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## Master memory

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### *Theme*

*On the multi-fuzzy sets*

By the students

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# Introduction

In 1965, Zadeh [25] has introduced the concept of "Fuzzy Sets" as an extension of the classical concept of sets. In the classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition, an element either belongs or does not belong to the set. While fuzzy sets theory permits the gradual assessment of the membership of elements in a set; this is described with the help of a membership function in the real unit interval  $[0, 1]$  or on a bounded poset. Fuzzy sets generalize classical sets, since the indicator functions (characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets if the latter only take the values 0 or 1. In the set theory, classical bivalent sets are usually called crisp sets.

Fuzzy sets are used in a wide range of domains in which the informations are incomplete or imprecise, such as mathematics, informatics, physics, chemistry, biology, economics, social sciences ... etc [3, 15, 26].

In 1983, Atanassov [1] has introduced the concept of "Intuitionistic Fuzzy Set" as a generalisation of fuzzy sets. Atanassov added new components that determine the degree of non-membership in the definition of fuzzy set. The fuzzy sets characterized by the degree of membership, while intuitionistic fuzzy sets characterized by both the degree of membership and the degree of non-membership. In 2013, Cuong [5] has introduced a new notion called "picture fuzzy sets (PFS)", which is a direct extension of the fuzzy sets and the intuitionistic fuzzy sets by considering three degrees (positive membership, neutral membership, negative membership). The above extensions "intuitionistic fuzzy sets" and "Picture fuzzy sets" are introduced based on the need for better modeling incomplete and imprecise information.

In 2011, Sebastian and Ramakrishnan [19, 20] introduced the concept of multi-fuzzy sets as an extension of the theory of fuzzy sets and their extensions. This theory considers general  $n$ -memberships degrees (given independently). It is also proposed for the aim to cover all the situations of incomplete and imprecise information that did not covered by the previous theories.

The aim of this dissertation is to study this concept of multi-fuzzy sets and their fundamental properties and operations.

The memory is divided into two chapters. In the first chapter, we give the fundamental concepts of fuzzy sets, operations of fuzzy sets, characteristics of fuzzy set, triangular norm and conorm,  $\alpha$ -cuts of fuzzy set and some extensions of fuzzy sets.

In the second chapter, we treat the notions of multi-fuzzy sets, extended operations on multi-fuzzy sets based t-norms and t-conorms. Also, we study special cases of multi-fuzzy sets and we present some applications of multi-fuzzy sets.

# Chapter 1

## Fuzzy sets and their generalizations

In this chapter, we recall some basic notions, properties and definitions of crisp sets, fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, pythagorean fuzzy sets, spherical fuzzy sets [25, 3, 15, 13, 1, 5].

### 1.1 Generalities on crisp sets

In this section, we recall the basic definitions and operations related to crisp sets. Further informations can be found [3, 4, 15].

#### 1.1.1 Basic definitions and operations of crisp sets

**Definition 1.1 (Universal set )** [15] *The letter  $X$  denotes the universe of discourse or universal set. This set contains all the possible elements of concern in each particular context or application from which sets can be formed.*

**Definition 1.2 (Crip set)** [3] *By a crisp set, or a classical set, or simply a set we mean a collection of distinct well-defined objects. These objects are said to be elements or members of the set. We usually denote the sets by capital letters  $A, B, C, \dots$  etc, and the members by  $a, b, c, \dots$  etc. To denote  $a$  is an element of  $A$  we write  $a \in A$ , the negation of  $a \in A$  is written  $a \notin A$  and means that  $a$  does not belong to  $A$ .*

**Example 1.1** *Let  $X = \mathbb{R}$  and  $A$  is the set of integers between 2 and 7 then*

$$A = \{2, 3, 4, 5, 6, 7\}, 4 \in A \text{ and } 8 \notin A$$

**Definition 1.3 ( Characteristic function)** [4] Let  $X$  be a universal set and  $A$  be a subset of  $X$  ( $A \subseteq X$ ) then the function

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the characteristic function of the set  $A$  in  $X$ .

• **Operations on crisp sets**

Let  $X$  be the universal set, we define operations on  $\mathcal{P}(X)$  and let  $A, B$  be two sets, for more details see [3, 15]

1. **Subset:** If every member of  $A$  is also a member of set  $B$  (i.e., if  $x \in A$  implies  $x \in B$ )  $A$  is called a subset of  $B$ , and this is written as

$$A \subseteq B$$

2. **Equality:** If  $A \subseteq B$  and  $B \subseteq A$ , then  $A$  and  $B$  contain the same members. They are called equal sets; this is denoted by

$$A = B$$

3. **Complement:** the complement of  $A$ , we mean the set of all elements in  $X$  which are not members of  $A$ , or

$$A^c = \{a \in X \mid a \notin A\}$$

4. **Union:** the union of sets  $A$  and  $B$  is defined to be the set of all elements which are members of  $A$  or  $B$  or both, or

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

5. **Intersection:** The intersection of sets  $A$  and  $B$  is defined to be the set of all elements which are members of both  $A$  and  $B$ , in notation

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

6. **Relative complement:** The relative complement of a set  $A$  with respect to set  $B$  is the set containing all the members of  $B$  that are not also members of  $A$ . This can be written  $B - A$ . Thus

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

**Example 1.2 :**

Let  $X = \{2, 3, 5, 7, 11, 13, 17, 19\}$  and let  $A, B$  be subsets of  $X$ .  $A = \{2, 11, 17\}$ ,  $B = \{3, 5, 7, 13, 17\}$

Then:

$$A^c = \{3, 5, 7, 13, 19\}$$

$$B^c = \{2, 11, 19\}$$

$$A \cup B = \{2, 3, 5, 7, 11, 13, 17\}$$

$$A \cap B = \{17\}$$

$$B - A = \{3, 5, 7, 13\}$$

**1.1.2 Properties of crisp sets**

<b>Involution</b>	$(A^c)^c = A$
<b>Commutativity</b>	$A \cup B = B \cup A$ $A \cap B = B \cap A$
<b>Associativity</b>	$A \cup (B \cap C) = (A \cup B) \cap C$ $A \cap (B \cup C) = (A \cap B) \cup C$
<b>Distributivity</b>	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
<b>Idempotence</b>	$A \cup A = A$ $A \cap A = A$
<b>Absorption</b>	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
<b>Absorption by <math>\emptyset</math> and <math>X</math></b>	$A \cup X = X$ $A \cap \emptyset = \emptyset$
<b>Identity</b>	$A \cup \emptyset = A$ $A \cap X = A$
<b>Law of contradiction</b>	$A \cap A^c = \emptyset$
<b>Law of excluded middle</b>	$A \cup A^c = X$
<b>De Morgan's laws</b>	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$

Table 1.1: Properties over crisp set

## 1.2 Fuzzy sets

The concept of a fuzzy set (FS) was used for the first time by Zadeh [25] to handle uncertainty in many fields of everyday life. Fuzzy set theory generalized the range values of classical set theory from the integer 0 and 1 to the interval  $[0, 1]$ . In addition, this concept has proven to be very useful in many different fields [4, 25, 26].

### 1.2.1 Notations and definitions

**Definition 1.4** [25] *Let  $X$  be a universe of discourse. A fuzzy set  $A = \{\langle x, \mu_A(x) \rangle \mid x \in X\}$  is characterized by a membership function  $\mu_A : X \rightarrow [0, 1]$ , where  $\mu_A(x)$  is interpreted as the degree of membership of the element  $x$  in the fuzzy subset  $A$  for any  $x \in X$ .*

**Notation 1.1 :**

- the set of all fuzzy sets on  $X$  is denoted by  $\mathbf{F}(X)$ .
- $A = \sum_{x \in X} \frac{\mu_A(x)}{x}$  if the universal set  $X$  is discrete and finite.
- $A = \int_X \frac{\mu_A(x)}{x}$  if the universal set  $X$  is continuous and infinite.

**Example 1.3** *Let  $X = \{a, b, c\}$  be a universal set,  $A = \{\langle a, 0.4 \rangle, \langle b, 0.1 \rangle, \langle c, 0.7 \rangle\}$  and  $B = \{\langle a, 0.5 \rangle, \langle b, 0.0 \rangle, \langle c, 0.3 \rangle\}$  be a fuzzy subsets in  $X$ .*

**Remark 1.1**

- (i) *In the particular case where  $\mu_A$  only takes values equal to 0 or 1, the fuzzy set  $A$  is a crisp set of  $X$ . A crisp set is therefore a special case of a fuzzy set.*
- (ii) *The empty (fuzz) set  $\emptyset$  is defined as  $\mu_{\emptyset}(x) = 0$  and the total set  $X$  is  $\mu_X(x) = 1$ , for all  $x \in X$ .*

### 1.2.2 Some basic operations on fuzzy sets

The concept of a fuzzy set of a set  $X$  begins a generalization of the notion of a classical set of  $X$ , these operations are chosen so as to be equivalent to classical operations of set theory when the membership function only takes the values 0 or 1.

Given two fuzzy sets  $A$  and  $B$  of  $X$ . For more details see [4]

- **Inclusion:** we say that A is included in B if and only if

$$\mu_A(x) \leq \mu_B(x)$$

In symbols :

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x)$$

- **Equality:** we say that  $A = B$ , if and only if

$$\mu_A(x) = \mu_B(x), \text{ for all } x \in X$$

- **Union:** the union of A and B is the fuzzy set C, where

$$\begin{aligned} \mu_C(x) &= \max\{\mu_A(x), \mu_B(x)\} \\ &= \mu_A(x) \vee \mu_B(x) \end{aligned}$$

With :  $C = A \cup B$

- **Intersection:** the intersection of A and B is the fuzzy set D, where

$$\begin{aligned} \mu_D(x) &= \min\{\mu_A(x), \mu_B(x)\} \\ &= \mu_A(x) \wedge \mu_B(x) \end{aligned}$$

With:  $D = A \cap B$

- **Complement:** the complement of a fuzzy set A is denoted  $CA$  ( or  $A^c$ ) is defined by:

$$\mu_{A^c}(x) = 1 - \mu_A(x), \text{ for all } x \in X$$

- **Addition ( $A + B$ ):**

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \rangle \mid x \in X\}$$

- **the multiplication ( $A \cdot B$ ):**

$$A \cdot B = \{\langle x, \mu_A(x) \cdot \mu_B(x) \rangle \mid x \in X\}$$

- **the difference ( $A - B$ ) :** the difference of two fuzzy sets A and B, is a fuzzy set  $A - B = A \cap CB$  whose the membership function is:

$$\mu_{A-B}(x) = \min\{\mu_A(x), 1 - \mu_B(x)\}$$

**Example 1.4** Let  $X = \{x, y, z\}$  a universal set and let  $A, B \in \mathbf{F}(x)$  given by:

$$A = \{\langle x, 0.2 \rangle, \langle y, 1 \rangle, \langle z, 0.6 \rangle\}$$

$$B = \{\langle x, 0.1 \rangle, \langle y, 0.7 \rangle, \langle z, 0.3 \rangle\}$$

So we get:

$$A^c = \{\langle x, 0.8 \rangle, \langle y, 0.0 \rangle, \langle z, 0.4 \rangle\}$$

$$B^c = \{\langle x, 0.9 \rangle, \langle y, 0.3 \rangle, \langle z, 0.7 \rangle\}$$

$$A \cup B = \{\langle x, 0.2 \rangle, \langle y, 1.0 \rangle, \langle z, 0.6 \rangle\}$$

$$A \cap B = \{\langle x, 0.1 \rangle, \langle y, 0.7 \rangle, \langle z, 0.3 \rangle\}$$

$$A + B = \{\langle x, 0.28 \rangle, \langle y, 1.0 \rangle, \langle z, 0.72 \rangle\}$$

$$A.B = \{\langle x, 0.02 \rangle, \langle y, 0.7 \rangle, \langle z, 0.18 \rangle\}$$

$$A - B = \{\langle x, 0.2 \rangle, \langle y, 0.3 \rangle, \langle z, 0.6 \rangle\}$$

**Proposition 1.1** Let  $X$  be a non-empty set, let  $A, B, C \in \mathbf{F}(X)$ , the following properties hold:

<b>Commutative</b>	$A \cap B = B \cap A, A \cup B = B \cup A$
<b>Associativity</b>	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$
<b>Distributivity</b>	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
<b>Idempotency</b>	$A \cap A = A, A \cup A = A$
<b>Identity</b>	$A \cup \emptyset = A, A \cap \emptyset = \emptyset$ $A \cup X = X, A \cap X = A$
<b>Transitivity</b>	If $A \subseteq B, B \subseteq C$ , then $A \subseteq C$
<b>Involution</b>	$(A^c)^c = A$
<b>De Morgan "laws"</b>	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$
<b>Law of Extended Middle</b>	$A \cup A^c \neq X$ (this property does not hold good in fuzzy set theory)
<b>Law of Contradiction</b>	$A \cap A^c \neq \emptyset$ (this property does not hold good in fuzzy set theory)
<b>Absorption</b>	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$

Table 1.2: Properties over fuzzy Set

### 1.2.3 Characteristics of fuzzy sets

The characteristics of a fuzzy set  $A$  of  $X$  most useful in describing it are that which shows how it differs from a classical set of  $X$ . For more detail see [26].

**Definition 1.5 (The support)** *The support of a fuzzy set  $A$  denoted by  $\text{supp}(A)$ . We mean all elements of  $X$  that belong to  $A$  to a nonzero degree, that is,  $\text{Supp}(A)$  is a crisp set defined by*

$$\text{Supp}(A) = \{x \in X : \mu_A(x) > 0\}$$

**Definition 1.6 (The kernel)** *The kernel of a fuzzy set  $A$  denoted by  $\text{Noy}(A)$ , is the set of elements of  $X$  for which the membership function of  $A$  is equal to 1*

$$\text{Ker}(A) = \{x \in X : \mu_A(x) = 1\}$$

**Definition 1.7 (The height)** *The height of a fuzzy set  $A$  denoted by  $H(A)$ , is the greatest value taken by its membership function*

$$H(A) = \sup\{\mu_A(x) : x \in X\}$$

**Definition 1.8** *A fuzzy set  $A$  is said to be normal whenever  $H(A)=1$ .*

**Definition 1.9 (The Cardinal of  $A$ )** *for a finite fuzzy set  $A$ . The cardinality  $|A|$  is defined as:*

$$|A| = \sum_{x \in X} \mu_A(x)$$

$\|A\| = \frac{|A|}{|X|}$  is called the relative cardinality of  $A$ .

**Example 1.5 (finished case)**

let  $X = \{a, b, c, d, e, f\}$ , let  $A = \{\langle a, 0.5 \rangle, \langle b, 0.2 \rangle, \langle c, 0.3 \rangle, \langle d, 0.4 \rangle, \langle e, 0.8 \rangle, \langle f, 0.6 \rangle\}$ , then

$H(A) = 0.8, \text{Supp}(A) = X, \text{Ker}(A) = \emptyset, |A| = 2.8, \|A\| = 0.46$

and  $B = \{\langle a, 0.0 \rangle, \langle b, 0.2 \rangle, \langle c, 1 \rangle, \langle d, 0.4 \rangle, \langle e, 1 \rangle, \langle f, 0.9 \rangle\}$ , then  $B$  is a fuzzy set normal, because,

$H(B) = 1, \text{Supp}(B) = \{b, c, d, e, f\}, \text{Ker}(B) = \{c, e\}, |B| = 3.5, \|B\| = 0.58$

**Example 1.6 (infinite case)**

Let  $X = [0, 35]$  (set of ages),  $\alpha \in [0, 1]$  and let  $A$  a fuzzy set of  $X$ . The ages young defined by:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in [20, 30] \\ 0 & \text{if } x \geq 35 \text{ and } x \leq 15 \\ \alpha & \text{if } x \in ]15, 20[ \text{ and } x \in ]30, 35[ \end{cases}$$

The kernel of A is  $Ker(A) = [20, 30]$  and the support of A is  $Supp(A) = ]15, 35[$  and the height of A is  $H(A)=1$ .

see figure1.1

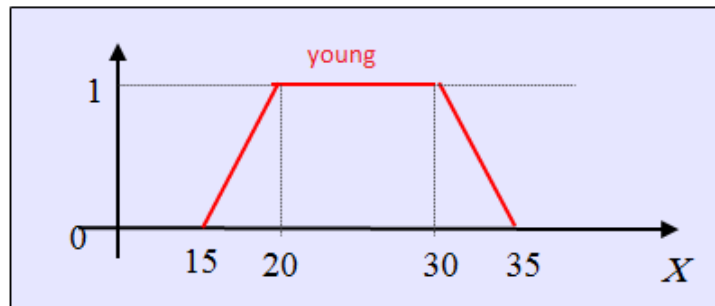


Figure 1.1: Fuzzy subset "young"

### 1.3 $\alpha$ -cuts of a fuzzy set

Two very important concepts of fuzzy set theory, which have been introduced by Zadeh[25], are the concepts of an  $\alpha$ -cut(pronounced alpha-cut), which is also known as  $\alpha$ -level, and of a strong  $\alpha$ -cut. An  $\alpha$ -cut is a crisp set that consists of those elements that have a membership degree greater than or equal to a specific value. More specifically:

**Definition 1.10** [15] Let A be a fuzzy set in X and let  $\alpha \in ]0,1]$ , the  $\alpha$ -cut of A, denoted by  $A_\alpha$ , we mean all elements of X that belong to A to a degree of least  $\alpha$ . That is,  $A_\alpha$  is a classical set defined by:

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$$

and  $A_\alpha$  is a ordinary subset of characteristic function:

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & \text{if } \mu_A(x) \geq \alpha \\ 0 & \text{sinon} \end{cases}$$

If A is a fuzzy set of a universe X, with membership function  $\mu_A$  we have :

$$\forall x \in X, \mu_A(x) = \sup_{\alpha \in ]0,1]} (\alpha \cdot \mu_{A_\alpha}(x)) \text{ (decomposition theorem)}$$

**Definition 1.11** Let  $A : X \rightarrow [0, 1]$  is a fuzzy set of  $X$ . Then, for any  $\alpha \in ]0, 1]$  the strong  $\alpha$ -cut  $A_\alpha^+$  is the crisp set defined by

$$A_\alpha^+ = \{x \in X : \mu_A(x) > \alpha\}$$

**Proposition 1.2** Let  $X$  be a non-empty set, let  $A, B \in \mathbf{F}(x)$ , for any  $\alpha, \beta \in ]0, 1]$  the  $\alpha$ -cuts of fuzzy sets verify :

- $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$
- $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$
- $A \subset B \iff A_\alpha \subset B_\alpha$ , for all  $\alpha \in ]0, 1[$
- If  $\alpha \leq \beta$ , then  $A_\beta \subseteq A_\alpha$
- $A_1 = \text{Noy}(A)$
- $A_0 = X$

**Example 1.7** Let  $X = \{1, 2, 3, 4, 5\}$  and let  $A, B \in \mathbf{F}(x)$ , then

$$A = \{\langle 1, 0.4 \rangle, \langle 2, 0.3 \rangle, \langle 3, 0.6 \rangle, \langle 4, 0.9 \rangle, \langle 5, 0.8 \rangle\}$$

$$B = \{\langle 1, 0.6 \rangle, \langle 2, 0.5 \rangle, \langle 3, 0.1 \rangle, \langle 4, 0.0 \rangle, \langle 5, 0.7 \rangle\}$$

Then

$$A \cup B = \{\langle 1, 0.6 \rangle, \langle 2, 0.5 \rangle, \langle 3, 0.6 \rangle, \langle 4, 0.9 \rangle, \langle 5, 0.8 \rangle\}$$

$$A \cap B = \{\langle 1, 0.4 \rangle, \langle 2, 0.3 \rangle, \langle 3, 0.1 \rangle, \langle 4, 0.0 \rangle, \langle 5, 0.7 \rangle\}$$

So we get

$$A_{0.5} = \{x \in X : \mu_A(x) \geq 0.5\} = \{3, 4, 5\}$$

$$B_{0.5} = \{x \in X : \mu_B(x) \geq 0.5\} = \{1, 2, 5\}$$

we also get

$$\begin{aligned}
 (A \cup B)_{0.5} &= \{x \in X : \mu_{(A \cup B)}(x) \geq 0.5\} \\
 &= A_{0.5} \cup B_{0.5} \\
 &= \{3, 4, 5\} \cup \{1, 2, 5\} \\
 &= \{1, 2, 3, 4, 5\} \\
 (A \cap B)_{0.5} &= \{x \in X : \mu_{(A \cap B)}(x) \geq 0.5\} \\
 &= A_{0.5} \cap B_{0.5} \\
 &= \{3, 4, 5\} \cap \{1, 2, 5\} \\
 &= \{5\}
 \end{aligned}$$

## 1.4 Triangular norms and conorms

Triangular norms and conorms generalize the basic connectives between fuzzy sets. They were first introduced in the theory of probabilistic metric spaces. Later these were found to be very suitable to be used with fuzzy sets. For more details see [17]

**Definition 1.12** (*t-norm*) A triangular norm (*t-norm*) is a function  $\mathcal{T} : [0, 1] \times [0, 1] \rightarrow [0, 1]$  verify:

1.  $\forall x \in [0, 1] : \mathcal{T}(x, 1) = x$  (boundary condition)
2.  $\forall x, y \in [0, 1] : \mathcal{T}(x, y) = \mathcal{T}(y, x)$  (commutativity)
3.  $\forall x, y_1, y_2 \in [0, 1] : \text{if } y_1 \leq y_2, \text{ then } \mathcal{T}(x, y_1) \leq \mathcal{T}(x, y_2)$  (monotonicity)
4.  $\forall x, y, z \in [0, 1] : \mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$  (associativity)

**Definition 1.13** (*t-conorm*) A triangular conorm (*t-conorm*) is a function  $\mathcal{S} : [0, 1] \times [0, 1] \rightarrow [0, 1]$  verify:

1.  $\forall x \in [0, 1] : \mathcal{S}(x, 0) = x$  (boundary condition)
2.  $\forall x, y \in [0, 1] : \mathcal{S}(x, y) = \mathcal{S}(y, x)$  (commutativity)
3.  $\forall x, y_1, y_2 \in [0, 1] : \text{if } y_1 \leq y_2, \text{ then } \mathcal{S}(x, y_1) \leq \mathcal{S}(x, y_2)$  (monotonicity)
4.  $\forall x, y, z \in [0, 1] : \mathcal{S}(x, \mathcal{S}(y, z)) = \mathcal{S}(\mathcal{S}(x, y), z)$  (associativity)

**Definition 1.14** A function  $c : [0, 1] \rightarrow [0, 1]$  is called a fuzzy complement operation, if it satisfies the following conditions :

- (i)  $c(0) = 1$  and  $c(1) = 0$ ,
- (ii) for all  $a, b \in [0, 1]$ , if  $a \leq b$ , then  $c(a) \geq c(b)$

**Definition 1.15** A  $t$ -norm  $\mathcal{T}$  and a  $t$ -conorm  $\mathcal{S}$  are dual with respect to a fuzzy complement operation  $c$  if and only if  $c(\mathcal{T}(a, b)) = \mathcal{S}(c(a), c(b))$  and  $c(\mathcal{S}(a, b)) = \mathcal{T}(c(a), c(b))$ , for all  $a, b \in [0, 1]$ .

**Remark 1.2**

- (i) The operator  $\min$  satisfies these properties, so we can define the intersection set by the operator  $t$ -norm  $A \cap_{\mathcal{T}} B$  such that  $\mu_{A \cap_{\mathcal{T}} B}(x) = \mathcal{T}(\mu_A(x), \mu_B(x))$ ;
- (ii) The operator  $\max$  satisfies these properties, so we can define the union set by the operator  $t$ -norm  $A \cup_{\mathcal{S}} B$  such that  $\mu_{A \cup_{\mathcal{S}} B}(x) = \mathcal{S}(\mu_A(x), \mu_B(x))$ ;
- (iii) Let  $\mathcal{T}$  a triangular norm, the application  $\mathcal{S}$  define by:

$$\mathcal{S} : [0, 1] \times [0, 1] \rightarrow [0, 1]$$

$$\mathcal{S}(x, y) = 1 - \mathcal{T}(1 - x, 1 - y)$$

Is the dual  $t$ -conorm of  $\mathcal{T}$ .

**Example 1.8** *Different triangular norms and conorms*

<i>t-norm</i>	<i>t-conorm</i>	<i>nom</i>
$\min(x, y)$	$\max(x, y)$	<i>Zadeh</i>
$\max(x + y - 1, 0)$	$\min(x + y, 1)$	<i>LuKasiewicz</i>
$xy$	$x + y - xy$	<i>Probabiliste</i>
$\frac{xy}{\gamma + (1 - \gamma)(x + y - xy)}$	$\frac{x + y + xy - (1 - \gamma)xy}{1 - (1 - \gamma)xy}$	<i>Hamacher</i> ( $\gamma > 0$ )
$\max(1 - ((1 - x)^p + (1 - y)^p)^{1/p}, 0)$	$\min((x^p + y^p)^{1/p}, 1)$	<i>Yager</i> ( $p > 0$ )
$\max((x + y - 1 + \lambda xy)/(1 + \lambda), 0)$	$\min(x + y + \lambda xy, 1)$	<i>Weber</i> ( $\lambda > -1$ )
$\begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{if no} \end{cases}$	$\begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 1 & \text{if no} \end{cases}$	<i>Drastique</i>

**Example 1.9** Let  $X = \{a, b, c\}$ , let  $A$  and  $B$  be a two fuzzy subsets of  $X$ , such that  $A = \{\langle a, 0.4 \rangle, \langle b, 0.8 \rangle, \langle c, 0.3 \rangle\}$ ,  $B = \{\langle a, 0.5 \rangle, \langle b, 0.2 \rangle, \langle c, 0.7 \rangle\}$ . We can use Lukasiewicz operators to define the union and the intersection by :

$$(i) \mu_{A \cap_{\mathcal{T}} B}(x) = \mathcal{T}(\mu_A, \mu_B) = \max(\mu_A(x) + \mu_B(x) - 1, 0), \forall x \in X;$$

$$(ii) \mu_{A \cup_{\mathcal{S}} B}(x) = \mathcal{S}(\mu_A, \mu_B) = \min(\mu_A(x) + \mu_B(x), 1), \forall x \in X.$$

So we get:

$$(i) A \cap_{\mathcal{T}} B = \{\langle a, 0.0 \rangle, \langle b, 0.0 \rangle, \langle c, 0.0 \rangle\};$$

$$(ii) A \cup_{\mathcal{S}} B = \{\langle a, 0.9 \rangle, \langle b, 1.0 \rangle, \langle c, 1.0 \rangle\}.$$

**Remark 1.3**

(i) The operator  $\min$  is the greatest of the  $t$ -norms.

$$\mathcal{T}(x, y) \leq \min(x, y), \text{ for any } t\text{-norm } \mathcal{T}$$

If the  $t$ -norm operator satisfies the idempotence therefore it is coincident the operator  $\min$  e.i. for all  $x$  and  $y$  in  $[0, 1]$ , we obtain

$$\mathcal{T}(x, y) = \min(x, y);$$

(ii) The operator  $\max$  is the smallest of the  $t$ -conorms such that for all  $x$  and  $y$  in  $[0, 1]$ , we obtain

$$\max(x, y) \leq \mathcal{S}(x, y).$$

If the  $t$ -conorm operator satisfies the idempotence therefore it is coincident the operator  $\max$  e.i. for all  $x$  and  $y$  in  $[0, 1]$ , we obtain

$$\mathcal{S}(x, y) = \max(x, y);$$

(iii) Any  $t$ -norm  $\mathcal{T}$  and any  $t$ -conorm  $\mathcal{S}$  satisfy for all  $x$  and  $y$  in  $[0, 1]$ , the following inequalities

$$\mathcal{T}_{drastic} \leq \mathcal{T}(x, y) \leq \min(x, y);$$

$$\max(x, y) \leq \mathcal{S}(x, y) \leq \mathcal{S}_{drastic}.$$

(iv) All  $t$ - norms coincide in the edge values of the interval  $[0, 1]$  e.i, (0 and 1)

$$\mathcal{T}(0, 0) = \mathcal{T}(0, 1) = \mathcal{T}(1, 0) = 0, \mathcal{T}(1, 1) = 1;$$

(v) All  $t$ - conorms coincide in the edge values of the interval  $[0, 1]$  e.i, (0 and 1)

$$\mathcal{S}(1, 1) = \mathcal{S}(1, 0) = \mathcal{S}(0, 1) = 1, \mathcal{S}(0, 0) = 0.$$

## 1.5 Extension of fuzzy sets

In this section, we study the fundamental definitions, operations and properties of intuitionistic fuzzy sets (IFS), picture fuzzy sets (PFS), spherical fuzzy sets(SFS) and pythagorean fuzzy sets (PYFS) [1, 5, 13, 12].

### 1.5.1 Intuitionistic fuzzy sets

**Definition 1.16** [1] *An intuitionistic fuzzy set (IFS)  $A$  in  $X$  is defined as an object of the following form*

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X\} \quad (1.1)$$

where the functions:

$$\mu_A : X \rightarrow [0, 1] \quad (1.2)$$

and

$$\nu_A : X \rightarrow [0, 1] \quad (1.3)$$

define the degree of membership and the degree of non-membership of the element  $x \in X$ , respectively, and for every  $x \in X$ :

$$0 \leq \mu_A + \nu_A \leq 1 \quad (1.4)$$

Obviously, each ordinary fuzzy set may be written as

$$\{\langle x, \mu_A, 1 - \mu_A \rangle \mid x \in X\}.$$

**Example 1.10** *Let  $X = \{a, b, c, d\}$ , and let  $A$  the intuitionistic fuzzy set in  $X$  where:*

$$A = \{\langle a, 0.4, 0.3 \rangle, \langle b, 0.2, 0.3 \rangle, \langle c, 1.0, 0.0 \rangle, \langle d, 0.9, 0 \rangle\}$$

**Definition 1.17** [1] *The value of*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \quad (1.5)$$

is called the degree of non-determinacy (or uncertainty) of the element  $x \in X$  to the intuitionistic fuzzy set  $A$ .

**Definition 1.18 (Characteristic function)** [1] *Let  $X^* = X \times [0, 1] \times [0, 1]$ , i.e.  $A \subset X^*$ . We define the characteristic function of an IFS  $A$*

$$\begin{aligned} \Omega_A : X^* &\rightarrow \{0,1\} \\ \Omega_A(\langle x, a, b \rangle) &= \begin{cases} 1 & \text{if } \mu_A = a \text{ and } \nu_A = b \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (1.6)$$

Therefore,  $A$  is a (standard) set in the sense of the set theory in the universe  $X^*$ .

### Intuitionistic fuzzy t-norms and t-conorms

Consider the set  $L^*$  and operation  $\leq_{L^*}$  defined by

$$L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2; \forall (x_1, x_2), (y_1, y_2) \in L^*$$

Then  $(L^*, \leq_{L^*})$  is a complete lattice, for each nonempty set  $A \subset L^*$

$$\sup A = \{\sup \{x_1 \mid x_1 \in [0, 1] \text{ and } (\exists x_2 \in [0, 1 - x_1])\}, \inf = \{x_2 \mid x_2 \in [0, 1] \text{ and } (\exists x_1 \in [0, 1 - x_2])\}\} \\ (x_1, x_2) \in A$$

$$\inf A = \{\inf \{x_1 \mid x_1 \in [0, 1] \text{ and } (\exists x_2 \in [0, 1 - x_1])\}, \sup = \{x_2 \mid x_2 \in [0, 1] \text{ and } (\exists x_1 \in [0, 1 - x_2])\}\} \\ (x_1, x_2) \in A$$

We denote its units by  $0_{L^*} = (0, 1)$  and  $1_{L^*} = (1, 0)$

Note that, if for  $x = (x_1, x_2), y = (y_1, y_2) \in L^*, x_1 \leq y_1$  and  $x_2 \leq y_2$  then  $x$  and  $y$  are incomparable w.r.t.  $\leq_{L^*}$ , denoted as  $x \parallel_{L^*} y$

**Definition 1.19** [10](**t-norm**) An intuitionistic fuzzy triangular norm is an  $(L^*)^2 \rightarrow L^*$  mapping  $\mathcal{T}$  satisfying the following conditions :

1.  $\forall x \in L^*, \mathcal{T}(x, 1_{L^*}) = x$  (border condition)
2.  $\forall (x, y) \in (L^*)^2, \mathcal{T}(x, y) = \mathcal{T}(y, x)$  (commutativity)
3.  $\forall (x, y, z) \in (L^*)^3, \mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$  (associativity)
4.  $\forall (x_1, x_2, y_1, y_2) \in (L^*)^4, x_1 \leq_{L^*} y_1 \text{ and } x_2 \leq_{L^*} y_2 \Rightarrow \mathcal{T}(x_1, x_2) \leq_{L^*} \mathcal{T}(y_1, y_2)$  (monotonicity)

**Example 1.11** [2] We give some basic examples of t-norm and t-conorm, for  $x, y \in L^*$

- $\mathcal{T}_M(x, y) = (\min(x_1, y_1), \max(x_2, y_2))$  (the minimum)
- $\mathcal{T}_L(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_2, y_2 + 1 - x_1))$  (Lukasiewicz)
- $\mathcal{T}(x, y) = (x_1 y_1, x_2 + y_2 - x_2 y_2)$

**Definition 1.20** [10](**t-conorm**) An intuitionistic fuzzy triangular conorm is  $(L^*)^2 \rightarrow L^*$  mapping  $\mathcal{S}$  satisfying the following conditions :

1.  $\forall x \in L^*, \mathcal{S}(x, 0_{L^*}) = x$  (border condition)
2.  $\forall (x, y) \in (L^*)^2, \mathcal{S}(x, y) = \mathcal{S}(y, x)$  (commutativity)
3.  $\forall (x, y, z) \in (L^*)^3, \mathcal{S}(x, \mathcal{S}(y, z)) = \mathcal{S}(\mathcal{S}(x, y), z)$  (associativity)

4.  $\forall (x_1, x_2, y_1, y_2) \in (L^*)^4, x_1 \leq_{L^*} y_1$  and  $x_2 \leq_{L^*} y_2 \Rightarrow \mathcal{S}(x_1, x_2) \leq_{L^*} \mathcal{S}(y_1, y_2)$  (monotonicity)

**Example 1.12** [2]

- $\mathcal{S}_M(x, y) = (\max(x_1, y_1), \min(x_2, y_2))$  (the minimum)
- $\mathcal{S}_L(x, y) = (\min(1, x_1 + 1 - y_2, y_1 + 1 - x_2), \max(0, x_2 + y_2 - 1))$  (Lukasiewicz)

**Proposition 1.3** [2] Let  $a, b \in L^*$  Then

1. If  $0_* \leq_{L^*} a$  and  $0_* \leq_{L^*} b$  then  $0 \leq_{L^*} \mathcal{T}(a, b)$ ,
2.  $0_{L^*} \leq_{L^*} \mathcal{T}_M(a, b)$  then  $(0_{L^*} \leq_{L^*} a$  and  $0_{L^*} \leq_{L^*} b)$
3.  $\mathcal{T}_M(a, a) = a$
4.  $a \leq_{L^*} \mathcal{S}(a, b)$
5.  $\mathcal{T}(a, b) \leq_{L^*} a$
6.  $a = \mathcal{T}_M(a, b)$  if and only if  $a \leq_{L^*} b$

### Some basic operations on intuitionistic fuzzy sets

Let  $X$  be the universal set and let  $A, B$  be two intuitionistic fuzzy sets as follows:

- **Subset** ( $A \subseteq B$ ):

$$A \subseteq B \iff \forall (x \in X) (\mu_A(x) \leq \mu_B(x)) (\nu_A(x) \geq \nu_B(x))$$

- **Equality** ( $A = B$ ):

$$A = B \iff (\forall x \in X) (\mu_A(x) = \mu_B(x)) (\nu_A(x) = \nu_B(x))$$

- **Complement** ( $A^c$ ):

$$A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$$

- **Intervention** ( $A \cap B$ ):

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\}$$

- **Union** ( $A \cup B$ ):

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\}$$

- **Addition** ( $A + B$ ):

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x) \rangle \mid x \in X\}$$

• **Multiplication** ( $A.B$ ):

$$A.B = \{ \langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x) \rangle \mid x \in X \}$$

**Example 1.13** Let  $X = \{a, b, c, d, e\}$ , let the IFSs  $A$  and  $B$  have the forms :

$$A = \{ \langle a, 0.01, 0.4 \rangle, \langle b, 0.02, 0.8 \rangle, \langle c, 0.9, 0.0 \rangle, \langle d, 0.1, 0.03 \rangle, \langle e, 0.0, 1.0 \rangle \}$$

$$B = \{ \langle a, 0.5, 0.3 \rangle, \langle b, 0.1, 0.4 \rangle, \langle c, 0.8, 0.1 \rangle, \langle d, 0.2, 0.4 \rangle, \langle e, 0.2, 0.6 \rangle \}$$

Then

$$A^c = \{ \langle a, 0.4, 0.01 \rangle, \langle b, 0.8, 0.02 \rangle, \langle c, 0.0, 0.9 \rangle, \langle d, 0.03, 0.1 \rangle, \langle e, 1.0, 0.0 \rangle \}$$

$$A \cap B = \{ \langle a, 0.01, 0.4 \rangle, \langle b, 0.1, 0.8 \rangle, \langle c, 0.8, 0.1 \rangle, \langle d, 0.1, 0.4 \rangle, \langle e, 0.0, 0.6 \rangle \}$$

$$A \cup B = \{ \langle a, 0.5, 0.3 \rangle, \langle b, 0.1, 0.4 \rangle, \langle c, 0.9, 0.0 \rangle, \langle d, 0.2, 0.03 \rangle, \langle e, 0.2, 0.6 \rangle \}$$

$$A + B = \{ \langle a, 0.5, 0.12 \rangle, \langle b, 0.11, 0.32 \rangle, \langle c, 0.98, 0.0 \rangle, \langle d, 0.28, 0.012 \rangle, \langle e, 0.2, 0.6 \rangle \}$$

$$A.B = \{ \langle a, 0.005, 0.58 \rangle, \langle b, 0.002, 0.88 \rangle, \langle c, 0.72, 0.1 \rangle, \langle d, 0.02, 0.41 \rangle, \langle e, 0.0, 1.0 \rangle \}$$

**Some properties of intuitionistic fuzzy sets**

The same properties are mentioned in the table 1.2 in addition to these :

<b>Commutativity</b>	$A + B = B + A$ $A.B = B.A$
<b>Associativity</b>	$A + (B + C) = (A + B) + C$ $A.(B.C) = (A.B).C$
<b>Distributivity</b>	$A + (B \cup C) = (A + B) \cup (A + C)$ $A + (B \cap C) = (A + B) \cap (A + C)$ $A.(B \cup C) = (A.B) \cup (A.C)$ $A.(B \cap C) = (A.B) \cap (A.C)$
<b>De Morgan's laws</b>	$(A.B)^c = A^c + B^c$ $(A + B)^c = A^c.B^c$

**Specific sets associated to a intuitionistic fuzzy set**

- **The universel set :**

$$X = \{ \langle x, 1, 0 \rangle \mid x \in X \}$$

- **The empty set :**

$$\emptyset = \{ \langle x, 0, 1 \rangle \mid x \in X \}$$

**Definition 1.21 (The support)** [18] The support of an intuitionistic fuzzy set  $A = (X, \mu_A(x), \nu_A(x))$  is defined as

$$Supp(A) = \{ x \in X : \mu_A(x) \neq 0 \text{ and } \nu_A(x) \neq 1 \}$$

In addition, the support length is  $SL(A) = | Supp(A) |$

**Example 1.14** Let  $X = \{ a, b, c, d \}$  and  $A = \{ \langle a, 0.1, 0.7 \rangle, \langle b, 0.9, 0 \rangle, \langle c, 0.3, 0.01 \rangle, \langle d, 0, 0.4 \rangle \}$   
Then,  $Supp(A) = \{ a, b, c \}$

**Definition 1.22 (the core)** [18] The core of an intuitionistic fuzzy set  $A = (X, \mu_A(x), \nu_A(x))$  is defined as

$$Core(A) = \{ x \in X : \mu_A(x) = 1 \text{ and } \nu_A(x) = 0 \}$$

In addition, the core length is  $CL(A) = | Core(A) |$

**Example 1.15** Let  $X = \{ a, b, c, d \}$  and  $A = \{ \langle a, 0.2, 0.5 \rangle, \langle b, 1, 0 \rangle, \langle c, 0.01, 0.2 \rangle, \langle d, 0.05, 0.5 \rangle \}$   
then,  $Core(A) = \{ b \}$

**Definition 1.23 (The height)** [18] The height of an intuitionistic fuzzy set  $A = (X, \mu_A, \nu_A)$  is defined as:

$$h(A) = \left\{ \sup_{x \in X} (\mu_A(x)), \min_{x \in X} (\nu_A(x)) \right\}$$

**Remark 1.4** [23]

If  $h(A) = \{ 1, 0 \}$  then the IFS A is said to be normal. Otherwise, A is said to be subnormal.

**Example 1.16** Let  $X = \{ a, b, c, d \}$  and  $A, B$  the subset of  $X$

$A = \{ \langle a, 0.02, 0.3 \rangle, \langle b, 1, 0 \rangle, \langle c, 0.2, 0.5 \rangle, \langle d, 0, 1 \rangle \}; h(A) = \{ 1, 0 \}; (A \text{ Normal})$

$B = \{ \langle a, 0.7, 0.01 \rangle, \langle b, 0.7, 0.1 \rangle, \langle c, 0.8, 0 \rangle, \langle d, 0.6, 0.3 \rangle \}; h(B) = \{ 0.8, 0 \}; (B \text{ Subnormal})$

**Definition 1.24 (The  $\alpha$ -cut)** [11] Let X be a universal set and A an IFS of X, let  $\alpha$  be a real number in the interval  $[0, 1]$ , The  $\alpha$ -cut of A is the set of those  $x \in X$  for which  $\mu_A(x) \geq \alpha$  and  $\nu_A(x) \leq 1 - \alpha$ . We will denote it by  $A_\alpha$

**Definition 1.25** [11] For an IFS  $A$  on  $X$ , the strong  $\alpha$ -cut of  $A$  is defined by

$$A_\alpha^+ = \{x \in X \mid \mu_A > \alpha \text{ and } \nu_A < 1 - \alpha\}$$

**Example 1.17** Let  $X = \{a, b, c, d, e, f, g\}$

$$A = \{\langle a, 0, 1 \rangle, \langle b, 0.2, 0.3 \rangle, \langle c, 0.5, 0.1 \rangle, \langle d, 0.3, 0.7 \rangle, \langle e, 0.4, 0.2 \rangle, \langle f, 0.6, 0.4 \rangle, \langle g, 1, 0 \rangle\}$$

$A_\alpha = \{x \in X \mid \mu_A \geq \alpha \text{ and } \nu_A \leq 1 - \alpha\}$ , then:

$$A_{0.6} = \{x \in X \mid \mu_A \geq 0.6 \text{ and } \nu_A \leq 0.4\} = \{\langle f, 0.6, 0.4 \rangle, \langle g, 1, 0 \rangle\}$$

$$A_{0.6}^+ = \{x \in X \mid \mu_A > 0.6 \text{ and } \nu_A < 0.4\} = \{\langle g, 1, 0 \rangle\}$$

**Proposition 1.4** [23] For every IFSs  $A, B$  and for every  $\alpha, \beta \in [0, 1]$  :

1.  $A_\alpha^+ \subseteq A_\alpha$
2.  $Supp(A)^c = X - Core(A)$
3.  $Core(A)^c = X - Supp(A)$
4.  $\alpha \leq \beta \implies A_\beta \subseteq A_\alpha$
5.  $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$
6.  $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$

### 1.5.2 Picture fuzzy sets

**Definition 1.26** [5] A picture fuzzy set  $A$  on a universe  $X$  is an object of the form

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x) \rangle \mid x \in X\}$$

where  $\mu_A(x) \in [0, 1]$  is called the degree of positive membership of  $x \in A$ ,  $\eta_A(x) \in [0, 1]$  is called the degree of neutral membership of  $x$  in  $A$  and  $\nu_A(x) \in [0, 1]$  is called the degree of negative membership of  $x$  in  $A$  and where  $\mu_A, \eta_A$  and  $\nu_A$  satisfy the following condition :

$$\forall x \in X, (\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1)$$

Then  $1 - \mu_A(x) + \eta_A(x) + \nu_A(x)$  could be called the degree of refusal membership of  $x$  in  $A$ .

**Example 1.18** Let us consider picture set  $A$  in  $X = \{a, b, c\}$

$$A = \{\langle a, 0.7, 0.1, 0 \rangle, \langle b, 0.6, 0.2, 0.1 \rangle, \langle c, 0.5, 0.2, 0 \rangle\}$$

**Remark 1.5** [9] Any intuitionistic fuzzy set  $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X\}$  may be identified with the picture fuzzy set in the form  $A = \{\langle x, \mu_A(x), 0, \nu_A(x) \rangle \mid x \in X\}$ .

### Picture fuzzy t-norms and t-conorms

We consider the set  $D^*$  defined by  $D^* = \{x = (x_1, x_2, x_3) \mid x \in [0, 1]^3, x_1 + x_2 + x_3 \leq 1\}$   
Consider the order relation  $\leq_1$  on  $D^*$  defined by: for all  $x \in D^*$

$$x \leq_1 y \iff ((x_1 \leq y_1) \wedge (x_3 \geq y_3)) \vee ((x_1 = y_1) \wedge (x_3 > y_3)) \vee ((x_1 = y_1) \wedge (x_3 = y_3) \wedge (x_2 \leq y_2))$$

We define the first, second and third projection mapping  $pr_1$ ,  $pr_2$  and  $pr_3$  on  $D^*$ , defined as  $pr_1(x) = x_1$ ,  $pr_2(x) = x_2$  and  $pr_3(x) = x_3$ , for all  $x \in D^*$ .

For each  $x, y \in D^*$ , we define

$$\inf(x, y) = \begin{cases} \min(x, y), & \text{if } x \leq_1 y \text{ and } y \leq_1 x \\ (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3), & \text{else} \end{cases}$$

$$\sup(x, y) = \begin{cases} \max(x, y), & \text{if } x \leq_1 y \text{ or } y \leq_1 x \\ (x_1 \wedge y_1, 0, x_3 \vee y_3), & \text{else} \end{cases}$$

Then,  $(D^*, \leq_1)$  is a complete lattice, for each non-empty  $A \subseteq D^*$ , we have

$$\inf A = (\inf pr_1 A, \inf pr_2 A, \inf pr_3 A)$$

where:

$$\begin{aligned} \inf pr_1 A &= \inf \{x_1 \in [0, 1] \mid \exists x = (x_1, x_2, x_3) \in A\} \\ \inf pr_3 A &= \sup \{x_3 \in [0, 1] \mid \exists x = (x_1, x_2, x_3) \in A\} \\ \inf pr_2 A &= \begin{cases} \inf \{x_2 : (\inf pr_1 A, x_2, \inf pr_3 A) \in A\} \\ 1 - \inf pr_1 A - \inf pr_3 A \end{cases} \quad \text{if } (\inf pr_1 A, z, \inf pr_3 A) \in A, \forall z \end{aligned}$$

and  $\sup A = (\sup pr_1 A, \sup pr_2 A, \sup pr_3 A)$ , where :

$$\begin{aligned} \sup pr_1 A &= \sup \{x_1 \in [0, 1] \mid \exists x = (x_1, x_2, x_3) \in A\} \\ \sup pr_3 A &= \inf \{x_3 \in [0, 1] \mid \exists x = (x_1, x_2, x_3) \in A\} \\ \sup pr_2 A &= \begin{cases} \sup \{x_2 : (\sup pr_1 A, x_2, \sup pr_3 A) \in A\} \\ 0 \end{cases} \quad \text{if } (\inf pr_1 A, z, \inf pr_3 A) \in A, \forall z \end{aligned}$$

We denote the units of  $D^*$  by  $1_{D^*} = (1, 0, 0)$  and  $0_{D^*} = (0, 0, 1)$ , if for  $x, y \in D^*$  that neither  $x \leq_1 y$  nor  $y \leq_1 x$ , then  $x$  and  $y$  are incomparable w.r.t.  $\leq_1$ , denoted as  $x \parallel_{\leq_1} y$

Let for each  $x$ , we denote

$$I(x) = \{y \in D^* : y = (x_1, y_2, x_3), 0 \leq y_2 \leq 1 - (x_1 + x_3)\}$$

**Definition 1.27** [7] A picture fuzzy t-norm is an  $(D^*)^2 \rightarrow D^*$  mapping  $\mathcal{T}$  satisfying the following conditions:

- (i)  $\mathcal{T}(x, y) = \mathcal{T}(y, x), \forall x, y \in D^*$  (commutativity)
- (ii)  $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z), \forall x, y, z \in D^*$  (associativity)
- (iii)  $\mathcal{T}(x, y) \leq_1 \mathcal{T}(x, z), \forall x, y, z \in D^*, y \leq_1 z$ , (monotonicity)
- (iv)  $\mathcal{T}(1_{D^*}, x) \in I(x), \forall x \in D^*$  (boundary condition)

**Example 1.19** [7] Some picture fuzzy t-norms, for all  $x, y \in D^*$

$$\mathcal{T}_{inf}(x, y) = \begin{cases} (x_1 \wedge y_1, 1 - (x_1 \wedge y_1) - (x_3 \vee y_3), x_3 \vee y_3), & x \parallel_{\leq_1} y \\ x \wedge y, & \text{otherwise} \end{cases}$$

$$\mathcal{T}_{02}(x, y) = \left( \begin{array}{l} \left( \begin{array}{l} x_1 \wedge y_1, \quad \text{if } x_1 \vee y_1 = 1 \\ 0, \quad \text{if } x_1 \vee y_1 < 1 \end{array} \right), \left( \begin{array}{l} x_2 \vee y_2, \quad \text{if } x_2 \wedge y_2 = 1 \\ 0, \quad \text{if } x_2 \wedge y_2 < 1 \end{array} \right), \left( \begin{array}{l} x_3 \wedge y_3, \quad \text{if } x_3 \vee y_3 = 0 \\ 1 \quad \text{if } x_3 \vee y_3 \neq 0 \end{array} \right) \end{array} \right)$$

$$\mathcal{T}_{min(x,y)} = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3).$$

$$\mathcal{T}(x, y) = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3).$$

$$\mathcal{T}(x, y) = (x_1 y_1, x_2 y_2, x_3 + y_3 - x_3 y_3).$$

**Definition 1.28** [7] A picture fuzzy t-conorm is an  $(D^*)^2 \rightarrow D^*$  mapping  $\mathcal{S}$  satisfying the following conditions:

- (i)  $\mathcal{S}(x, y) = \mathcal{S}(y, x), \forall x, y \in D^*$  (commutativity);
- (ii)  $\mathcal{S}(x, \mathcal{S}(y, z)) = \mathcal{S}(\mathcal{S}(x, y), z), \forall x, y, z \in D^*$  (associativity);
- (iii)  $\mathcal{S}(x, y) \leq_1 \mathcal{S}(x, z), \forall x, y, z \in D^*, y \leq_1 z$ , (monotonicity);
- (iv)  $\mathcal{S}(0_{D^*}, x) \in I(x), \forall x \in D^*$  (boundary condition).

**Example 1.20** [7]

$$\mathcal{S}_{sup}(x, y) = \begin{cases} (x_1 \vee y_1, 0, x_3 \wedge y_3), & x \parallel_{\leq_1} y \\ x \vee y, & \text{otherwise} \end{cases}$$

$$\mathcal{S}_{02}(x, y) = \left( x_1 \vee y_1, \left( \begin{array}{l} x_2 \wedge y_2, \quad \text{if } x_2 \vee y_2 = 1 \\ 0, \quad \text{if } x_2 \vee y_2 < 1 \end{array} \right), x_3 \wedge y_3 \right)$$

$$\mathcal{S}_{03}(x, y) = \left( \begin{array}{l} \left( \begin{array}{l} x_1 \vee y_1 \quad x_1 \wedge y_1 = 0 \\ 1 \quad \text{if } x_1 \wedge y_1 \neq 0 \end{array} \right), x_2 \wedge y_2, \left( \begin{array}{l} x_3 \wedge y_3 \quad \text{if } x_3 \vee y_3 = 1 \\ 0 \quad \text{if } x_3 \vee y_3 < 1 \end{array} \right) \end{array} \right)$$

$$\mathcal{S}_{max}(x, y) = (x_1 \vee y_1, x_2 \wedge y_2, x_3 \wedge y_3).$$

$$\mathcal{S}(x, y) = (x_1 + y_1 - x_1y_1, x_2y_2, x_3y_3).$$

**Proposition 1.5** *For any picture fuzzy t-norm and picture fuzzy t-conorm*

$$1. \mathcal{T}_{02}(x, y) \leq_1 \mathcal{T}(x, y) \leq_1 \mathcal{T}_{min}(x, y), \forall x, y \in D^*$$

$$2. \mathcal{S}_{02}(x, y) \leq_1 \mathcal{S}(x, y) \leq_1 \mathcal{S}_{03}(x, y), \forall x, y \in D^*$$

### Operations over picture fuzzy sets

Let  $X$  be a universal set and  $A, B$  two picture fuzzy sets of  $X$ , the consequent operations of PFSs are given as follows:

- **The subset** ( $A \subseteq B$ ):

$$A \subseteq B \iff \forall (x \in X) (\mu_A(x) \leq \mu_B(x), \eta_A(x) \leq \eta_B(x) \text{ and } \nu_A(x) \geq \nu_B(x))$$

- **The equality** ( $A = B$ ):

$$A = B \iff (\forall x \in X) (A \subseteq B \text{ and } B \subseteq A)$$

- **The Complement** ( $A^c$ ):

$$A^c = \{\langle x, \nu_A(x), \eta_A(x), \mu_A(x) \rangle \mid x \in X\}$$

- **The union** ( $A \cup B$ ):

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\}$$

- **The intersection** ( $A \cap B$ ):

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\}$$

- **Addition** ( $A + B$ ):

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \eta_A(x)\eta_B(x), \nu_A(x)\nu_B(x) \rangle \mid x \in X\}$$

- **Multiplication** ( $A.B$ ):

$$A.B = \{\langle x, \mu_A(x)\mu_B(x), \eta_A(x) + \eta_B(x) - \eta_A(x)\eta_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x) \rangle \mid x \in X\}$$

For more information, see [16]

**Example 1.21** Let  $X$  be a universal set and  $A, B$  be two PFSs in  $X$  have the forms:

$$X = \{x_1, x_2, x_3\}$$

and

$$A = \{\langle x_1, 0.3, 0.3, 0.2 \rangle, \langle x_2, 0.7, 0.1, 0.1 \rangle, \langle x_3, 0.4, 0.3, 0.2 \rangle\}$$

$$B = \{\langle x_1, 0.3, 0.4, 0.1 \rangle, \langle x_2, 0.6, 0.2, 0.1 \rangle, \langle x_3, 0.4, 0.3, 0.1 \rangle\}$$

then

$$A^c = \{\langle x_1, 0.2, 0.3, 0.3 \rangle, \langle x_2, 0.1, 0.1, 0.7 \rangle, \langle x_3, 0.2, 0.3, 0.4 \rangle\}$$

$$B^c = \{\langle x_1, 0.1, 0.4, 0.3 \rangle, \langle x_2, 0.1, 0.2, 0.6 \rangle, \langle x_3, 0.1, 0.3, 0.4 \rangle\}$$

$$A \cup B = \{\langle x_1, 0.3, 0.3, 0.1 \rangle, \langle x_2, 0.7, 0.1, 0.1 \rangle, \langle x_3, 0.4, 0.3, 0.1 \rangle\}$$

$$A \cap B = \{\langle x_1, 0.3, 0.3, 0.2 \rangle, \langle x_2, 0.6, 0.1, 0.1 \rangle, \langle x_3, 0.4, 0.3, 0.2 \rangle\}$$

$$A + B = \{\langle x_1, 0.51, 0.12, 0.02 \rangle, \langle x_2, 0.88, 0.02, 0.01 \rangle, \langle x_3, 0.64, 0.09, 0.02 \rangle\}$$

$$A.B = \{\langle x_1, 0.09, 0.58, 0.28 \rangle, \langle x_2, 0.42, 0.28, 0.19 \rangle, \langle x_3, 0.16, 0.51, 0.28 \rangle\}$$

### Specific sets associated to a picture fuzzy sets

- The universal set:

$$X = \{\langle x, 1, 0, 0 \rangle \mid x \in X\}$$

- The empty set:

$$\emptyset = \{\langle x, 0, 0, 1 \rangle \mid x \in X\}$$

**Definition 1.29 (The height)** [8] Let  $A$  be a PFS of  $X$  then height is defined as:

$$h(A) = \left\{ \sup_{x \in X} (\mu_A(x)), \min_{x \in X} (\eta_A(x)), \min_{x \in X} (\nu_A(x)) \right\}$$

**Example 1.22** Let  $X$  be a universal set and  $A$  be PFS in  $X$  have the forms:

$$X = \{u_1, u_2, u_3\}$$

$$A = \{\langle u_1, 0.4, 0.2, 0.3 \rangle, \langle u_2, 0.7, 0.1, 0.1 \rangle, \langle u_3, 0.6, 0.2, 0.2 \rangle\}$$

$$h(A) = \{0.7, 0.1, 0.1\}$$

**Definition 1.30 (The  $\alpha$ -cut)** [9] Let  $A$  be a picture fuzzy set of the set  $X$ , let  $\alpha$  be real number in the interval  $[0, 1]$ . The  $\alpha$ -cut of  $A$  (or level  $\alpha$  of  $A$ ) is the crisp set  $A_\alpha$  defined by:

$$A_\alpha = \{x \in X : \mu_A(x) + \eta_A(x) \geq \alpha \text{ and } \nu_A(x) \leq 1 - \alpha\}$$

**Example 1.23**  $A = \{\langle a, 0.8, 0.05, 0.1 \rangle, \langle b, 0.7, 0.1, 0.2 \rangle, \langle c, 0.05, 0.01, 0.4 \rangle\}$  is a picture fuzzy set on the universe  $X = \{a, b, c\}$ . Then

$$A_{0.2} = \{x \in X \mid \mu_A(x) + \eta_A(x) \geq 0.2 \text{ and } \nu_A(x) \leq 0.8\} = \{a, b\}$$

**Proposition 1.6** Let  $A$  be a PFS on  $X$ . Then the  $\alpha$ -cut  $A_\alpha$  is a PFS and  $A \subset A_\alpha$

### 1.5.3 Spherical fuzzy sets

**Definition 1.31** [13] Spherical fuzzy sets (SFS)  $A$  of the universe  $X$  is given by:

$$A = \{\langle x, \mu_A(x), \nu_A(x), \pi_A(x) \rangle \mid x \in X\}$$

where  $\mu_A : X \rightarrow [0, 1]$ ,  $\nu_A : X \rightarrow [0, 1]$ ,  $\pi_A : X \rightarrow [0, 1]$  and

$$0 \leq (\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 \leq 1, \forall x \in X$$

For each  $x$ , the numbers  $\mu_A(x)$ ,  $\nu_A(x)$  and  $\pi_A(x)$  are degree of membership, non membership and hesitancy of  $x$  to  $A$ , respectively.

**Example 1.24** Let  $X = \{a, b, c, d, e\}$  be the universal set and  $A$  be spherical fuzzy set of  $X$  have the form:

$$A = \{\langle a, 0.7, 0.1, 0 \rangle, \langle b, 0.5, 0.2, 0.1 \rangle, \langle c, 0.3, 0.2, 0.8 \rangle, \langle d, 0.4, 0.3, 0.2 \rangle, \langle e, 0.2, 0.1, 0.9 \rangle\}$$

#### Operations over the spherical fuzzy sets

Let  $X$  be a universal set and  $A, B$  two spherical fuzzy sets of  $X$ , the consequent operations of SFS are given as follows :

- **Union** ( $A \cup B$ ):

$$A \cup B = \{\langle x, \max\{\mu_A, \mu_B\}, \min\{\nu_A, \nu_B\}, \max\{1 - ((\max\{\mu_A, \mu_B\})^2 + (\min\{\nu_A, \nu_B\})^2), \max\{\pi_A, \pi_B\}\}\rangle\}$$

- **Intersection**( $A \cap B$ ):

$$A \cap B = \{ \langle x, \min\{\mu_A, \mu_B\}, \max\{\nu_A, \nu_B\}, \min\{1 - ((\min\{\mu_A, \mu_B\})^2 + (\max\{\nu_A, \nu_B\})^2), \min\{\pi_A, \pi_B\}\} \rangle \}$$

- **Addition** ( $A + B$ ):

$$A + B = \{ \langle x, (\mu_A^2 + \mu_B^2 - \mu_A^2 \mu_B^2)^{1/2}, \nu_A \nu_B, ((1 - \mu_B^2)\pi_A^2 + (1 - \mu_A^2)\pi_B^2 - \pi_A^2 \pi_B^2)^{1/2} \rangle \}$$

- **Multiplication**( $A.B$ ):

$$A.B = \{ \langle x, \mu_A \mu_B, (\nu_A^2 + \nu_B^2 - \nu_A^2 \nu_B^2)^{1/2}, ((1 - \nu_B^2)\pi_A^2 + (1 - \nu_A^2)\pi_B^2 - \pi_A^2 \pi_B^2)^{1/2} \rangle \}$$

**Example 1.25** Let  $X = \{a, b, c\}$  be the universal set and  $A, B$  two spherical fuzzy sets of  $X$  represented as follows:

$$A = \{ \langle a, 0.5, 0.6, 0.2 \rangle, \langle b, 0.8, 0.4, 0.3 \rangle, \langle c, 0.1, 0.3, 0.6 \rangle \}$$

$$B = \{ \langle a, 0.2, 0.5, 0.6 \rangle, \langle b, 0.9, 0, 0.2 \rangle, \langle c, 0.7, 0.4, 0.3 \rangle \}$$

then

$$A \cup B = \{ \langle a, 0.5, 0.5, 0.6 \rangle, \langle b, 0.9, 0, 0.3 \rangle, \langle c, 0.7, 0.3, 0.6 \rangle \}$$

$$A \cap B = \{ \langle a, 0.2, 0.6, 0.2 \rangle, \langle b, 0.8, 0.4, 0.2 \rangle, \langle c, 0.1, 0.4, 0.3 \rangle \}$$

$$A + B = \{ \langle a, 0.52, 0.3, 0.246 \rangle, \langle b, 0.96, 0, 0.06 \rangle, \langle c, 0.49, 0.12, 0.24 \rangle \}$$

$$A.B = \{ \langle a, 0.1, 0.72, 0.12 \rangle, \langle b, 0.72, 0.16, 0.34 \rangle, \langle c, 0.07, 0.23, 0.59 \rangle \}$$

#### 1.5.4 Pythagorean fuzzy sets

**Definition 1.32** [12] Let  $X$  be a universal set. Then a Pythagorean fuzzy set  $A$ , which is a set of ordered pairs over  $X$ , is defined by the following:

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$$

where the functions

$$\mu_A : X \rightarrow [0, 1] \text{ and } \nu_A : X \rightarrow [0, 1]$$

define the degree of membership and the degree of nonmembership, respectively, of the element  $x \in X$  to  $A$  which is a subset of  $X$ , and for every  $x \in X$ :

$$0 \leq (\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$$

Supposing  $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$  then there is a degree of indeterminacy of  $x \in X$  to  $A$  defined by  $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\nu_A(x))^2]}$  and  $\pi_A(x) \in [0, 1]$ . In what follows,  $(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1$ . Otherwise,  $\pi_A(x) = 0$  whenever  $(\mu_A(x))^2 + (\nu_A(x))^2 = 1$

We denote the set of all PFSs over  $X$  by  $PFS(X)$

**Example 1.26** [12] Let  $A \in PFS(X)$ . Suppose that  $\mu_A(x) = 0.7$  and  $\nu_A(x) = 0.5$  for  $X = \{x\}$ . Clearly,  $0.7 + 0.5 \not\leq 1$  but  $(0.7)^2 + (0.5)^2 \leq 1$ . Thus,  $\pi_A(x) = 0.5099$ , and hence,  $(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1$ .

### Operations over pythagorean fuzzy sets

Let  $X$  be a universal set and  $A, B$  two pythagorean fuzzy sets of  $X$ , the consequent operations of (PYFS) are given as follows:

- **Subset** ( $A \subseteq B$ ):

$$A \subseteq B \iff \forall(x \in X)(\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x))$$

- **Equality** ( $A = B$ ):

$$A = B \iff (\forall x \in X)(\mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x))$$

- **Complement** ( $A^c$ ):

$$A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$$

- **Union** ( $A \cup B$ ):

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\}$$

- **Intersection** ( $A \cap B$ ):

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\}$$

- **Addition**(A+B):

$$A + B = \left\{ \left\langle x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \nu_A(x)\nu_B(x) \right\rangle \mid x \in X \right\}$$

- **Multiplication**(A.B):

$$A.B = \left\{ \left\langle x, \mu_A(x), \mu_B(x), \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \right\rangle \mid x \in X \right\}$$

**Example 1.27** Let  $A, B \in PFS(X)$  for  $X = \{x_1, x_2, x_3\}$ . Suppose that

$$A = \{\langle x_1, 0.6, 0.2 \rangle, \langle x_2, 0.4, 0.6 \rangle, \langle x_3, 0.5, 0.3 \rangle\}$$

$$B = \{\langle x_1, 0.8, 0.1 \rangle, \langle x_2, 0.7, 0.3 \rangle, \langle x_3, 0.6, 0.1 \rangle\}$$

Then

$$A^c = \{\langle x_1, 0.2, 0.6 \rangle, \langle x_2, 0.6, 0.4 \rangle, \langle x_3, 0.3, 0.5 \rangle\}$$

$$A \cup B = \{\langle x_1, 0.8, 0.1 \rangle, \langle x_2, 0.7, 0.3 \rangle, \langle x_3, 0.6, 0.1 \rangle\}$$

$$A \cap B = \{\langle x_1, 0.6, 0.2 \rangle, \langle x_2, 0.4, 0.6 \rangle, \langle x_3, 0.5, 0.3 \rangle\}$$

$$A + B = \{\langle x_1, 0.87, 0.02 \rangle, \langle x_2, 0.75, 0.18 \rangle, \langle x_3, 0.72, 0.03 \rangle\}$$

$$A.B = \{\langle x_1, 0.48, 0.22 \rangle, \langle x_2, 0.28, 0.64 \rangle, \langle x_3, 0.3, 0.31 \rangle\}$$

# Chapter 2

## Multi-fuzzy sets

In this chapter, we will use the following notations.  $X, N, I$  and  $I^X$  stand for; a nonempty set called a universal set, the set of all natural numbers, the unit interval  $[0, 1]$  and the set of all functions from  $X$  to  $I$  respectively.  $I^k$  stands for  $I \times I \times \dots \times I$  ( $k$ -times), for any positive integer  $k$ .

### 2.1 Generalities on multi-fuzzy sets

In this section, we introduce basic notions of the multi-fuzzy sets and operations of multi-fuzzy. Our definition of multi-fuzzy set is different from other definitions mentioned in the chapter 1.

#### 2.1.1 Definitions and examples

**Definition 2.1** [14] Let  $X$  be a nonempty set,  $N$  the set of all natural numbers and  $\{L_i : i \in N\}$  a family of complete lattices. A multi-fuzzy set  $A$  in  $X$  is a set of ordered sequences

$$A = \{\langle x, \mu_1(x), \mu_2(x), \dots, \mu_i(x), \dots \rangle : x \in X\} \quad (2.1)$$

where  $\mu_i \in L_i^X$ , for  $i \in N$ .

**Notation 2.1** Let  $L_i = L$ ; for  $i = 1, \dots, k$ .  $(\prod L_i)^X = (L^X)^K$ , the set of all multi-fuzzy membership functions of multi-fuzzy sets in  $X$  of dimension  $k$ .

**Remark 2.1** [6] The function  $\mu_A = \langle \mu_1, \mu_2, \dots \rangle$  is called a multi-membership function of multi-fuzzy set  $A$ . If the sequences of the membership functions have only  $k$ -terms (finite number of

terms),  $k$  is called the dimension of  $A$ . Let  $L_i = [0, 1]$  (for  $i = 1, 2, \dots, k$ ), then the set of all multi-fuzzy sets in  $X$  of dimension  $k$  is denoted by  $M^kFS(X)$ .

**Example 2.1** [19] For this we define multi-fuzzy membership functions with linearly dependent infinite coordinates. That is,

$$A = \left\{ \langle x, \mu_1(x), \mu_2(x), \dots, \mu_i(x), \dots \rangle : x \in X, \sum_{i=1}^{\infty} \mu_i(x) = 1 \right\}$$

where  $\mu_i \in I^X$  (for  $i = 1, 2, \dots$ ). Here  $\mu_1(x)$  and  $\nu_2(x)$  are the membership and nonmembership values of  $x$  in  $A$  and  $\pi(x) = \langle \mu_3(x), \mu_4(x), \dots \rangle$  is the membership value of non-specificity or non-determinacy of  $x$ . Clearly  $\pi(x)$  itself is a multi-fuzzy membership function of infinite dimension.

**Example 2.2** Let  $A$  multi-fuzzy set in  $X$  of dimension 2 with value domain  $L_1 \times L_2$  where  $L_1 = L_2 = [0, 1]$  represented as follows:

$$X = \{x_1, x_2\}$$

$$A = \{\langle x_1, 0.1, 0.8 \rangle, \langle x_2, 0.3, 0.6 \rangle\}$$

**Example 2.3** [19] Suppose a color image is approximated by an  $m \times n$  matrix of pixels. Let  $X$  be the set of all pixels of the color image. For any pixel  $x \in X$ , the membership values  $\mu_r(x), \mu_g(x), \mu_b(x)$  being the normalized red value, green value and blue value of the pixel  $x$  respectively. So the color image can be approximated by the collection of pixels with the multi-membership function  $(\mu_r(x), \mu_g(x), \mu_b(x))$  and it can be represented as a multi-fuzzy set

$$A = \{\langle x, \mu_r(x), \mu_g(x), \mu_b(x) \rangle : x \in X\}$$

In a two dimensional image, color of pixels cannot be characterized by a membership function of an ordinary fuzzy set, but it can be characterized by a three dimensional membership function  $(\mu_r(x), \mu_g(x), \mu_b(x))$ . In fact, a multi-fuzzy set can be understood to be a more general fuzzy set using ordinary fuzzy sets as its building blocks.

**Remark 2.2** [24]

- (i) If  $\sum_{i=1}^k \mu_i(x) \leq 1 \forall x \in X$ , then the multi-fuzzy set of dimension  $k$  is called a normalized multi-fuzzy set. If  $\sum_{i=1}^k \mu_i(x) = l > 1$ , for some  $x \in X$ , we redefine the multi-membership degree  $(\mu_1(x), \mu_2(x), \dots, \mu_k(x))$ , then the non-normalized multi-fuzzy set can be changed into a normalized multi-fuzzy set
- (ii) **The absolute multi-fuzzy set:** Let  $A \in M^k FS(X)$  if  $A = \{\langle x, 1, 1, 1, \dots, 1 \rangle : x \in X\}$  then  $A$  is called the absolute multi-fuzzy set of dimension  $k$ , denoted by  $1_k$
- (iii) **The null multi-fuzzy set:** Let  $A \in M^k FS(X)$  if  $A = \{\langle x, 0, 0, 0, \dots, 0 \rangle : x \in X\}$  then  $A$  is called the null multi-fuzzy set of dimension  $k$ , denoted by  $\emptyset_k$

### 2.1.2 Operations on multi-fuzzy sets and their $\alpha$ -cuts

Let  $\{L_i : i \in N\}$  be a family of complete lattices

$$A = \{\langle x, \mu_1, \dots, \mu_i(x) \rangle : x \in X, \mu_i \in L_i^X, i \in N\}$$

and

$$B = \{\langle x, \nu_1, \dots, \nu_i(x) \rangle : x \in X, \nu_i \in L_i^X, i \in N\}$$

be multi-fuzzy sets in a non-empty set  $X$ . We define the following relations and operations:

- **Subset**( $A \sqsubseteq B$ ):

$$A \sqsubseteq B \iff \mu_i(x) \leq \nu_i(x), \forall x \in X \text{ and } \forall i \in N$$

- **Equality**( $A = B$ ):

$$A = B \iff \mu_i(x) = \nu_i(x), \forall x \in X \text{ and } \forall i \in N$$

- **Classical complement**( $A^c$ ):

$$\begin{aligned} A^c &= \{\langle x, (1 - \mu_i(x))_{i \in N} \rangle : x \in X\} \\ &= \{\langle x, 1 - \mu_1(x), 1 - \mu_2(x), \dots, 1 - \mu_i(x), \dots \rangle : x \in X\} \end{aligned}$$

- **Union**( $A \sqcup B$ ):

$$A \sqcup B = \{\langle x, \mu_1(x) \vee \nu_1(x), \mu_2(x) \vee \nu_2(x), \dots, \mu_i(x) \vee \nu_i(x), \dots \rangle : x \in X\}$$

- **Intersection**( $A \sqcap B$ ):

$$A \sqcap B = \{ \langle x, \mu_1(x) \wedge \nu_1(x), \mu_2(x) \wedge \nu_2(x), \dots, \mu_i(x) \wedge \nu_i(x), \dots \rangle : x \in X \}$$

- **Addition**( $A + B$ ):

$$A + B = \{ \langle x, \mu_1(x) + \nu_1(x) - \mu_1(x) \cdot \nu_1(x), \dots, \mu_i(x) + \nu_i(x) - \mu_i(x) \cdot \nu_i(x), \dots \rangle : x \in X \}$$

- **Multiplication**( $A \cdot B$ ):

$$A \cdot B = \{ \langle x, \mu_1(x) \cdot \nu_1(x), \mu_2(x) \cdot \nu_2(x), \dots, \mu_i(x) \cdot \nu_i(x), \dots \rangle : x \in X \}$$

**Example 2.4** Let  $X$  be a universal set  $A, B$  be two multi-fuzzy sets in  $X$  of dimension 6 have the forms:

$$X = \{x_1, x_2, x_3\}$$

and

$$A = \{ \langle x_1, 0.5, 0.8, 0.6, 0.7, 0.4 \rangle, \langle x_2, 0.2, 0.1, 0.3, 0.5, 0.6 \rangle, \langle x_3, 0.3, 0.4, 0.1, 0.5, 0.4 \rangle \}$$

$$B = \{ \langle x_1, 0.3, 0.1, 0.2, 0.4, 0.6 \rangle, \langle x_2, 0.5, 0.7, 0.6, 0.4, 0.8 \rangle, \langle x_3, 0.2, 0.3, 0.5, 0.4, 0.6 \rangle \}$$

then :

$$A^c = \{ \langle x_1, 0.5, 0.2, 0.4, 0.3, 0.6 \rangle, \langle x_2, 0.8, 0.9, 0.7, 0.5, 0.4 \rangle, \langle x_3, 0.7, 0.6, 0.9, 0.5, 0.6 \rangle \}$$

$$A \sqcup B = \{ \langle x_1, 0.5, 0.8, 0.6, 0.7, 0.6 \rangle, \langle x_2, 0.5, 0.7, 0.6, 0.5, 0.8 \rangle, \langle x_3, 0.3, 0.4, 0.5, 0.5, 0.6 \rangle \}$$

$$A \sqcap B = \{ \langle x_1, 0.3, 0.1, 0.2, 0.4, 0.4 \rangle, \langle x_2, 0.2, 0.1, 0.3, 0.4, 0.6 \rangle, \langle x_3, 0.2, 0.3, 0.1, 0.4, 0.4 \rangle \}$$

$$A+B = \{ \langle x_1, 0.65, 0.82, 0.68, 0.82, 0.76 \rangle, \langle x_2, 0.6, 0.73, 0.72, 0.7, 0.92 \rangle, \langle x_3, 0.44, 0.58, 0.55, 0.7, 0.76 \rangle \}$$

$$A \cdot B = \{ \langle x_1, 0.15, 0.08, 0.12, 0.28, 0.24 \rangle, \langle x_2, 0.1, 0.07, 0.18, 0.2, 0.48 \rangle, \langle x_3, 0.06, 0.12, 0.05, 0.2, 0.24 \rangle \}$$

**The  $\alpha$ - cut:**

**Definition 2.2** [20] For every  $A \in (\Pi L_i)^X$  and  $\alpha \in \Pi L_i$  an  $\alpha$ -cut of  $A$  is the crisp set

$$A_{[\alpha]} = \{x \in X : \alpha \leq \mu_A(x), \alpha \in \Pi L_i\}$$

**Properties of  $\alpha$ - cut of multi-fuzzy sets :**

If  $A, B \in (\prod L_i)^X$  and  $\alpha, \beta \in \prod L_i$  :

1.  $\alpha \leq \beta$  implies  $A_{[\beta]} \subseteq A_{[\alpha]}$
2.  $A \sqsubseteq B$  if and only if  $A_{[\alpha]} \subseteq B_{[\alpha]}$
3.  $A = B$  if and only if  $A_{[\alpha]} = B_{[\alpha]}$
4.  $(A \sqcap B)_{[\alpha]} = A_{[\alpha]} \cap B_{[\alpha]}$
5.  $A_{[\alpha]} \cup B_{[\alpha]} \subseteq (A \sqcup B)_{[\alpha]}$

## 2.2 Extended operations on multi-fuzzy sets based t-norms and t-conorms

In this section, we introduce the operations  $\mathcal{S}, \mathcal{T}$  and  $\mathcal{M}$  on multi-fuzzy sets. In this section,  $A, B$  and  $C$  will denote the following multi-fuzzy sets in  $X$  of dimension  $k$  (where  $k$  is a fixed positive integer) and  $L_i = [0, 1]$  for  $i= 1, \dots, k$ .

$$A = \{ \langle \mu_1(x), \mu_2(x), \dots, \mu_k(x) \rangle : x \in X \}$$

$$B = \{ \langle \nu_1(x), \nu_2(x), \dots, \nu_k(x) \rangle : x \in X \}$$

$$C = \{ \langle \eta_1(x), \eta_2(x), \dots, \eta_k(x) \rangle : x \in X \}$$

**Definition 2.3** [19] Let  $A$  and  $B$  be multi-fuzzy sets in  $X$ . The  $\mathcal{T}$  operations,  $\mathcal{S}$  operations and  $\mathcal{M}$  operations can be defined as:

$$\mathcal{T}(A, B) = \{ \langle x, t_1(\mu_1(x), \nu_1(x)), t_2(\mu_2(x), \nu_2(x)), \dots, t_n(\mu_n(x), \nu_n(x)), \dots \rangle : x \in X \}$$

$$\mathcal{S}(A, B) = \{ \langle x, s_1(\mu_1(x), \nu_1(x)), s_2(\mu_2(x), \nu_2(x)), \dots, s_n(\mu_n(x), \nu_n(x)), \dots \rangle : x \in X \}$$

$$\mathcal{M}(A, B) = \{ \langle x, m_1(\mu_1(x), \nu_1(x)), m_2(\mu_2(x), \nu_2(x)), \dots, m_n(\mu_n(x), \nu_n(x)), \dots \rangle : x \in X \}$$

where  $n$  is a positive integer ;  $t_i, s_i$  and  $m_i$  are different t-norm, s-norm and aggregation operation from  $I \times I$  into  $I$  respectively, for  $1 \leq i, j, l \leq k$ .

**Example 2.5** Let  $A$  and  $B$  be multi-fuzzy sets in  $X$  of dimension  $k$ . Hamacher  $\mathcal{T}$  operations and  $\mathcal{S}$  operations of  $A$  and  $B$  are:

$$\mathcal{T}^H(A, B) = \{ \langle x, t_{r_1}(\mu_1(x), \nu_1(x)), \dots, t_{r_k}(\mu_k(x), \nu_k(x)) \rangle : x \in X \}$$

where 
$$t(a, b) = \frac{ab}{r + (1-r)(a+b-ab)}, r > 0;$$

$$\mathcal{S}^H(A, B) = \{ \langle x, s_{r_1}(\mu_1(x), \nu_1(x)), \dots, s_{r_k}(\mu_k(x), \nu_k(x)) \rangle : x \in X \}$$

where 
$$s(a, b) = \frac{a+b+(r-2)ab}{1+(r-1)ab}, r > 0;$$

**Theorem 2.1** [19] Let  $A, B$  and  $C$  be multi-fuzzy sets in  $X$  of dimension  $k$ .

(a)  $\mathcal{T}(A, \mathbf{1}) = A$  where  $\mathbf{1} = \{ \langle x, 1, 1, \dots, 1 \rangle : x \in X \}$

(b)  $\mathcal{T}(A, B) = \mathcal{T}(A, B)$

(c)  $\mathcal{T}(A, \mathcal{T}(B, C)) = \mathcal{T}(\mathcal{T}(A, B), C)$

(d)  $B \sqsubseteq C$  implies  $\mathcal{T}(A, B) \sqsubseteq \mathcal{T}(A, C)$

**Proof**

Proof (a)

$$\begin{aligned} \mathcal{T}(A, \mathbf{1}) &= \{ \langle x, t_1(\mu_1(x), 1), t_2(\mu_2(x), 1), \dots, t_k(\mu_k(x), 1) \rangle : x \in X \} \\ &= \{ \langle x, \mu_1(x), \mu_2(x), \dots, \mu_k(x) \rangle : x \in X \} \\ &= A \end{aligned}$$

Proofs of (b), (c) and (d) are similar to (a). ■

**Remark 2.3** Similarly,  $\mathcal{S}$  operations are commutative, associative and non-decreasing operations with  $\mathcal{S}(A, \mathbf{0}) = A$  where  $\mathbf{0} = \{ \langle x, 0, 0, \dots, 0 \rangle : x \in X \}$ .

In the remaining part, we will use  $t_i$  and  $s_i$  as different t-norm and s-norm respectively, for  $1 \leq i, j \leq k$ . Here we introduce the notion of p-conjugates of  $\mathcal{T}$  and  $\mathcal{S}$  Operations, which are very useful links between multi-fuzzy sets and Atanassov intuitionistic fuzzy sets  $s_p$  be the dual of t-norm  $t_p$  with respect to the fuzzy complement operation  $c$ . For a fixed.

**Definition 2.4** [19] Let  $A$  and  $B$  be multi-fuzzy sets in  $X$  of dimension  $k$  and let s-norm  $s_p$  be the dual of t-norm  $t_p$  with respect to the fuzzy complement operation  $c$ . For a fixed  $p \in \{1, 2, \dots, k\}$ , the p-conjugate of  $\mathcal{T}(A, B)$  with respect to  $c$  is the multi-fuzzy set  $\mathcal{T}_p^c(A, B)$  obtained from  $\mathcal{T}(A, B)$  by replacing  $t_p(\mu_p(x), \nu_p(x))$  as  $s_p(\mu_p(x), \nu_p(x))$  and the remaining coordinates kept to be the some as in  $\mathcal{T}(A, B)$ . That is

$$\mathcal{T}(A, B)_p^c = \{\langle x, t_1(\mu_1(x), \nu_1(x)), \dots, t_{p-1}(\mu_{p-1}(x), \nu_{p-1}(x)), s_p(\mu_p(x), \nu_p(x)), t_{p+1}(\mu_{p+1}(x), \nu_{p+1}(x)), \dots \rangle : x \in X\}$$

**Definition 2.5** [19] Let  $A$  and  $B$  be multi-fuzzy sets in  $X$  of dimension  $k$  and let t-norm  $t_p$  be the dual of s-norm  $s_p$  with respect to the fuzzy complement operation  $c$ . For a fixed  $p \in \{1, 2, \dots, k\}$ , the p-conjugate of  $\mathcal{S}(A, B)$  with respect to  $c$  is the multi-fuzzy set  $\mathcal{S}_p^c(A, B)$  obtained from  $\mathcal{S}(A, B)$  by replacing  $s_p(\mu_p(x), \nu_p(x))$  as  $t_p(\mu_p(x), \nu_p(x))$  and the remaining coordinates kept to be the some as in  $\mathcal{T}(A, B)$ . That is

$$\mathcal{S}(A, B)_p^c = \{\langle x, s_1(\mu_1(x), \nu_1(x)), \dots, s_{p-1}(\mu_{p-1}(x), \nu_{p-1}(x)), t_p(\mu_p(x), \nu_p(x)), s_{p+1}(\mu_{p+1}(x), \nu_{p+1}(x)), \dots \rangle : x \in X\}$$

**Example 2.6** Let  $A$  and  $B$  be multi-fuzzy sets in  $X$  of dimension 2,  $c$  be the classical fuzzy complement. Let

$$\mathcal{S}(A, B) = \{\langle x, \mu_1(x) + \nu_1(x) - \mu_1(x).\nu_1(x), \mu_2(x) + \nu_2(x) - \mu_2(x).\nu_2(x) \rangle : x \in X\}$$

Then

$$\mathcal{S}_p^c(A, B) = \{\langle x, \mu_1(x) + \nu_1(x) - \mu_1(x).\nu_1(x), \mu_2(x).\nu_2(x) \rangle : x \in X\}$$

**Remark 2.4** q-conjugate of  $\mathcal{T}_p^c(A, B)$  with respect to the fuzzy complement operation  $d$  is denoted by  $\mathcal{T}_{(p,q)}^{(c,d)}(A, B)$  for every  $p, q \in \{1, 2, \dots, k\}$ . That is

$$\mathcal{T}_{(p,q)}^{(c,d)}(A, B) = \mathcal{T}_q^d(\mathcal{T}_p^c(A, B))$$

**Proposition 2.1** Let  $A, B$  be multi-fuzzy sets in  $X$  of dimension  $k$  and let  $\mathcal{T}, \mathcal{S}$  be fixed operators. If  $p_1 p_2$  and  $p_3$  are distinct integers in  $\{1, 2, \dots, k\}$ , then

1.  $\mathcal{T}_{(p,p)}^{(c,c)}(A, B) = \mathcal{T}(A, B)$
2.  $\mathcal{T}_{(p_1,p_2)}^{(c_1,c_2)}(A, B) = \mathcal{T}_{(p_1,p_2)}^{(c_1,c_2)}(B, A)$
3.  $\mathcal{T}_{((p_1,p_2),p_3)}^{((c_1,c_2),c_3)}(A, B) = \mathcal{T}_{(p_1,(p_2,p_3))}^{(c_1,(c_2,c_3))}(B, A)$

where  $c_1, c_2, c_3$  and  $c$  are fuzzy complement operations.

### Proof

$\mathcal{T}_p^c(A, B)$  is obtained from  $\mathcal{T}(A, B)$  by replacing  $t_p(\mu_p(x), \nu_p(x))$  as  $s_p(\mu_p(x), \nu_p(x))$  and remaining coordinates are unchanged. Similarly  $\mathcal{T}_{(p,p)}^{(c,c)}(A, B)$  is obtained from  $\mathcal{T}_p^c(A, B)$  by replacing  $s_p(\mu_p(x), \nu_p(x))$  as  $t_p(\mu_p(x), \nu_p(x))$  and remaining coordinates are unchanged. Hence

$$\mathcal{T}_{(p,p)}^{(c,c)}(A, B) = \mathcal{T}(A, B)$$

(c) and (b) similar to the proof of (a). ■

## 2.3 Special cases of multi-fuzzy sets

### (i) Fuzzy sets as multi-fuzzy sets of dimension 1: [22]

Clearly, a multi-fuzzy set  $A$  of  $X$  with  $L_1 = [0, 1]$  have the form  $A = \{\langle x, \mu_A(x) \rangle : x \in X\}$  is multi-fuzzy set of dimension 1. If  $\mu_A(x)$  is the degree membership of  $x$  in  $A$ , then  $A$  is a Zadeh fuzzy set. That is every fuzzy set in  $X$  is a multi-fuzzy set in  $X$  of dimension 1.

**Example 2.7** Let  $X = \{x_1, x_2, x_3\}$  be a universal set, and  $A$  be a multi-fuzzy set have the form :

$$A = \{\langle x_1, 0.1 \rangle, \langle x_2, 0.5 \rangle, \langle x_3, 0.6 \rangle\}$$

### (ii) Intuitionistic fuzzy sets as multi-fuzzy sets of dimension 2 :

Consider multi-fuzzy set  $A$  of  $X$  with value domain  $L_1 \times L_2$  where  $L_1 = L_2 = [0, 1]$ . That is,  $A = \{\langle x, \mu_1(x), \mu_2(x) \rangle : x \in X\} = (\mu_1(x), \mu_2(x))$  is multi-fuzzy set of dimension 2. If  $\mu_1(x)$  and  $\mu_2(x)$  are the membership and nonmembership degrees of  $x$  in  $A$  respectively and if  $\mu_1(x) + \mu_2(x) \leq 1, \forall x \in X$  then  $A$  is an intuitionistic fuzzy set. That is, every intuitionistic fuzzy set in  $X$  is a multi-fuzzy set in  $X$  of dimension 2.

**Example 2.8** *If  $\mu_1 + \mu_2 \leq 1$ , then  $A$  is intuitionistic fuzzy set, but multi-fuzzy set need not be an intuitionistic fuzzy set, for example the multi-fuzzy set  $A = \{ \langle x, \mu_1, \mu_2 \rangle : \mu_1 = 0.9, \mu_2 = 0.8 \}$  is not an intuitionistic fuzzy set*

**(iii) Picture fuzzy sets as multi-fuzzy sets of dimension 3 :**

Let  $A$  be a multi-fuzzy set in  $X$  with value domain  $L_1 \times L_2 \times L_3$  where  $L_1 = L_2 = L_3 = [0, 1]$ ,  $A = \{ \langle x, \mu_A(x), \eta_A(x), \nu_A(x) \rangle \mid x \in X \}$  is multi-fuzzy set of dimension 3. If  $\mu_A(x)$ ,  $\eta_A(x)$  and  $\nu_A(x)$  are the degree of positive membership, the degree of neutral membership and the degree of negative membership values of  $x$  in  $A$  respectively and if  $\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$  then  $A$  is an picture fuzzy set. That is every picture fuzzy set in  $X$  is a multi-fuzzy set in  $X$  of dimension 3.

**Example 2.9** *Not every picture fuzzy set is a multi-fuzzy set, for example  $A = \{ \langle a, 0.6, 0.5, 0.4 \rangle, \langle b, 0.3, 0.4, 0.2 \rangle, \langle c, 0.9, 0.4, 0.1 \rangle \}$  is multi-fuzzy set of dimension 3 but it is not picture fuzzy set car  $0.6 + 0.5 + 0.4 \geq 1$*

## 2.4 Some applications of multi-fuzzy sets

Multi-fuzzy set theory is useful to characterize the problems in the elds of image processing, taste recognition, pattern recognition, decision making and approximation of vague data.

### 2.4.1 Image Processing Using Multi-Fuzzy Sets

**Application 2.4.1** [21] A digital image is a representation of a two-dimensional (2D) image of finite set of digital values, called pixels (or picture elements). Pixel is the smallest addressable area of a display. Any image can be approximated by an  $m \times n$  matrix of pixels. The resolution of an image is described as the number of pixels horizontally times the number of pixels vertically. We will refer to a pixel by the number of its row and the number of its column. By this convention, the x-axis is vertical and the y-axis is horizontal. Pixel values typically represent gray levels, colors, heights, etc. Digital image processing focuses on two major tasks: improvement of pictorial information for human interpretation and processing of image data for storage, transmission, and representation for autonomous machine perception (see [Plataniotiset al.(1995)(1998)]). The primary colors red, green, and blue are combined to

reproduce other colors. Color images are usually described in the additive color space RGB. In the RGB color space, a color is represented by a triplet (R, G, B), where R, G, and B denote the intensities of the red component, the green component, and the blue component, respectively. Note that, the intensities of the pixels are integers in the interval  $[0, p]$  (usually  $p = 255$ ). The digital form of a color image, is a mapping from a 2D grid of uniformly spaced discrete points  $(i, j) : i \in \mathbb{N}_m, j \in \mathbb{N}_n$  into  $[0, p]^3$ , where  $\mathbb{N}_m$  is the set of all positive integers less than or equal to  $m$ . That is,  $(R(i, j), G(i, j), B(i, j))$  is the  $p+1$ -scale color value of the pixel  $(i, j)$ . Theory of multi-fuzzy sets has practical relevance in image processing. In multi-fuzzy image processing method, the pixel value  $(R(i, j), G(i, j), B(i, j))$  of the given image is converted into its multi-fuzzy counterpart,  $(r(i, j), g(i, j), b(i, j))$ , for all  $i \in \mathbb{N}_m$  and  $j \in \mathbb{N}_n$ . See one of the ordered triple of useful conversion formulae:

$$r(i, j) = \left(1 + \frac{\bar{r} - R(i, j)}{\rho}\right)^{-u} \quad (2.2)$$

$$g(i, j) = \left(1 + \frac{\bar{g} - G(i, j)}{\gamma}\right)^{-v} \quad (2.3)$$

$$b(i, j) = \left(1 + \frac{\bar{b} - B(i, j)}{\beta}\right)^{-w} \quad (2.4)$$

where  $\bar{r}, \bar{g}, \bar{b} \in [0, p]$  are the reference constants defining the degrees of brightness and  $\rho, \gamma, \beta, u, v, w$  are the appropriate positive real numbers for the conversion (see the fuzzy image-processing problem in the book by [Klir and Yuan (1995)]). Color of a pixel cannot be represented by a membership function of an ordinary fuzzy set, but it is possible to characterize by a 3D membership function  $(\mu_r, \mu_g, \mu_b)$ ; where  $\mu_r, \mu_g$  and  $\mu_b$  are the membership functions from  $\{1, \dots, m\} \times \{1, \dots, n\}$  into  $[0, 1]$ . Therefore, a color image can be approximated by a collection of pixels with a multi-membership function  $(\mu_r, \mu_g, \mu_b)$ . The multi-fuzzy set

$$\{\langle x, \mu_1(x), \mu_2(x), \mu_3(x) \rangle : x = (i, j) \in X, \mu_1(x) = r(i, j), \mu_2(x) = g(i, j), \mu_3(x) = b(i, j), i \in \mathbb{N}_m, j \in \mathbb{N}_n\}$$

represents the color image .

### 2.4.2 How to reconstruct a color image from three linearly independent gray images of a picture?

Using 3-dimensional multi-fuzzy membership functions, we can characterize the color of a pixel in a color image. Suppose an image is approximated by an  $m \times n$  matrix of pixels with multi-fuzzy membership function  $(\mu_r, \mu_g, \mu_b)$ . The membership values  $\mu_r(x), \mu_g(x), \mu_b(x)$  being the

normalized red value, green value and blue value of the pixel  $x \in X$  respectively, where  $X$  is the set consisting of the  $mn$  pixels. So the color image can be approximated by the collection of pixels with the multi-membership function  $(\mu_r(x), \mu_g(x), \mu_b(x))$  and it can be represented as a multi-fuzzy set

$$A \{ \langle x, \mu_r(x), \mu_g(x), \mu_b(x) \rangle : x \in X \}$$

Construct 3 different gray images of the color image  $A$  as follows. Let  $x$  be an arbitrary pixel of the image, for  $i, j = 1, 2, 3$ ;  $a_{(i,j)} \geq 0$  and  $a_{(i,1)} + a_{(i,2)} + a_{(i,3)} = 1$ .

Define gray values  $h_1(x), h_2(x)$  and  $h_3(x)$  of  $x$  as

$$h_i(x) = a_{(i,1)}\mu_r(x) + a_{(i,2)}\mu_g(x) + a_{(i,3)}\mu_b(x).$$

The fuzzy sets  $H_i = \{ \langle x, h_i(x) \rangle : x \in X \}$ ; (for  $i = 1, 2, 3$ ) are the gray images (with different gray tones) of the color image  $A$ . If the coefficient matrix of  $h_i(x)$ , for  $i = 1, 2, 3$  is invertible, then using matrix inversion we can reconstruct the original colour image from the 3 gray images. Let  $PA(x) = H(x)$  be the matrix representation of the pixel  $x \in X$  of the 3 gray images, where

$$P = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} \end{bmatrix}$$

$$A(x) = (\mu_r(x), \mu_g(x), \mu_b(x))^T$$

and

$$H(x) = (h_1(x), h_2(x), h_3(x))^T$$

If  $P$  is invertible, then we have  $A(x) = P^{-1}H(x)$ . Hence we can reconstruct the color membership value  $\mu_A(x) = (\mu_r(x), \mu_g(x), \mu_b(x))$  of the pixel  $x$  from 3 linearly independent gray values.

We hope that this technique is useful in image processing.

# Conclusion

In this master memory, we have studied the notion of multi-fuzzy sets as a generalization of the notion of fuzzy sets. More specifically, we have presented the most important definitions, examples, operations and properties. We have finished this work by some applications of this notion of multi-fuzzy sets.

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

## ملخص:

موضوع المذكرة مكتوب في اطار امتداد المجموعات الضبابية، ندرس المفاهيم والخصائص الاساسية للمجموعات متعددة الضبابية، وكذلك فئات معينة من هذا المفهوم

## كلمات مفتاحية:

مجموعة هش، مجموعة ضبابية، تمديد مجموعات غامضة ، مجموعة متعددة ضبابية.

## Abstract :

The subject of memoire is written within the framework of the extention of fuzzy sets, we study the basic concepts and characteristics of multi-fuzzy sets. Also, particular classes of this concept.

## Key words :

Crisp set, Fuzzy set, Extension of fuzzy sets, Multi-fuzzy set.

## Résumé :

Le sujet du mémoire est écrit dans le cadre de l'extention des ensembles flous, nous étudions les concepts de base et les caractéristiques des ensembles multi-flous. Aussi, des classes particulière de ce concept.

## Mot-clés :

Ensemble classic, Ensemble flou , Extension des ensembles flous, Ensembl multi-flou.