^{1,p}(\Omega)\right)$ is a closed subspace of E since E is reflexive, B_E is compact in the weak topology $\sigma\left(E, E^\star\right)$, and M is closed in the topology $\sigma\left(E, E^\star\right)$ Therefore, B_M is compact in $\sigma\left(E, E^\star\right)$, and Therefore $T\left(W^{1,p}\right)$ is reflexive .As a consequence, $W^{1,p}$ is also reflexive .

(iii) Under the notation of (ii), and taking into account that E is separable , it follows that $T\left(W^{1,p}(\Omega)\right)$ is separable and therefore $W^{1,p}(\Omega)$ is also separable . □

[1] As mentioned in the Preface, whenever a black circle precedes some content, this content is original.

Under some conditions, one can think of a function $u \in W^{1,p}(\Omega)$ as a function in $u \in C(\Omega)$. Indeed, if $u \in W^{1,p}(\Omega)$, for a certain $1 \leq p \leq \infty$,and $u_{x_i} \in C(\Omega)$ for each $1 \leq i \leq\ N$, where the partial derivative is a weak partial derivative, then it can be proven that there exists $v \in C^1(\Omega)$ suth that $u = v$ a.e .

Moreover, we can stablish the following density result, although we will later prove a stronger result under some more assumptions. But first, we need to introduce the following lemma.

Lemma 1.2.2. *[12][4] Set $\rho \in L^1(\mathbb{R}^N), v \in W^{1,p}(\mathbb{R}^N)$ with $1 \leq p \leq \infty$. Then, $\rho \star v \in W^{1,p}(\mathbb{R}^N)$ and for each $i = 1; ......, N$, we must have that $(\rho \star v)_{xi} = \rho \star (v)_{x_i}$.*

Proposition 1.2.3 (Friedrichs)**.** *[12][4] Let $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$.Then there exists a sequence $\{u_n\}_{n \in N} \subset C_c^\infty(\mathbb{R}^N)$ such that*

$$u_n|_\Omega \to u \in L^p(\omega)$$

and

$$\nabla u_n|_\omega \to \nabla u|_\omega \quad in \;\; L^p(\omega)^N \quad for\ all \;\; \omega \subset\subset \Omega$$

If $\Omega = \mathbb{R}^N$, then there exists a sequence $\{u_n\}_{n \in N} \subset C_c^\infty(\mathbb{R}^N)$ so that

$$u_n \to u \quad in \quad L^p(\mathbb{R}^N)$$

and

$$\nabla u_n \to \nabla u \;\; in \quad L^p(\mathbb{R}^N)^N$$

Proof. Set $\overline{u} = u\chi_\Omega$, and take $v_n = \rho_n \star \overline{u}$ with ρ_n a sequence of mollifiers.Then,$v_n \in C^\infty(\mathbb{R}^N)$ and, moreover,$v_n \to \overline{u}$ in $L^p(\mathbb{R}^N)$. We must see that for each $\omega \subset\subset \Omega$,$\nabla v_n|_\omega \to \nabla u|_\omega$ in $L^p(\omega)^N$. Let $\omega \subset\subset \Omega$,and take a function $\alpha \in C_c^1(\Omega)$ such that $1 \leq \alpha \leq \infty$ and $\alpha|\omega = 1$.It is easy to check that such function exists. Then, for n large enough,

$$\mathrm{supp}(\rho_n \star (\overline{\alpha u}) - \rho_n \star \overline{u}) = \mathrm{supp}(\rho_n \star (1-\overline{\alpha})\overline{u}) \subset \overline{\mathrm{supp}\,\rho_n + \mathrm{supp}(1-\overline{\alpha})\overline{u})} \subset \overline{B(0,1/n) + \mathrm{supp}(1-\overline{\alpha})} \subset (\omega)^c$$

And, therefore,

$$\rho_n \star (\overline{\alpha u}) = \rho_n \star \overline{u} \text{ on } \omega \tag{1.3}$$

Using the previous lemma, we have

$$(\rho_n \star \overline{\alpha u})_{x_i} = \rho_n \star (\overline{\alpha u})_{x_i} = \rho_n \star (\overline{\alpha u_{x_i} + \alpha_{x_i} u})$$

The last equality follows from the fact that for each $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \overline{\alpha u}\varphi_{x_i} = \int_\Omega \alpha u \varphi_{x_i} = \int_\Omega u[(\alpha u)_{x_i} - \alpha_{x_i}\varphi] = -\int_\Omega (u_{x_i}\alpha\varphi + u\alpha_{x_i}\varphi) = -\int_{\mathbb{R}^N} (\overline{\alpha u_{x_i} + \alpha_{x_i} u})\varphi$$

As a consequence, it follows that

$$(\rho_n \star \overline{\alpha u})_{x_i} \to \overline{\alpha u_{x_i} + \alpha_{x_i} u} \quad \text{in } \ L^p(\mathbb{R}^N)$$

After a restriction to ω, we have that

$$(\rho_n \star \overline{\alpha u})_{x_i} \to u_{x_i} \quad \text{in } \ L^p(\omega)$$

We shall define a certain sequence of cut-off functions ζ_n now.Fix a function $\zeta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$, and

$$\zeta(x) = \begin{cases} 1 & if \quad |x| < 1 \\ 0 & if \quad |x| \ge 2 \end{cases}$$

We define the sequence of cut-offs $\zeta_n(x) = \zeta(x/n)$. Now, using the dominated convergence theorem the sequence $u_n = \zeta_n v_n$ satisfies that $u_n \to u$ in $L^p(\Omega)$, and $\nabla u_n \to \nabla u$ in $(L^p(\omega))^N$ .if $\Omega = \mathbb{R}^N$,the sequence defined by $u_n = \zeta_n(\rho_n \star u)$ satifies the desired properties .

□

The following proposition offers a characterization of the elements of $W^{1,p}$

Proposition 1.2.4. *[12][4] Let $u \in L^p(\Omega)$, with $1 < p \le \infty$. The following properties are equivalent:*

(i) $u \in W^{1,p}(\Omega)$

(ii) There exists a constant $C > 0$ *such that*

$$\left|\int_\Omega u\varphi_{x_i}\right| \leq C \left\|\varphi\right\|_{L^{p'}(\Omega)} \qquad \forall\varphi \in C_c^\infty(\Omega), \qquad \forall i = 1, 2....., N$$

(iii) there exists a constant $C > 0$ *so that for every* $\omega \subset\subset \Omega$ *and* $h \in \mathbb{R}^N$ *such that* $|h| < dist\,(\omega, \delta\Omega)$ *we have*

$$\left\|\tau_h u - u\right\|_{L^p(\omega)} \leq C\,|h|$$

Where τ_h *is defined by* $\tau_h u(x) = u(x+h)$*.*

If $\Omega = \mathbb{R}^N$*, we have*

$$\left\|\tau_h u - u\right\|_{L^p(\mathbb{R}^N)} \leq |h| \left\|\nabla u\right\|_{L^p(\mathbb{R}^N)}$$

Proof. $(i) \Rightarrow (ii)$ Since $u \in W^{1,p}(\Omega)$, for each $\varphi \in C_c^\infty(\Omega)$ $i = 1; ....; N$,

$$\left|\int_\Omega u\varphi_{x_i}\right| = \left|\int_\Omega u_{x_i}\varphi\right| \leq \left\|u_{x_i}\right\|_p \left\|\varphi\right\|_{p'}$$

$(ii) \Rightarrow (i)$ Given $i \in \{1, ..., N\}$, Consider the linear functional

$$\varphi \in C_c^\infty(\Omega) \mapsto \int_\Omega u\varphi_{x_i}$$

This linear functional is defined on a dense subspace of $L^{p'}$,since $p' < \infty$ as $1 < p$. Moreover, this functional is continuous for the norm in $L^{p'}$ because of (ii). Thus, we may apply Hahn-Banach theorem in order to extend this functional to a bounded linear functional F that is defined in all of $L^{p'}$ Applying the Riesz representation theorem, there must exist a function $g \in L^p$

$$\langle F, \varphi\rangle = \int_\Omega f\varphi \qquad \forall\varphi \in L^{p'}$$

In particular,

$$\int_\Omega u\varphi_{x_i} = \int_\Omega g\varphi \qquad \forall\varphi \in C_c^\infty$$

And therefore $u \in W^{1,p}$.

$(i) \Rightarrow (iii)$.Suppose that $u \in C_c^\infty(\mathbb{R}^N)$ .A density argument will beused to prove the general case.Set $h \in \mathbb{R}^N$, and set $v(t) = u(x+th)$, for each $t \in \mathbb{R}$ Clearly $v'(t) = h\nabla u(x+th)$, and hence

$$u(x+h) - u(x) = v(1) - v(0) = \int_0^1 v'(t)dt = \int_0^1 h.\nabla u(x+th)dt$$

Therefore, for each $1 \leq p < \infty$

$$|\tau_h u(x) - u(x)|^p \leq |h|^p \int_0^1 |\nabla u(x+th)|^p \, dt$$

Integrating on ω, we reach to

$$\int_\omega |\tau_h u(x) - u(x)|^p \, dx \leq |h|^p \int_0^1 dt \int_\omega |\nabla u(x+th)|^p = |h|^p \int_0^1 dt \int_{\omega+th} |\nabla u(y)|^p \, dy$$

Now, take $|h| < dist(\omega, \partial\Omega)$.Then, clearly there exists an open set $\omega' \subset\subset \Omega$ such that $\omega + th \subset \omega'$ for all $t \in [0,1]$,Therefore,

$$\|\tau_h u - u\|_{L^p(\omega)}^p \leq |h|^p \int_\omega |\nabla u|^p$$

Which proves the case where $u \in C_c^\infty(\mathbb{R}^N), 1 < p < 1$. Using proposition 1.2.3, the general case follows.

$(iii) \Rightarrow (ii)$. Take $\varphi \in C_c^\infty(\Omega)$. We may pick an open set ω that is contained in between $\operatorname{supp}\varphi$,and Ω that is, $\operatorname{supp}\varphi \subset \omega \subset\subset \Omega$.Now, we proceed as follows: pick $h \in \mathbb{R}^N$,with $|h| < dist(\omega, \partial\Omega)$.We are now under the hypotheses of (iii), so that

$$\left| \int_\Omega (\tau_h u - u)\, \varphi \right| \leq |h\, \| | \, \varphi\|_{L^{p'}(\Omega)}$$

Now, we have that

$$\int_\Omega (u(x+h) - u(x))\varphi(x)dx = \int_\Omega u(y) \left(\varphi\left(y-h\right) - \varphi\left(y\right)\right) dy$$

We conclude that

$$\int_\Omega u(y) \frac{(\varphi(y-h) - \varphi(y))}{|h|} dy \leq C \, \|\varphi\|_{L^{p'}(\Omega)}$$

(ii) follows from letting $h = te_i$ and taking $t \to 0$.

Weak derivatives enjoy some properties that are analogous to the case of C^1 functions, such as the differentiation of a product and composition, and the change of variables formula. The following results, whose proofs are omitted, state these properties (See H. Brezis [4] , Chapter 9, for the corresponding proofs) □

Weak derivatives enjoy some properties that are analogous to the case of C^1 functions, such as the differentiation of a product and composition, and the change of variables formula. The following results, whose proofs are omitted, state these properties (See H. Brezis [4], Chapter 9, for the corresponding proofs)

Proposition 1.2.5. *[12][4] Let $1 \leq p \leq \infty$. Then,$W^{1,P}(\Omega) \cap L^{\infty}(\Omega)$ is closed under multiplication, that is, for every $u, v \in W^{1,P}(\Omega) \cap L^{\infty}(\Omega)$, its product $uv \in W^{1,P}(\Omega) \cap L^{\infty}(\Omega)$,. Moreover,*

$$(uv)_{x_i} = u_{x_i} v + u v_{x_i}, \quad i = 1, 2...., N$$

Proposition 1.2.6. *Let $G \in C^1(\mathbb{R})$ be a differentiable continuous function such that $G(0) = 0$ and $\|G'\|_{\infty} \leq M$, with $M \geq 0$.Then, for each $u \in W^{1,P}(\Omega)$ $(1 < p \leq \infty)$the composition of G and u belongs to $W^{1,P}$.That is, $G \circ u \in W^{1,P}(\Omega)$,and moreover*

$$(G \circ u)_{x_i} = (G' \circ u) u_{x_i}, \qquad i = 1, 2...., N$$

Proposition 1.2.7. *[12][4] Let $\Omega', \Omega \subset \mathbb{R}^N$ be two open sets, and $H : \Omega' \to \Omega$ a bijective map of class C^1, such that $H^{-1} \in C^1(\Omega)$,$jacH \in L^{\infty}(\Omega')$ and $jacH^{-1} \in L^{\infty}(\Omega)$ where Jac denotes the Jacobian matrix. Then, $u \circ H \in W^{1,P}(\Omega')$*

$$(u(H(y)))_{y_j} = \sum_i u_{x_i}(H(y))(H_i(y))_{y_j}, \qquad j = 1, 2...., N$$

1.3 $W^{m,p}(\Omega)$spaces

After defining the $W^{1,p}(\Omega)$,spaces, we can define the general $W^{m,p}$ spaces recursively. Let $m \geq 2$ be an integer, and $1 \leq p \leq \infty$ .Then, we define

$$W^{m,p}(\Omega) = \{u \in W^{m-1,p}(\Omega) \left| u_{x_i} \in W^{m-1,p}(\Omega) \right., \qquad \forall i = 1, 2...., N\}$$

An equivalent way of defining these Sobolev spaces is defning $W^{m,p}$ as

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\omega) \middle| {}^{\forall \alpha \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \text{such that}}_{\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \quad \forall \varphi \in C_c^\infty(\Omega)} \right\}$$

The multi-index notation have been used. That is,$\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ and $|\alpha| = \sum_i \alpha_i$. Moreover,

$$D^\alpha \varphi = \frac{\delta^{|\alpha|} \varphi}{\delta x_1^{\alpha_1} ....\delta x_N^{\alpha_N}}$$

We denote $D^\alpha u = g_\alpha$.Then, it can be proved (although its proof will be omitted) that $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{m,p}} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p \tag{1.4}$$

Just like in the $W^{1,2}$ case, $W^{m,2}(\Omega)$,is a Hilbert space that is denoted by $H^m(\Omega)$, and its scalar product is

$$(u, v)_{H^m} = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2}$$

Again, the norm arising from this scalar product is equivalent to 1.4.

1.4 Extension Operators and Sobolev Inequalities

One may wonder if given a function $u \in W^{1,p}(\Omega)$ with $\Omega \subsetneq \mathbb{R}^N$, there exists an extension $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$.That is, a function $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$ such that $\widetilde{u}_{|\Omega} = u$ a.e. This is not true in general, unless we require more hypotheses to Ω . If the domain is smooth enough, a concept that will be defined now, the result is actually true.

Notation. Let $x \in \mathbb{R}^N$.We write x as $x = (x', x_N)$,with $x' \in \mathbb{R}^{N-1}$ Moreover, we denote $|x'| = \|(x_1, ..., x_{N-1})\|_2$, where $\|.\|$ is the euclidean norm of $\mathbb{R}^{N-1}$. Finally, we define the following sets:

(i) $\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^N \,|x_N > 0\}$

(ii) $Q = \{(x', x_N) \in R^N \,||x'| < 1 and \ |x_N| < 1\}$

(iii) $Q = Q \cap \mathbb{R}^N_+$

(iv) $Q = \{(x', 0) \in \mathbb{R}^N \,||x'| < 1\}$

Definition 1.4.1. An open set $\Omega \subset \mathbb{R}^N$ is said to be of class C^1 if for every $x \in \partial\Omega = \Gamma$ there exists a neighborhood U of x in $\mathbb{R}^N$ and a bijective map $H : Q \to U$ such that

$$H \in C^1(\overline{Q}), \qquad H^{-1} \in C^1(\overline{U}), \qquad H\left(Q_+\right) = U \cap Q, \text{ and } H\left(Q_0\right) = U \cap \Gamma,$$

Under these conditions, H is said to be a local chart.

The following theorem assures the existence of an extension operator that extends any function $u \in W^{1,p}(\Omega)$ to a function $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$, as long as Ω is of class C^1:

Theorem 1.4.2. *[12][4] Let $\Omega \in \mathbb{R}^N$be a domain of class C^1 with $\Gamma = \partial\Omega$ bounded or $\Omega = \mathbb{R}^N_+$.Then, there exists a linear operator*

$$P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

with $1 \leq p \leq \infty$,that fulfills the following properties for each $u \in W^{1,p}(\Omega)$

(i) $(Pu)_{|\Omega} = u$,

(ii) $\|Pu\|_{L^p(\mathbb{R}^N)} \leq C\,\|u\|_{L^p(\Omega)}$,

(iii) $\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \leq C\,\|u\|_{W^{1,p}(\Omega)}$,

with $C \geq 0$ a constant that depends only on Ω. P is the extension operator we mentioned before.

Using this result, we may prove a density result regarding $W^{1,p}$ spaces.

Corollary 1.4.3. *[12][4] Suppose Ω is of class C^1. Then, the restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^N)$ form a dense subspace of $W^{1,p}(\Omega)$.*

Proof. Let $u \in W^{1,p}(\Omega)$ First, we will assume that Γ is bounded. Using the previous theorem, there exists an extension operator P. Let

$$u_n = \zeta_n\left(\rho_n \star Pu\right)$$

With ζ_n the cut-off functions previously mentioned, and ρ_n a sequence of molliffers. Then,$u_n \in C_c^\infty(\mathbb{R}^N)$ $u_{n_{|\Omega}} \to u$ in $W^{1,p}(\Omega)$.

If, on the other hand, Γ is not bounded, we consider the sequence $\zeta_n u$. Then, $\zeta_n u \to u$ in $W^{1,p}(\Omega)$ so we may pick $n_0 \geq 1$ so that $\|\zeta_{n_0} u - u\|_{W^{1,p}} < \epsilon$. Using the case where Γ is bounded, we can construct an extension $v \in W^{1,p}(\mathbb{R}^N)$ of ζ_{n_0}.Finally, using 1.2.3 we pick $w \in C_C^\infty(\mathbb{R}^N)$ such that $\|w - v\|_{W^{1,p}(\mathbb{R}^N)} \leq \epsilon$.Then,

$$\begin{aligned} \left\|w_{|\Omega} - u\right\|_{W^{1,p}(\Omega)} &\leq \left\|w_{|\Omega} - \zeta_{n_0} u\right\|_{W^{1,p}(\Omega} + \|\zeta n_0 u - u\|_{W^{1,p}(\Omega} \\ &\leq \|w - v\|_{W^{1,p}(\mathbb{R}^N)} + \epsilon < 2\epsilon \end{aligned}$$

□

The following corollary generalizes a classical result of C^1 functions: if u is of class C^1 and its partial derivatives vanish in an open connected set U, then u is constant on U.

Corollary 1.4.4. *[12][4] Let $\Omega \subset \mathbb{R}^N$ be a domain such that $\Omega = \mathbb{R}^N$,or Ω is of class C^1 with Γ bounded.If $u \in W^{1,p}(\Omega)$ satisfies that*

$$u_{x_i} = 0 \quad on \;\; U \subset \Omega, \quad \forall 1 \leq i \leq N$$

with $U \subset \Omega$ an open connected set, then $u_{|U}$ is constant.

Proof. Assume that $\Omega = \mathbb{R}^N$.First. Let $\{\rho_n\}$ be a sequence of mollifiers such that $\rho_n \star u \to u$ in $W^{1,p}(\mathbb{R}^N)$. Using Lemma 1.2.2,

$$(\rho_n \star u)_{x_i} = \rho_n \star (u_{x_i}) = 0 \quad on \;\; U, \quad \forall 1 \leq i \leq N, n \geq 1$$

Since $(\rho_n \star u) \in C^\infty(\mathbb{R}^N)$,$\rho_n \star u$has to be constant on U, and taking into account that $(\rho_n \star u)_{|U} \to u_{|U}$,u is constant on U.

if $\Omega \subset \mathbb{R}^N$ is a domain of class C^1 with Γ bounded,using the Extension Operator Theorem 1.4.2, we extend u to a function in $W^{1,p}(\mathbb{R}^N)$ using the extension operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$.Then,$(Pu)_{x_i|U} = u_{x_i|U} = 0$.

Thus, using the case where $\Omega = \mathbb{R}^N$ we just proved, necessarily $Pu_{|U}$ has to be constant, and as a consequence $u_{|U}$ is constant. □

In numerous occasions, it is useful to embed Sobolev spaces into other spaces. Namely, it is important to know if we can embed a Sobolev space in some L^q space, or even in the space of continuous functions. Moreover, it is useful to determine when these embeddings are continuous or compact.
The dimension of the space will play a key role here, and the space where the Sobolev space is embedded will in general depend on the dimension of the space. Whether Ω is a proper subset of $\mathbb{R}^N$ or not will also be important.

1.4.1 When $\Omega = \mathbb{R}^N$

Lemma 1.4.5. *[12][4] Let $N \geq 2$,and set $f_1, ..., f_N \in L^{N-1}(\mathbb{R}^{N-1})$.Given $x \in \mathbb{R}^N$,we denote $\widetilde{x_i}$ as*

$$\widetilde{x_i} = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_N) \in \mathbb{R}^{N-1}$$

Then,

$$f(x) = f_1(\widetilde{x_1})...f_N(\widetilde{x}_N) \in L^1(\mathbb{R}^N)$$

And we have the estimate

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}$$

Now, the following theorem gives us a first result about when a Sobolev space is included in a L^p space.

Theorem 1.4.6 (Sobolev, Gagliardo, Nirenberg)**.** *[12][4] For each $1 \leq p \leq N$,let $p^\star$ be the unique number that is defined by $\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}$. Then, we have the inclusion*

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^\star}(\mathbb{R}^N) and p^* = \frac{PN}{N-P}$$

Moreover, there exists a constant C that only depends on p and N, such that

$$\|u\|_{p^\star} \leq C \|\nabla u\|_p$$

Proof. Assume that $p = 1$ and $u \in C_c^1(\mathbb{R}^N)$ First. Then,

$$|u(x_1, x_2, ..., x_N)| = \left| \int_{-\infty}^{x_1} u_{x_1}(t, x_2, ..., x_N) dt \right| \leq \int_{-\infty}^{\infty} |u_{x_1}(t, x_2, ..., x_N)| \, dt \tag{1.5}$$

Let $f_i(\widetilde{x}_i)$=$\int_{-\infty}^{\infty} |u_{x_i}(x_1, ..., x_{i-1}, t, x_{i+1}; ..., x_N)| \, dt$. Then, proceeding as in 1.5, we have that

$$|u(x_1, x_2, ..., x_N)| \leq f_i(\widetilde{x}_i), \quad i = 1, ..., N$$

It follows, using the previous lemma, that

$$\int_{\mathbb{R}^N} |u(x)|^{N/(N-1)} \, dx \leq \prod_{i=1}^{N} \|f_i\|_{L^1(\mathbb{R}^{N-1})}^{1/N-1} = \prod_{i=1}^{N} \|u_{x_i}\|_{L^1(\mathbb{R}^N)}^{1/N-1}$$

That is,

$$\|u\|_{L^{N/(N-1)}} (\mathbb{R}^N) \leq \prod_{i=1}^{N} \|u_{x_i}\|_{L^1(\mathbb{R}^N)}^{1/N} \tag{1.6}$$

Which is precisely what we wanted to prove, since if $p = 1$,$p^\star = N/(N-1)$.For the general case $1 < p < N$ (although with $u \in C_c^1(\mathbb{R}^N)$),we proceed as follows. Let $m \geq 1$.Applying 1.6 to $|u|^{m-1} u$, we have that

$$\|u\|_{mN/(N-1)}^{m} \leq m \prod_{i=1}^{N} \left\| |u|^{m-1} u_{x_i} \right\|_1^{1/N} \leq m \, \|u\|_{P'(m-1)}^{m-1} \prod_{i=1}^{N} \|u\|_p^{1/N} \tag{1.7}$$

Since m is arbitrary, we may pick m so that $mN/(N-1) = P'(m-1)$,obtaining $m = (N-1)p^\star/N$. Notice that $m \geq 1$ since $1 < p < N$. Then,

$$\|u\|_{p^\star} \leq m \prod_{i=1}^{N} \|u_{x_i}\|_p^{1/N}$$

And thus $\|u\|_{p^\star} \leq C \, \|\nabla u\|_p$ The conclusion is immediate: Using Fatou's lemma, $u \in L^{p^\star}$ and $\|u\|_{p^\star} \leq C \, \|\nabla u_n\|_p$ □

Remark 1.4.7. In the theorem, we can take $C = C(p, N) = (N-1)p = (N-P)$. However, this is not the optimal constant. It is possible to calculate the optimal one, although the procedure is not simple at all.

Corollary 1.4.8. *[12][4] If $1 \leq p < N$, then*

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \quad \forall q \in [p, p^\star]$$

with a continuous injection.

Proof. Let $q \in [p, p^\star]$. Write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^\star}, \text{ for some } \alpha \in [0,1]$$

It can be checked that $\|u\|_q \le \|u\|_p^\alpha \|u\|_{p^\star}^{1-\alpha} \le \|u\|_p \|u\|_{p^\star}$. Young's inequality has been used in here. Therefore, using the theorem that was just proved,

$$\|u\|_q \le C \|u\|_{W^{1,p}} \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

□

A good question that arises from the Sobolev, Gagliardo and Nirenberg is what happens when $p = N$. The following corollary answer this question:

Corollary 1.4.9. *[12][4] The following embedding holds:*

$$W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \quad \forall q \in [N, +\infty)$$

Proof. As usual, we assume that $u \in C_c^1(\mathbb{R}^N)$ first. We can apply 1.7 with $p = N$, obtaining

$$\|u\|_{mN/(N-1)}^m \le m \|u\|_{(m-1)N/(N-1)}^{m-1} \|\nabla u\|_N \quad \forall m \ge 1$$

Using Young's inequality,

$$\|u\|_{mN/(N-1)} \le C\Big(\|u\|_{(m-1)N/(N-1)} + \|\nabla u\|_N \Big) \quad \forall m \ge 1$$

In the previous equation we can pick $m = N$. Then,

$$\|u\|_{N^2/(N-1)} \le \|u\|_{W^{1,N}}$$

Using Gagliardo-Nirenberg interpolaiton inequality,we conclude that

$$\|u\|_q \le C \|u\|_{W^{1,p}}$$

For every q such that $N \le q \le N^2/(N-1)$.We repeat the argument with $m = N+1, N+2, ...$and we finally get

$$\|u\|_q \le \|u\|_{W^{1,N}} \quad \forall u \in C_c^1(\mathbb{R}^N)$$

For every $q \ge N$. Repeating the usual density argument, the corollary is proved. □

Lastly, we have the following embedding result by Morrey, for the case where $p > N$.

Proposition 1.4.10 (Morrey)**.** *[12][4] If $p > N$, then*

$$W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$$

The injection is continuous, and moreover for every $u \in W^{1,p}(\mathbb{R}^N)$, and if we define α as $\alpha = 1 - N/p$, we have

$$|u(x) - u(y)| \leq C\,|x-y|^\alpha \,\|\nabla u\|_p \quad a.e.\ x, y \in \mathbb{R}^N \tag{1.8}$$

where C is constant and depends only on p and N.

For a proof of this theorem, see H. Brezis [4].

Let us emphasize an implication of the previous theorem. Let $\Lambda \subset \mathbb{R}^N$be a set of zero measure such that the inequality 1.8 is satisfied in $\mathbb{R}^N \setminus \Lambda$ Then, we can extend the function $u_{|R^N \setminus A}$ to a continuous function in $\mathbb{R}^N$,and given the fact that $\mathbb{R}^N \setminus \Lambda$ is dense in $\mathbb{R}^N$ this extension is unique. That is, we can replace u by a continuous representative. Using repeatedly the theorems and corollaries that were stated previously :

Corollary 1.4.11. *[12][4] Let $m \in N$ and $1 \leq p < \infty$. Then, we have the following continuous injections:*

$$\begin{aligned} W^{m,p}(\mathbb{R}^N) &\subset L^q(\mathbb{R}^N), \quad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{N}, \ \text{if}\ \frac{1}{p} - \frac{m}{N} > 0 \\ W^{m,p}(\mathbb{R}^N) &\subset L^q(\mathbb{R}^N), \quad \forall \quad p \leq q < \infty \ \text{if}\ \frac{1}{p} - \frac{m}{N} = 0 \\ W^{m,p}(\mathbb{R}^N) &\subset L^\infty(\mathbb{R}^N), \quad \text{if} \quad \frac{1}{p} - \frac{m}{N} < 0 \end{aligned} \tag{1.9}$$

Moreover, we have that $W^{m,p}(\mathbb{R}^N) \subset C^k(\mathbb{R}^N)$ with $k = [m - (N/p)]$

1.4.2 When $\Omega \subset \mathbb{R}^N$

In what follows, Ω will be considered to be a domain of class C^1, with $\Gamma = \partial\Omega$ bounded, or $\Omega = \mathbb{R}^N_+$. In this section the Theorem 1.4.2 will play a crucial role. The general idea that will appear in the proofs of this section is to extend the functions of $W^{1,p}(\Omega)$ to a function of$W^{1,p}(\mathbb{R}^N)$ using Theorem 1.4.2, in order to use the results from the previous section.

Proposition 1.4.12. *[12][4] For each $1 \leq p \leq \infty$,we have the following continuous injections:*

$$W^{1,p}(\Omega) \subset L^{p^\star}(\Omega), \quad \text{where} \quad \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, \ \text{if } p < N$$

$$W^{1,p}(\Omega) \subset L^{q}(\Omega), \quad \forall \quad p \leq q < \infty \ \text{ if } p = N \tag{1.10}$$

$$W^{1,p}(\Omega) \subset L^{\infty}(\Omega), \qquad\qquad \text{if} \quad p > N$$

Proof. Using Theorem 1.4.2, we take the extension operator
$P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$.Then, we apply the different results from the previous section where we studied the case of $\mathbb{R}^N$, in order to conclude the desired results after a restriction to Ω . □

Corollary 1.4.13. *[12][4] If $p > N$, for every $u \in W^{1,p}(\Omega)$ we have*

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,p}} |x - y|^{\alpha} \quad a.e. \ x, y \in \Omega \tag{1.11}$$

with $\alpha = 1 - \frac{N}{p}$ and C is a constant depending on Ω, p and N. Thus, $W^{1,p} \subset C(\overline{\Omega})$.

Proof. . In a similar way as in the proof of the previous theorem, take the extension operator P and apply Morrey's Theorem 1.4.10 □

Corollary 1.4.14. *[12][4] The conclusions of Corollary 1.4.11 are still true if $\mathbb{R}^N$ is replaced by Ω. The following result shows different cases in which the injection is compact, instead of just continuous. It will be very useful in the second part of the Dissertation, where PDEs will be studied.*

Theorem 1.4.15 (Rellich-Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^N$be a bounded domain of class C^1. Then, the following injections are compact:*

$$\begin{aligned} &W^{1,p}(\Omega) \subset L^{q}(\Omega) \quad \forall q \in [1, p^\star), \quad \text{where} \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, \ \text{if } p < N \\ &W^{1,p}(\Omega) \subset L^{q}(\Omega) \quad \forall q \in [p, +\infty), \ \text{if } p = N \\ &W^{1,p}(\Omega) \subset C(\overline{\Omega}), \qquad \text{if } p > N \end{aligned} \tag{1.12}$$

Proof. Let $p > N$. Let $\mathcal{H}$ be the unit ball in $W^{1,p}(\Omega)$.Using Proposition 1.4.12 the injection $W^{1,p}(\Omega) \subset L^{\infty}(\Omega)$ is continuous. Thus, there exists a constant $M > 0$ such that $\|u\|_{\infty} \leq M \|u\|_{W^{1,p}}$. Therefore, the set $\mathcal{H}$ is bounded. Now, we will prove that $\mathcal{H}$ is equicontinuous. Given $\epsilon > 0$, set $\delta = (\epsilon/C)^{1/\alpha}$,where C is the constant mentioned by 1.11. Then, for each $x, y \in \Omega$ such that $|x - y| < \delta$,

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,p}} |x - y|^{\alpha} \quad a.e. \, \forall u \in \mathcal{H}$$

So, indeed, $\mathcal{H}$ is an equicontinuous family. Using Ascoli-Arzelàs theorem, $\mathcal{H}$ has compact closure and therefore the the injection is compact.

The case $p = N$ reduces to the case $p < N$, so we will study the ase where $p < N$ now.

Again, we denote by $\mathcal{H}$ the unit ball in $W^{1,p}(\Omega)$.Using the Theorem 1.4.2, we consider the extension operator P. Set $\mathcal{F} = P(\mathcal{H})$, so that $\mathcal{H} = \mathcal{F}_{|\Omega}$ We will use Kolmogorov Riesz compactness theorem to prove that $\mathcal{H}$ has compact closure in $L^p(\Omega)$, for $q \in [1, p^{\star})$.We can assume that $q \geq p$, since Ω is bounded. Clearly,F is bounded in $W^{1,p}(\mathbb{R}^N)$,for (the prove is similar to the case $p > N$), and therefore it is also bounded in $L^q(\mathbb{R}^N)$ by Corollary 1.4.8. In order to use Kolmogorov Riesz's theorem, we have to check that

$$\lim_{|h| \to 0} \|\tau_h f - f\|_{L^q(\mathbb{R}^N)} = 0 \text{ uniformly in f } \in \mathcal{F} \tag{1.13}$$

Using Proposition (1.2.4),

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^N)} \leq |h| \|\nabla f\|_{L^p(\mathbb{R}^N)} \quad f \in \mathcal{F}$$

We write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^{\star}} \text{ for some } \alpha \in (0, 1]$$

Since $p \leq q < p^{\star}$. Using Gagliardo-Nirenberg interpolation inequality we conclude that

$$\|\tau_h f - f\|_{L^q(\mathbb{R}^N)} \leq \|\tau_h f - f\|^{\alpha}_{L^p(\mathbb{R}^N)} \|\tau_h f - f\|^{1-\alpha}_{L^{p^{\star}}(\mathbb{R}^N)} \tag{1.14}$$

$$\leq |h|^{\alpha} \|\nabla f\|^{\alpha}_{L^p(\mathbb{R}^N)} (2 \|f\|_{L^{p^{\star}}(\mathbb{R}^N)})^{1-\alpha} \leq C |h|^{\alpha}$$

Where C is a constant that does not depend on $\mathcal{F}$, because as we have proved $\mathcal{F}$ is bounded in $W^{1,p}$. Then, 1.13 holds, and thus using Kolmogorov-Riesz's compactness theorem the injection is compact. □

Corollary 1.4.16. *Let $\{u_n\}$ be a bounded sequence in $W^{1,p}(\Omega)$, $1 \leq p < \infty$,such that $u_n \to u$ weakly on $W^{1,p}(\Omega)$,with Ω bounded andof class C^1,Then, there exists a subsequence that converges strongly to u in $L^p(\Omega)$and in particular there exists a subsequence that converges a.e. to u.*

Proof. .Using Theorem (1.13)[4], the injection $W^{1,p}(\Omega) \subset L^p(\Omega)$ is compact. That is, the injection operator $i : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. Thus and taking into account that $\{u_n\}$ is bounded, $i(u_n) = u_n$ has a convergent subsequence in $L^p(\Omega)$. Therefore, due to the uniqueness of the limit, we must have that $u_n \to u$ strongly in $L^p(\Omega)$. Moreover, since $u_n \to$ in $L^p(\Omega)$,we can extract again a subsequence that converges a.e. to u. □

1.5 $W_0^{m,p}(\Omega)$ space and its dual

Definition 1.5.1. [12] We denote by $W_0^{1,p}(\Omega)$ with $1 \leq p < \infty$ the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$.We write $H_0^1(\Omega) := W_0^{1,2}(\Omega)$.We can equip the space $W_0^{1,p}$ with the norm of $W^{1,p}$. Then, the space is a separable Banach space,and it is reflexive for $1 < p < \infty$. if $p = 2$, the space H_0^1 is a Hilbert space, equipped with the scalar product of H^1. An immediate observation is that given the fact that $C_0^1(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we obtain $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$

Similarly, we define de space $W_0^{m,p}(\Omega)$ as the closure of $C_c^m(\Omega)$ in $W_0^{m,p}(\Omega)$
Intuitively, a function of $W_0^{1,p}(\Omega)$ vanishes on $\Gamma = \partial\Omega$. This is noT accurate at all, since a function $u \in W^{1,p}(\Omega)$ is well defined up to a set of zero measure $(itisdefineda.e.)$, so it does not make sense to say that u vanishes on Γ. However, the next result will formalize this idea. Similarly we can think of a function $u \in W_0^{m,p}(\Omega)$ as a function $u \in W^{m,p}$ such tat $D^\alpha u = 0$,on Γ, for every multi-index α with $|\alpha| \leq m - 1$.

Theorem 1.5.2. *[12][4] Let Ω be a domain of class C^1,and let $1 \leq p < 1$. Set $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Then, $u = 0$ on Γ if and only if $u \in W_0^{1,p}(\Omega)$.*

The proof will be omitted[2]. This kind of results belong to the Theory of traces. Roughly speaking, the trace of u on Γ , denoted by $u_{|\Gamma}$ is a linear operator,

defined in an appropiate way from $W^{1,p}(\Omega)$ into $L^p(\Gamma)$. A corollary of this result is Poincare's inequality, that estimates the norm of a function $u \in W_0^{1,p}(\Omega)$.in terms of its gradient:

Corollary 1.5.3 (Poincare's inequality). *Let $1 \leq p < 1$, and a bounded domain. Then, there exists a constant C such that* [2]

$$\|u\|_p \leq C \|\nabla u\|_p \qquad \forall u \in W_0^{1,p}(\Omega)$$

The constant C depends only on Ω and p. Therefore, $\|\nabla u\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$.equivalent to 1.1. For the Hilbert space $H_0^1(\Omega)$, the scalar product $\sum_i \int_\Omega u_{x_i} v_{x_i}$ induces the norm $\|\nabla u\|_2$ and it is equivalent to $\|u\|_{H^1}$.

The dual of $W_0^{1,p}(\Omega)$ will be denoted by $W_0^{-1,p'}(\Omega)$. If $p = 2$, we write $H^{-1}(\Omega) := W^{-1,2}(\Omega)$.if Ω is bounded, we have the following continuous and dense injections:

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W_0^{-1,p'}(\Omega) \qquad if \;\; 2N/(N+2) \leq p < \infty$$

If Ω is not bounded the injections are still continuous and dense, but only if $2N/(N+2) \leq p < \infty$. Therefore, in particular

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

With continuous and dense injections, for every domain $\Omega \subset R^N$. the following proposition gives a better insight of the elements of $W^{-1,p'}$.

Proposition 1.5.4. *[12][4] Given $f \in W_0^{-1,p'}(\Omega)$ there exist $f_0, f_1, ..., f_N \in L^{p'}(\Omega)$ such that*

$$\langle f, v\rangle = \int_\Omega f_0 v + \sum_{i=1}^{N} \int_\Omega f_i v_{x_i} \qquad \forall v \in W_0^{1,p}(\Omega)$$

and $\|f\| = max_{0 \leq i \leq N} \|f_i\|_{p'}$. Moreover, we may pick $f_0 = 0$ if Ω is bounded.

[2]See H. Brezis [1] for a proof of the theorem.

Proof. Set $E = L^p(\Omega)^{N+1}$. E is a Banach space equipped with the norm

$$\|h\| = \sum_{i=1}^{N} \|h_i\|_p, \qquad h = (h_0, h_1, ...h_N)$$

We define the following map:

$$\begin{aligned} T : W_0^{1,p}(\Omega) &\rightarrow E \\ u &\mapsto (u, u_{x_1}, u_{x_2}, ..., u_{x_N}) \end{aligned} \tag{1.15}$$

Taking into account (1.1), T is a isometry. Set $G = T(W_0^{1,p}(\Omega))$,and set $S = T^{-1} : G \rightarrow W^{1,p}(\Omega)$.The map $h \in G \mapsto \langle f, Sh \rangle$ is a bounded linear functional on G. We may use Hahn-Banach theorem now, and extend it to a bounded linear functional ϕ that is defined on all of E, and $\|\Phi\|_{E^\star} = \|F\|$.

Using Riesz representation theorem, there exist functions $f_0, f_1, ..., f_N \in L^{P'}$ such that

$$\langle \Phi, h \rangle = \sum_{i=0}^{N} \int_\Omega f_i h_i \quad \forall h \in E$$

Moreover, $\|\Phi\|_{E^\star} = max_{0 \le i \le N} \|f_i\|_{p'}$.Therefore, if $u \in W_0^{1,p}(\Omega)$,

$$\langle \Phi, Tu \rangle = \langle f, u \rangle = \int_\Omega f_0 u + \sum_{i=0}^{N} \int_\Omega f_i u_{x_i} \tag{1.16}$$

If Ω is bounded we may equip the space $W_0^{1,p}(\Omega)$ with the following norm:

$$\|u\|_{W^{1,p}} = \sum_{i=1}^{N} \|u_{x_i}\|_p$$

and repeating the same argument with $E = L^P(\Omega)^N$,we conclude that we can take f_0 in (1.16) □

CHAPTER 2

SOBOLEV SPACES WITH VARIABLE EXPONENTS

2.1 Basic properties

Definition 2.1.1. [5][6][10] Assume that $u \in L^1_{loc}(\Omega)$.Let $\alpha := (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ be a multi-index. If there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial^{\alpha_1 + ... + \alpha_n} \psi}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n} dx = (-1)^{\alpha_1, ..., \alpha_n} \int_\Omega \psi g dx$$

for all $\psi \in C_0^\infty(\Omega)$, then g is called a weak partial derivative of u with respect to α . The function g is denoted by $\partial_\alpha u$ or by $\frac{\partial^{\alpha_1 + ... + \alpha_n} u}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}$. Moreover, we write ∇u to denote the weak gradient $\left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$ of u and we write short $\partial_j u$ for $\frac{\partial u}{\partial x_j}$ with $j = 1, ..., n$. More generally we write $\nabla^k u$ to denote the tensor with entries $\partial_\alpha u, |\alpha| = k$

If a function u has classical derivatives then they are also weak derivatives of u. Also by definition $\nabla u = 0$ almost everywhere in an open set where $u = 0$.

Definition 2.1.2. [6][10][11] The function $u \in L^{p(\cdot)}(\Omega)$ belongs to the space $W^{k,p(\cdot)}(\Omega)$,where $k \in \mathbb{N}_0$ and $p \in \mathcal{P}(\Omega)$,if its weak partial derivatives $\partial_\alpha u$ with $|\alpha| \leq k$ exist and belong to $L^{p(\cdot)}(\Omega)$ We define a semimodular on $W^{k,p(\cdot)}$ by

$$\varrho_{W^{k,p(\cdot)}(\Omega)}(u) := \sum_{0 \leq |\alpha| \leq K} \varrho_{L^{p(\cdot)}(\Omega)}(\partial_\alpha u)$$

which induces a norm by

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \varrho_{W^{k,p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \leq 1\}.$$

For $k \in \mathbb{N}$ the space $W^{k,p(\cdot)}(\Omega)$ is called Sobolev space and its elements are called Sobolev functions. Clearly $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$

Remark 2.1.3. [11] It is also possible to define the semimodular $\varrho_{W^{k,p(\cdot)}(\Omega)}$ on the larger set $W^{k,1}_{loc}(\Omega)$ or even $L^1_{loc}(\Omega)$. Then $W^{k,p(\cdot)}(\Omega)$ is just the corresponding semimodular space.

We define local Sobolev spaces as usual:

Definition 2.1.4. [6][10][11] A function u belongs to $W^{k,p(\cdot)}_{loc}(\Omega)$ if $u \in W^{k,p(\cdot)}(U)$ for every open $U \subset\subset \Omega$. We equip $W^{k,p(\cdot)}_{loc}(\Omega)$ with the initial topology induced by the embeddings $W^{k,p(\cdot)}_{loc}(\Omega) \hookrightarrow W^{k,p(\cdot)}(\Omega)$ for all open $U \subset\subset \Omega$

Sobolev functions, as Lebesgue functions, are defined only up to measure zero and thus we identify functions that are equal almost everywhere. If the set Ω is clear from the content, we abbreviate $\|u\|_{W^{k,p(\cdot)}(\Omega)}$ to $\|u\|_{k,p(\cdot)}$ and $\varrho_{W^{k,p(\cdot)}(\Omega)}$ to $\varrho_{k,p(\cdot)}$.

Remark 2.1.5. (i) Note that in $W^{k,p(\cdot)}(\Omega)$

$$\sum_{m=0}^{k} \varrho_{L^{p(\cdot)}(\Omega)}(|\nabla^m u|) \qquad and \qquad \sum_{m=0}^{k} \||\nabla^m u|\|_{L^{p(\cdot)}(\Omega)}$$

define a semimodular and a norm equivalent to the Sobolev semimodular and the Sobolev norm, respectively. We abbreviate $\||\nabla^m u|\|_{L^{p(\cdot)}(\Omega)}$ as $\|\nabla^m u\|_{L^{p(\cdot)}(\Omega)}$, $m \in N$

(ii) One easily proves, using Lemma 3.2.4 (see [6]), that for each 1-finite partition $(\Omega_i)_{i\in N}$ of Ω we have

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} \leq \sum_{i=1}^{\infty} \|u\|_{W^{k,p(\cdot)}(\Omega_i)}$$

for all $u \in W^{k,p(\cdot)}(\Omega)$.

Theorem 2.1.6. *[6][10][11] let $p \in \mathcal{P}(\Omega)$.The space $W^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if $1 < p^- \leq p^+ < \infty$.*

Proof. we proof only the case $k = 1$, the proof for the general case is similar. We first show that the Sobolev spaces is a Banach space, for that let (u_i) be a Cauchy sequence in $W^{1,p(\cdot)}(\Omega)$. We have to show that there exists $u \in W^{1,p(\cdot)}(\Omega)$ such that $u_i \to u$ in $W^{1,p(\cdot)}(\Omega)$. as $i \to \infty$. Since the Lebesgue space $L^{p(\cdot)}(\Omega)$ is a Banach space $(Theorem 3.2.7)$ (see [6]), there exist $u, g_1, ..., g_n \in L^{p(\cdot)}(\Omega)$ such that $u_i \to u$ and $\partial_j u_i \to g_j$ in $L^{p(\cdot)}(\Omega)$ for every $j = 1, ..., n$. Let $\psi \in C_0^\infty(\Omega)$. Since u_i is in $W^{1,p(\cdot)}(\Omega)$. we have

$$\int_\Omega u_i \partial_j \psi dx = -\int_\Omega \psi \partial_j u_i dx$$

Strong convergence in $L^{p(\cdot)}(\Omega)$ implies weak convergences and hence we have

$$\int_\Omega u_i \partial_j \psi dx \to \int_\Omega u \partial_j \psi dx \quad and \quad \int_\Omega \psi \partial_j u_i dx \to \int_\Omega \psi g_j dx$$

as $i \to \infty$. Thus the right-hand sides on the previous line yield that $(g_1, ..., g_n)$is the weak gradient of$u$. It follows that $u \in W^{1,p(\cdot)}$ and $u_j \to u$ in $W^{1,p(\cdot)}$.

By Theorem 3.4.4 (see [6]) , $L^{p(\cdot)}(\Omega)$ is separable if $p^+ < \infty$ and by Theorem 3.4.7 (see [6]), $L^{p(\cdot)}(\Omega)$ is reflexive if $1 < p^- \leq p^+ < \infty$. By the mapping $u \mapsto (u, \nabla u)$, the space $W^{1,p(\cdot)}(\Omega)$ is a closed subspace of $L^{p(\cdot)}(\Omega) \times (L^{p(\cdot)}(\Omega))^n$.thus $W^{1,p(\cdot)}(\Omega)$ is separable if $p^+ < \infty$, and reflexive if $1 < p^- \leq p^+ < \infty$ by Proposition 1.4.4.

For the uniform convexity we note that $W^{1,p(\cdot)}(\Omega)$ satisfies the Δ_2- condiction provided that $p^+ < \infty$.The $L^{p(\cdot)}$-modular is uniformly convex for $p^- > 1$ by Theorem 2.4.11 and the proof of Theorem 3.4.9. Thus $\varrho_{W^{k,p(\cdot)}}$ is uniform convex as a sum of uniform convex modulars $(Lemma 2.4.16)$ (see [6]).Thus $W^{1,p(\cdot)}(\Omega)$ is uniform convex with its own norm by Theorem 2.4.14 (see [6]). □

A normed space X has the Banach-Saks property if $\frac{1}{m}\sum_{i=1}^m u_i \to u$ when ever $u_i \rightharpoonup u$.

Corollary 2.1.7. *[6][10][5] Let $p \in \mathcal{P}(\Omega)$ with $1 < p^- \leq p^+ < \infty$.Then the Sobolev space $W^{k,p(\cdot)}(\Omega)$ has the Banach-Saks property.*

Lemma 2.1.8. *[5][6][10] Let $p \in \mathcal{P}(\Omega)$ with $W^{k,p(\cdot)}(\Omega) \hookrightarrow W^{k,p^-}_{loc}(\Omega)$. if $|\Omega| < \infty$,then $W^{k,p(\cdot)}(\Omega) \hookrightarrow W^{k,p^-}(\Omega)$.*

Proof. This follows immediately from the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ see Corollary 3.3.4. □

A (real valued) function space is a lattice if the point-wise minimum and maximum of any two of its elements belong to the space. Next we show that the variable exponent Sobolev space of first order has this property.

Proposition 2.1.9. *[5][6][11] Let* $p \in \mathcal{P}(\Omega)$*. If* $u, v \in W^{1,p(\cdot)}(\Omega)$ *then* $\max\{u, v\}$ *and* $\min\{u, v\}$ *are in* $W^{1,p(\cdot)}(\Omega)$ *with*

$$\nabla \max(u,v)(x) = \begin{cases} \nabla u(x), & \textit{for almost every } x \in \{u \geq v\}\,; \\ \nabla v(x), & \textit{for almost every } x \in \{v \geq u\}\,; \end{cases}$$

$$\nabla \min(u,v)(x) = \begin{cases} \nabla u(x), & \textit{for almost every } x \in \{u \leq v\}\,; \\ \nabla v(x), & \textit{for almost every } x \in \{v \leq u\}\,; \end{cases}$$

In particular,$|u|$ *belongs to* $W^{1,p(\cdot)}(\Omega)$ *and* $|\nabla\, |u|| = |\nabla u|$ *almost every where in* Ω

Proof. It suffices to prove the assertions for $\max\{u, v\}$ since $\min\{u, v\} = -\max\{-u, -v\}$. By Lemma 2.1.8 we know that $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,1}_{loc}(\Omega)$ and so the formulas for $\nabla \max(u, v)$ and $\nabla \min(u, v)$. We next note that $\|\max\{u \leq v\}\|_{p(\cdot)} \leq \|u\|_{p(\cdot)} + \|v\|_{p(\cdot)}$ and $\|\nabla \max\{u \leq v\}\|_{p(\cdot)} \leq \|\nabla u\|_{p(\cdot)} + \|\nabla v\|_{p(\cdot)}$.Thus it follows that $\max\{u, v\} \in W^{1,p(\cdot)}(\Omega)$.Analogously, we get $\min\{u, v\} \in W^{1,p(\cdot)}(\Omega)$.The claims for $|u|$ follow by noting that $|u| = \max\{u, 0\} - \min\{u, 0\}$. ☐

Note that the previous proposition yields that $\nabla u = 0$almost everywhere in a set where u is constant.

We close this section by defining Sobolev spaces with zero boundary values and proving basic properties for them.

Definition 2.1.10. [6][10][11] let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. The Sobolev space $W^{k,p(\cdot)}_0(\Omega)$ with zero boundary values is the closure of the set of $W^{k,p(\cdot)}(\Omega)$-functions with compact support, i.e.

$$\left\{u \in W^{k,p(\cdot)}(\Omega) : u = u\chi_k \text{ for a compact } k \subset \Omega\right\}$$

in $W^{k,p(\cdot)}(\Omega)$.

Remark 2.1.11. The closure of $C_0^\infty(\Omega)$ in the space $W^{k,p(\cdot)}(\Omega)$ is denoted by $H^{k,p(\cdot)}_0(\Omega)$.

Clearly $C_0^\infty(\Omega) \subset W^{k,p(\cdot)}(\Omega)$.Later in Section 11.2 we will study in more detail Sobolev functions with zero boundary values. We will show in Proposition (see [6]) that if p is bounded and smooth functions are dense in the Sobolev space then $W^{k,p(\cdot)}_0(\Omega) = H^{k,p(\cdot)}_0(\Omega)$.In particular we will obtain that if $p \in \mathcal{P}^{log}(\Omega)$ is bounded, then $W^{k,p(\cdot)}_0(\Omega) = H^{k,p(\cdot)}_0(\Omega)$ (Corollary $11.2.4$).

Remark 2.1.12. [11] In contrast to $H_0^{1,p(\cdot)}(\Omega)$, the space $W_0^{1,p(\cdot)}(\Omega)$ has the following fundamental property: if $u \in W^{1,p(\cdot)}(\Omega)$ and v is a Lipschitz continuous function with compact support in Ω , then $uv \in W^{1,p(\cdot)}(\Omega)$.Next we will see that for certain exponents p the product uv need not to be in $H_0^{1,p(\cdot)}(\Omega)$,and thus it may hold that $H_0^{1,p(\cdot)}(\Omega) \subsetneqq W_0^{1,p(\cdot)}(\Omega)$.

Theorem 2.1.13. *[6][10][11]*

Let $p \in \mathcal{P}(\mathbb{R}^n)$. The space $W_0^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if $1 < p^- \leq p^+ < \infty$.

Proof. Since $W_0^{1,p(\cdot)}(\Omega)$is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, the properties, follow by Proposition 1.4.4(see [6]) and Theorem 2.1.6. $\square$

Lemma 2.1.14. *[5][6][10] Let $p \in \mathcal{P}(\mathbb{R}^n)$.and $u \in W_0^{k,p(\cdot)}(\Omega)$. Then u extended by zero to $\mathbb{R}^n \setminus \Omega$ belongs to $W^{k,p(\cdot)}(\mathbb{R}^n)$*

Proof. Let $u \in W^{k,p(\cdot)}(\Omega)$ with compact support, i.e. there exists a compact set $K \subset \Omega$ such that $u = \chi_K u$ almost everywhere. We define εu to be the extension of u (as a measurable function) by zero outside of Ω . that $\varepsilon u \in W^{k,p(\cdot)}(\mathbb{R}^n)$ and $\partial_\alpha \varepsilon(u) = \varepsilon(\partial_\alpha u)$ almost everywhere for $|\alpha| \leq k$. Choose $\eta \in C_0^\infty(\Omega)$ such that $\chi_K \leq \eta \leq \chi_\Omega$ .Then for all $\psi \in C_0^\infty(\mathbb{R}^n)$ and $|\alpha| \leq k$ we have

$$\int_{\mathbb{R}^n} \varepsilon u \partial_\alpha \psi dx = \int_\Omega u \partial_\alpha(\psi\eta) dx = (-1)^{|\alpha|} \int_\Omega (\partial_\alpha u)\psi\eta dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} (\varepsilon(\partial_\alpha u))\psi dx,$$

where we used that $u = 0$ and $\partial_\alpha u = 0$ outside of K and $\eta = 1$ on K.

This proves $\partial_\alpha \varepsilon(u) = \varepsilon(\partial_\alpha u)$ .Since $\varepsilon(\partial_\alpha u) \in L^{p(\cdot)}(\mathbb{R}^n)$, it follows that $\varepsilon u \in W^{k,p(\cdot)}(\mathbb{R}^n)$. Moreover, $\|u\|_{W^{k,p(\cdot)}(\Omega)} = \|\varepsilon u\|_{W^{k,p(\cdot)}(\mathbb{R}^n)}$,so ε is a isometry on the set of compactly supported $W^{k,p(\cdot)}(\mathbb{R}^n)$ functions. Since those functions are by definition dense in $W_0^{k,p(\cdot)}(\Omega)$ to $W^{k,p(\cdot)}(\mathbb{R}^n)$.In particular, $u \in W_0^{k,p(\cdot)}(\Omega)$ implies $\varepsilon u \in W^{k,p(\cdot)}(\mathbb{R}^n)$. $\square$

2.2 Poincaré inequalities

We start this section by showing that for log-Hölder continuous exponents we get the Poincaré inequality with a constant proportional to diam (Ω). After that we give a relatively mild condition on the exponent for the Poincaré inequality to hold. We also show that this condition is, in a certain sense, the best possible.

Concerning the regularity of the domain we consider in particular bounded John domains (cf. Definition 7.4.1) (see[6]) . The constant exponent Poincaré inequality is known to hold for more irregular domains but the inequality is mostly used in John domains.

We recall the following well-known lemma that estimates u in terms of the Riesz potential, due to.For completeness we provide a proof. Recall that I1 denotes the Riesz potential operator (cf. Definition 6.1.1)(see[6]); note also the convention that $I_1 f$ denotes $I_1(\chi_\Omega f)$ if the function f is defined only in Ω.

Lemma 2.2.1. *[5][6][11]*

(a) For every $u \in W_0^{1,1}(\Omega)$, the inequality

$$|u| \leq cI_1 |\nabla u|$$

holds a.e. in Ω with the constant c depending only on the dimension n.

(b) if $\Omega \subset \mathbb{R}^n$ is a bounded α-John domain, then there exists a ball $B \subset \Omega$ and a constant c such that

$$|u(x) - \langle u \rangle_B| \leq cI_1 |\nabla u| (x)$$

holds a.e. in Ω for every $W^{1,1}(\Omega)$. The ball B satisfies $|B| \leq |\Omega| \leq c' |B|$ and the constants c and c' depend only on the dimension n and α.

Proof. We prove only (b). The proof of (a) is similar.

We consider first claim (b) when Ω is a ball. Assume that $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$. We have

$$u(x) - u(y) = -\int_0^{|x-y|} \nabla u \left(x + r\frac{y-x}{|y-x|} \right) . \frac{y-x}{|y-x|} dr$$

Integrating with respect to y over Ω and dividing the result by $|\Omega|$, we obtain

$$u(x) - \langle x \rangle_\Omega = -\frac{1}{\Omega} \int_\Omega \int_0^{|x-y|} \nabla u \left(x + r\frac{y-x}{|y-x|} \right) . \frac{y-x}{|y-x|} drdy$$

Using the notation

$$D(z) = \begin{cases} \nabla u(z), & \text{if } z \in \Omega; \\ 0, & \text{if } z \notin \Omega; \end{cases}$$

we obtain

$$\begin{aligned}|u(x)-\langle u\rangle_\Omega| &\leq \frac{1}{|\Omega|}\int_{\{x-y\leq diam(\Omega)\}}\int_0^\infty \left|D\Big(x+r\frac{y-x}{|y-x|}\Big)\right| drdy\\ &= \frac{1}{|\Omega|}\int_0^\infty\int_{\partial B(0,1)}\int^{diam(\Omega)} |D(x+rw)|\,\varrho^{n-1}d\varrho dwdr\\ &\leq \int_0^\infty\int_{\partial B(0,1)} |D(x+rw)|\,dwdr\\ &= \int_\Omega \frac{|D(y)|}{|x-y|^{n-1}}dy = I_1\,|\nabla u|\,(x)\end{aligned}$$

This concludes the proof in the ball when u is smooth. For $u \in W^{1,1}(\Omega)$, we take smooth approximations ψ_i such that $\psi_i \to u$ in $W^{1,1}(\Omega)$ and almost everywhere. Then $\langle \psi_i\rangle_\Omega \to \langle u\rangle_\Omega$ and $I_1\,|\nabla\psi_i|\,(x) \to I_1\,|\nabla u|\,(x)$,where we also used the continuity of the Riesz potential in $L^1(\Omega)$. This yields the claim for u.

Suppose now that Ω is a bounded α-John domain. Then Ω satisfies the emanating chain condition with constants depending only on α.let Q_0 be the central emanating ball. If $x \in Q_0$, then the claim follows by what was just proved. Otherwise, let $(Q_j)_{j=0}^m$ be the emanating chain connecting x and Q_0. Let B_j be the balls in the intersection $Q_j \cap Q_{j+1}$ as in the Definition 7.4.1 (see[6]) Then

$$\begin{aligned}\left|u(x)-\langle u\rangle_{Q_0}\right| &\leq \left|u(x)-\langle u\rangle_{Q_m}\right| + \sum_{j=0}^{m-1}\left|\langle u\rangle_{Q_{j+1}}-\langle u\rangle_{B_j}\right| + \left|\langle u\rangle_{B_j}-\langle u\rangle_{Q_j}\right|\\ &\leq I\,|\nabla u|\,(x) + 2\sum_{j=0}^{m-1}\left|\langle u\rangle_{B_j}-\langle u\rangle_{Q_j}\right|\end{aligned}$$

Let us estimate the second term:

$$\begin{aligned}\left|\langle u\rangle_{B_j}-\langle u\rangle_{Q_j}\right| &\leq \fint_{B_j}\left|u-\langle u\rangle_{Q_j}\right| dy\\ &\leq \sigma_2^n \fint_{Q_j}\left|u-\langle u\rangle_{Q_j}\right| dy\\ &\leq c\,\sigma_2^n diam(Q_j)\fint_{B_j}|\nabla u|\,dy.\end{aligned}$$

Since$|x-y| \leq \sigma_2\, diam(Q_j)$ and (Q_j) has overlap less than or equal to σ_1 ,we obtain

$$\sum_{j=0}^{m-1} diam(Q_j)\fint_{Q_j}|\nabla u|\,dy \leq c\sigma_2^{n-1}\sum_{j=0}^{m-1}\int_{Q_j}\frac{|\nabla u|}{|x-y|^{n-1}}dy \leq c\sigma_1\sigma_2^{n-1} I\,|\nabla u|\,(x)$$

. The assertion follows when we combine the previous three estimates. □

Remark 2.2.2. [11] One easily checks that the assertions in Lemma 2.2.1 also holds for $u \in L^1_{loc}(\Omega)$ with $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

Lemma 2.2.3. *[5][6][16] Let Ω be a bounded α-John domain and let $p \in \mathcal{P}^{log}(\Omega)$.If $A \subset \Omega$ has positive finite measure, then*

$$c\frac{|A|}{|\Omega|} \left\| u - \langle u \rangle_A \right\|_{L^{p(\cdot)}(\Omega)} \leq \left\| u - \langle u \rangle_\Omega \right\|_{L^{p(\cdot)}(\Omega)} \leq c \left\| u - \langle u \rangle_A \right\|_{L^{p(\cdot)}(\Omega)}$$

For $u \in L^{p(\cdot)}(\Omega)$, where c depends on the dimension n, $c_{loc}(p)$ and α.

Proof. By the triangle inequality, $\left\| u - \langle u \rangle_\Omega \right\|_{L^{p(\cdot)}(\Omega)} \leq \left\| u - \langle u \rangle_A \right\|_{L^{p(\cdot)}(\Omega)} + \left\| \langle u \rangle_A - \langle u \rangle_\Omega \right\|_{L^{p(\cdot)}(\Omega)}$ We estimate the second term by Holder's inequality:

$$\begin{aligned} \left\| \langle u \rangle_A - \langle u \rangle_\Omega \right\|_{L^{p(\cdot)}(\Omega)} &= \left| \langle u \rangle_A - \langle u \rangle_\Omega \right| \left\| 1 \right\|_{L^{p(\cdot)}(\Omega)} \\ &= |\Omega|^{-1} \left\| u - \langle u \rangle_A \right\|_{L^1(\Omega)} \left\| 1 \right\|_{L^{p(\cdot)}(\Omega)} \\ &\leq c\frac{\left\| 1 \right\|_{L^{p'(\cdot)}(\Omega)} \left\| 1 \right\|_{L^{p(\cdot)}(\Omega)}}{|\Omega|} \left\| u - \langle u \rangle_A \right\|_{L^1(\Omega)} . \end{aligned}$$

Since $p \in \mathcal{P}^{log}(\Omega)$,the fraction in the last estimate is bounded by a constant according to Lemma 7.4.5 (see[6]). The lower bound is proved similarly. □

Theorem 2.2.4 (Poincaré inequality)**.** *[6][13][10] let $p \in \mathcal{P}^{log}(\Omega)$,or $p \in \mathcal{A}$*

(a) *for every $u \in W_0^{1,p(\cdot)}(\Omega)$, the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c\, diam(\Omega) \left\| \nabla u \right\|_{L^{p(\cdot)}(\Omega)}$$

holds with a constant c depending only on the dimension n and $c_{log}(p)$.

(b) *If Ω is a bounded α-John domain, then*

$$\left\| u - \langle u \rangle_\Omega \right\|_{L^{p(\cdot)}(\Omega)} \leq c diam(\Omega) \left\| \nabla u \right\|_{L^{p(\cdot)}(\Omega)}$$

for $u \in W^{1,p(\cdot)}(\Omega)$, The constant c depends only on the dimension n, α and $c_{log}(p)$.

Proof. We prove only the latter case. The proof for the first case is similar;

the only difference is to use in Lemma 2.2.1 case (a) instead of case (b). We note that $p \in \mathcal{P}^{log}(\Omega)$ can be extended to $\mathbb{R}^n$ so that $p \in \mathcal{A}$ (Proposition 4.1.7 and Theorem 4.4.8)(see[6]).

By Lemma 2.2.1 (b) and Lemma 6.1.4 we obtain

$$|u(x) - \langle u \rangle_B| \leq I_1 |\nabla u|(x) \leq c\, diam(\Omega) \sum_{k=1}^{\infty} 2^{-k} T_{k+k_0} |\nabla u|(x)$$

for every $u \in W^{1,p(\cdot)}(\Omega)$ and almost every $x \in \Omega$ where $k_0 \in \mathbb{Z}$ is chosen such that $2^{-k_0-1} \leq diam(\Omega) \leq 2^{-k_0}$. Since $p \in \mathcal{A}$, the averaging operator T_{k+k_0} is bounded on $L^{p(\cdot)}(\Omega)$ (cf. Remark 6.1.3)(see[6]). Using also the triangle inequality, we obtain

$$\begin{aligned} \|u - \langle u \rangle_B\|_{L^{p(\cdot)}(\Omega)} &\leq c\, diam(\Omega) \sum_{k=1}^{\infty} 2^{-k} \|T_{k+k_0} |\nabla u|\|_{L^{p(\cdot)}(\Omega)} \\ &\leq c\, diam(\Omega) \|\nabla u\|_{L^{p(\cdot)}(\Omega)}. \end{aligned}$$

The estimate for $\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)}$follows from this and Lemma 2.2.3. □

In the case that $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $p^+ < \infty$ we give an alternative proof for Theorem 2.2.4 (b) based on the decomposition Theorem 7.4.9. The proof is not self-contained since it uses the variable exponent Poincaré inequality in cubes.

Proof. of Theorem 8.2.4(b) for $p \in \mathcal{P}^{log}(\Omega)$ with $p^+ < \infty$ let $p \in \mathcal{P}^{log}(\Omega)$ with $p^+ < \infty$.Let $\mathcal{Q}$ be the chain covering of Ω. Let $u \in W^{1,p(\cdot)}(\Omega)$ with $\int_\Omega u dx = 0$. Then Proposition 7.4.11(see[6]), the Poincaré inequality in cubes, $diam(Q) \leq diam(\Omega)$ for $Q \in \mathcal{Q}$,and Theorem 7.3.22(see[6]) yield

$$\begin{aligned} \|u\|_{L_0^{p(\cdot)}(\Omega)} &\leq c \left\| \sum_{Q \in \mathcal{Q}} \chi_Q \frac{\left\|u - \langle u \rangle_Q\right\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_Q\|_{p(\cdot)}} \right\|_{L^{p(\cdot)}(\Omega)} \\ &\leq c \left\| \sum_{Q \in \mathcal{Q}} \chi_Q \frac{diam(Q) \|\nabla u\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_Q\|_{p(\cdot)}} \right\|_{L^{p(\cdot)}(\Omega)} \\ &\leq c\, diam(\Omega) \left\| \sum_{Q \in \mathcal{Q}} \chi_Q \frac{\|\nabla u\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_Q\|_{p(\cdot)}} \right\|_{L^{p(\cdot)}(\Omega)} \\ &\leq c\, diam(\Omega) \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \end{aligned}$$

□

Theorem 2.2.4(a) immediately yields

Corollary 2.2.5. *[5][6][11] Let Ω be bounded and $p \in \mathcal{P}^{log}(\Omega)$ or $p \in \mathcal{A}$.For every $u \in W_0^{1,p(\cdot)}(\Omega)$ the inequality*

$$\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \leq \|u\|_{W^{1,p(\cdot)}(\Omega)} \leq (1 + c\, diam(\Omega))\, \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds with constant c depending only on the dimension n, and $c_{log}(p)$.

Corollary 2.2.6. *[5][6][11] Let be a bounded α-John domain and let $p \in \mathcal{P}^{log}(\Omega)$ or $p \in \mathcal{A}$. Furthermore, let $A \subset \Omega$ be such that $|A| \approx |\Omega|$. Then*

$$\|u - \langle u\rangle_A\|_{L^{p(\cdot)}} \leq c\, diam(\Omega)\, \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for $u \in L^1_{loc}(\Omega)$ with $|\nabla u| \in L^{p(\cdot)}(\Omega)$. The constant c depends only on the dimension n, α and $c_{log}(p)$.

Let us next consider modular versions of the Poincaré inequality. In the constant exponent case there is an obvious connection between modular and norm versions of the inequality, which does not hold in the variable expo- nent context. Indeed, the following one-dimensional example shows that the Poincaré inequality can not, in general, hold in a modular form.

Example 2.2.7. Let $p : (-2, 2) \to [2, 3]$ be a Lipschitz continuous exponent that equals 3 in $(-2,-1) \cup (1, 2), 2$ in $(\frac{-1}{2}, \frac{1}{2})$ and is linear elsewhere. Let u_λ be a Lipschitz function such that $u_\lambda(\pm 2) = 0, u_\lambda = \lambda$ in $(-1, 1)$ and $|u'_\lambda| = \lambda$ in $(-2,-1) \cup (1, 2)$. Then

$$\frac{\overline{\varrho}_{p(\cdot)}(u_\lambda)}{\overline{\varrho}_{p(\cdot)}(u'_\lambda)} = \frac{\int_{-2}^{2} |u_\lambda|^{p(x)}\, dx}{\int_{-2}^{2} |u'_\lambda|^{p(x)}\, dx} \leq \frac{\int_{-1/2}^{1/2} \lambda^2 dx}{\int_{-2}^{-1} \lambda^3 dx} = \frac{1}{2\lambda} \to \infty$$

as $\lambda \to 0^+$.

Proposition 2.2.8. *[6][13][16] Let $p \in \mathcal{P}^{log}(\Omega)$ be a bounded exponent.*

(a) *Let be bounded. For $m > 0$ there exist a constant c depending on the dimension n, $c_{log}(p)$, m, and p^+ such that*

$$\int_\Omega \Big(\frac{|u|}{diam(\Omega)}\Big)^{p(x)} dx \leq c \int_\Omega |\nabla u|^{p(x)}\, dx + c \int_{B(z,diam(\Omega))} (e + |x|)^{-m} dx$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \leq 1$ and all $z \in \Omega$

(b) *Let be a bounded α-John domain. For $m > 0$ there exist a constant c depending on the dimension n, $c_{log}(p), m, p^+, \alpha, |\Omega|$ and $diam(\Omega)$ such that*

$$\int_\Omega \Big(\frac{|v - \langle v\rangle_B|}{diam(\Omega)}\Big)^{p(x)} dx \leq c \int_\Omega |\nabla u|^{p(x)}\, dx + c \int_{B(z,diam(\Omega))} (e + |x|)^{-m} dx$$

for all $u \in W^{1,p(\cdot)}(\Omega)$ with $\|\nabla v\|_{L^{p(\cdot)}(\Omega)} \leq 1$ and all $z \in \Omega$ The ball B is from Lemma 2.2.1.

Proof. Assume that $u \in W_0^{1,p(\cdot)}(\Omega)$ is extended by zero outside (Lemma 8.1.14)(see[6]). By Lemmas 2.2.1 (a) and 6.1.4 (see[6]) we obtain

$$|u(x)| \leq c \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \leq c\ diam(\Omega)\ \sum_{k=0}^{\infty} 2^{-k} T_{k+k_0} |\nabla u|(x),$$

where $k_0 \in \mathbb{Z}$ is chosen such that$2^{-k_0-1} \leq diam(\Omega) \leq 2^{-k_0}$. Exactly the same estimate holds with u replaced by $v - \langle v\rangle_B$ using Lemma 2.2.1 (b), so it suffices to derive the estimate of the first claim involving u.

We divide by $diam(\Omega)$ and raise both sides of this inequality to the power $p(x)$, integrate over, and use $p^+ < \infty$ to obtain

$$\begin{aligned}\int_\Omega \Big(\frac{|u|}{diam(\Omega)}\Big)^{p(x)} dx &\leq c \int_\Omega \Big(\sum_{k=0}^{\infty} 2^{-k} T_{k+k_0} |\nabla u|\Big)^{p(x)} dx \\ &\leq c \sum_{k=0}^{\infty} 2^{-k} \int_\Omega \Big(T_{k+k_0} |\nabla u|\Big)^{p(x)} dx,\end{aligned}$$

where we used convexity in the second step. Since $\varrho_{L^{p(\cdot)}(\Omega)}(\nabla u) \leq 1$ by the unit ball property, we may use Corollary (4.2.5) (see[6]) for $|\nabla u|_{\chi\Omega}$ on the ball $B(z, diam(\Omega))$ and get

$$\int_\Omega \Big(T_{k+k_0} |\nabla u|\Big)^{p(x)} dx \leq \int_\Omega \Big(|\nabla u|\Big)^{p(x)} dx + c \leq \int_{B(z,diam(\Omega))} (e + |x|)^{-m} dx$$

, where we also used that the dyadic cubes $2Q$ are locally $N-$finite. Combining the last two inequality proves the claim. □

Remark 2.2.9. If $p \in \mathcal{P}^{log}(\Omega)$ with no restriction on p^+, then the first estimate in Proposition 2.2.8 reads

$$\int_\Omega \varphi_{p(x)}\Big(\beta \frac{|u|}{diam(\Omega)}\Big) dx \leq \int_\Omega \varphi_{p(x)}(|\nabla u|) dx + \int_{B(z,diam(\Omega))} (e + |x|)^{-m} dx$$

for some $\beta \in (0,1)$ and all $u \in W^{1,p(\cdot)}(\Omega)$ with $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \leq 1$. The con- stant β depends only on $c_{\log}(p)$, m and n. The second estimate in Proposi- tion 2.2.8 has to be changed accordingly. The proof is the same.

Remark 2.2.10. It is possible to replace $\langle u \rangle_B$ in Lemma 2.2.1 (b) and Proposition 2.2.8 by $\langle u \rangle_\Omega$ or even $\langle u \rangle_A$, where $A \subset \Omega$ is a set with $|A| \approx |\Omega|$. Indeed, it follows from Jensen's inequality and Corollary 2.2.6 for $p = 1$ that

$$|\langle u \rangle_A - \langle u \rangle_B| \leq \fint_A |u - \langle u \rangle_B| \, dy \leq c \fint_A |u - \langle u \rangle_B| \, dy \leq c \;\; diam(\Omega) \fint_\Omega |\nabla u| \, dy$$

for all $u \in W^{1,1}(\Omega)$. In particular, this implies

$$|\langle u \rangle_A - \langle u \rangle_B| \leq c \int_\Omega \frac{|\nabla u(y)|}{(diam(\Omega))^{n-1}} dy \leq c I_1(\nabla u)(x).$$

for any $x \in \Omega$. This proves the modified version of Lemma 2.2.1 (b). With this new estimate we get the modified version of Proposition 2.2.8 with no change in the proof.

The following improvement of Proposition 2.2.8 is useful in the study of $p(\cdot)$-minimizers and is the starting point for reverse Hölder estimates. The result is from we will study next.

Proposition 2.2.11. *[6][11] Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$ and let $s \leq p^-$ satisfy $s \in [1, \frac{n}{n-1})$. Then for every $m > 0$ there exist a constant c depending on n, $c_{log}(p)$, m, and s such that*

$$\begin{aligned} \fint_{B_R} \Big(\frac{|u|}{R}\Big)^{p(x)} dx &\leq c\Big(\fint_{B_R} |\nabla u|^{\frac{p(\cdot)}{s}} \, dx\Big)^s + c \fint_{B_R} (e + |x|)^{-ms} dx \\ \fint_{B_R} \Big(\frac{|v - \langle v \rangle_{B_R}|}{R}\Big)^{p(x)} dx &\leq c\Big(\fint_{B_R} |\nabla v|^{\frac{p(\cdot)}{s}} \, dx\Big)^s + c \fint_{B_R} (e + |x|)^{-ms} dx \end{aligned}$$

for every ball B_R with radius R, and every $u \in W_0^{1,\frac{p(\cdot)}{s}}(B_R), v \in W^{1,\frac{p(\cdot)}{s}}(B_R)$ with $\|\nabla u\|_{L^{p(\cdot)}/s + L^\infty}, \|\nabla v\|_{L^{q(\cdot)}/s + L^\infty} \leq 1$

Proof. By Jensen's inequality the case $s > 1$ implies the case $s = 1$ and thus we may assume that $s > 1$. By Lemma 2.2.1 we have, for $x \in B_R$,

$$|u(x)| \leq c \int_{B_R} \frac{|\nabla u(y)|}{|x - y|^{n-1}} dy, \quad |v(x) - \langle v \rangle_{B_R}| \leq c \int_{B_R} \frac{|\nabla v(y)|}{|x - y|^{n-1}} dy$$

Starting from here the proofs for u and v are the same, so we just present the estimate for u. With the previous estimate, the help of Lemma 6.1.12 applied to $p(\cdot)/s$ and $p^+ < \infty$ we get

$$\begin{aligned}(I) &:= \fint_{B_R} \Big(\frac{|u(x)|}{R}\Big)^{p(x)} dx \le c(p^+) \fint_{B_R} \Big(\int_{B_R} \frac{|\nabla u(y)|}{R|x-y|^{n-1}} dy\Big)^{p(x)} dx \\ &\le c \fint_{B_R} \Big(\int_{B_R} \frac{|\nabla u(y)|^{\frac{p(y)}{s}}}{R\,|x-y|^{n-1}} dy\Big)^{s} dx + c \fint_{B_R} (M((e+|\cdot|^{-m}))(x))^s dx \\ &=: (II) + (III).\end{aligned}$$

In order to estimate (II) we set $J := \int_{B_R} |\nabla u|^{p(\cdot)/s}\, dx$. We can assume $J > 0$ in the following, since otherwise $\nabla u = 0$ and there is nothing to estimate. We apply Jensen's inequality for the probability measure $\mu := |\nabla u|^{p(\cdot)/s}/J$ and the convex function $t \mapsto t^s$, then we use Fubini?s theorem to change the integration order and $\int_{B_R} \frac{dx}{|x-y|^{s(n-1)}} \approx R^{-s(n-1)+n}$ for a.e. $y \in B_R$ using $s < \frac{n}{n-1}$:

$$\begin{aligned}(II) &\le c \fint_{B_R} J^{s-1} \int_{B_R} \frac{|\nabla u(y)|^{\frac{p(y)}{s}}}{R^s\,|x-y|^{s(n-1)}} dy dx \\ &\le c J^{s-1} R^{n-sn} \fint_{B_R} |\nabla u(y)|^{\frac{p(y)}{s}} dy \le c\Big(\fint_{B_R} |\nabla u(y)|^{\frac{p(y)}{s}} dy\Big)^{s}.\end{aligned}$$

Since $s > 1$, we can use the boundedness of M on $L^s(B_R)$ to conclude

$$(III) \le c \fint_{B_R} (e+|x|)^{-ms} dx$$

Combining the estimates for (I)?(III) we obtain the result. □

Lemma 2.2.12. *[6][11] Let $\Omega \subset \mathbb{R}^n$ be a bounded α-John domain. If $1 \le r < n$ and $r \le q \le r^*$ or if $r \ge n$ and $q < \infty$, then*

$$\|u - \langle u\rangle_\Omega\|_{L^q(\Omega)} \le c\,|\Omega|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{r}}\, \|\nabla u\|_{L^r(\Omega)}.$$

for all $u \in W^{1,r}(\Omega)$. In the first case the constant c depends only on n, r and α, while in the second case it depends also on q.

Using the previous constant exponent Sobolev-Poincaré inequality we are able to prove the Poincaré inequality in bounded John domains for variable exponents.

Lemma 2.2.13. *[6][11] Let $\Omega \subset \mathbb{R}^n$ be a bounded α-John domain. If $p \in \mathcal{P}(\Omega)$ is bounded with $p^+ \leq (p^-)^*$ or $p^- \geq n$, then there exists a constant c depending on the dimension n, p^-, p^+, and α such that for every $u \in W^{1,p(\cdot)}(\Omega)$ we have*

$$\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq c(1 + |\Omega|)^2 \, |\Omega|^{\frac{1}{n} + \frac{1}{p^+} - \frac{1}{p^-}} \, \|\nabla u\|_{L^{p(\cdot)}(\Omega)} .$$

Proof. Assume first that $p^+ \leq (p^-)^*$. Since $p(x) \leq p^+ \leq (p^-)^*$ we deduce from Corollary 3.3.4 (see[6]) and Lemma 2.2.12 that

$$\begin{aligned}
\|u - \langle u \rangle_\Omega\|_{p(\cdot)} &\leq 2(1 + |\Omega|) \, \|u - \langle u \rangle_\Omega\|_{p^+} \\
&\leq c(n, p^-, \alpha)(1 + |\Omega|) \, |\Omega|^{\frac{1}{n} + \frac{1}{p^+} - \frac{1}{p^-}} \, \|\nabla u\|_{p^-} \\
&\leq c(n, p^-, \alpha)(1 + |\Omega|)^2 \, |\Omega|^{\frac{1}{n} + \frac{1}{p^+} - \frac{1}{p^-}} \, \|\nabla u\|_{p(\cdot)}
\end{aligned}$$

Let $p^- \geq n$. We choose a constant $q \in [1, n)$ such that $p^+ = q^*$. We obtain the result by using similar arguments than in the previous case. The only difference is that the constant in the second inequality in the above chain of inequalities is $c(n, p^+, \alpha)$. □

Lemma 2.2.14. *[6][11] For a constant $q \in (1, n)$, arbitrary $x \in \mathbb{R}^n$ and $R > r > 0$ we have*

$$\inf \int_{B(x,R)} |\nabla u|^q \, dx = c \left| \frac{q - n}{q - 1} \right|^{q-1} \left| R^{\frac{q-n}{q-1}} - r^{\frac{q-n}{q-1}} \right|^{q-1},$$

where the infimum is taken over all $u \in C_0^\infty(B(x, R))$ with $u\,|_{B(x,r)} = 1$. Here the constant c depends only on the dimension n.

The following proposition shows that for general non-constant exponents the Poincaré inequality does not hold.

Proposition 2.2.15. *[6] [11] Let B be a unit ball in the plane. For every $q_1 \in [1, 2)$ and $q_2 \in (2, \infty)$ there exists $p \in \mathcal{P}(B)$ with $p^- = q_1$ and $p^+ = q_2$ such that the norm-version of the PoincarÃfÂ© inequality,*

$$\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(B)} \leq c \, \|\nabla u\|_{L^{p(\cdot)}(B)} ,$$

does not hold.

Proof. Our aim is to construct a sequence of functions in $B \subset \mathbb{R}^2$ for which the constant in the Poincaré inequality goes to infinity.Let $e_1 = (1,0), B_i := B(2^{-i}e_1, \frac{1}{4}2^{-i}) \subset B$ and $B_i' := B(2^{-i}e_1, \frac{1}{8}2^{-i^2}) \subset B$ for every $i \in \mathbb{N}$.Let $u_i \in C_0^\infty(Bi)$ with $u_i\,|_{\overline{B'}} = 1$. Define $p := q_2$ in ever B_i' and $p := q_1$ otherwise in B with positive first coordinate. Since $\nabla u_i = 0$ in B_i' we obtain

$$\|\nabla u_i\|_{L^{p(\cdot)}(B_i)} = \|\nabla u_i\|_{L^{q_1}(B_i)}$$

Let $\tilde{B}_i := B(-2^{-i}e_1, \frac{1}{4}2^{-i})$. We extend u_i to B as an odd function of the first coordinate in $\tilde{B}_i$ and by zero elsewhere. We extend p to B as an even function of the first coordinate. We denote these extensions by $\tilde{u}_i$ and $\frac{p^*}{n'}$.

By Lemma 2.2.14 we may choose the functions ui such that

$$\|\nabla \tilde{u}_i\|_{L^{\frac{p^*(\cdot)}{n'}}(B)} \leq c(q_1) \left| \left(\frac{1}{4}2^{-i}\right)^{\frac{q_1-2}{q_1-1}} - \left(\frac{1}{8}2^{-i^2}\right)^{\frac{q_1-2}{q_1-1}} \right|^{\frac{1-q_1}{q_1}}.$$

For large i, the right-hand side is approximately equal to $c(q_1)2^{-i^2\frac{2-q_1}{q_1}}$. Since $\langle \tilde{u}_i \rangle_B = 0$, we obtain

$$\|\tilde{u}_i - \langle \tilde{u}_i \rangle_B\|_{L^{\frac{p^*(\cdot)}{n'}}(B)} = \|\tilde{u}_i\|_{L^{\frac{p^*(\cdot)}{n'}}(B)} \geq |B_i'|^{\frac{1}{q_2}} \geq c2^{-i^2\frac{2}{q_2}}$$

Combining the previous two inequalities, we find that

$$\frac{\|\tilde{u}_i - \langle \tilde{u}_i \rangle_B\|_{L^{\frac{p^*(\cdot)}{n'}}(B)}}{\|\nabla \tilde{u}_i\|_{L^{\frac{p^*(\cdot)}{n'}}(B)}} \geq c(q_1)2^{i^2(\frac{2}{q_1}-1-\frac{2}{q_2})} \to \infty$$

as $i \to \infty$ if $\frac{2}{q_1} - 1 - \frac{2}{q_2} > 0$, that is, if $q_2 > \frac{2q_1}{2-q_1} > 2$.

□

The following theorem shows that the condition $p^+ \leq (p^-)^*$ in Lemma 2.2.13 can be replaced by a set of local conditions.

Theorem 2.2.16. *[6][10] Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain and $p \in \mathcal{P}(\Omega)$ be bounded. Assume that there exist John domains $D_i, i = 1, ..., j$, such that $\Omega = \cup_{i=1}^{j} D_i$ and either $p_{D_i}^+ \leq (p_{D_i}^-)^*$ or $p_{D_i}^- \geq n$ for every i. Then there exists a constant c such that*

$$\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq \|\nabla u \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)}$$

for every $u \in W^{1,p(\cdot)}(\Omega)$.The constant c depends on n, $diam(\Omega), |D_i|, p$ and the John constants of Ω and $D_i, i = 1, ..., j$.

Proof. Using the triangle inequality we obtain

$$\begin{aligned} \|u - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} &\leq \sum_{i=1}^{j} \|u - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(D_i)} \\ &\leq \sum_{i=1}^{j} \|u - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)} + \sum_{i=1}^{j} \|\langle u\rangle_\Omega - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)}. \end{aligned}$$

We estimate the first part of the sum using Lemma 2.2.13. This yields for every $i = 1, ..., j$

$$\|u - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)} \leq c \|\nabla u\|_{L^{p(\cdot)}(D_i)} \leq c \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

with constants depending on $n, p^+_{D_i}, p^-_{D_i}, |D_i|, \alpha_i$, where α_i is the John constant of D_i. Therefore it remains only to estimate the sum of the terms $\|\langle u\rangle_\Omega - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)}$. We use the constant exponent Poincaré inequality (in the third inequality):

$$\begin{aligned} \|\langle u\rangle_\Omega - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)} &\leq \|1\|_{L^{p(\cdot)}(D_i)} \fint_{D_i} |u(x) - \langle u\rangle_\Omega| \, dx \\ &\leq \|1\|_{L^{p(\cdot)}(D_i)} |D_i|^{-1} \int_{D_i} |u(x) - \langle u\rangle_\Omega| \, dx \\ &\leq c(n, diam(\Omega), \alpha) |D_i|^{-1} \|1\|_{L^{p(\cdot)}(D_i)} \|\nabla u\|_{L^1(\Omega)} \\ &\leq c(n, diam(\Omega), \alpha) |D_i|^{-1} \|1\|_{L^{p(\cdot)}(D_i)} \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \end{aligned}$$

for every $i = 1, ..., j$. Here α is the John constant of Ω.By Corollary 3.3.4 (see[6]) $\|1\|_{L^{p(\cdot)}(D_i)}$ depends only on p and $|D_i|$, which completes the proof. □

Next we prove the Poincaré inequality for Sobolev functions with zero boundary values using Lemma 2.2.13.

Theorem 2.2.17. *[6][10][11] Let Ω be bounded. Assume that $p \in \mathcal{P}(\Omega)$ and there exists $\delta > 0$ such that for every $x \in \Omega$ either*

$$p^+_{B(x,\delta)\cap\Omega} \leq \frac{np^-_{B(x,\delta)\cap\Omega}}{n - p^-_{B(x,\delta)\cap\Omega}} \ \textit{or} \ p^-_{B(x,\delta)\cap\Omega} \geq n \tag{2.1}$$

Alternatively, assume that p is uniformly continuous in Ω. Then the inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq \|\nabla u\|_{L^{p(\cdot)}(\Omega)} .$$

holds for every $u \in W_0^{1,p(\cdot)}(\Omega)$. Here the constant c depends on p, Ω, δ and the dimension n. .

Proof. Note that if p is continuous in $\overline{\Omega}$ or uniformly continuous in Ω, then p satisfies the first set of conditions of the theorem for some $\delta > 0$. By the assumptions there exist $x_1, ..., x_j$ and $\delta > 0$ such that

$$\Omega \subset \bigcup_{i=1}^{j} B(x_i, \delta)$$

and each ball $B(x_i, \delta)$ satisfies either of the two inequalities in (2.1). We write $B_i := B(x_i, \delta)$ and denote by χ_i the characteristic function of $B_i \cap \Omega$.In each B_i we define $p_i(x) := p(x)\chi_i + p^-_{B_i\cap\Omega}(1-\chi_i)$. Then in each B_i either $p_i^+ \leq (p_i^-)^*$ or $p_i^- > n$. Let $\tilde{u}$ be the zero extension of $u \in W_0^{1,p(\cdot)}(\Omega)$ to $\mathbb{R}^n \setminus \Omega$ (Lemma 2.2.13). By the triangle inequality we obtain

$$\begin{aligned} \|u\|_{L^{p(\cdot)}(\Omega)} &\leq \left\| \tilde{u} \sum_{i=1}^{j} \chi_i \right\|_{L^{p(\cdot)}(\Omega)} \leq \sum_{i=1}^{j} \|\tilde{u}\|_{L^{p_i(\cdot)}(B_i)} \\ &\leq \sum_{i=1}^{j} \|\tilde{u} - \langle \tilde{u} \rangle_{B_i}\|_{L^{p_i(\cdot)}(B_i)} + \sum_{i=1}^{j} |\langle \tilde{u} \rangle_{B_i}| \, \|1\|_{L^{p_i(\cdot)}(B_i)} . \end{aligned}$$

We estimate the first sum on the right-hand side of the previous inequality. By Lemma 2.2.13 we obtain

$$\begin{aligned} \|\tilde{u} - \langle \tilde{u} \rangle_{B_i}\|_{L^{p_i(\cdot)}(B_i)} &\leq c(1 + |B_i|)^2 \, |B_i|^{\frac{1}{n} + \frac{1}{p_i^+} - \frac{1}{p_i^-}} \|\nabla \tilde{u} -\|_{L^{p_i(\cdot)}(B_i)} \\ &\leq c(1 + |B_i|)^2 \, |B_i|^{\frac{1}{n} + \frac{1}{p_i^+} - \frac{1}{p_i^-}} \|\nabla u\|_{L^{p(\cdot)}(\Omega)} . \end{aligned}$$

for every $i = 1, ...j$. To estimate the second sum, $\sum_{i=1}^{j} |\langle \tilde{u} \rangle_{B_i}| \, \|1\|_{L^{p_i(\cdot)}(B_i)}$, in the above inequality we use Lemma 3.2.12 (see[6]) to estimate $\|1\|_{L^{p_i}(B_i)}$ by a constant depending only on p and δ. Further, the constant exponent Poincaré inequality implies that

$$\begin{aligned} |\langle \tilde{u} \rangle_{B_i}| &\leq \frac{c(n)}{\delta^n} \int_\Omega |u| \, dx \leq \frac{c}{\delta^n} diam(\Omega) \int_\Omega |\nabla u| \, dx \\ &\leq \frac{c(n)}{\delta^n} diam(\Omega)(1 + |\Omega|) \, \|\nabla u\|_{L^{p(\cdot)}(\Omega)} , \end{aligned}$$

again for every $i = 1, ...j$. Combining the last three estimates yields the assertion.

Remark 2.2.18. [6][11] Assume that Ω is convex and $p \in \mathcal{P}(\Omega)$ is uniform continuous (or $p \in C(\overline{\Omega})$). As in the proof of Theorem 2.1.8 we may cover by Ω finitely many balls $B(x_i, \delta)$ so that (2.1) holds. Since is convex so is $B(x_i, \delta) \cap \Omega$ and thus it is a John domain. Hence Theorem 2.2.16 yields that the Poincaré inequality

$$\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq c \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds for every $u \in W^{1,p(\cdot)}(\Omega)$.

□

2.3 Poincaré inequalities and embeddings

In this section we assume that the exponent p is log-Hölder continuous with $1 \leq p^- \leq p^+ < n$. We prove that the Sobolev-Poincaré inequality holds for general Sobolev functions in bounded John domains and for Sobolev functions with zero boundary values in open sets. Bounded John domains are almost the right class of irregular domains for the constant exponent Sobolev-Poincaré inequality. We give an example which shows that Sobolev embed- dings do not hold for every continuous p. We close this section by studying the Sobolev embedding in the case $p^- > n$. Sobolev embeddings in the case $p^+ = n$ need a target space that is not a variable exponent Lebesgue space, and are studied in Section 8.6 together with the other limit case $p^- = n$. We define the Sobolev conjugate exponent point-wise, i.e.

$$p^* := \frac{np(x)}{n - p(x)}$$

when $p(x) < n$ and $p^*(x) = \infty$ otherwise.

Theorem 2.3.1. *[6][10][11] Let $p \in \mathcal{P}^{log}(\Omega)$ satisfy $1 \leq p^- \leq p^+ < n$.*

(a) *For every $u \in W_0^{1,p(\cdot)}(\Omega)$, the inequality*

$$\|u\|_{L^{p^*(\cdot)}(\Omega)} \leq c \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds with a constant c depending only on the dimension n, $c_{log}(p)$, and p^+.

(b) *If is a bounded $\alpha-$John domain, then*

$$\|u - \langle u\rangle_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq c\,\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for $u \in W^{1,p(\cdot)}(\Omega)$. The constant c depends only on the dimension $n, \alpha, c_{log}(p) and p^+$.

If we add the extraneous assumption $p^- > 1$, then we immediately obtain a proof using results on operators that we proved earlier: the inequalities follow from Lemma 2.2.1, Lemma 2.2.3 and Theorem 6.1.9 (see[6]); the constant in this case also depends on p^-.

We obtain the Sobolev embedding as a corollary.

Corollary 2.3.2. *[6] [11] Let Ω be a bounded $\alpha-$John domain and let $p \in \mathcal{P}^{log}(\Omega)$. Let $q \in \mathcal{P}(\Omega)$ be bounded and assume that $q \leq p^*$. Then*

$$W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega),$$

where the embedding constant depends only on $\alpha, |\Omega|, n, c_{log}(p) and q^+$.

Proof. Let $r \in (1, n)$ be such that $r^* \geq q^+$. Corollary 3.3.4 and Lemma 3.2.12 (see[6]) yield

$$\begin{aligned}\|u\|_{q(\cdot)} &\leq \|u - \langle u\rangle_\Omega\|_{q(\cdot)} + \|\langle u\rangle_\Omega\|_{q(\cdot)}\\ &\leq 2(1+|\Omega|)\,\|u - \langle u\rangle_\Omega\|_{\min\{p^*(\cdot),r^*\}} + \max\{|\Omega|^{\frac{1}{q+}-1}, 1\}\,\|u\|_1\\ &\leq 2(1+|\Omega|)\max\{|\Omega|^{\frac{1}{q+}-1}, 1\}(\|u - \langle u\rangle_\Omega\|_{\min\{p^*(\cdot),r^*\}} + \|u\|_{p(\cdot)}).\end{aligned}$$

Since $\min\{p^*(\cdot), r^*\} \in \mathcal{P}^{log}(\Omega)$, Theorem 2.3.1 (b) and Corollary 3.3.4 (see[6]) yield

$$\|u - \langle u\rangle_\Omega\|_{\min\{p^*(\cdot),r^*\}} \leq c\,\|\nabla u\|_{\min\{p(\cdot),r\}} \leq c(1+|\Omega|)\,\|\nabla u\|_{p(\cdot)}$$

The claim follows by combining these two inequalities. □

Now we move on to the complete proof, covering also the case $p^- = 1$. In this case the Riesz potential is not strong type $p(\cdot)$, i.e. Theorem 6.1.9 (see[6]) is not available. Our proof is based on the weak type estimate for the Riesz potential. We first give a proof of Theorem 2.3.1 (b) in which the constant additionally depends on $diam(\Omega)$.

Lemma 2.3.3. *[6][11] Let Ω be a bounded Ω $\alpha-$John domain and let $p \in \mathcal{P}^{log}(\Omega)$ satisfy $1 \leq p^- \leq p^+ < n$. Then*

$$\|u - \langle u\rangle_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq c\,\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for every $u \in W^{1,p(\cdot)}(\Omega)$.The constant c depends only on the dimension $n, p^+, c_{log}(p), \alpha and diam(\Omega)$.

Proof. By a scaling argument we may assume without loss of generality that $(1+|\Omega|)\,\|\nabla u\|_{p(\cdot)} \leq 1$.We need to show that $\varrho_{L^{p^*(\cdot)}(\Omega)}(|u - \langle u\rangle_\Omega|)$ is uniformly bounded. For every $j \in \mathbb{Z}$ we set $\Omega_j := \{x \in \Omega : 2^j < |u(x) - \langle u\rangle_\Omega| \leq 2^{j+1}\}$ and $v_j := \max\{0, \min\{|u - \langle u\rangle_\Omega| - 2^j, 2^j\}\}$. From Proposition 2.1.9 follows $v_j \in W^{1,p(\cdot)}(\Omega)$. By Lemma 2.2.1 (b) we have

$$|v_j(x) - \langle v_j\rangle_B| \leq cI_1\,|\nabla v_j|\,(x)$$

for almost every $x \in \Omega$.Here the radius of $B \subset \Omega$ depends on α. We obtain by the pointwise inequality $v_j \leq |u - \langle u\rangle_\Omega|$ and by the constant exponent Poincaré inequality Lemma 2.2.12 that

$$\begin{aligned} v_j &\leq |v_j(x) - \langle v_j\rangle_B| + \langle v_j\rangle_B \leq cI_1\,|\nabla v_j|\,(x) + \fint_B |u - \langle u\rangle_\Omega|\,dx \\ &\leq cI_1\,|\nabla v_j|\,(x) + c\int_\Omega |u - \langle u\rangle_\Omega|\,dx \leq cI_1\,|\nabla v_j|\,(x) + c\int_\Omega |\nabla u|\,dx \\ &\leq cI_1\,|\nabla v_j|\,(x) + c(1+|\Omega|)\,\|\nabla u\|_{p(\cdot)} \leq c_1(I_1\,|\nabla v_j|\,(x) + 1). \end{aligned} \tag{2.2}$$

For the rest of this proof we fix the constant c_1 to denote the constant on the last line.It depends only on n, α and $diam(\Omega)$.

Using the definition of Ω_j we get

$$\begin{aligned} \int_\Omega |u(x) - \langle u\rangle_\Omega|^{p^*(x)}\,dx &= \sum_{j=-\infty}^{\infty} \int_{\Omega_j} |u(x) - \langle u\rangle_\Omega|^{p^*(x)}\,dx \\ &\leq \sum_{j=-\infty}^{\infty} \int_{\Omega_j} 2^{(j+1)p^*(x)} dx. \end{aligned}$$

For every $x \in \Omega_{j+1}$ we have $v_j(x) = 2^j$ and thus obtain by (2.2) the pointwise inequality $c_1 I_1\,|\nabla v_j|\,(x) + c_1 > 2^j$ for almost every $x \in \Omega_{j+1}$. Note that if $a + b > c$, then $a > \frac{1}{2}c$ or

$b > \frac{1}{2}c$. Thus

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\int_{\Omega_j}2^{(j+1)p^*(x)}dx &\leq \sum_{j=-\infty}^{\infty}\int_{\{x\in\Omega_j:c_1I_1|\nabla v_j|(x)+c_1>2^{j-1}\}}2^{(j+1)p^*(x)}dx\\ &\leq \sum_{j=-\infty}^{\infty}\int_{\{x\in\Omega:c_1I_1|\nabla v_j|(x)>2^{j-2}\}}2^{(j+1)p^*(x)}dx+\sum_{j=-\infty}^{\infty}\int_{\{x\in\Omega_j:c_1>2^{j-2}\}}2^{(j+1)p^*(x)}dx.\end{aligned}$$

Since $(1+|\Omega|)\left\|\nabla u\right\|_{p(\cdot)}\leq 1$, we obtain by Theorem 6.1.11 (see[6]) for the first term on the right-hand side that

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\int_{\{x\in\Omega:c_1I_1|\nabla v_j(y)|(x)+>2^{j-2}\}}2^{(j+1)p^*(x)}dx &\leq c\sum_{j=-\infty}^{\infty}\Big(\int_{\Omega_j}|\nabla v_j(y)|^{p(y)}\,dy+|0<|\nabla v_j|\leq 1|\Big)\\ &\leq c\sum_{j=-\infty}^{\infty}\Big(\int_{\Omega_j}|\nabla u|^{p(y)}\,dy+|\Omega_j|\Big)\\ &\leq \int_{\Omega}|\nabla u|^{p(y)}\,dy+c\,|\Omega|\end{aligned}$$

Let j_0 be the largest integer satisfying $c_1 > 2^{j_0-2}$. Hence

$$\sum_{j=-\infty}^{\infty}\int_{\{x\in\Omega_j:c_1>2^{j-2}\}}2^{(j+1)p^*(x)}dx\leq\int_{\Omega}\sum_{j=-\infty}^{j_0}2^{(j+1)p^*(x)}dx\leq c\,|\Omega|\,,$$

which concludes the proof. □

Next we use the local-to-global trick to generalize the previous lemma, removing the dependence of the constant on $diam(\Omega)$.

Proof. of Theorem 2.3.1 We prove only part (a) of the theorem the second part follows by essentially identical arguments.

Let $u \in W_0^{1,p(\cdot)}(\Omega)$ and extend it by 0 to $\mathbb{R}^n\setminus\Omega$ (Lemma 2.1.14).By a scaling argument we may assume that $\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^n)}=1$. Let (Q_j) be a partition of Rn into unit cubes. As was noted in Example 7.4.2, every Qj is a John domain with the same constant. Thus Lemma 2.3.3 implies that

$$\begin{aligned}\|u\|_{L^{p^*(\cdot)}(Q_j)} &\leq \left\|u-\langle u\rangle_{Q_j}\right\|_{L^{p^*(\cdot)}(Q_j)}+\left|\langle u\rangle_{Q_j}\right|\|1\|_{L^{p(\cdot)}(Q_j)}\\ &\leq c\,\|\nabla u\|_{L^{p(\cdot)}(Q_j)}+\left|\langle u\rangle_{Q_j}\right|.\end{aligned}$$

Next we apply Corollary 7.3.23, the previous inequality, and the triangle inequality in $\ell^{p^*_\infty}$:

$$\begin{aligned}\|u\|_{L^{p^*(\cdot)}(\mathbb{R}^n)} &\leq c\Big(\sum_j \|u\|^{p^*_\infty}_{L^{p^*(\cdot)}(Q_j)}\Big)^{1/p^*_\infty} \\ &\leq c\Big(\sum_j (\|\nabla u\|_{L^{p(\cdot)}(Q_j)} + \big|\langle u\rangle_{Q_j}\big|)^{p^*_\infty}\Big)^{1/p^*_\infty} \\ &\leq c\Big(\sum_j (\|\nabla u\|^{p^*_\infty}_{L^{p(\cdot)}(Q_j)}\Big)^{1/p^*_\infty} + c\Big(\sum_j \big|\langle u\rangle_{Q_j}\big|^{p^*_\infty}\Big)^{1/p^*_\infty}\end{aligned}$$

Note that we end up with the wrong power after the inequality for using Corollary 7.3.23 (see[6]) a second time : we would want the norm to be raised to the power of p_∞ instead of p^*_∞. However, since $\|\nabla u\|_{L^{p(\cdot)}(Q_j)} \leq \|u\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} = 1$ and $p_\infty \leq p^*_\infty$, we conclude that $\|\nabla u\|^{p^*_\infty}_{L^{p(\cdot)}(Q_j)} \leq \|\nabla u\|^{p_\infty}_{L^{p(\cdot)}(Q_j)}$. Then we can use Corollary 7.3.23 (see[6]) again:

$$\sum_j \|\nabla u\|^{p^*_\infty}_{L^{p(\cdot)}(Q_j)} \leq \sum_j \|\nabla u\|^{p_\infty}_{L^{p(\cdot)}(Q_j)} \approx \|\nabla u\|^{p_\infty}_{L^{p(\cdot)}(\mathbb{R}^n)} = 1.$$

It remains to control $\sum_j \big|\langle u\rangle_{Q_j}\big|^{p^*_\infty}$. For this we define an auxiliary function $v := |u| * \chi_{(0,1/2)}$. Then $\big|\langle u\rangle_{Q_j}\big| \leq c\big|\langle v\rangle_{Q_j}\big|$, so it suffices to consider the sum over $\big|\langle v\rangle_{Q_j}\big|$. Using also Hölder's inequality, we conclude that

$$\sum_j \big|\langle u\rangle_{Q_j}\big|^{p^*_\infty} \leq c\sum_j \big|\langle v\rangle_{Q_j}\big|^{p^*_\infty} \leq c\sum_j \int_{Q_j} |v(x)|^{p^*_\infty}\,dx \leq \int_{\mathbb{R}^n} |v(x)|^{p^*_\infty}\,dx$$

Then it follows from the constant exponent Sobolev inequality that

$$\Big(\sum_j \big|\langle u\rangle_{Q_j}\big|^{p^*_\infty}\Big)^{1/p^*_\infty} \leq c\,\|v\|_{L^{p^*_\infty}(\mathbb{R}^n)} \leq c\,\|\nabla v\|_{L^{p_\infty}(\mathbb{R}^n)}.$$

Next we notice that

$$|\nabla v(x)| \leq \int_{B(x,1)} |\nabla\,|u(y)||\,dy \leq 2\,\|\nabla u\|_{p(\cdot)}\,\|1\|_{L^{p'(\cdot)}(B(x,1))} \leq c < \infty,$$

so $|\nabla v| \in L^\infty(\mathbb{R}^n)$. Since $L^{p_\infty} \cap L^\infty \cong L^{p(\cdot)} \cap L^\infty$ (Lemma 3.3.12)(see[6]), it follows that $\|\nabla v\|_{L^{p_\infty}(\mathbb{R}^n)} \leq c\,\|\nabla v\|_{L^{p(\cdot)}(\mathbb{R}^n)} + c \leq c\,\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} + c \leq c$, where we used the boundedness of convolution (Lemma 4.6.1)(see[6]) in the second step. □

Remark 2.3.4. [6][11] Using Hölder inequality we see that the Sobolev-Poincaré inequality implies the Poincaré inequality also in the variable exponent context. By a suitable choice of intermediate exponent, we can relax the condition $1 \leq p^- \leq p^+ < n$ to arbitrary bounded exponents in this case.

The following proposition is due to Kovàcik and Ràkosník [14]. The exponent p^* is the best possible for constant p,next will study; using this we show that it is also the best possible for a variable continuous exponent p.

Proposition 2.3.5. *[6] Let $p, q \in \mathcal{P}(\Omega) \cap C(\Omega)$ satisfy $p^+ < n$. If $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, then $q \leq p^*$.*

Proof. Suppose that $q(x) > p^*(x)$ for some $x \in \Omega$. By the continuity of p and q, there exist $s \in (1, n), t \in (1, \infty)$ and $r > 0$ such that

$$p^*(y) < s^* < t < q(y)$$

for every $y \in B(x, r)$. By Corollary 3.3.4 (see [6]), $W^{1,s}(B(x, r) \hookrightarrow W^{1,p(\cdot)}(B(x, r))$ and $L^{q(\cdot)}(B(x, r)) \hookrightarrow L^t(B(x, r))$. Since $s^* < t$ we have $W^{1,s}(B(x, r)) \not\hookrightarrow L^t(B(x, r))$. Thus $W^{1,s}(B(x, r) \not\hookrightarrow L^{q(\cdot)}(B(x, r))$, which is a contradiction, and so $q(x) \leq p^*(x)$. □

Next we construct a continuous exponent for which the Sobolev embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ does not hold.

Proposition 2.3.6. *[6][11] Let $\Omega \subset \mathbb{R}^2$ be the intersection of the upper half-plane and the unit disk. There exists a continuous exponent $p \in \mathcal{P}(\Omega)$ with $1 < p^- \leq p^+ < 2$ such that*

$$W^{1,p(\cdot)}(\Omega) \not\hookrightarrow L^{p^*(\cdot)}(\Omega)$$

Proof. Fix t and s such that $1 < t < s < 2$ and define $f(\tau) := 2(\frac{\tau}{t} - 1)$ for $\tau \in [t, 2]$. Denoting by (r, ψ) spherical coordinates in Ω (with $\psi \in (0, \pi)$) we define the variable exponent p as follows:

$$p(r, \psi) = \begin{cases} t, & \text{if } \psi \geq r^{f(t)} = 1; \\ \tau, & \text{if } \tau \in (t, s) \text{ satisfying } \psi = r^{f(\tau)}; \\ s, & \text{if } \psi \leq r^{f(s)}. \end{cases}$$

We consider the function $u(x) = |x|^\mu$, where $\mu := \frac{s-2}{t}$. Note that u does not belong to $L^{p^*(\cdot)}(\Omega)$, because

$$\begin{aligned} \overline{\varrho}_{p^*(\cdot)}(u) &= \int_\Omega |x|^{\mu p^*(x)}\, dx \geq \int_0^1 \int_0^{r^{f(s)}} r^{\frac{2s\mu}{2-s}} r\, d\psi dr \\ &= \int_0^1 r^{\frac{2s\mu}{2-s} + f(s) + 1} dr = \infty. \end{aligned}$$

The last equality follows since $\frac{2s\mu}{2-s} + f(s) + 1 = -1$. However, u belongs to $W^{1,p(\cdot)}(\Omega)$. We easily calculate that $|\nabla u(x)| = |\mu| \, |x|^{\mu-1}$. Since $|\mu| < 1$, we find that

$$\int_\Omega |\nabla u(x)|^{p(x)} \, dx < \int_0^1 \int_0^\pi r^{(\mu-1)p(r,\psi)} d\psi r dr.$$

We first estimate the parts of the domain where $p(x) = t$ or $p(x) = s$:

$$\int_0^1 \int_0^\pi r^{(\mu-1)t} d\psi r dr = (\pi - 1) \int_0^1 r^{(\mu-1)t+1} dr < \infty,$$

since $(\mu - 1)t + 1 > -1$, and

$$\int_0^1 \int_0^{r^{f(s)}} r^{(\mu-1)s} d\psi r dr = \int_0^1 r^{(\mu-1)s+f(s)+1} dr < \infty$$

since $(\mu - 1)t + 1 > -1$. Let us denote the integral over these parts by $K < \infty$.

In the remaining part we have $p(r, \psi) = \tau$ and $\psi = r^{f(\tau)}$. Solving this equation for τ we find that $p(r, \psi) = (\frac{1}{2} \frac{\log \psi}{\log r} + 1)t$. Thus we have

$$\begin{aligned} \int_0^1 \int_0^\pi r^{(\mu-1)p(r,\psi)} d\psi r dr &\leq K + \int_0^1 \int_0^1 e^{(\mu-1)(\frac{1}{2}\log\psi + \log r)t} d\psi r dr \\ &= K + \int_0^1 \int_0^1 \psi^{(\mu-1)t/2} d\psi r^{(\mu-1)t+1} dr \\ &= K + \frac{2}{s-t} \frac{1}{s-t} < \infty. \end{aligned}$$

So we have shown that $|\nabla u| \in L^{p(\cdot)}(\Omega)$.For our function u we find that $|u(x)| = \frac{1}{|\mu|} |\nabla u(x)| \, |x| \leq \frac{1}{|\mu|} |\nabla u(x)|$ and so it also follows that $u \in L^{p(\cdot)}(\Omega)$, and we are done. $\square$

Theorem 2.3.7. *[6][10][11] We write $\delta(x) := \min\{1, dist\{x, \partial\Omega\}\}$. Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $p^- > n$. Then there exists a constant c such that*

$$\sup_{y \in B(x,\delta(x))} \frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{p(x)}}} \leq c \, \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for every $u \in W^{1,p(\cdot)}(\Omega)$ and for every $x \in \Omega$. The constant depends only on the dimension n, p^- and $c_{\log}(p)$.

Proof. Let $x \in \Omega$. If $|x - y| < \delta(x)$, then there exists $r < \delta(x)$ such that $\frac{1}{2} r < |x - y| < r$. Denote $B := B(x, r)$. Since

$$W^{1,p(\cdot)}(B) \hookrightarrow W^{1,p_B^-}(B),$$

we obtain by the constant exponent result , that

$$
\begin{aligned}
|u(x)-u(y)| &\leq cr^{1-\frac{n}{p_B^-}} \|\nabla u\|_{L^{p_B^-}(B)} \\
&\leq c(1+|B(0,1)|)r^{1-\frac{n}{p_B^-}} \|\nabla u\|_{L^{p(\cdot)}(\Omega)} .
\end{aligned}
$$

Since $r<1$ the log-Hölder continuity of p implies that $r^{1-\frac{n}{p_B^-}} \leq cr^{1-\frac{n}{p(x)}}$. Using $r<2|x-y|$ we obtain the claim. □

2.4 Compact embeddings

We start this section by examining compact embeddings of $W^{1,p(\cdot)}(\Omega)$ into $L^{p(\cdot)}(\Omega)$. We first prove a quantitative version of Theo rem 4.6.4 (b) (see [6]) for Sobolev functions.

Lemma 2.4.1. *[6][11] Let $p \in \mathcal{A}$ be bounded, and let ψ be a standard mollifier. There exists $A>0$, depending only on the $\mathcal{A}$-constant of p, such that*

$$
\|u*\psi_\varepsilon - u\|_{p(\cdot)} \leq \varepsilon A \|\nabla u\|_{p(\cdot)}
$$

for all $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and all $\varepsilon>0$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$. Using the properties of the mollifier, Fubini's theorem and a change of variables we deduce

$$
\begin{aligned}
u*\psi_\varepsilon(x)-u(x) &= \int_{\mathbb{R}^n}\int_0^1 \psi_\varepsilon(y)\nabla u(x-ty)\cdot y dt dy \\
&= \int_0^1\int_{\mathbb{R}^n} \psi_{\varepsilon t}(y)\nabla u(x-y)\cdot \frac{y}{t} dy dt.
\end{aligned}
$$

This yields the pointwise estimate

$$
\begin{aligned}
|u*\psi_\varepsilon(x)-u(x)| &\leq \varepsilon\int_0^1\int_{\mathbb{R}^n} |\psi_{\varepsilon t}(y)|\,|\nabla u(x-y)|\, dy dt \\
&= \varepsilon\int_0^1 |\nabla u| * |\psi_{\varepsilon t}|(x) dt
\end{aligned}
$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Since spt $\psi_{\varepsilon t} \subset B(0,\varepsilon)$ for all $t \in [0,1]$, the estimate is of a local character. Due to the density of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,1}_{loc}(\mathbb{R}^n)$ the same estimate holds almost everywhere for all

$u \in W^{1,1}_{loc}(\mathbb{R}^n)$. Hence, it holds in particular for all $u \in W^{1,p(\cdot}(\mathbb{R}^n)$. The pointwise estimate thus yields a norm inequality, which due to the properties of the Bochner integral implies

$$\|u * \psi_\varepsilon - u\|_{p(\cdot)} \leq \varepsilon \left\| \int_0^1 |\nabla u| * |\psi_{\varepsilon t}| \, dt \right\|_{p(\cdot)} \leq \varepsilon \int_0^1 \| |\nabla u| * |\psi_{\varepsilon t}| \|_{p(\cdot)} \, dt.$$

By Lemma 4.6.3 we obtain $\| |\nabla u| * |\psi_{\varepsilon t}| \|_{p(\cdot)} \leq K \|\psi\|_1 \|\nabla u\|_{p(\cdot)}$.Now, the claim follows due to $\|\psi\|_1 = 1$. □

Theorem 2.4.2. *[6][10][11] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $p \in \mathcal{P}^{\log}(\Omega)$ or $p \in \mathcal{A}$ be bounded. Then*

$$W^{1,p(\cdot)}_0(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$$

Proof. By Theorem 4.4.8 (see [6]) we always have $p \in \mathcal{A}$. Let $u_k \rightharpoonup u$ in $W^{1,p(\cdot)}_0(\Omega)$. We write $v_k := u_k - u$ and hence $v_k \rightharpoonup 0$ in $W^{1,p(\cdot)}_0(\Omega)$. Thus $\|v_k\|_{1,p(\cdot)}$ is uniformly bounded. Furthermore, we extend the functions v_k by zero outside of Ω (Theorem 2.1.14). We have to show that $v_k \to 0$ in $L^{p(\cdot)}(\Omega)$. Let ψ_ε be the standard mollifier. Then we have $v_k(x) = (v_k - \psi_\varepsilon * v_k)(x) + \psi_\varepsilon * v_k(x)$ and Lemma 2.4.1 implies

$$\begin{aligned} \|v_k\|_{p(\cdot)} &\leq \|v_k - v_k * \psi_\varepsilon\|_{p(\cdot)} + \|v_k * \psi_\varepsilon\|_{p(\cdot)} \\ &\leq c\,\varepsilon \|\nabla v_k\|_{p(\cdot)} \|v_k * \psi_\varepsilon\|_{p(\cdot)} . \end{aligned} \tag{2.3}$$

Since $v_k \rightharpoonup 0$ and $\varepsilon > 0$ is fixed we obtain

$$v_k * \psi_\varepsilon(x) = \int_{\mathbb{R}^n} \psi_\varepsilon(x-y) v_k(y) dy \longrightarrow 0$$

as $k \to \infty$. Let $\Omega_\varepsilon := \{x \in \mathbb{R}^n : dist(x, \Omega) \leq \varepsilon\}$. Thus $v_k * \psi_\varepsilon(x) = 0$ for all $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$. By Hölder's inequality we obtain for all $x \in \Omega_\varepsilon$ that

$$|v_k * \psi_\varepsilon(x)| = \left| \int_{\mathbb{R}^n} \psi_\varepsilon(x-y) v_k(y) dy \right| \leq c \|v_k\|_{p(\cdot)} \|\psi_\varepsilon(x - \cdot)\|_{p'(\cdot)} .$$

Since $\psi \in C^\infty_0(\mathbb{R}^n)$ we have $|\psi| \leq c$ and thus $|\psi_\varepsilon| \leq c\varepsilon^{-n}$. This yields $\|\psi_\varepsilon(x - \cdot)\|_{p'(\cdot)} \leq c\,\varepsilon^{-n} \|\chi_{\Omega_\varepsilon}\|_{p'(\cdot)} \leq c(\varepsilon, p)$ independently of $x \in \mathbb{R}^n$ and k. Using the uniform boundedness of v_k in $L^{p(\cdot)}$ we all together proved

$$|v_k * \psi_\varepsilon(x)| \leq c(\varepsilon, p) \chi_{\Omega_\varepsilon}(x)$$

for all $x \in \mathbb{R}^n$. Since $\chi_{\Omega_\varepsilon} \in L^{p(\cdot)}(\mathbb{R}^n)$ and $v_k * \psi_\varepsilon(x) \to 0n$almost everywhere, we obtain by the theorem of dominated convergence that $v_k * \psi_\varepsilon(x) \to 0$ in $L^{p(\cdot)}(\mathbb{R}^n)$ as $k \longrightarrow \infty$. Hence it follows from (2.3) that

$$\limsup_{k\to\infty} \|v_k\|_{p(\cdot)} \leq c\,\varepsilon \limsup_{k\to\infty} \|\nabla v_k\|_{p(\cdot)} .$$

Since $\varepsilon > 0$ was arbitrary and $\|\nabla v_k\|_{p(\cdot)}$ was uniformly bounded this yields that $v_k \to 0$ in $L^{p(\cdot)}(\mathbb{R}^n)$ and thus $u_k \to u$ in $L^{p(\cdot)}(\mathbb{R}^n)$. □

In fact, it is easy to improve the previous result to deal with higher exponents in the Lebesgue space:

Corollary 2.4.3. *[5][6][11] Let Ω be a bounded domain and let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $p^+ < n$. Then*

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p^*(\cdot)-\varepsilon}(\Omega)$$

for every $\varepsilon \in (0, n')$.

We give two alternative proofs. The first one is based on interpolation. The later one is a straight forward, but it uses Corollary 3.3.4 (see [6]) for the exponent $\frac{\alpha p^*(\cdot)(p^*(\cdot)-\varepsilon)}{\varepsilon}$ that is less than 1. However, it is easy to check that Corollary 3.3.4 (see [6]) holds also in this case.

Proof. 1 We extend the exponent p to the whole space by Proposition 4.1.7 (see [6]). Due to Theorem 2.3.1 and Theorem 2.4.2 we have $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\Omega)}(\Omega)$, respectively. Therefore, by complex interpolation, see Theorem 7.1.2, $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q_\theta(\cdot)}(\Omega)$, where $q_\theta \in \mathcal{P}(\Omega)$ is defined for $\theta \in (0,1)$ by $\frac{1}{q_\theta} = \frac{1-\theta}{p^*} + \frac{\theta}{p} = \frac{1}{p} - \frac{1-\theta}{n}$. Instead of interpolation, one can also use Hölder's inequality in this argument. For θ small enough, we have $p^* - \varepsilon \leq q_\theta \leq p^*$ and therefore by Corollary 3.3.4 (see [6]) $L^{q_\theta(\cdot)}(\Omega) \hookrightarrow L_\theta^{p^*(\cdot)-\varepsilon}(\Omega)$. The claim follows as the composition of a compact and a bounded embedding is compact.

□

Proof. 2 Fix $\varepsilon \in (0, n')$.As in the previous proof, it suffices to show that $u_k \to 0$ in $L^{p^*(\cdot)-\varepsilon}(\Omega)$ whenever $u_k \rightharpoonup 0$ in $W_0^{1,p(\cdot)}(\Omega)$. Define $\alpha := \varepsilon((p^+)^*)^{-2}$. Then an application of Hölder's inequality (3.2.22)(see [6]) yields

$$\|u_k\|_{p^*(\cdot)-\varepsilon} \le 2\left\||u_k|^{\alpha}\right\|_{\frac{p^*(\cdot)(p^*(\cdot)-\varepsilon)}{\varepsilon}} \left\||u_k|^{1-\alpha}\right\|_{p^*(\cdot)}$$

By Corollary 3.3.4(see [6]) and 2.3.1 $\left\||u_k|^{1-\alpha}\right\|_{p^*(\cdot)} < \infty$. On the other hand,

$$\left\||u_k|^{\alpha}\right\|_{\frac{p^*(\cdot)(p^*(\cdot)-\varepsilon)}{\varepsilon}} = \|u_k\|^{\alpha}_{\frac{\alpha p^*(\cdot)(p^*(\cdot)-\varepsilon)}{\varepsilon}} \le 2(1+|\Omega|)\|u_k\|_1 \to 0$$

since$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$. □

When p is (almost) continuous, it is possible to prove Theorem 2.4.2 by a different argument.We present the method here for general Sobolev functions.

Theorem 2.4.4. *[5][6][10] Let $\Omega \subset \mathbb{R}^n$ be a bounded convex open set. Let $p \in \mathcal{P}(\Omega)$ and assume that there exists $\delta > 0$ such that*

$$p^+_{B(x,\delta)\cap\Omega} < (p^-_{B(x,\delta)\cap\Omega})^* \quad \textit{or} \quad p^-_{B(x,\delta)\cap\Omega} \ge n$$

for every $x \in \Omega$. Then

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$$

Proof. We may cover Ω by finitely many balls B_i, with radius δ, such that either $p^+_{B_i\cap\Omega} < (p^-_{B_i\cap\Omega})^*$ *or* $p^-_{B_i\cap\Omega} \ge n$. Let $\varepsilon > 0$ be so small that $p \le (p^-_{B_i\cap\Omega})^* - \varepsilon$ in each $B_i \cap \Omega$. We obtain

$$\begin{aligned} W^{1,p(\cdot)}(B_i\cap\Omega) &\hookrightarrow W^{1,p^-_{B_i\cap\Omega}}(B_i\cap\Omega) \\ &\hookrightarrow\hookrightarrow L^{(p^-_{B_i\cap\Omega})^*-\varepsilon}(B_i\cap\Omega) \hookrightarrow L^{p(\cdot)}(B_i\cap\Omega), \end{aligned}$$

where the convexity of $B_i \cap \Omega$ is used in the second embedding. Since there was only finitely many balls, we obtain the claim. □

Theorem 2.4.5. *[5][6][10] Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ be bounded, let $q \in \mathcal{P}(\mathbb{R}^n)$ be bounded and suppose that*

$$q(x) \le p^*(x) - \frac{\omega(|x|)}{\log(e + 1/|x|)}$$

where $\omega : [0,\infty) \to [0,\infty)$ is increasing and continuous with $\omega(0) = 0$. Then $W^{1,p(\cdot)}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\mathbb{R}^n)$.

Let us point out that the exponent q in the previous theorem cannot be locally log-Hölder continuous, i.e. $q \notin \mathcal{P}^{\log}(\mathbb{R}^n)$.

2.5 Extension operator

In this section we study extension operators for variable exponent Sobolev functions. We show that for certain domains Ω there exists a bounded extension operator ε from $W^{m,p(\cdot)}(\Omega)$ to $W^{m,p(\cdot)}(\mathbb{R}^n)$ for every $m \in \mathbb{N}_0$ and all $p \in \mathcal{A}$.

Definition 2.5.1. [6][10][11] A domain $\Omega \subset \mathbb{R}^n$ is called an (ε,∞)-domain, $0 < \varepsilon \leq 1$, if every pair of points $x, y \in \Omega$ can be joined by a rectifiable path γ parametrize by arc-length such that

$$\ell(\gamma) \leq \frac{1}{\varepsilon} |x - y|$$

,

$$B\Big(\gamma(t), \frac{\varepsilon\, |x - \gamma(t)|\, |y - \gamma(t)|}{|x - y|}\Big) \subset \Omega \quad \text{for all} \quad t \in [0, \ell(\gamma)]\,,$$

where $\ell(\gamma)$ is the length of γ.

Theorem 2.5.2. *[6][10] Let Ω be an (ε,∞)-domain and suppose that $p \in \mathcal{A}$ with $1 < p^- \leq p^+ < \infty$. If $u \in W^{k,p(\cdot)}(\Omega)$, then it can be extended to a function in $W^{k,p(\cdot)}(\mathbb{R}^n)$, with*

$$\|u\|_{W^{k,p(\cdot)}(\mathbb{R}^n)} \leq c\, \|u\|_{W^{k,p(\cdot)}(\Omega)}$$

where the constant c depends on n, ε, p^+, p^- and the $\mathcal{A}$-constant of p.

Proof. Let $\Lambda : W^{k,1}(\Omega) \hookrightarrow W^{k,1}(\mathbb{R}^n)$ be the extension operator of Jones. Chua generalized Jones? extension result to the weighted case:

$$\int_{\mathbb{R}^n} |\partial_\alpha(\Lambda u)|\, w dx \leq c \int_{\Omega} |g|\, w dx$$

for all $w \in A_1$, where $|\alpha| \leq k, g := \sum_{|\beta|\leq k} |\partial_\beta u|$ and the constant depends on w only through $\|w\|_{A_1}$. Note also that M is bounded by Theorem 5.7.2 (see [6]) since $p \in \mathcal{A}$. Hence the extrapolation theorem, Theorem 7.2.1 (see [6]), implies that

$$\|\partial_\alpha(\Lambda u)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c\,\|g\|_{L^{p(\cdot)}(\Omega}.$$

This applies to every $|\alpha| \leq k$, so by the triangle inequality we obtain

$$\sum_{|\alpha|\leq k} \|\partial_\alpha(\Lambda u)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c\,\|g\|_{L^{p(\cdot)}(\Omega)} \leq \sum_{|\beta|\leq k} \|\partial_\beta u\|_{L^{p(\cdot)}(\Omega)}$$

which implies the claim. □

Proposition 2.5.3 (Dyadic Whitney decomposition). *[5][6] Let $\Omega \subsetneq \mathbb{R}^n$ be an open non-empty set. Then there exists a countable family $\mathcal{F}$ of dyadic cubes such that*

(a) $\bigcup_{Q\in\mathcal{F}} \overline{Q} = \Omega$ *and the cubes from $\mathcal{F}$ are pairwise disjoint.*

(b) $\sqrt{n}\ell(Q) < dist(Q, \Omega^{\complement}) \leq 4\sqrt{n}\ell(Q)$ *for all $Q \in \mathcal{F}$.*

(c) *if $\overline{Q}, \overline{Q'} \in \mathcal{F}$ intersect, then*

$$\frac{1}{4} \leq \frac{\ell(Q)}{\ell(Q')} \leq 4.$$

(d) *For given $Q \in \mathcal{F}$, there exists at most 12^n cubes $Q' \in \mathcal{F}$ touching Q.*

Lemma 2.5.4. *[5][6][11] Let $\Omega \subsetneq \mathbb{R}^n$ be a non-empty (ε, ∞)-uniform domain, and let $\mathcal{W}_1$ and $\mathcal{W}_2$ denote the dyadic Whitney decomposition of and $\mathbb{R}^n \setminus \Omega$. Further, let*

$$\mathcal{W}_3 := \Big\{Q \in \mathcal{W}_2 : \ell(Q) \leq \frac{\varepsilon\ diam(\Omega)}{16n}\Big\}.$$

Then for every $Q \in \mathcal{W}_3$ there exists a reflected cube $Q^* \in \mathcal{W}_1$ such that

$$1 \leq \ frac\ell(Q^*)\ell(Q) \leq 4,$$

$$dist(Q, Q^*) \leq c\ell(Q),$$

where $c = c(n,\varepsilon)$. *Moreover, if* $Q, Q_2 \in \mathcal{W}_3$ *touch, then there exists a chain* $F_{Q,Q_2} = \{Q^* = S_1, S_2, ..., S_{jQ} = Q_2^*\}$ *of touching cubes in* $\mathcal{W}_2$ *with* $j_Q \leq j_{\max}(\varepsilon, n)$ *connecting* Q^* *and* Q_2^* *with*

$$\frac{1}{4} \leq \frac{\ell(S_j)}{\ell(S_{j+1})} \leq 4 \ \textit{ for } \ j = 1, ..., j_Q - 1.$$

Observe that all cubes in a chain F_{Q,Q_2} *are of comparable size.*

Observe that all cubes in a chain F_{Q,Q_2} *are of comparable size. We need some preparation before we can define our extension operator. Let* $\omega \in C_0^\infty([0,1]^n)$ *with* $\omega \geq 0$ *and* $\int_{\mathbb{R}^n} \omega(x)dx = 1$. *For* $Q \in \mathcal{W}_1$ *let* $\omega_Q \in C_0^\infty(Q)$ *be defined by* $\omega_Q := \ell(Q)^{-n}\omega \circ L_Q^{-1}$, *where* L_Q *is the affine linear mapping from* $[0,1]^n$ *onto* Q. *In particular,* $\omega_Q \geq 0$ *and* $\int_{\mathbb{R}^n} \omega(x)dx = 1$. *For* $m \in \mathbb{N}_0$ *and* $v \in W^{m,1}(Q)$ *let* $\Pi_Q^m v$ *denote the* Q*-averaged Taylor polynomial of degree* m, *i.e.*

$$(\Pi_Q^m v)(x) = \sum_{|\alpha| \leq m} \int_Q (y) \nabla^\alpha v(y) \frac{(x-y)^\alpha}{\alpha!} dy \tag{2.4}$$

In the definition of Π_Q^m *it suffices to assume* $v \in L^1_{loc}(Q)$, *if all derivatives are moved by partial integration to* $\omega(y)$ *and* $(x-y)^\alpha$ *using also that* $\omega \in C_0^\infty(Q)$. *If* $\alpha \in \mathbb{N}_0^n$ *with* $0 \leq |\alpha| \leq m$, *then* $\partial_\alpha \Pi_Q^m v = \Pi_Q^{m-|\alpha|}(\partial_\alpha v)$. *The averaged Taylor polynomial has the following nice properties [75, Theorem 4.7]. If* $Q \in \mathcal{W}_1, 1 \leq q \leq \infty$ *and* $0 \leq |\beta| \leq k \leq m$, *then*

$$\begin{aligned} \left\|\Pi_Q^m v\right\|_{L^q(Q)} &\leq c\left\|v\right\|_{L^q(Q)} \quad \textit{for all} \quad v \in L^q(Q), \\ \left\|\partial_\beta(v - \Pi_Q^m v)\right\|_{L^q(Q)} &\leq c\ell(Q)^{k-|\beta|}\left\|\nabla^k v\right\|_{L^q(Q)} \quad \textit{for all} \quad v \in W^{1,q}(Q). \end{aligned} \tag{2.5}$$

We only need the case $q = 1$.

We need a partition of unity for $\mathcal{W}_3$, *see Theorem 1.4.7. For each* $Q \in \mathcal{W}_3$ *choose* $\varphi_Q \in C_0^\infty(\frac{17}{16}Q)$ *with* $0 \leq \varphi_Q \leq 1$ *such that*

$$\begin{aligned} \sum_{Q \in \mathcal{W}_3} \varphi_Q &= 1 \qquad \textit{for all} \quad x \in \bigcup_{Q \in \mathcal{W}_3} Q, \\ 0 \leq \sum_{Q \in \mathcal{W}_3} \varphi_Q &\leq 1, \\ \left|\nabla^k \varphi_Q\right| &\leq c\ell(Q)^{-k} \ \textit{ for all } \ 0 \leq k \leq m. \end{aligned}$$

Then $\varphi_Q \varphi_{Q'} \neq 0$ *if and only if* $\overline{Q} \cap \overline{Q'} \neq \emptyset$.

We can now define our extension operator $\varepsilon^m : L^1(\Omega) \to L^1_{loc}(\mathbb{R}^n)$ by

$$\varepsilon^m = \begin{cases} v, & on\ \Omega, \\ \sum_{Q\in\mathcal{W}_3} \varphi_Q \Pi^m_Q v, & on\ \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{2.6}$$

That the operator ε^m defines a bounded extension operator from $W^{m,q}(\Omega)$ to $W^{m,q}(\mathbb{R}^n)$ for every $1 \le q < \infty$. It is important for our considerations that the extension operator ε^m is of local type in the sense that for every ball $B_1 \subset \mathbb{R}^n$, there exists a ball $B_2 \subset \mathbb{R}^n$ such that $\varepsilon u^m u = \varepsilon^m v$ on B_1 if $u = v$ on $B_2 \cap \Omega$. Since $W^{m,p(\cdot)}(\Omega \cap B_2) \hookrightarrow W^{m,1}(\Omega \cap B_2)$ for every ball B_2, we can apply Chua?s result for $q = 1$ to conclude that $\varepsilon^m v \in W^{m,1}_{loc}(\mathbb{R}^n)$ for every $v \in W^{m,p(\cdot)}(\Omega)$. So we already know that $\varepsilon^m v$ has weak derivatives up to order m in L^1_{loc} and it remains to prove the estimates of these derivatives in $L^{p(\cdot)}(\mathbb{R}^n)$. This saves us the trouble of approximation by smooth functions.

We begin with some local estimates.

Lemma 2.5.5. *[6] If $Q \in \mathcal{W}_3$, then*

$$\left\| \nabla^k \varepsilon^m v \right\|_{L^\infty(Q)} \le c\, |Q|^{-1} \left\| \nabla^k v \right\|_{L^1(F(Q))}$$

for all $0 \le k \le m$ with $c = c(\varepsilon, n, m)$, where

$$F(Q)) := \bigcup_{\substack{Q_2\in\mathcal{W}_3 \\ Q \text{ and } Q_2 \text{ touch}}} \bigcup_{S\in F_{Q,Q_2}} S.$$

Proof. Let $Q \in \mathcal{W}_3$ and $k \in \mathbb{N}^n_0$ with $0 \le |k| \le m$. Using $\sum\limits_{Q_2\in\mathcal{W}_3} \varphi_Q = 1$ on Q, the product rule and the estimates for $\partial_\beta \varphi_{Q_2}$, we get

$$\begin{aligned} \left\| \nabla^k \varepsilon^m v \right\|_{L^\infty(Q)} &= \left\| \nabla^k \sum_{Q_2\in\mathcal{W}_3} \varphi_{Q_2} \Pi^m_{Q_2} v \right\|_{L^\infty(Q)} \\ &\le \left\| \nabla^k \sum_{Q_2\in\mathcal{W}_3 : \overline{Q_2}\cap\overline{Q}\neq\emptyset} \varphi_Q (\Pi^m_{Q_2} v - \Pi^m_Q v) \right\|_{L^\infty(Q)} + \left\| \nabla^k \Pi^m_Q v \right\|_{L^\infty(Q)} \\ &\le c \sum_{j=0}^{k} \ell(Q)^{j-k} \sum_{Q_2\in\mathcal{W}_3 : \overline{Q_2}\cap\overline{Q}\neq\emptyset} \left\| \nabla^j (\Pi^m_{Q_2} v - \Pi^m_Q v) \right\|_{L^\infty(Q)} + \left\| \nabla^k \Pi^m_Q v \right\|_{L^\infty(Q)} \end{aligned}$$

The terms in the norms are polynomials of order at most m. For any polynomial z of order m, $\|z\|_{L^\infty(Q)} \le c\, |Q|^{-1} \|z\|_{L^1(Q_2)}$ if Q and Q_2 are cubes of similar size and with a maximal distance of Q and Q2 comparable to the size of Q. This is a consequence of the fact that all norms on a finite dimensional space are equivalent. The independence of c of Q and Q_2 follows by a scaling argument. We use this fact and $\partial_\alpha \Pi_Q^m v = \Pi_Q^{m-|\alpha|}(\partial_\alpha v)$ for $0 \leq |\alpha| \leq m$, to

$$\left\| \nabla^j (\Pi_{Q_2}^m v - \Pi_Q^m v) \right\|_{L^\infty(Q)} \leq c \, |Q|^{-1} \left\| \Pi_{Q_2}^{m-j}(\nabla^j v) - \Pi_Q^{m-j}(\nabla^j v) \right\|_{L^1(Q)}$$

estimate considering also that Q and Q_2 are of comparable size. Let $F_{Q,Q_2} = \{S_1, ..., S_{j_Q}\}$ denote the chain connecting Q^* and Q_2^*. Then

$$\begin{aligned} \left\| \Pi_{Q_2}^{m-j}(\nabla^j v) - \Pi_Q^{m-j}(\nabla^j v) \right\|_{L^1(Q)} &\leq \sum_{i=1}^{j_Q - 1} \left\| \Pi_{S_{i+1}}^{m-j}(\nabla^j v) - \Pi_{S_i}^{m-j}(\nabla^j v) \right\|_{L^1(Q)} \\ &\leq 2 \sum_{i=1}^{j_Q} \left\| \Pi_{S_i}^{m-j}(\nabla^j v) - P_i \right\|_{L^1(Q)} \end{aligned}$$

for any function P_i. We then set $P_i := \Pi_{S_i \cup S_{i+1}}^{m-j}(\nabla^j v)$.

Again we use the fact that we are working with polynomials. Namely let P be a polynomial of degree m and let E and F be measurable subsets of a cube Q with $|E|, |F| > \gamma |Q|$, for some $\gamma > 0$, then $\|P\|_{L^1}(E) \leq c(\gamma, m) \|P\|_{L^1}(F)$. For the proof of this fact see [223, Lemma 2.1]. Using this in the first step, and (2.5) in the last step, we get

$$\begin{aligned} \left\| \Pi_{S_i}^{m-j}(\nabla^j v) - P_i \right\|_{L^1(Q)} &\leq c \left\| \Pi_{S_{i+1}}^{m-j}(\nabla^j v) - \Pi_{S_i \cup S_{i+1}}^{m-j}(\nabla^j v) \right\|_{L^1(S_i)} \\ &\leq c \left\| \nabla^j v - \Pi_{S_{i+1}}^{m-j}(\nabla^j v) \right\|_{L^1(S_i)} + \left\| \nabla^j v - \Pi_{S_i \cup S_{i+1}}^{m-j}(\nabla^j v) \right\|_{L^1(S_i \cup S_{i+1})} \\ &\leq c \ell(Q)^{k-j} \left\| \nabla^k v \right\|_{L^1(\cup_i S_i)} \end{aligned}$$

where we have also used that all cubes in the chain F_{Q,Q_2} are of comparable size. On the other hand it follows from the $L^\infty - L^1$ estimate for polynomials and (2.5) that

$$\left\| \nabla^k \Pi_Q^m v \right\|_{L^\infty(Q)} \leq c \, |Q| \left\| \Pi_Q^{m-k} \nabla^k v \right\|_{L^1(Q)} \leq c \, |Q| \left\| \nabla^k v \right\|_{L^1(Q)}.$$

Combining the above estimates we get

$$\left\| \nabla^k \varepsilon^m v \right\|_{L^\infty(Q)} \leq c \, |Q|^{-1} \sum_{Q_2 \in \mathcal{W}_3 : \overline{Q_2} \cap \overline{Q} \neq \emptyset} \left\| \nabla^k v \right\|_{L^1(\cup F_{Q,Q_2})} + c \, |Q|^{-1} \left\| \nabla^k v \right\|_{L^1(Q)},$$

which yields the claim by the definition of $F(Q)$. □

In order to use that $p \in \mathcal{A}$, we have to reformulate the previous lemma in terms of averaging operators.

Corollary 2.5.6. *[6] There exists $c = c(\varepsilon, n, m) > 0$ such that*

$$\sum_{Q \in \mathcal{W}_3} \chi_Q \left|\nabla^k \varepsilon^m v\right| \leq c \sum_{Q \in \mathcal{W}_3} \chi_Q M_{Q^*} \Big(\underbrace{T_{\frac{17}{16}\mathcal{W}_1} \circ \ldots \circ T_{\frac{17}{16}\mathcal{W}_1}}_{(j_{\max}+1)-times} \circ T_{\mathcal{W}_1}(\nabla^k v) \Big)$$

where $j_{\max}$ is the maximal chain length in Lemma 2.5.4.

Proof. We show that it follows from the definition of $F(Q)$ that

$$\frac{1}{|Q|} \int_{F(Q)} \left|\nabla^k v\right| dx \leq c \Big(\underbrace{T_{\frac{17}{16}\mathcal{W}_1} \circ \ldots \circ T_{\frac{17}{16}\mathcal{W}_1}}_{(j_{\max}+1)-times} \circ T_{\mathcal{W}_1}(\nabla^k v) \Big)(z)$$

for all $z \in Q^*$. First, the application of $T_{\mathcal{W}_1}$ ensures that the L^1-averages over all cubes participating in a chain starting from Q are calculated.Then these averages are accumulated at Q^* by passing them along the chains by the $(j_{\max}+1)$-fold application of $T_{\frac{17}{16}\mathcal{W}_1}$, where we use that the neighboring cubes in $\frac{17}{16}\mathcal{W}_1$ have sufficiently big overlap. Finally, this value is transported by the operator $\sum_{Q \in \mathcal{W}_3} \chi_Q M_{Q^*}$ from Q^* to Q. Together with Lemma 2.5.5 the claim follows. □

We turn to the case $Q \in \mathcal{W}_2 \setminus \mathcal{W}_3$.

Lemma 2.5.7. *[6][11] Let $Q \in \mathcal{W}_2 \setminus \mathcal{W}_3$, then*

$$\left\|\nabla^k \varepsilon^m v\right\|_{L^\infty(Q)} \leq c\, |Q|^{-1} \sum_{j=1}^{k} \ell(Q)^{j-k} \sum_{Q_2 \in \mathcal{W}_3 : \overline{Q_2} \cap \overline{Q} \neq \emptyset} \left\|\nabla^j v\right\|_{L^\infty(Q)}$$

for all $0 \leq k \leq m$ with $c = c(\varepsilon, n, m)$.

Proof. Let $Q \in \mathcal{W}_2 \setminus \mathcal{W}_3$ and $k \in \mathbb{N}_0^n$ with $0 \leq k \leq m$. Using the product rule and the estimates for $\partial_\beta \varphi_{Q_2}$, we get

$$\begin{aligned}
\left\|\nabla^k \varepsilon^m v\right\|_{L^\infty(Q)} &= \left\| \nabla^k \sum_{Q_2 \in \mathcal{W}_3} \varphi_{Q_2} \Pi^m_{Q_2} v \right\|_{L^\infty(Q)} \\
&\leq c \sum_{j=1}^{k} \ell(Q)^{j-k} \sum_{Q_2 \in \mathcal{W}_3 : \overline{Q_2} \cap \overline{Q} \neq \emptyset} \left\|\nabla^j \Pi^m_{Q_2} v\right\|_{L^\infty(Q_2)}
\end{aligned}$$

using also that Q and Q_2 are of comparable size. Now, the claim follows as in the proof of Lemma 2.5.5. $\square$

As before we reformulate this in terms of averaging operators.

Corollary 2.5.8. *[6] Let $Q \in \mathcal{W}_2 \setminus \mathcal{W}_3$, then*

$$\sum_{Q \in \mathcal{W}_2 \setminus \mathcal{W}_3} \chi_Q \left| \nabla^k \varepsilon^m v \right| \leq c \sum_{j=0}^{k} \ell(Q)^{j-k} T_{\frac{17}{16}\mathcal{W}_1} \circ T_{\frac{17}{16}\mathcal{W}_1} \Big(\sum_{Q \in \mathcal{W}_3} \chi_Q M_{Q^*}(\nabla^j v) \Big)$$

for all $0 \leq k \leq m$ with $c = c(\varepsilon, n, m)$.

Proof. The proof is similar to the one of Corollary 2.5.6. The application of $\sum\limits_{Q \in \mathcal{W}_3} \chi_Q M_{Q^*}$ ensures that the L^1-averages over all cubes in $\mathcal{W}_1$ are calculated and transported from Q^* to Q. Then the values of the neighboring cubes Q_2 are transported by two applications of $T_{\frac{17}{16}\mathcal{W}_1}$ to Q. Together with Lemma 2.5.7 the claim follows. $\square$

We are now prepared to prove our extension result.

Theorem 2.5.9. *[6][10][11] Let Ω be an (ε, ∞)-domain and $m \in \mathbb{N}_0$.Then for every $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ the operator ε^m defined in (2.6) is a bounded extension operator from $W^{m,p(\cdot)}(\Omega)$ to $W^{m,p(\cdot)}(\mathbb{R}^n)$. In particular,*

$$\|\varepsilon^m v\|_{W^{m,p(\cdot)}(\mathbb{R}^n)} \leq c \|v\|_{W^{m,p(\cdot)}(\Omega)} \tag{2.7}$$

for all $v \in W^{m,p(\cdot)}(\Omega)$, where c only depends on ε, $diam(\Omega), n, m,$ and $c_{\log}(p)$. Moreover, if Ω is unbounded, then

$$\left\|\nabla^k \varepsilon^m v\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \left\|\nabla^k v\right\|_{L^{p(\cdot)}(\Omega)} \tag{2.8}$$

for all $v \in W^{m,p(\cdot)}(\Omega)$ and $0 \leq k \leq m$, where c only depends on $\varepsilon, n, m,$ and $c_{\log}(p)$.

Proof. Let $v \in W^{m,p(\cdot)}(\Omega)$. Then $\varepsilon^m v \in W^{m,p(\cdot)}_{loc}(\mathbb{R}^n)$ by the discussions before Lemma 2.5.5. It follows from Proposition 2.5.3 that the families $\mathcal{W}_1, \frac{17}{16}\mathcal{W}_1$ and $\frac{17}{16}\mathcal{W}_2$ are locally $(12^n + 1)$-finite. By Theorem 4.4.8 (see [6]) the corresponding averaging operators $T\mathcal{W}_1, T\frac{17}{16}\mathcal{W}_1, T\frac{17}{16}\mathcal{W}_2$ are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Moreover,it follows from $p \in P^{\log}(\mathbb{R}^n)$ and Theorem 4.4.15 (see [6]) that the operator $f \mapsto \sum_{Q \in \mathcal{Q}} \chi_Q M_{Q^*} f$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.Now, the estimates for $\nabla^k \varepsilon^m v$

on $\mathbb{R}^n \setminus \Omega$ follow from Corollary 2.5.6 and 2.5.8 using also that $\ell(Q) > \frac{\varepsilon}{16n} diam(\Omega)$ for all $Q \in \mathcal{W}_2 \setminus \mathcal{W}_3$.The estimates on follow from $\varepsilon^m v = v$ on Ω.If Ω is unbounded, then $\bigcup_{Q \in \mathcal{W}_3} Q = \mathbb{R}^n \setminus \Omega$ up to measure zero. So in this case it suffices to rely on Corollary 2.5.6, which results in the sharper estimates and the independence of the constants of $diam(\Omega)$. $\square$

Remark 2.5.10. The dependence of the constant in (2.7) on $diam(\Omega)$ is similar as in Chua's paper [75]: the constant blows up as $diam(\Omega) \to 0$.

Remark 2.5.11. [6] The previous theorem can be directly generalized to the case of (ε, δ)-domains These sets are similar to (ε, ∞)-domains, except that (ε, δ)-domains do not have to be connected and the conditions on γ in Definition 2.5.1 have to be checked only for $x, y \in \Omega$ with $|x - y| \leq \delta$. Theorem 2.5.9 remains valid for such (ε, δ)-domains whose components have a diameter bounded away from zero.

2.6 Examples

Definition 2.6.1. [1] We say that a function $h : \overline{\Omega} \to \mathbb{R}$ is log-Hölder continuous on $\overline{\Omega}$ if and only if there exists $c > 0$ such that

$$|h(x) - h(y)| \leq \frac{c}{-\ln|x - y|}, \quad \text{for all} \quad x, y \in \overline{\Omega}, 0 < |x - y| \leq \frac{1}{2}. \tag{2.9}$$

It is worth mentioning that, in general,$W_0^{1,p(\cdot)}(\Omega) \subsetneq \mathring{W}^{1,p(\cdot)}(\Omega)$ and $C_0^\infty(\Omega)$ is not necessarily dense in $\mathring{W}^{1,p(\cdot)}(\Omega)$. Nevertheless, we have the following result.

Proposition 2.6.2. *[1] If $p \in C_+(\overline{\Omega})$ satisfies (2.9), then $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{1,p(\cdot)}(\Omega)$ in particular, the spaces $W_0^{1,p(\cdot)}(\Omega)$ and $\mathring{W}^{1,p(\cdot)}(\Omega)$ coincide.*

We recall the famous Poincaré inequality.

Proposition 2.6.3. *[1] Let $p \in C_+(\overline{\Omega})$, then there exists a finite constant $C > 0$ such that for every $u \in \mathring{W}^{1,p(\cdot)}(\Omega)$*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{2.10}$$

where C depends only on Ω and $p(\cdot)$.

In view of Poincaré's inequality (2.10), it is possible to define the equivalent norm of the space $\mathring{W}^{1,p(\cdot)}(\Omega)$ *by the relation (see [15])*

$$\|u\|_{\mathring{W}^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}\,.$$

We point out that the above norm is equivalent to the following norm

$$\sum_{i=1}^{N} \|D_i u\|_{L^{p(\cdot)}(\Omega)}$$

Remark 2.6.4. Note that the following inequality

$$\int_\Omega |f|^{p(x)}\,dx \leq \int_\Omega |\nabla f|^{p(x)}\,dx,$$

in general does not hold . But by Proposition 1.18 (see [1]) and (2.10) we have

$$\int_\Omega |f|^{p(x)}\,dx \leq C\max\{\|f\nabla\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|f\nabla\|^{p^-}_{L^{p(\cdot)}(\Omega)}\}. \tag{2.11}$$

Let us denote by

$$p^* = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \text{any number from } [1,\infty) & \text{if } p(x) \geq N. \end{cases}$$

the Sobolev conjugate exponent.

Theorem 2.6.5. *[1] Let* $p, q \in C_+(\overline{\Omega})$ *such that* $q(x) < p^*(x)$ *in* $\overline{\Omega}$*, then for every* $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\|u\|_{L^{q(\cdot)}(\Omega)} \leq C\,\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

with a constant C *depending on* N*,* p *and* Ω*. The embedding* $\mathring{W}^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact. Moreover, if* p *satisfies the log-Hölder continuity assumption (2.9) and* $p^+ < N$*, then the Sobolev embedding holds also for the critical case* $q(\cdot) = p^*(\cdot)$ *i.e. the embedding* $\mathring{W}^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ *is continuous.*

Remark 2.6.6. [1] In the constant exponent case it is well-known that the conclusion of Proposition 2.6.3 is true for $W_0^{1,p(\cdot)}(\Omega)$ with an arbitrary bounded open set Ω. Similarly, the conclusion of Theorem 2.6.5 is true for $W_0^{1,p(\cdot)}(\Omega)$ with an arbitrary bounded open set .

CHAPTER 3

ANISOTROPIC CONSTANT AND VARIABLE EXPONENTS SOBOLEV SPACES

3.1 Anisotropic constant exponents Sobolev spaces

3.1.1 Anisotropic Sobolev spaces

Proposition 3.1.1. *Let Y be a real vector space and $\rho : Y \to [0, +\infty]$. ρ is called a convex modular in Y if ρ satisfies the following conditions:*

(1) $\rho(u) = 0$ *if and only if* $u = 0$;

(2) $\rho(-u) = \rho(u), \forall u \in Y$;

(3) $\rho(\alpha u + \beta v) \leq \alpha\rho(u) + \beta\rho(v), \forall u, v \in Y, \forall \alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$.

Let $p_0, p_1, ..., p_N$ be $N + 1$ real numbers with $p_i \geq 1, i = 0, 1, ..., N$. We denote

$$\vec{p} = \{p_i, i = 0, 1, 2..., N\}, \underline{p} = \min\{p_i, i = 0, 1, 2..., N\}, D^0 u = u \text{ and } D^i u = \frac{\partial u}{\partial x_i}$$

for $i = 1, ..., N$. With a slight abuse of the notation, we introduce the anisotropic Sobolev space

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega), D^i u \in L^{p_i}(\Omega), i = 1, 2..., N \right\}$$

under the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \left\| D^i u \right\|_{L^{p_i}(\Omega)}. \tag{3.1}$$

We define also $W_0^{1,\vec{p}}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (3.1)(see [2]). The dual of $W_0^{1,\vec{p}}(\Omega)$ is denoted by $W^{-1,\vec{p'}}(\Omega)$,where $\vec{p'} = \left\{ p'_i, i = 0, 1, ..., N \right\}$, $p'_i = \frac{p_i}{p_i - 1}$ and $p_i > 1$.

Remark 3.1.2. [2] Arguing as (R.Adams Sobolev space), it can be easily seen that $W^{1,\vec{p}}(\Omega)$ is as separable Banach space and reflexive if $1 < p_i < \infty$ for all $i = 0, 1, 2, ..., N$.

Lemma 3.1.3. *[2][18] Let Ω be a bounded open set in $\mathbb{R}^n$ Then the following embeddings are compact*

- *if $\underline{p} < N$ then $W_0^{1,\vec{p}}(\Omega) \to L^q(\Omega), \qquad \forall q \in \left[\underline{p}, p^*\right[$, where $\frac{1}{p^*} = \frac{1}{\underline{p}} - \frac{1}{N}$,*
- *if $\underline{p} = N$ then $W_0^{1,\vec{p}}(\Omega) \to L^q(\Omega), \forall q \in \left[\underline{p}, +\infty\right[$,*
- *if $\underline{p} > N$ then $W_0^{1,\vec{p}}(\Omega) \to L^\infty(\Omega) \cap C^k(\overline{\Omega})$, where $k = E\left(1 - \frac{N}{\underline{p}}\right)$.*

Herein,

$$E(x) = n \text{ for } x \in [n, n+1), n \in N,$$

Theorem 3.1.4. *[1] Let $\alpha_i \leq 1, i = 1, ..., N$, we pose $\overrightarrow{\alpha} = (\alpha_1, ..., \alpha_N)$. Suppose that $u \in W_0^{1,\alpha}(\Omega)$, and set*

$$\frac{1}{\overline{\alpha}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\alpha_i}, \quad r = \begin{cases} \overline{\alpha}^* = \frac{N\overline{\alpha}}{N-\overline{\alpha}} & \text{if } \overline{\alpha} < N; \\ anynumber\,from[1,\infty) & \text{if } \overline{\alpha} \geq 0). \end{cases}$$

Then, there exists a positive constant C depending on $N, p_1, ..., p_N$ if $\overline{\alpha} < N$ and also on r and meas(Ω) if $\overline{\alpha} \geq N$, such that

$$\|u\|_{L^r(Q)} \leq C\prod_{i=1}^{N}(\|u\|_{L^{\alpha_i}(Q)} + \|D_i u\|_{L^{\alpha_i}(Q)})^{\frac{1}{N}} \tag{3.2}$$

Proof. see([1]) □

Proposition 3.1.5. *[7][17][9]*

(1) *Let $\Omega \subset \mathbb{R}^N$ b e a rectangular domain and $\overline{p} = (p_1, p_2, ..., p_N) \in [1, \infty)^N$.*

  (1) *If $\overline{p} < N$, then $W^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{\frac{N\overline{p}}{N-\overline{p}}}(\Omega)$, and $W^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ provided $q \in [1, \frac{N\overline{p}}{N-\overline{p}})$.*

  (2) *If $\overline{p} = N$, then $W^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $q \in [1, \infty)$.*

  (3) *If $\overline{p} > N$, then $W^{1,\overrightarrow{p}}(\Omega) \hookrightarrow C^{0,\beta}(\overline{\Omega})$ with*

$$0 < \beta = \frac{\alpha}{N/p^\wedge + \alpha} < 1, \quad \alpha = 1 - \frac{N}{\overline{p}}.$$

(2) *The same statements as in part 1 are true for any bounded domain* $\Omega \subset \mathbb{R}^N$ *if* $W_0^{1,\overrightarrow{p}}(\Omega)$ *is used instead of* $W^{1,\overrightarrow{p}}(\Omega)$.

Lemma 3.1.6 (Anisotropic Sobolev inequality[18])**.** *Let* Q *be a cube of* $\mathbb{R}^N$ *with faces parallel to the coordinate planes. Suppose* $p_i \geq 1, i = 1, ..., N$ *and* $u \in \bigcap_{i=1}^{N} W^{1,p_i}(Q)$*. Then*

$$\|u\|_{L^s(Q)} \leq K \prod_{i=1}^{N} (\|u\|_{L^{p_i}(Q)} + \|D_i u\|_{L^{p_i}(Q)})^{\frac{1}{N}}, \tag{3.3}$$

where $s = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ *if* $\overline{p} < N$ *with* $\overline{p}$ *given by* $\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$*. The constant* K *depends on* N *and* p_i*, Furthermore, if* $\overline{p} \geq N$*, the inequality (3.3) is true for all* $s \geq 1$*, and K depends on s and* $|Q|$*.*

3.1.2 An imbedding from $W^{E,\overline{p}}(\Omega)$ into $L_q(\Omega)$

let N denote the set of non-negative integers and n any positive integer .For multi-indices $\alpha, \beta \in \mathbb{N}^n$, $\alpha = (\alpha_1, ..., \alpha_n)$,$\beta = (\beta_1, ..., \beta_n)$, the length of $\alpha\, is\, |\alpha| = \alpha_1 + ... + \alpha_n$,$D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n} \left(D_i = \frac{\partial}{x_i} \right)$ is a corresponding diffrerential opirator and $\alpha \leqq \beta$ means $\alpha_i \leqq \beta_i$ For all $i = 1, ..., n$ .througtout this paper $\Omega \in \mathbb{R}^n$ is a bounded domain . we shall consider the Banach spaces $L_p(\Omega)$,$(1 \leq p < \infty)$ of integrable functions and $C^{k,\mu}(\overline{\Omega})$ $(k \in N, 0 \leqq \mu \leqq 1)$ Hölder-continuouse with the norms

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/P}$$

and

$$\|u\|_{\mathbf{C}^k(\overline{\Omega})} = \|u\|_{\mathbf{C}^{k,0}(\overline{\Omega})} = \sum_{|\alpha| \leq K} \sup_{x \in \Omega} |D^\alpha u(x)| \quad (if\ \mu = 0)$$

or

$$\|u\|_{\mathcal{C}^{k,\mu}(\overline{\Omega})} = \|u\|_{\mathbf{C}^k(\overline{\Omega})} + \sum_{|\alpha| = k} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha} (\text{if} \mu = 0),$$

respectively.

In accordance with the definitions (see [17]) we shall say that $\Omega \subset \mathbb{R}^n$ is of class $\Re^{k,\mu}$ $(k \in N, 0 \leqq \mu \leqq 1)$,if there existe numbers $\alpha, \beta > 0, M$ Cartesiam coordinate systems $(x_{r1}, ..., x_{rn}) = \left(x_r^{'}, ..., x_{rn} \right)$ obtained from the original system $(x_1, ..., x_n)$.

Theorem 3.1.7. *[17] Let Ω be of class $\mathcal{C}(H)$ for some $H > 0$.let $E = \Big\{\alpha \in \mathbb{N}^n; |\alpha| \leqq 1\Big\}, \overline{p} = \{p_i\}_{i=0}^n, 1 \leqq p_i \leqq p_0 < \infty$ for $i = 1, ..., n$ then there existe such a constant $c > 0$ that*

$$\|u\|_{L_p(\Omega)} \leqq c\,\|u\|_{W^{E,\overrightarrow{p}}(\Omega)} \tag{3.4}$$

For every $u \in W^{E,\overrightarrow{p}}(\Omega)$

$$\frac{1}{q} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{1}{p_i} - 1\right) \tag{3.5}$$

or $1 \leqq q < \infty$ arbitrary , if the term in 3.5 is positive or non-positive ,respactively .

Proof. see [17]. □

Theorem 3.1.8. *[17] Let the assumpations of 3.1.7 be fulfilled .Let $1 \leqq r < q$ .where q satisfies the conditions of the statement of 3.1.7.Then the imbedding $W^{E,\vec{p}}(\Omega) \to L_r(\Omega)$ is compact .*

Proof. Firstly we using the Reisz thoerm on characterization of compact sets in $L_p(\Omega)$ (we will next study) one proves the compactness of imbedding $W^{E,\vec{p}}(\Omega) \to L_1(\Omega)$ .Secondly ,one uses the fact that there exist a continuous imbedding $W^{E,\vec{p}}(\Omega) \to L_q(\Omega)$. Now , Holder's inequality yields the compact imbedding $W^{E,\vec{p}}(\Omega) \to L_r(\Omega)\ for\ 1 \leqq r < q$

□

Imbedding into space of continuous functions

Lemma 3.1.9. *[17] Let Ω be of class $\Re^{0,1}$, $E = \Big\{\alpha \in N^n; |\alpha| \leq 1\Big\}, \overline{p} = \{p_i\}_n^{i=0}$ with $n \leqq p_0 < \infty$ for $i = 1, ..., n$ then there exist a continuous imbedding*

$$W^{E,\vec{p}}(\Omega) \to \mathbf{C}(\overline{\Omega})$$

.

Proof. Setting $p = \min\limits_{1 \leq i \leq n} p_i$ and using Hölder's inequality we obtain

$$\|u\|_{W^{1,p}(\Omega)} \leqq c\,\|u\|_{W^{E,\overline{p}}(\Omega)}\,,$$

where the constant $c > 0$ depends only on Ω and p For $p > n$ itt suffices to use the will-known imbedding theorem on $W^{1,p}(\Omega) \to \mathbf{C}(\overline{\Omega})$ □

Definition 3.1.10. [17] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. $\overline{\mu} = \{p_i\}_n^{i=0}$ with $0 \leq \mu < 1$ for $i = 1, ..., n$ we define the anisotropic space $\mathbf{C}^{0,\overline{\mu}}(\overline{\Omega})$ Hölder-continuous functions as the class of $u \in \mathbf{C}(\overline{\Omega})$ with a finite norm

$$\|u\|_{\mathbf{C}^{0,\overline{\mu}}(\overline{\Omega})} = c\,\|u\|_{\mathbf{C}(\overline{\Omega})} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|^{\mu_i}}{\sum\limits_{i=1}^{n} |x - y|^{\alpha}} \tag{3.6}$$

Remark 3.1.11. The space $\mathbf{C}^{0,\overline{\mu}}(\overline{\Omega})$ with the norm $(3.6)$is a Banach space

Definition 3.1.12. [17] Let $H > 0, \delta > 0$.we say that a domain $\Omega \subset \mathbb{R}^n$ is of class $\mathbf{C}(H, \delta)$ whose edges are parallel to the coordinte axes and one vertex is at the origin, such that $y + c \subset \Omega$ for all $y \in \Omega, |y - x\| < \delta$.

Lemma 3.1.13. *[17] Let $\Omega \subset \mathbb{R}^n$ be of class $\mathbf{C}(H, \delta)$ for some $H > 0, \delta > 0$ then Ω is of class $\Re^{0,1}$*

Proof. Let $K = \min\left(\frac{\delta}{2}, \frac{H}{4}\right)$.Given $x_0 \in \delta\Omega$,we denote by B the ball of rdius K with the center at x_0 let $x \in B \cap \Omega$ and let C be the cube discribed in $3.1.12$ we have $\|x - y| \leqq 2k \leqq \delta$ for $y \in B \cap \Omega$ and so $y + C \in \Omega$.Hence the set $B \cap \Omega$ possesses the same properties as the set $B \cap \Omega$ in the proof of lemma 3.1.9.thus, repeating that proof exacly we obtain that Ω is of class $\Re^{0,1}$. □

3.1.3 Traces of function in $W^{E,\overline{p}}(\Omega)$ on the boundray of Ω

A domain $\Omega \subset \mathbb{R}^n$ will be called an i-domain for some $i = 1, .., n$ if there exist numbers $a_j, b_j, -\infty < a_j < b_j < \infty$ for $i = 1, .., n$ and fonctions $\varphi\psi$ continuouns and bounded on

$$\Delta_i = \left\{x^{'} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\right\}; a_j < x_j b_j,\ \ for j \neq i\right\}$$

such that $\varphi(x^{'}) < a_j, b_j < \psi(x^{'})$ for $x^{'} \in \Delta_i$ and

$$\Omega = \left\{(x', x_j); x' \in \Delta_i, \varphi(x') < x_i < \psi(x')\right\}$$

the set

$$\Gamma_i = \left\{(x', x_j); x' \in \Delta_i, x_i = \psi(x')\right\}$$

or $\Gamma_i = \Big\{(x', x_j); x' \in \Delta_i, x_i = \varphi(x')\Big\}$ will be called an i boundary of Ω we shall say that a function u is defined a.e on Γ_i if $u(x', \psi(x'))$ is defined for a.e.$x' \in \Delta_i$By $L_p(\Gamma_i)(1 \leqq P < \infty$ we denote the class of function u defined a.e on Γ_i with a finit norm

$$\|u\|_{L_p(\Gamma_i)} = \Big(\int_{\Delta_i} \|u(x', \psi(x'))|^p \, dx'\Big)^{1/p} \tag{3.7}$$

the class with the norm is 3.7a Banach space.

Theorem 3.1.14. *[17] Let $\Omega \subset \mathbb{R}^n$ be i-domain for all $i = 1, .., n$.Let $E = \Big\{\alpha \in N^n; |\alpha| \leqq 1\Big\}, \vec{p} = \Big\{p_i\Big\}_{i=0}^{n}$ with $1 \leqq p_i \leqq p_0 < \infty$ for $i = 1, .., n$ denote by Γ_i the i-boundary of Ω and set*

$$q_i = 1 + \left(1 - \frac{1}{p_i}\right) \tag{3.8}$$

then there exist a uniquely determined continuouns linear mapping

$$\Re_i = W^{E,\vec{p}}(\Omega) \to L_i(\Gamma_i)$$

such that $\Re_i(u) = u \mid_{r_i}$ for all $u \in \mathbf{c}^{\infty}(\overline{\Omega})$.

Proof. see [17]. □

Remark 3.1.15. [17] It fallows from $3.1.14$ (see[17]) that

$$\Re_i u(x_i) = \lim_{x_i \to \psi(x')} (x', x_i) \quad \text{for a.e} \quad x' \in \Delta_i.$$

set $p = \min p_i$. An application of Hölder′s inequality gives the imbedding from $W^{E,\vec{p}}(\Omega)$ into $W^{1,p}(\Omega)$.Henec by the theory of isotropic Sobolev space (see [17])then there exist a uniquely determined continuouns operator $\Re : W^{E,\vec{p}}(\Omega) \to L_q(\delta\Gamma), q = \frac{(n-1)p}{(n-p)}$, such that $\Re u = u \mid_{\delta\Omega}$ for all $\mathbf{c}^{\infty}(\overline{\Omega})$. if Ω is not a parallelepided ,then there exist a relativly open subset Γ of $\delta\Omega$ which is a subset of any i-boundray Γ_i for all $i = 1, .., n$ thus on Γ there exists both the i-trace $\Re u \in L_{q_i}(\Gamma)$ for $i = 1, .., n$ and the "global"trace $\Re u \in L_{q_i}(\Gamma)$of $u \in W^{E,\vec{p}}(\Omega)$. on can easily prov that $q \leqq \min_{1 \leqq i \leqq n} q_i$ the fallwing lemma shows what is the coherence betwin

$$\Re_i u \quad \text{and} \quad \Re u$$

.

Lemma 3.1.16. *[17] Let the assumptions of treorem (see[17]) be satisfied. Suppose* $\Gamma \subset \Lambda_r$ *from some* $r = 1, ...M$ *and set*

$$M_i = \Big\{x' \in \Delta_i; \Re_i u(x', \psi(x')) \neq \Re u(y), \Lambda_r^{-1}(x', \psi(x'))\Big\}$$

for $i = 1, .., n$ *.then* $maes_{n-1}(M) = 0$ *.*

Proof. (See[17]) □

Definition 3.1.17. [17] Let $\Omega \subset \mathbb{R}^n$ be a domain ,$E \subset N^n$ a finits set of multiindices,$D \in E$ and $\vec{p} = \Big\{p_\alpha\Big\}_{\alpha \in E}$ with $1 \leqq p \leqq \infty$ for $\alpha \in E$.By $W_{E,\vec{p}}(\Omega)$ we denote the calsure of the set $\mathbf{C}_0^\infty(\Omega)$ in topology of $W^{E,\vec{p}}(\Omega)$.

Remark 3.1.18. [17] the set $W_0^{E,\vec{p}}(\Omega)$ with the norm of $W^{E,\vec{p}}(\Omega)$.forma a separable Banach,wich is reflixive if $p_\alpha > 1$ for all $\alpha \in E$.

Proof. let $\Omega \subset \mathbb{R}^n$ be an i-domain , for all $E \subset N^n$ be a finits convex set of multiindices,(i.e $E = ch_n(E) \cap \mathbb{R}^n$,where $ch_n(A)$ denote the convex hull the $\mathbb{R}^n$ of the set A such that $\beta \in E$, provided $\alpha \in E$,$\beta \in N^n$ and $\beta \leqq \alpha$.Let $\vec{p} = \big\{p_\alpha\big\}_{\alpha \in E} 1 \leqq p_\alpha \leqq p_\beta < \infty$ if $\alpha, \beta \in E, \alpha < \beta$. it follows from the proof of theorem (see[17]) that for each $i = 1, .., n$ and $\alpha \in E$ such that $\alpha^{(i)} = (\alpha_1, ..., \alpha_{\alpha-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_n) \in E$ there exist a uniquely determined continuouns operator $\Re_{i,\alpha} : W^{E,\vec{p}}(\Omega) \to L_1(\Gamma_i)$ such that $\Re_{i,\alpha} = D^\alpha(u) \mid_{\Gamma_i}$ for all $u \in \mathbf{c}^\infty(\overline{\Omega})$ Denote by $\mathbf{W}^{E,\vec{p}}(\Omega)$ the class of all functions $u \in W^{E,\vec{p}}(\Omega)$ with $\Re_{i,\alpha} = 0$ for all admissible i and α. □

Example 3.1.19. [17] let $n = 3\ p_1 = p_2 = 1$ and $p_0 = 3$.then give $q = \frac{c}{4}$ and $k = 2$, recpectvely and so $\tau = 2$,$v_1 = v_2 = 1$. we take $t = 2,\ \lambda = -\frac{8}{9}$. we set $u(x) = (x_1^2 + x_2^2)^{-4/9}$ and intoduce the polar cordinates (ϱ, ϑ).then $D_3 u = 0$and for$i = 1, 2$ we have

$$\int_\Omega \|D_i u(x)\|\, dx \leqq \frac{8}{9} int_0^{\sqrt{2}} \varrho^{-8/9} d\varrho < \infty,$$

while

$$\int_\Omega \|u(x)\|\, dx \geqq int_0^1 \varrho^{-1} d\varrho = \infty,$$

3.2 Anisotropic variable exponents Sobolev spaces

Proposition 3.2.1. *(see 3.1.1) Let ρ be a convex modular in Y. Define*

$$X = Y_\rho = \Big\{u \in Y : \lim_{\lambda \to 0^+} \rho(\lambda u) = 0\Big\}$$

and

$$\|u\|_\rho = \inf\{\lambda > 0 : \rho\Big(\frac{u}{\lambda}\Big) \leq\} \text{ for } \; u \in X$$

Then X is a linear subspace of Y and $\|\cdot\|_\rho$ is a norm on X.$(X, \|\cdot\|_\rho)$ is called a Nakano's modular space generated by the modular ρ, and $\|\cdot\|_\rho$ is called the Luxemburg norm.

Let $\Omega \subset \mathbb{R}^N$ be an open domain. A function $\varphi : \Omega \subset \mathbb{R} \to \mathbb{R}$ is called a Musielak-Orlicz function if φ satisfies the following conditions:

(1) *φ is a Caratheodory function, that is, $\varphi(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $x \in \Omega$ and $\varphi(\cdot, t) : \Omega \to \mathbb{R}$ is measurable for every $t \in \mathbb{R}$.*

(2) *$\varphi(x, t) \geq 0$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$; $\varphi(x, t) = 0$ if and only if $t = 0$.*

(3) *$\varphi(x, -t) = \varphi(x, t)$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.*

(4) *The function $\varphi(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is convex for a.e. $x \in \Omega$.*

In the case that $\varphi(x, t) = \varphi(t)$ without x, a Musielak-Orlicz function φ is called an Orlicz function (or a Young function).

Define $S(\Omega) = \{u | u : \Omega \to \mathbb{R} \;\; \text{is measurable}\}$ and

$$\Phi = \{\varphi | \varphi : \Omega \times \mathbb{R} \to \mathbb{R} \;\; \text{Musielak Orlicz function}\}$$

Let $\varphi \in \Phi$. Define

$$\rho_\varphi(u) \int_\Omega \varphi(x, u(x))dx \;\; \forall u \in S(\Omega).$$

Then ρ_φ is a convex modular in $S(\Omega)$. We denote by $L^\varphi(\Omega)$ the modular space $(S(\Omega))_{\rho_\varphi}$ generated by the modular ρ_φ, namely

$$L^\varphi(\Omega) = \Big\{u \in S(\Omega) : \lim_{\lambda \to 0^+} \rho_\varphi(\lambda u) = 0\Big\}$$

. The Luxemburg norm on $L^{\varphi}(\Omega)$ is denoted by $\|\cdot\|_{\rho_\varphi}$.$(L^{\varphi}(\Omega), \|\cdot\|_{\rho_\varphi})$ is called a Musielak ,Orlicz space, which is a special case of Nakano's modular spaces.

Proposition 3.2.2. *[7] [3] For any $\varphi \in \Phi$ the normed linear space $(L^{\varphi}(\Omega), \|\cdot\|_{\rho_\varphi})$ is complete, that is,$(L^{\varphi}(\Omega), \|\cdot\|_{\rho_\varphi})$ is a Banach space.*

Let $\varphi \in \Phi$. We say that φ satisfies condition (Δ_2) if there exist a positive constant K and a nonnegative function $h \in L^1(\Omega)$ such that

$$\varphi(x, 2t) \leq K\varphi(x, t) + h(x) \text{ for a:e } x \in \Omega \text{ and } t \in \mathbb{R}.$$

Proposition 3.2.3. *[7][17] Let $\varphi \in \Phi$.. If φ satisfies condition (Δ_2), then the following statements hold, where ρ_φ is written simply by ρ.*

$\mathcal{Q}_1$ $\rho(2v) \leq K\rho(v) + C, \forall v \in S(\Omega),$ *where C is a positive constant.*

$\mathcal{Q}_2$ $L^{\varphi}(\Omega) = \{v \in S(\Omega) : \rho(\frac{v}{\lambda}) < \infty \text{ forall } \lambda \neq 0\} = \{v \in S(\Omega) : \rho(v) < +\infty\}.$

$\mathcal{Q}_3$ *Let $v \in L^{\varphi}(\Omega) \setminus \{0\}$. Then $\rho(\alpha v)$, as a function of $\alpha \in [0, 1)$, is continuous and strictly increasing in $\alpha \in [0, 1)$. Moreover $\rho(\alpha v) = 1$ if and only if $|\alpha| = \frac{1}{\|v\|_\rho}$.*

$\mathcal{Q}_4$ $\rho(v) < 1(\textit{resp.} = 1; > 1) \Leftrightarrow \|v\|_\rho < 1(\textit{resp.} = 1; > 1),$ *where $v \in L^{\varphi}(\Omega)$.*

$\mathcal{Q}_5$ $\rho(v_n) \to 0(resp. + \infty), \|v_n\|_\rho \to 0(resp. + \infty) asn \to \infty,$ *where $v_n \in L^{\varphi}(\Omega)$.*

$\mathcal{Q}_6$ $\rho(v_n) \to 1 \Leftrightarrow \|v_n\|_\rho \to 1$

$\mathcal{Q}_7$ *The modular functional ρ is continuous on the space $(L^{\varphi}(\Omega), \|\cdot\|_\rho)$, that is, $\rho(v_n) \to \rho(v)$ if $v_n \to v$ in $(L^{\varphi}(\Omega), \|\cdot\|_\rho)$.*

We denote by $\mathbb{Z}_+$ the set of nonnegative integers.For a multi-index $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{Z}_N^+, |\alpha| = \alpha_1 + ... + \alpha_N \in \mathbb{Z}_N^+$ is the length of α, and $D^\alpha = D_1^{\alpha_1}...D_N^{\alpha_N}$ is respective differential operator, where $D_i = \frac{\partial}{\partial \chi_i}$.

Let $E \subset \mathbb{Z}_N^+$ be a finite set of multi-indices with $0 \in E$, and let $\varphi(E) = \{\varphi_\alpha : \alpha \in E\} \subset \Phi$.Define the anisotropic Musielak-Orlicz-Sobolev space $W^{E,\varphi(E)}(\Omega)$ by

$$W^{E,\varphi(E)}(\Omega) = \{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^{\varphi_\alpha}(\Omega) \text{ for } \alpha \in E\}$$

with the norm $\|u\|_{W^{E,\varphi(E)}(\Omega)} = \sum_{\alpha\in E} \|D^\alpha u\|_{\rho_{\varphi_\alpha}}$.

In the case where $\varphi_\alpha = \varphi$ *for all* $\alpha \in E, W^{E,\varphi(E)}(\Omega) = W^{E,\varphi}(\Omega)$ *is an isotropic Musielak-Orlicz-Sobolev space.In the case where each* $\varphi_\alpha(\alpha \in E)$ *is an Orlicz function, that is,* φ_α *is independent of* $x \in \Omega$, $W^{E,\varphi(E)}(\Omega))$ *is an anisotropic Orlicz-Sobolev space.*

In this article we restrict ourselves to the case that $\Omega \subset \mathbb{R}^N$ *is a bounded domain,* $E = \{\alpha \in \mathbb{Z}_N^+ : |\alpha| \leq \alpha\} \longleftrightarrow \{0, (1, 2, ..., N)\}, \varphi(E) = \{\varphi_0, (\varphi_1, .., \varphi_N)\}$ *and* $\varphi_i(x,t) = |t|^{p_i(x)}(x)$ *for* $i = 0, 1, ..., N$, *where*

$$p_i \in L_+^\infty(\Omega) := \{p \in L^\infty(\Omega) : p(x) \geq 1 \text{ for a.e. } x \in \Omega\}.$$

In this case $W^{E,\varphi(E)}$ *is written by* $W^{1,p_0(\cdot),(p_1(\cdot),...,p_N(\cdot))}(\Omega)$ *or* $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$, *where*

$$\overrightarrow{p}(x) = (p_1(x), ..., p_N(x)).$$

$W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ *is called an anisotropic variable exponent Sobolev space. In the case where* $p_i(x) = p(x)$ *for* $i = 0, 1, ..., N, W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega) = W^{1,p(\cdot)}(\Omega)$ *is the usual (isotropic) variable exponent Sobolev space.In the case where each* $p_i(\Omega)$ *is a constant* $p_i \in [1,\infty), W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega) = W^{1,\{p_0,\overrightarrow{p}\}}(\Omega)$ *is the anisotropic (constant exponent) Sobolev space.*

For any $p \in L_+^\infty(\Omega)$,*setting* $\varphi(x,t)\,|t|^{p(x)}$ *for* $x \in \Omega$ *and* $t \in \mathbb{R}$, *then* φ *is a Musielak-Orlicz function satisfying condition* (Δ_2).*Thus, Proposition3.2.3 is valid for such* φ. *The Luxemburg norm on* $L^{p(\cdot)}(\Omega)$ *is denoted simply by* $\|\cdot\|_{p(\cdot)}$.

For the Banach spaces X *and* Y, *we denote by* $X \hookrightarrow Y$ *that* X *is continuously embedded into* Y, *and by* $X \hookrightarrow\hookrightarrow Y$ *that* X *is compactly embedded into* Y. *For* $\{u_n\} \subset X$ *and* $u_0 \in X, u_n \to u_0$ *denotes that* $\{u_n\}$ *converges strongly to* u_0 *in* X, *and* $u_n \rightharpoonup u_0$ *denotes that* $\{u_n\}$ *converges weakly to* u_0 *in* X.

It is obvious that $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ *because* $p_i(x) \leq 1$. *From Proposition 3.2.2 and the completeness of* $W^{1,1}(\Omega)$, *we immediately have the following theorem.*

Theorem 3.2.4. *[7]* $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ *is a Banach space.*

Let $p_i \in L_+^\infty(\Omega)$ for $i = 0, 1, ..., N$. Define

$$\tilde{p}(u) = \int_\Omega |u|^{p_0(x)}\,dx + \sum_{i=1}^N \int_\Omega |D_i u|^{p_i(x)}\,dx, \;\; \forall u \in L^1_{loc}(\Omega).$$

Then $\tilde{p}$ is a convex modular in $Y = L^1_{loc}(\Omega)$ and $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega) = Y_{\tilde{p}}$ is the modular space generated by the modular $\tilde{p}$.The Luxemburg norm $\|\cdot\|_{\tilde{p}}\, on W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$, defined by $\|u\|_{\tilde{p}} = \inf\{\lambda > 0 : \tilde{p}\Big(\frac{u}{\lambda} \leq 1\Big)\}$ is equivalent to the norm $\|u\|_{W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)} = \|u\|_{p_0(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}$. It is easy to see that the corresponding properties $(\mathcal{Q}_1) - (\mathcal{Q}_7)$ stated in Proposition 3.2.3 are also true if we use $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega), \tilde{p}$ and $\|\cdot\|_{\tilde{p}}$ instead of$L^{\varphi}(\Omega)$, ρ and $\|\cdot\|_{\rho}$, respectively.

For $q \in L^{\infty}_{+}(\Omega)$ and $G \subset \Omega$, define

$$q_-(G) = \operatorname*{ess\,inf}_{x \in G} q(x) \text{ and } q_+(G) = \operatorname*{ess\,sup}_{x \in G} q(x).$$

For simplicity we write q_- and q_+ instead of $q_-(\Omega)$ and $q_+(\Omega)$, respectively.

Theorem 3.2.5. *[7] Let $p_i \in L^{\infty}_{+}(\Omega)$ for $i = 0, 1, ..., N$. If $p_{i,-} > 1$ for $i = 0, 1, ..., N$, then the Banach space $(W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega), \|\cdot\|_{\tilde{p}})$ is uniformly convex, and consequently, it is reflexive.*

Proof. Define $\rho : (S(\Omega))^{N+1} \to [0, +\infty]$ by

$$\rho(v_0, v_1, ..., v_N) = \sum_{i=1}^{N} \int_{\Omega} |v_i|^{p_i(x)}\, dx, \quad \forall (v_0, v_1, ..., v_N) \in (S(\Omega))^{N+1}.$$

Then ρ is a convex modular in $Y = (S(\Omega))^{N+1}$, and the modular space Y_ρ generated by ρ is the Cartesian product $\Pi_{i=1}^{N} L^{p_i(\cdot)}(\Omega)$. For each $i = 0, 1, ..., N$, since $p_{i,-} > 1, (L^{p_i(\cdot)}(\Omega), \|\cdot\|_{p_i(\cdot)})$ is uniformly convex, it follows that the Banach space $(\Pi_{i=1}^{N} L^{p_i(\cdot)}(\Omega), \|\cdot\|_{\rho})$ is uniformly convex. Define $I(u) = (u, D_1 u, ..., D_N u))$ for $u \in (W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$. It is obvious that

$$I : \Big(W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega), \tilde{p}\Big) \to (\Pi_{i=1}^{N} L^{p_i(\cdot)}(\Omega), \|\cdot\|_{\rho})$$

is an isometric embedding. Because every linear subspace of a uniformly convex linear normed space is also uniformly convex, $\Big(W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega), \|\cdot\|_{\tilde{p}}\Big)$ is uniformly convex. □

We say that $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ is a $C^1(\overline{\Omega})$-module if whenever $u \in W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ and $\psi \in C^1(\overline{\Omega})$, it follows that $\psi u \in W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$. The $C^1(\overline{\Omega})$-module is a 'good' property. Let us analyse that under what condition $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ is surely a $C^1(\overline{\Omega})$-module.

Let $i \in \{1, 2, ..., N\}$ be fixed, and let $p_0, p_i \in L^{\infty}_{+}(\Omega)$. Define

$$W^{1_i,\{p_0(\cdot),p_i(\cdot)\}}(\Omega) = \{u \in L^{p_0(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega)\}.$$

Lemma 3.2.6. *[7][17][9] $W^{1_i,\{p_0(\cdot),p_i(\cdot)\}}(\Omega)$ is a $C^1(\overline{\Omega})$-module if*

$$p_0(x) \geq p_i(x) \quad \textit{for a.e.} \quad x \in \Omega. \tag{3.9}$$

Proof. Let condition (3.10) hold.Then $L^{p_0(\cdot)}(\Omega) \hookrightarrow L^{p_i(\cdot)}(\Omega)$. Let $u \in W^{1_i,\{p_0(\cdot),p_i(\cdot)\}}(\Omega)$ and $\psi \in C^1(\overline{\Omega})$. Put $v = \psi u$. Then, obviously $v \in L^{p_0(\cdot)}(\Omega)$.Note that$D_i v = \psi D_i u + u D_i \psi . Since D_i u \in L^{p_i(\cdot)}(\Omega)$ and $\psi \in C^0(\overline{\Omega})$ we have $\psi D_i u \in L^{p_i(\cdot)}(\Omega)$. Since $u \in L^{p_0(\cdot)}(\Omega) \hookrightarrow L^{p_i(\cdot)}(\Omega)$ and $D_i \psi \in C^0(\overline{\Omega})$ we have $u D_i \psi \in L^{p_i(\cdot)}(\Omega)$. Thus $D_i v \in L^{p_i(\cdot)}(\Omega)$ and $v \in W^{1_i,\{p_0(\cdot),p_i(\cdot)\}}(\Omega)$. This shows that $W^{1_i,\{p_0(\cdot),p_i(\cdot)\}}(\Omega)$ is a $C^1(\overline{\Omega})$-module. □

From Lemma 3.2.6 we immediately have the following theorem.

Theorem 3.2.7. *[7][14] $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ is a $C^1(\overline{\Omega})$-module if*

$$p_0(x) \geq p_i(x) \quad \textit{for a.e.} \quad x \in \Omega \quad \textit{and} \quad i = 1, ..., N. \tag{3.10}$$

For $\overrightarrow{p}(\cdot) = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (L_+^\infty(\Omega))^N$,define

$$p^\vee(x) = \max\{p_1(x), ..., p_N(x)\} \quad \textit{and} \quad p^\wedge(x) = \min\{p_1(x), ..., p_N(x)\}, \quad \forall x \in \Omega$$

From Theorem 3.2.7 we see that, for $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ to be a $C^1(\overline{\Omega})$-module, an 'exact' case is to take $p_0(x) = p^\vee(x)$. In what follows, the space $W^{1,\{p^\vee(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ is written simply by $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, namely

$$\begin{aligned} W^{1,\overrightarrow{p}(\cdot)}(\Omega) &= \{u \in L^{p^\vee(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega) \quad \textit{for} \quad i = 1, ..., N\} \\ &= \{u \in L^1_{loc}(\Omega) : u \in L^{p_i(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega) \quad \textit{for} \quad i = 1, ..., N\}. \end{aligned}$$

Below we focus our discussion on the space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

We denote by $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and write $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$. Let Ω have a Lipschitz boundary $\partial\Omega$. For $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset W^{1,1}(\Omega), u|_{\partial\Omega}$ denotes the trace on $\partial\Omega$ of u in $W^{1,1}(\Omega)$. Then

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \{u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) : u|_{\partial\Omega} = 0\}.$$

Obviously, $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. It is well-known that in the constant exponent case, that is, when $\overrightarrow{p}(\cdot) = \overrightarrow{p} \in [1, \infty)^N, W_0^{1,\overrightarrow{p}}(\Omega) = \mathring{W}^{1,\overrightarrow{p}}(\Omega)$. However in the variable exponent case, in *general,* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \neq \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$*, and* $C^\infty(\overline{\Omega})$ *need not be dense in* $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$*. For the density of smooth functions in* $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *we have the following result.*

Theorem 3.2.8. *[7][14] Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with Lipschitz boundary and* $\overrightarrow{p}(\cdot) \in (L_+^\infty(\Omega))^N$*. Suppose that for each* $i = 1, 2, ..., N, p_i : \overline{\Omega} \to \mathbb{R}$ *is log-Holder continuous, that is, there exists a positive constant Lsuch that*

$$|p_i(x) - p_i(y)| \leq \frac{L}{-\ln|x-y|} \quad \textit{for} \quad x, y \in \overline{\Omega} \quad \textit{with} \quad |x-y| \leq \frac{1}{2}.$$

Then

(a) $C_0^\infty(\Omega)$ *is dense in* $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$*, and thus* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$*.*

(b) $C_0^\infty(\Omega)$ *is dense in* $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *if* Ω *is a rectangular domain with edges parallel to the coordinate axes.*

Proof. Statement (1) immediately follows from the known result on $C_0^\infty(\Omega)$-density for the variable exponent Lebesgue?Sobolev spaces [9]. To prove statement (2), let Ω be a rectangular domain with edges parallel to the coordinate axes. Denote by 3Ω the rectangular domain with same centre as Ω and three times the side, and let η be a C^∞ cut-off function between Ω and 3Ω. We extend the function p_i by reflections to a function on $\overline{3\Omega}$, denoted still by p_i. For $\tilde{u} \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$,we may extend it by reflections to a function $\tilde{u} \in W^{1,\overrightarrow{p}(\cdot)}(3\Omega)$ .Set $v = \eta\tilde{u}$. Then $v \in W_0^{1,1}(3\Omega) \cap W^{1,\overrightarrow{p}(\cdot)}(3\Omega)$. By statement (1), v can be approximated by the functions in $C_0^\infty(3\Omega)$, and thus u can be approximated by the functions in $C^\infty(\overline{\Omega})$. Statement (2) is proved. □

We now turn to consider the embeddings of the anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. For $\overrightarrow{p}(\cdot) = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (L_+^\infty(\Omega))^N$ and $x \in \overline{\Omega}$, we define

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)}},$$

$$\overline{p}^*(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)} & \text{if } \overline{p}(x) < N; \\ +\infty & \text{if } \overline{p}(x) \geq N. \end{cases}$$

In what follows, for brevity, a rectangular domain (or a cube) with edges parallel to the coordinate axes is simply said to be a rectangular domain (or a cube). For $x \in \mathbb{R}^N$ and $\varepsilon > 0$, we denote by $K(x, \varepsilon)$ the N-dimensional open cube with centre x and edge length ε. Define

$$C^0_+(\overline{\Omega}) = \{u \in C^0(\overline{\Omega}) : u(x) \geq 1 \quad \text{forall} \quad x \in \overline{\Omega}\}.$$

For $\overrightarrow{p}(\cdot) = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (L^\infty_+(\Omega))^N$ and $G \subset \Omega$, denote

$$\overrightarrow{p_-(G)} = (p_{1,-}(G), p_{2,-}(G), ..., p_{N,-}(G).$$

Note that, by our notation, $\overline{p_-(G)}$ denotes the harmonic average for $\overrightarrow{p_-(G)}$.

We say that Ω is a rectangular-like domain if Ω is a union of finitely many rectangular domains. It is obvious that in statement (2) of Theorem 3.2.8 and in part 1 of Proposition 3.1.5 the condition that Ω is a rectangular domain can be replaced by that Ω is a rectangular-like domain.

For $W^{1,\overrightarrow{p}}(\Omega)$ we give the following compact embedding theorem.

Theorem 3.2.9. *[7]*

(1) *Let $\Omega \subset \mathbb{R}^N$ be a rectangular-like domain and $\overrightarrow{p}(\cdot) = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (C^0_+(\overline{\Omega}))^N$.*

- **(1)** *If $q \in C^0_+(\overline{\Omega})$ and*

$$q(x) < \max\{\overline{p}^*(x), p^\vee(x)\} \quad \text{forall} \quad x \in \overline{\Omega} \tag{3.11}$$

  then $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$

- **(2)** *If $\overline{p}(x) > N$ for all $x \in \overline{\Omega}$, then there exists $\beta \in (0, 1)$ such that $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow C^{0,\beta}(\overline{\Omega})$,and consequently, $W^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$.*

(2) *The same statements as in part 1 are true for any bounded domain $\Omega \subset \mathbb{R}^N$ if use $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ or $W^{1,\overrightarrow{p}(\cdot)}_0(\Omega)$ instead of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

In order to prove statement (1) in part 1 of Theorem 3.2.9, we first give the following two lemmas.

Lemma 3.2.10. *[7] Let $\Omega \subset \mathbb{R}^N$ be a bounded domains and $\overrightarrow{p}(\cdot) \in (L^\infty_+(\Omega))^N$ Suppose that $r \in L^\infty_+(\Omega)$ and $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega)$. Then $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q \in L^\infty_+(\Omega)$ satisfying condition*

$$\operatorname*{ess\,inf}_{x \in \Omega}(r(x) - q(x)) > 0$$

Proof. Let $q \in L_+^\infty(\Omega) and \operatorname{ess\,inf}_{x\in\Omega}(r(x)-q(x)) = \varepsilon > 0$. Note that this implies that $r_- \geq 1+\varepsilon$. Without loss of generality, we may assume $q_- > 1$, otherwise we use $\max\{q(x), 1+\varepsilon/2\}$ instead of $q(x)$. It is obvious that $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega$. Let $\{u_n\} \subset W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a bounded sequence. Then, because $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ and $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ we can choose a subsequence of $\{u_n\}$, denoted still by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ (weakly) in $L^{r(\cdot)}(\Omega)$ and $u_n \rightharpoonup u_0$ (strongly) in $L^1(\Omega)$. Set$v_n = u_n - u_0$. By the Holder inequality for the variable exponent Lebesgue spaces ([14] [9]) we have

$$\int_\Omega |v_n|^{q(x)}\,dx = \int_\Omega |v_n|^{\frac{r(x)-q(x)}{r(x)-1}}\,|v_n|^{\frac{r(x)(q(x)-1)}{r(x)-1}}\,dx \leq 2\left\| |v_n|^{\frac{r(x)-q(x)}{r(x)-1}} \right\|_{L^{\frac{r(x)-1}{r(x)-q(x)}}(\Omega)} \cdot \left\| |v_n|^{\frac{r(x)(q(x)-1)}{r(x)-1}} \right\|_{L^{\frac{r(x)-1}{q(x)-1}}(\Omega)}.$$

Since $v_n \to 0$ in $L^1(\Omega)$, we can obtain that $\left\| |v_n|^{\frac{r(x)-q(x)}{r(x)-1}} \right\|_{L^{\frac{r(x)-1}{r(x)-q(x)}}(\Omega)} \to 0$, and since $v_n \rightharpoonup 0$ in $L^{r(\cdot)}(\Omega)$, we can see that $\left\{\left\| |v_n|^{\frac{r(x)(q(x)-1)}{r(x)-1}} \right\|_{L^{\frac{r(x)-1}{q(x)-1}}(\Omega)}\right\}$ is bounded, and consequently, $\int_\Omega |v_n|^{q(x)}\,dx \to 0$, which implies $v_n \to 0$ in $L^{q(\cdot)}(\Omega)$, that is,$u_n \to u_0$ in $L^{q(\cdot)}(\Omega)$. Lemma 3.2.10 is proved. □

From Lemma 3.2.10 and the fact that $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{p^\vee(\cdot)}(\Omega)$ we have the following corollary.

Corollary 3.2.11. *[7] Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (L_+^\infty(\Omega))^N$. Then $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q \in L_+^\infty(\Omega)$ satisfying condition*

$$\operatorname*{ess\,inf}_{x\in\Omega}(p^\vee(x) - q(x)) > 0.$$

Lemma 3.2.12. *[7] Let Ω, $\overrightarrow{p}(\cdot)$ and $q(\cdot)$ satisfy the assumptions given in part 1(1) of Theorem 3.2.9. Then, given any $x_0 \in \overline{\Omega}$, there exists an open cube $K(x_0, \varepsilon_{x_0})$ such that*

$$W^{1,\overrightarrow{p}(\cdot)}(K(x_0,\varepsilon_{x_0}) \cap \Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(K(x_0,\varepsilon_{x_0}) \cap \Omega). \tag{3.12}$$

Proof. Let any $x_0 \in \overline{\Omega}$ be given. Under assumption (2.3), two cases may happen: either $q(x_0) < p^\vee(x_0)$ or $q(x_0) < \overline{p}^*(x_0)$. In the case that $q(x_0) < p^\vee(x_0)$, by the continuity of $\overrightarrow{p}$ and q, there exists a cube $K(x_0,\varepsilon_{x_0})$ such that $q(x) < p^\vee(x)$ for all $x \in \overline{K(x_0,\varepsilon_{x_0}) \cap \Omega}$. By Corollary 3.2.11, (3.12) holds.Now let $q(x_0) < \overline{p}^*(x_0)$. We consider the following two cases, respectively.

case (i) $\overline{p}(x_0) < N$. By the continuity of $\overrightarrow{p}(\cdot)$ and $q(\cdot)$, since $q(x_0) < \overline{p}^*(x_0)$, there exists a cube $K(x_0, \varepsilon_{x_0})$ such that

$$q_+(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega}) < \left(\overline{p_-(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})}\right)^*.$$

Note that $\overline{p_-(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})} \leq \overline{p}(x_0) < N$. Applying Proposition 3.1.5 to $W^{1,\overrightarrow{p_-(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})}}(K(x_0,\varepsilon_{x_0}) \cap \Omega)$,we have that

$$W^{1,\overrightarrow{p_-(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})}}(K(x_0,\varepsilon_{x_0}) \cap \Omega) \hookrightarrow\hookrightarrow L^{q_+(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})}(K(x_0,\varepsilon_{x_0}) \cap \Omega)$$

As

$$W^{1,\overrightarrow{p}(\cdot)}(K(x_0,\varepsilon_{x_0}) \cap \Omega) \hookrightarrow W^{1,\overrightarrow{p_-(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})}}(K(x_0,\varepsilon_{x_0}) \cap \Omega)$$

And

$$L^{q_+(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})}(K(x_0,\varepsilon_{x_0}) \cap \Omega) \hookrightarrow L^{q(\cdot)}(K(x_0,\varepsilon_{x_0}) \cap \Omega)$$

hence (3.12) holds.

Case (ii) $\overline{p}(x_0) \geq N$ Since $q \in C^0$, we have $q_+ < \infty$. It is easy to see that there exists $\delta > 0$ such that

$$q_+ < r^* \quad \text{if} \quad r > N - \delta.$$

Note that, by our notation,$r^* = +\infty if r \geq N$. From $\overline{p}(x_0) \geq N$ and the continuity of $\overrightarrow{p}(\cdot)$ and $\overline{p}(\cdot)$, we can see that there exists a cube $(K(x_0, \varepsilon_{x_0})$ such that

$$\overline{p_-(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})} > N - \delta$$

and consequently,

$$q_+ < \left(\overline{p_-(\overline{K(x_0,\varepsilon_{x_0}) \cap \Omega})}\right)^*$$

From this we can see that (3.12) holds. Lemma 3.2.12 is proved. □

Proof of Theorem 3.2.9. Based on Lemma 3.2.12, using the finite covering theorem for the compact set $\overline{\Omega}$, we can prove statement (1) in part 1 of Theorem 3.2.9. Applying conclusion (3) of Proposition 3.1.5 to a small rectangular-like neighbourhood of each point $x \in \overline{\Omega}$ and using the finite covering theorem for the compact set $\overline{\Omega}$, we can prove statement (2) in part 1 of Theorem 3.2.9. To prove part 2 of Theorem 3.2.9, let us consider the space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(or $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$), where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Take a rectangular domain $G \subset \mathbb{R}^N$ such that $\Omega \subset G$. For each $i = 1, 2, ..., N$, there exists a continuous extension of the function $p_i(\cdot)$ to $\overline{G}$, denoted still by $p_i(\cdot)$, such that $p_{i,-}(G) = p_{i,-}(\Omega)$ and $p_{i,+}(G) = p_{i,+}(G)$. For any $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(or $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$) we additionally define $u(x) = 0$ for $x \in \overline{G} \setminus \Omega$. Then $u \in W^{1,\overrightarrow{p}(\cdot)}(G)$. Applying the results in part 1 of Theorem 3.2.9 to $W^{1,\overrightarrow{p}(\cdot)}(G)$, we can complete the proof of part 2 of Theorem 3.2.9. □

The following theorem is a consequence of Theorem 3.2.9.

Theorem 3.2.13. *[7] Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overline{p}(\cdot) = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (C_+^0(\overline{\Omega}))^N$. Suppose that*

$$p^{\vee}(x) < \overline{p}^*(x) \quad \textit{for all} \quad x \in \overline{\Omega}. \tag{3.13}$$

Then the following Poincare-type inequality holds:

$$\|u\|_{L^{p^{\vee}(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)} \quad \textit{for all} \quad u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)(or W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)) \tag{3.14}$$

where C is a positive constant independent of $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.Thus, $\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$or $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proof. Theorem 3.2.13 immediately follows from Theorem 3.2.9 and the Jaak Peetre's lemma on the equivalence of norms [42]. In addition, here we give a conventional proof of Theorem 3.2.13. Arguing by contradiction, assume that inequality (3.14) does not hold. Then there exists a sequence $\{u_n\} \subset \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)(or W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)))$ such that

$$\|u_n\|_{L^{p^{\vee}(\cdot)}(\Omega)} \geq n \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)}$$

Without loss of generality, we can assume that $\|u_n\|_{L^{p^{\vee}(\cdot)}(\Omega)} = 1$. Then

$$\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)} \leq \frac{1}{n} \quad \text{for} \quad n = 1, 2....,$$

and $\{u_n\}$ is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(or$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$).By Theorem 3.2.9,condition (3.13) implies that $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(or$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$)$\hookrightarrow\hookrightarrow L^{p^{\vee}(\cdot)}(\Omega)$, and consequently there exists a subsequence of $\{u_n\}$, denoted still by $\{u_n\}$, such that $\{u_n\}$ is convergent in $L^{p^\vee(\cdot)}(\Omega)$. Thus,$\{u_n\}$ is a Cauchy sequence in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(or$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$),and hence there exists $u_0 \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(or $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$) such that $u_n \to u_0$ in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(or $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$) as $n \to \infty$. Since $\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)} \leq \frac{1}{n}$ and $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u_0}{\partial x_i}$ in $L^{p_i(\cdot)}(\Omega)$ for $i = 1, ..., N$, we have

$$\sum_{i=1}^{N} \left\| \frac{\partial u_0}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)} = \lim_{n\to\infty} \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)} = 0,$$

and consequently $\nabla u_0 = 0$. It follows from $u_0 \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset W_0^{1,1}(\Omega)$ that $u_0 = 0$, which contradicts with that $\|u_0\|_{L^{p^\vee(\cdot)}(\Omega)} = \lim_{n\to\infty} \|u_n\|_{L^{p^\vee(\cdot)}(\Omega)} = 1$. The proof is complete. □

Now let us consider the case that $p_0 = 1$. In this case, $W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega) = W^{1,\{1,\overrightarrow{p}(\cdot)\}}(\Omega)$. It is well-known that, in the constant exponent case, when $\overrightarrow{p} = (p_1, p_2, ..., p_N)$ and $\overline{p} < N$, in general, the embedding $W^{1,\{1,\overrightarrow{p}\}}(\Omega) \hookrightarrow L^{\overline{p}^*}(\Omega)$ may not be true even if Ω is a cube.

Proposition 3.2.14. *[7]*

(1) *Let $\Omega \subset \mathbb{R}^N$ be a rectangular domain and $\overrightarrow{p} = (p_1, p_2, ..., p_N) \in [1, \infty)^N$ satisfy condition*

$$p_i < \overline{p}^* \quad \textit{for} \quad i = 1, 2, ..., N.$$

Then $W^{1,\{1,\overrightarrow{p}\}}(\Omega) \hookrightarrow L^{\overline{p}^*}(\Omega)$ if $\overline{p} < N$, and $W^{1,\{1,\overrightarrow{p}\}}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, \infty)$ if $\overline{p} \geq N$.

(2) *If $W_0^{1,\{1,\overrightarrow{p}\}}(\Omega)$ is used instead of $W^{1,\{1,\overrightarrow{p}\}}(\Omega)$ in statement (1), then the corresponding embedding results for $W_0^{1,\{1,\overrightarrow{p}\}}(\Omega)$ are true for the generic bounded open set Ω, not only for a rectangular domain.*

Based on Proposition 3.2.14 we give the following theorem.

Theorem 3.2.15. *[7]*

(1) *Let $\Omega \subset \mathbb{R}^N$ be a rectangular-like domain and $\overrightarrow{p} = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (C_+^0(\overline{\Omega}))^N$. Suppose that (3.13) holds. Then $W^{1,\{1,\overrightarrow{p}(\cdot)\}}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and consequently, by Theorem 3.2.9,$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\{p_0(\cdot),\overrightarrow{p}(\cdot)\}}(\Omega)$ for any $p_0 \in C_+^0(\overline{\Omega})$ satisfying condition $p_0(x) < \overline{p}^*$ for $x \in \overline{\Omega}$.*

(2) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and (3.13) holds.Then $\mathring{W}^{1,\{1,\overrightarrow{p}(\cdot)\}}(\Omega) = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $W_0^{1,\{1,\overrightarrow{p}(\cdot)\}}(\Omega) = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

The proof of Theorem 3.2.15 is omitted here because it can easily be completed by applying Proposition 3.2.14 to a small rectangular-like neighbourhood of each point $x \in \overline{\Omega}$ and using the finite covering theorem for the compact set $\overline{\Omega}$.

Let $\overrightarrow{p} = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (L_+^\infty(\overline{\Omega}))^N$ .We denote by $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i(\cdot)}(\Omega)}$. It is obvious that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset \mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. In the constant exponent case, that is, when $\overrightarrow{p} = (p_1, p_2, ..., p_N) \in \mathbb{R}^N$ with $p^\wedge \geq 1$, it is well-known that the following Poincare-type inequality holds :

$$\|u\|_{L^{p_i}(\Omega)} \leq C \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \quad \text{for all} \quad u \in C_0^\infty(\Omega), (i = 1, ..., N) \tag{3.15}$$

and consequently,$W_0^{1,\overrightarrow{p}}(\Omega) = \mathcal{D}_0^{1,\overrightarrow{p}}(\Omega)$ and $\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$ is an equivalent norm on $W_0^{1,\overrightarrow{p}}(\Omega) For \mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we have the following theorem.

Theorem 3.2.16. *[7] Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\overrightarrow{p} = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in (C_+^0(\overline{\Omega}))^N$ and (3.13) hold. Then $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

Proof. Since $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow \mathcal{D}_0^{1,1}(\Omega) = W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ we have that $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,\{1,\overrightarrow{p}(\cdots)\}}(\Omega)$. By Theorem 3.2.15, $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. □

Remark 3.2.17. It is well-known that $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ if $1 \leq p < N$. It is also known that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ if $p(x) < N$ for all $x \in \overline{\Omega}$ and $p(\cdot)$ is Lipschitz continuous (or log-Holder continuous) on $\overline{\Omega}$. Theorem 3.2.9 gives a compact embedding result for $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, however, the following question on the critical embedding for $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is open.

Theorem 3.2.18. *Let $\Omega \subset \mathbb{R}^N$ be a rectangular domain. Does the continuous embedding $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{\overline{p}^*(\cdot)}(\Omega)(or W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{\overline{p}^*(\cdot)}(\Omega))$ hold if $\overline{p}(x) < N$ for all $x \in \overline{\Omega}$ and $p_i(\cdot)$ is Lipschitz continuous (or log-Holder continuous) on $\overline{\Omega}$ for $i = 1, 2, ..., N$?*

Remark 3.2.19. In Theorem 3.2.16 condition (3.13) is assumed; however, in the constant exponent case, inequality (3.15) holds without condition (3.13). We do not know whether the corresponding inequality like (3.15) for the Lipschitz variable exponent case holds without (3.13), namely the following question is open.

Lemma 3.2.20. *Does the equality $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ hold for the Lipschitz variable exponent $\overrightarrow{p}(\cdot)$ without assumption (3.13)?*

Theorem 3.2.21. *[7] Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p_i(\cdot) > 1$ are continuous functions. Suppose that*

$$p_i(x) < \overline{p}^*(x). \tag{3.16}$$

Then the following Poincar-type inequality holds:

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \forall u \in \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega).$$

where C is a positive constant independent of u. Thus,$\sum\limits_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega)$.

3.2.1 Examples

Let$\Omega \subset \mathbb{R}^N$, $N \leq 2$, be a bounded open domain with Lipschitz boundary.

Let $\overrightarrow{p}(\cdot) : \overline{\Omega} \subset \mathbb{R}^N$ be a vector-valued function defined by

$$\overrightarrow{p}(\cdot) = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)),$$

with $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, ..., N.\}$

Define

$$p_+(x) = \max(p_1(x), ..., p_1(x)), \forall x \in \overline{\Omega}.$$

The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$\begin{aligned} W^{1,\overrightarrow{p}(\cdot)}(\Omega) &= \{u \in L^{p_+(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega), \quad \text{for} \quad i = 1, ..., N\} \\ &= \{u \in L^{1_{loc}}(\Omega) : u \in L^{p_i(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)} \quad \text{for} \quad i = 1, ..., N\} \end{aligned}$$

endowed with the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_+(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|D_i u\|_{L^{p_+(\cdot)}(\Omega)}. \tag{3.17}$$

We denote by $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and we introduce a natural generalization of the variable exponent Sobolev space $\mathring{W}^{1,p(\cdot)}(\Omega)$ we write $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$. As in isotropic case, since is supposed to be a bounded open domain with Lipschitz boundary $\partial\Omega$, then

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \{u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) : u|_{\partial\Omega} = 0\}$$

. The spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega), W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ are separable and reflexive Banach spaces when they are supplied with the norm defined in (3.17). By $\left(\mathring{W}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$ we understand the dual space of $\mathring{W}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

It is well-known that in the constant exponent case, that is, when $\overrightarrow{p}(\cdot) = \overrightarrow{p} \in (1,+\infty)^N, W_0^{1,\overrightarrow{p}}(\Omega) = \mathring{W}^{1,\overrightarrow{p}}(\Omega)$. However in the variable exponent case, in general $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \subsetneq \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and the smooth functions are in general not dense in $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Nevertheless, if p_i is log-Holder continuous over $\overline{\Omega}$ for each $i \in \{,...,N\}$ then $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and thus $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

We put for all $x \in \overline{\Omega}$

$$\overline{p}^*(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)} & \text{if } \overline{p}(x) < N, \\ \infty & \text{if } \overline{p}(x) \leq N. \end{cases} \quad \text{where} \quad \overline{p}(x) = \frac{N}{\sum\limits_{i=1}^{N} \frac{1}{p_i(x)}}$$

In what follows we introduce some versions of the anisotropic embedding theorems.

Theorem 3.2.22. *[1] Let $\Omega \subset \mathbb{R}^N$ an open bounded domain and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. If $q \in C_+(\overline{\Omega})$ with $q(x) < \max(p_+(x), \overline{p}^*(x))$ for all $x \in \overline{\Omega}$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega), \tag{3.18}$$

is compact. The same statement is true if we use $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ instead of $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Theorem 3.2.23. *[1] Let $\Omega \subset \mathbb{R}^N$ an open bounded domain and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \Omega, p_+(x) < \overline{p}^*(x). \tag{3.19}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)} \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \;\; \text{or} \;\; \left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right) \tag{3.20}$$

where C is a positive constant independent of u. Thus $\sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ or $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)$.

Let $\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot)) \in (C_+(\overline{\Omega}))^N$. We let $\mathcal{D}^{1,\overrightarrow{p}_0(\cdot)}$ be the variable exponent anisotropic Sobolev space defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}} = \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}.$$

the space $\mathcal{D}_0^{1,\overrightarrow{p}_0(\cdot)}(\Omega)$ equipped with the above norm is a reflexive Banach space. It is obvious that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \subseteq \mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. In the constant exponent case, that is, when $\overrightarrow{p}(\cdot) = \overrightarrow{p} = (p_1, ..., p_N) \leq (1, \infty)^N$, it is well-known that the following Poincaré-type inequality :

$$\|u|_{L^{p_i}(\Omega)} \leq C \|D_i u|_{L^{p_i}(\Omega)} \quad \text{for all} \quad u \in C_0^\infty(\Omega), i = 1, ..., N$$

The sufficient condition for the coincidence of the spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathcal{D}^{1,\overrightarrow{p}_0(\cdot)}(\Omega)$ is given by the following proposition

Proposition 3.2.24. *[1] Let $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$ and (1.13) hold. Then $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

In order to facilitate the manipulation of the space $\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we introduce the following notations

$$\overrightarrow{p}_- = (p_1^-, ..., p_N^-), p_+^- = \max\{p_1^-, ..., p_N^-\} \;\; \text{and} \;\; p_-^- = \max\{p_1^-, ..., p_N^-\}.$$

By $\overline{p}^-$ we denote the harmonic mean of $\overrightarrow{p}_-$, that is,

$$\frac{1}{\overline{p}^-} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i^-}.$$

If $1 < \overline{p}^- < N$, we define $(\overline{p}^-)^* \in \mathbb{R}^+$ and $p-, \infty \in \mathbb{R}^+$ by

$$(\overline{p}^-)^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1} = \frac{N\overline{p}^-}{N - p^-}, \quad \text{and} \quad p-, \infty = \max\{(\overline{p}^-)^*, p_+^-\}.$$

Theorem 3.2.25. *[1] Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with Lipschitz boundary. Assume that*

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{3.21}$$

Then, for any $q \in C_+(\overline{\Omega})$ satisfying

$$q(x) < p-, \infty \quad \textit{for all} \quad x \in \overline{\Omega},$$

the embedding

$$\mathcal{D}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega), \tag{3.22}$$

is continuous and compact.

Theorem 3.2.26. *[15] Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p_i(\Omega) > 1$ are continuous functions. Suppose that*

$$p_i(x) < \overline{p}^*(x).$$

where

$$\overline{p}^*(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \textit{if } \overline{p}(x) < N; \\ +\infty & \textit{if } \overline{p}(x) \geq N. \end{cases}$$

and $\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum\limits_{i=1}^{N} \frac{1}{p_i(x)}$. Then the following PoincarÃƒÂ©-type inequality holds:

$$\|u\|_{L^{p_+(\cdot}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot}(\Omega)}, \quad \forall u \in \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega).$$

where C is a positive constant independent of u and $p_+(\Omega) = \max\{p_1(x), ..., p_N(x)\}x \in \Omega$. Thus,$\sum\limits_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot}(\Omega)}$ is an equivalent norm on $\bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega)$.

Lemma 3.2.27. *[15] Let $\Omega \subset \mathbb{R}^N, Q_T = (0,T) \times \Omega$, and $p_i(\cdot) :\rightarrow (1,\infty)$ be a continuous function. We have the following continuous embeddings*

$$L^{p_i^+}(0,T; L^{p_i(\cdot)}(\Omega)) \hookrightarrow L^{p_i(\cdot)}(Q_T) \hookrightarrow L^{p_i^-}(0,T; L^{p_i(\cdot)}(\Omega)) \tag{3.23}$$

Proof. Let $v \in L^{p_i(\cdot)}(Q_T)$ and Hölder inequality, we find the estimate

$$\begin{aligned}\int_0^T \|v(t,\cdot)\|^{p_i^-}_{L^{p_i(\cdot}(\Omega)}\,dt &\leq \int_\Omega |v(t,x)|^{p_i(x)}\,dx, \Big(\int_\Omega |v(t,x)|^{p_i(x)}\,dx\Big)^{p_i^-/p_i^+}\Big\}dt \\ &\leq \int_0^T\int_\Omega |v(t,x)|^{p_i(x)}\,dxdt + T^{1-p_i^-/p_i^+}\Big(\int_0^T\int_\Omega |v(t,x)|^{p_i(x)}\,dxdt\Big)^{p_i^-/p_i^+}\end{aligned}$$

we obtain

$$\int_0^T \|v(t,\cdot)\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)}\,dt \leq \max\Big\{\, \|v\|^{p_i^-}_{L^{p_i(\cdot)}(QT)}, \|v\|^{p_i^+}_{L^{p_i(\cdot)}(QT)}\Big\} + T^{1-p_i^-/p^+} \max\Big\{\, \|v\|^{(p_i^-)^2/p^+}_{L^{p_i(\cdot)}(QT)}, \|v\|^{p_i^-}_{L^{p_i(\cdot)}(QT)}\Big\}.$$

Therefore, $v \in L^{p_i^-}(0,T;L^{p_i(\cdot)}(\Omega))$, and the embedding of $L^{p_i(\cdot)}(QT)$ into $L^{p_i^-}(0,T;L^{p_i(\cdot)}(\Omega))$ is continuous.The first embedding in (3.23) can be proved in a similar way. □

Conclusion

In this memory, we presented a study on some results about an anisotropic variable exponents Sobolev spaces, and their basic properties. This work raises a number of questions that deserve to be addressed for example [7], Anisotropic variable exponents Sobolev spaces and $\overline{p}(x)$ Laplacian equations. Elliptic Equations. For this reason we think that the memory will be useful also for researchers interested in the anisotropic variable exponents Sobolev spaces case only.

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