REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

Université Mohamed Boudiaf de M'sila
Faculté des Mathématiques et de l'Informatique
Département des Mathématiques

Mémoire de Master

Domaine : Mathématiques et Informatique
Filière : Mathématiques
Option : Analyse Mathématique et Numérique

Thème

Comparaison entre Runge-Kutta et méthode de Simpson Modifiée Pour un Probléme Initiale

Présentée par :
ZERKAOUI Ouidad

Devant le jury composé de :

DILMI Mustapha M.C.B . Univ de M'sila **Président**.

Mostefa NADIR Prof . Univ de M'sila **Encadreur** .

Gagui Bachir M.C.A . Univ de M'sila **Examinateur**.

Année universitaire 2019/2020

الملخص

في هذا العمل, قمنا بحل المعادلة التفاضلية الخطية من الدرجة الاولى ذات القيم الابتدائية باستعمال طريقة رونج كيتا, ثم نقوم بتحويل المعادلة التفاضلية الى معادلة تكاملية من النوع فولتيرا لنقوم بحلها بطريقة سيمبسون المعادلة لنجد ان الطريقتين تؤديان الى نفس النتائج.

الكلمات المفتاحية:

طريقة رونج كيتا.

طريقة سيمبسون المعدلة.

المعادلة التفاضلية من الدرجة الاولى.

المعادلة التكاملية من نوع فولتيرا.

Acknowledgements

First and foremost, All praises goes to Allah the most Merciful for giving me strength and

capacity to complete this work.

I would like to express my sincere gratitude to my supervisor Prof. Mostefa NADIR for the continuous support and help. I am indebted to him for his excellent guidance, caring and providing. It has been an honor to have such a kind and competent person as a supervisor.

I would like also to express special thanks to all the teachers of the department for their direct and indirect support and encouragement.

Words fail to express our gratitude to our families: parents, brothers, and sisters who have provided us with their encouragement, love and understanding.

Dedications

This work is specially dedicated to my parents and all my extended family who have provided me with their encouragment, love and patience .

To all my freinds.To every soul that has been a source of inspiration to me.

I didicate this work.

Contents

Introduction

Many problems of mathematical physics can began started in the form of initial and boundary value problems associated with the ordinary and partial differential equations.These equations also occur as reformulations of other mathematical problems such as integral equations can be cast into the Volterra integral equations.

This introduction contains a brief description of the objective, approach, and organization of the thesis. The objective of this work is to explain the most numerical methods. For example, Range-Kutta .

The current study tend to investigate the numerical solutions of the differential equation and some linear integral equations with continuous kernel. This thesis is divided into three chapters:

> In The first Chapter: is a discussion of the nations and a reminder on compact operators.
>
> In The second Chapter:is devoted to the Existence and Uniqueness theorem of the first order differential equation, in which I try to apply the numerical method of Runge_Kutta for solving differential equation to approximate the exact solution by programing. Moreover, I use the preferred language in numerical analysis in Matlab. The chapter deals also with the way by which we can transform a first order differential equation to Integral equation.
>
> In The third Chapter: I explain the concept of Volterra Integral equation accordingly with the Simpson method. Moreover, I try to solve the first order Volterra integral equation by the Simpson method to obtain approximate solution of the linear integral equation.

Chapter 1

operator compact

1.1 compact linear operators

(see [1])

A linear operator A defined from a normed space E into normed space F is called a linear compact operator or completely continuous linear operator if for every bounded subset G of E, the image $A(G)$ is relatively compact in F. In other words, the closure$\overline{A(G)}$is compact.

Theorem 1.1.1 *(Compactness criterion)*

A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if and only if for every bounded sequence φ_n in E, the sequence $A\varphi_n$ in F has a convergence.

Let φ_n be a bounded sequence in E, since the operator A is compact, then the set $\{A\varphi_n\}$ is relatively compact in F where this property shows that $A\varphi_n$ contains a convergent subsequence.

Conversely, let us consider any bounded subset G in E and let $\psi_n$be any sequence in $A\varphi_n$. Then there exists a bounded sequence φ_n in G, such that

$$\psi_n = A\varphi_n$$

By assumption, $A\varphi_n = \psi_n$ contains a convergent subsequence ψ_{n_k} in F. Thus $A(G)$ is relatively compact, because for any bounded sequence ψ_n in $A(G)$ there exists a convergent subsequence ψ_{n_k} in F.In other words ,for all bounded set $G \subset E.$ the set $A(G)$ is relatively compact in F. Hence A is compact.

The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators $A_1$and A_2 is a compact operator, for every scalars α and β .

Let φ_n be a bounded sequence in F and let $A\varphi_n$ be a sequence in F, then

$$A\varphi_n(x) = \alpha A_1\varphi_n(x) + \beta A_2\varphi_n(x), \qquad \text{with } \ \varphi_n \in E, n \in \mathbb{N}.$$

The operators A_1 and A_2 are compact, one can extract from $A_1\varphi_n$ and $A_2\varphi_n$ two convergent subsequences which give by their sum a convergent subsequence of $A\varphi_n$. Hence A is compact.

The product AB of two bounded operators A and B is compact if either of operators A or B is compact.

Let φ_n be a bounded sequence in E, then if we consider $B\varphi_n$ as a bounded, and from the compactness of the operator A gives a convergent subsequence $A(B\varphi_n(x))$ of $A(B\varphi_n(x))$.

Hence the operator AB is compact.

On the other hand, if we consider B as a compact, one can extract from a convergent subsequence $B\varphi_{n(k)}(x)$, and from boundedness of the operator A gives the convergence of the sequence $A(B\varphi_{n(k)}(x))$. Hence the operator AB is compact.

A sequence $A_n$of compact operators dened from a normed space E into a Banach space F converges uniformly to an operator A, say,

$$\lim_{n\to\infty} \parallel A_n - A \parallel = 0$$

Then the limit operator A is compact.

Let φ_n be bounded sequence in E, the operators A_1 is compact, then one can extract form the sequence $A_1\varphi_n$ aconvergent subsequence, say φ_n^1 a subsequence from φ_n such that $A_1\varphi_n^1$ coverges .

In the same way, we can extract from the sequence $A_2\varphi_n^1$ n a convergent subsequence, sayφ_n^1 a subsequence from φ_n^1 such that $A_2\varphi_n^1$ n converges.

Noting that, we obtain from the bounded sequence φ_n a subsequence φ_n^1 n such that $A_1\varphi_n^2$ and $A_2\varphi_n^2$ both converge.

Continuing in this way, we see that, for the compact operators $A_1, A_2, ..., A_P$ there exists a nested subsequences

$$\varphi_n^p \subset ...\varphi_n^2 \subset \varphi_n^1 \subset \varphi_n$$

such that, the sequence $A_k\varphi_n^p$ converge for all $k = 1, 2, ..., p$

In order to show the compactness of the operator limit A, we must use the completeness of the space $F$and showing that the sequence $A\varphi_n^p$ is Cauchy sequence.

Noting that the sequence φ_n is bounded, say $\| \varphi_n \| \leq M$ for all n. Hence $\| \varphi_n^p \| \leq M$ for each n and p. choose $n = p$ so that

$$\| A_n - A \| < \frac{\varepsilon}{3}.$$

Hence we obtain

$$\begin{aligned}
\| \quad & A\varphi_n^p - A\varphi_n^q \| = \| A\varphi_n^p - A_n\varphi_n^p + A_n\varphi_n^p - A_n\varphi_n^q + A_n\varphi_n^q - A\varphi_n^q \| \\
\leq \quad & \| A\varphi_n^p - A_n\varphi_n^p \| + \| A_n\varphi_n^p - A_n\varphi_n^q \| + \| A_n\varphi_n^q - A\varphi_n^q \| \\
\leq \quad & \| A_n - A \| \| \varphi_n^p \| + \| A_n\varphi_n^p - A_n\varphi_n^q \| + \| A_n - A \| \| \varphi_n^q \| \\
\leq \quad & \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}M = \varepsilon.
\end{aligned}$$

Remembering that, due to the completeness of the space F, the Cauchy sequence $A\varphi_n^p$ converges as a subsequence of $A\varphi_n$ where φ_n^p is a subsequence of an arbitrary bounded sequence φ_n. Hence the compactness of the operator A.

Theorem 1.1.2 *(finite dimensional range)*

Let A be a bounded operators defined from E into F with the range $A(E)$ has a finite dimension $\dim A(E) < \infty$, then the operator A is compact.

Indeed, for all bounded set G in E, the range $A(G)$ is a bounded set in finite dimensional space $A(E)$. Hence $A(G)$ is relatively compact, is follows that A is a compact operator.

Theorem 1.1.3 *(finite dimensional domain)*

Let A be a bounded operator defined from E into F with the domain E has a finite dimension $\dim E < \infty$, then the operator A is compact .

Indeed, the space E has a finite dimension $\dim E < \infty$ implies the finite dimensional range $A(E)$, say

$$\dim A(E) \leq \dim E$$

it follows that A is compact operator.

Let G be a closed subspace in the normed space E such that, $G \neq E$ then there exists an element $\varphi \in E$ with $\| \varphi \|= 1$ such that, for all $\psi \in G$, we have

$$\| \varphi - \psi \| \geq \alpha, \qquad \text{with } 0 < \alpha < 1$$

Indeed, let f be an element of E such that $f \notin G$ then, we get

$$\inf_{h \in G} \| f - h \|= \beta > 0,$$

choosing an element g belongs to G such that,

$$\beta \leq \| f - g \| \leq \frac{\beta}{\alpha}.$$

Define the vector φ by

$$\varphi = \frac{f-g}{\| f - g \|},$$

this vector φ has a unit norm$\| \varphi \|= 1$,besides, for all $\psi \in G$ we get

$$\begin{aligned}
\| \quad \varphi - \psi \| &= \| \frac{f-g}{\| f - g \|} - \psi \| \\
&= \frac{1}{\| f - g \|} \| f - [g + (\| f - g \| \psi] \| \\
&\geq \frac{\beta}{\| f - g \|} \geq \alpha.
\end{aligned}$$

The identity operator I defined from a normed space E into E is compact if and only if the space E has a finite dimension let φ_1 be an element of E, such that $\| \varphi_1 \|= 1$, then the set of finite dimension $G_1 = span\{\varphi_1\}$ represents a closed subspace of E. So there exists an element $\varphi_2 \in E$, such that $\| \varphi_2 \|= 1$ and $\| \varphi_1 - \varphi_2 \| \geq \frac{1}{2}$. By the same way we take a closed subspace $G_2 = span\{\varphi_1, \varphi_2\}$ and finding an element $\varphi_2 \in E$ such that $\| \varphi_2 \|= 1$ with $\| \varphi_2 - \varphi_3 \| \geq \frac{1}{2}$. One repeat the same procedure until the obtaining of sequence φ_n verifying $\| \varphi_n \|= 1$and $\| \varphi_m - \varphi_n \| \geq \frac{1}{2}$,for all $m \neq n$.

Noting that, the sequence φ_n is bounded but does not contain any convergent subsequence. Hence the operator $I\varphi_n = \varphi_n$ is not compact .

The closed unit ball $B(0,1)$ in the normed space E of infinitely dimensional is not compact .

Indeed, $B(0,1)$ is bounded but cannot be compact ;thus

$$I(B(0,1) = B(0,1) = \overline{B(0,1)})$$

is not relatively compact.

A bounded operator A in a normed space E is not generally a compact operator.

Indeed, see the identity operator $A = I$ in the infinitely dimensional normed space E .

The integral operator A defined from $C(G)$ into $C(G)$

$$A\varphi(x) = \int_x k(x,y)\varphi(y)dy, \qquad x, y \in G$$

With continuous kernel $k(x,y)$ is a compact operator .

Let E be a bounded set of $C(G)$ then, for each $\varphi \in E$, we have

$$\| \varphi \| \leq M,$$

besides, for all $x \in G$ and $\varphi \in E$, we get

$$\begin{aligned} | \quad A\varphi(x) | &= | \int_G k(x,y)\varphi(y)dy | \\ \leq \quad M | G | &\max_{x,y\in G} | k(x,y) | . \end{aligned}$$

It follows that $A(E)$ is bounded .

By assumption, the kernel $k(x,y)$ is continuous over the compact $G \times G$, thus it is uniformly continuous and therefore

$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y, z \in \Omega, | x - y | < \delta \Longrightarrow | k(x,z) - k(y,z) | < \frac{\varepsilon}{M\ mes(\Omega)}$

Hence, for each $\varphi \in G$and $x, y \in \Omega$, with $| x - y | < \delta$

$$\begin{aligned} | \quad A\varphi(x) - A\varphi(y) | &= | \int_\Omega (k(x,z) - k(y,z))\varphi(z)dz | \\ < \quad \frac{\varepsilon}{M\ mes(\Omega)} &M\ mes(\Omega) = \varepsilon. \end{aligned}$$

this relation expresses that$A(G)$ is equicontinuous. Hence $A(G)$ is relatively compact, so by $Arzela\_Ascoli's$ theorem A is compact .

Weakly singular kernel

The kernel $k(x,y)$ is said to be weakly singular if it is defined continuous on $\Omega \times \Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ for all $x \neq y$ and there exist a positive constants M and $\alpha \in]0,n]$ such that

$$| k(x,y) |< \frac{M}{| x-y |^{n-\alpha}}, \quad x,y \in \Omega \quad ,x \neq y.$$

In other words,

$$\forall x,y \in \Omega, x \neq y, \exists M > 0, | k(x,y) |< \frac{M}{| x-y |^{n-\alpha}} \quad ,0 < \alpha \leq n$$

The integral operator A defined from $C(\Omega)$ into$C(\Omega)$ with weakly continuous kernel is a compact operator.

Noting that, the integral operator

$$A\varphi(x) = \int_G k(x,y)\varphi(y)dy, \quad x,y \in \Omega$$

exists as an improper integral, due to the weakly continuous kernel

$$| k(x,y) | \varphi(y) \leq M \ \| \varphi \|| x-y |^{n-\alpha},$$

further,

$$\int_G | x-y |^{n-\alpha} dy \leq w_n \int_0^d \rho^{\alpha-n}\rho^{n-1}d\rho = \frac{w_n}{\alpha}d^{\alpha},$$

where w_n designates the surface area of the unit sphere in $\mathbb{R}^n$ and d the diameter of the set$G$.

Let us construct a sequence of compact operators A, which converges to the integral operator A, such that

$$\lim_{n\to\infty} \| A_p - A \|= 0.$$

choosing now a linear continuous function h defined on $[0,\infty[$ into $\mathbb{R}$,by

$$h(t) = \begin{cases} 0 & if \ 0 \leq t \leq \frac{1}{2} \\ 2t-1 & if \ \frac{1}{2} \leq t \leq 1 \\ 1 & if \ 1 \leq t < \infty \end{cases}$$

The function $k_p(x,y)$ defined on $G \times G$ into $\mathbb{R}$, by

$$k_p(x,y) = \begin{cases} \quad h(p \mid x-y \mid)k(x,y) & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

is a continuous kernel for each $p \in \mathbb{N}$. Hence the integral operators A_p such that

$$A_p\varphi(x) = \int_G k_p(x,y)\varphi(y)dy, \qquad x,y \in G.$$

are compact.

Besides, for all $x \in G$, we get

$$\begin{aligned} \mid \quad & A_p\varphi(x) - A\varphi(x) \mid = \mid \int_G [k_p(x,y) - k(x,y)]\varphi(y)dy \mid \\ = \quad & \mid \int_{G \cap |x-y| < \frac{1}{p}} \{h(p \mid x-y \mid\}k(x,y)\varphi(y)dy \mid \\ \leq \quad & M \parallel \varphi \parallel w_n \int_0^{\frac{1}{p}} \rho^{\alpha-n}\rho^{n-1}d\rho \\ \leq \quad & M \parallel \varphi \parallel \frac{w_n}{\alpha\rho^\alpha}. \end{aligned}$$

It is simple to see that the convergence $A_p\varphi$ to $A\varphi$ is uniform, so it follows that,

$$\parallel A - A_p \parallel \leq M\frac{w_n}{\alpha\rho^\alpha} \to 0, \qquad \text{when } p \to \infty,$$

and thus$A$is compact operator.

The integral operator A dened from the normed space$C(\partial G)$ into $C(\partial G)$ with continuous or weakly continuous kernel is a compact operator, where under ∂G we designate a regular boundary of the set G.

Chapter 2

The resolution of the initial value problems

2.1 First-order linear differential equations

(see [2])

is an equation of the form

$$a(x)y' + b(x)y = c(x) \tag{2.1.1}$$

where $a(x), b(x)$and $c(x)$ are given continuous functions of x on same interval. $I \subset \mathbb{R}$ and we ask $\forall x \in I; a(x) \neq 0$, we divide both sides of equation (2.1.1) by the leading coefficient $a(x)$ to reduce it to a more useful form, usually called the standard form of a linear equation:

$$y' + b_1(x)y = c_1(x) \tag{2.1.2}$$

where $b_1(x) = \frac{b(x)}{a(x)}, c_1(x) = \frac{c(x)}{a(x)}$.

2.2 Existence and uniqueness of solutions

(see [3])

In this section we formulate results regarding the existence and uniqueness of solution $y' = f(x, y)$. We begin with the classical *Peano* existence theorem for the system

$$\begin{cases} y' = f(x,y) \\ y(x_0) = \alpha \end{cases} \tag{2.2.1}$$

which assumes only that the function f is cotinuous in the same domain.

Theorem 2.2.1 *(Existence and uniqueness):*

A first order differential equation is an equation of the form

$$y' = f(x,y) \tag{2.2.2}$$

Assume that the function $f(x,y)$ is continuous in the domain

$$D = \{(x,y) :\mid x - x_0 \mid\leq a, \parallel y - y_0 \parallel\leq b\} \tag{2.2.3}$$

and that there exists a constant M such that$\parallel f(x,y) \parallel \leq M$ for $(x,y) \in D$ .then system (2.2.1) has at least one solution $y = \varphi(x)$ defined for

$$\mid x - x_0 \mid\leq a_1 := \min\{a, b/M\}$$

and passing through the point (x_0, y_0).

Afunction $f(x,y)$ satisfies a*Lipschitz*condition in D with a *Lipschitz* constant L if

$$\parallel f(x,y_1) - f(x,y_2) \parallel\leq\parallel y_1 - y_2 \parallel \tag{2.2.4}$$

for all $(x,y_1),(x,y_2) \in D$.

Lemma 2.2.1

Assume that the function $f(x,y)$ is continuous and satisfies a *Lipschitz* condition (2.2.4) in the domain D defined by (2.2.3) set

$$M = \max\{\parallel f(x,y) \parallel: (x,y) \in D\}$$

the function $f(x,y)$ through by (x_0, y_0).

Proof. (existence)

First consider the interval $[x_0, x_0+a_1]$,the proof for the interval $[x_0-a_1, x_0]$ is analogous. Define the integral operator

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(s, \varphi(s))ds \tag{2.2.5}$$

$x \in [x_0, x_0 + a_1]$.put $Y = \{y \in \mathbb{R}^m : \| y - y_0 \| \leq b\}$ and denote by $C([x_0, x_0 + a_1], Y)$ the space of continuous functions form $[x_0, x_0 + a_1]$ into Y with a uniform norm. Observe that if $y \in C([x_0, x_0 + a_1], Y)$, then

$$\| \varphi(x) - y_0 \| \leq \int_{x_0}^{x} \| f(s, \varphi(s)) \| ds \leq M \ a_1 \leq M \ \frac{b}{M} = b.$$

and it follows that the operator ϕ takes the function form $C([x_0, x_0 + a_1], Y)$ into $C([x_0, x_0 + a_1], Y)$:

$$\phi : C([x_0, x_0 + a_1], Y) \to C([x_0, x_0 + a_1], Y)$$

define the sequence of function $\varphi_i(x) \in C([x_0, x_0 + a_1], Y)$ by the formula

$$\varphi_{i+1}(x) = \phi(\varphi_i(x)) = y_0 + \int_{x_0}^{x} f(s, \varphi_i(s))ds,$$

i=0,1,... where $\varphi_0(x) = \varphi_0, x \in [x_0, x_0 + a_1]$ then we have the bound

$$\| \varphi_i(x) - \varphi_{i-1}(x) \| \leq \frac{ML^{i-1}(x - x_0)^i}{i!} \tag{2.2.6}$$

we prove (2.2.6) by induction with the respect to i. Since

$$\| \varphi_1(x) - \varphi_0(x) \| \leq \int_{x_0}^{x} \| f(s, \varphi_0(s) \| ds \leq M(x - x_0)$$

this bound is true for $i = 1$. Assuming now that (2.2.6) is true for i we have

$$\begin{aligned}
\| \quad \varphi_{i+1}(x) - \varphi_i(x) \| &\leq \int_{x_0}^{x} \| f(s, \varphi_i(s)) - f(s, \varphi_{i-1}(x)) \| ds. \\
\leq \quad L \int_{x_0}^{x} \| \varphi_i(s) - \varphi_{i-1}(s) \| ds. \\
\leq \quad L \int_{x_0}^{x} \frac{ML^{i-1}(s - x_0)^i}{i!} ds = \frac{ML^i(x - x_0)^{i+1}}{(i+1)!}.
\end{aligned}$$

which is equivalent to (2.2.6) with i replaced by $i + 1$. By replacing (2.2.6) we have

$$\sum_{i=0}^{\infty} \| \varphi_i(x) - \varphi_{i-1}(x) \| \leq \frac{M}{L} \sum_{i=1}^{\infty} \frac{(L(x - x_0))^i}{i!}$$

and since the serie on the right -hand side is uniformly convergent $Me^{L(x-x_0)-1}/L$.

where i the partial sum is equal to $\varphi_i(x)$ it is also uniformly convergent on the interval $[x_0, x_0 + a_1]$. Denoted by $\overline{\varphi}(x)$ the limit of the sequence $\varphi_i(x)$. Since

$$\| f(s, \varphi_m(s)) - f(s, \varphi_n(s)) \| \leq \| \varphi_m(s) - \varphi_n(s) \| .$$

the integral $\int_{x_0}^{x} f(s, \varphi_n(s))ds$ is uniformly convergent for $x \in [x_0, x_0 + a_1]$ passing to the limit in (1.3.6) as $i \to \infty$ it follows that $\overline{\varphi}(x)$ satisfies the integral equation

$$\overline{\varphi}(x) = y_0 + \int_{x_0}^{x} f(s, \overline{\varphi}(s))ds,$$

$x \in [x_0, x_0 + a_1]$.Hence, $\overline{\varphi}(x)$ also satisfies the equivalent initial value problem (2.2.1).

Uniqueness:

to prove uniqueness, we assume that there are two solutions $\psi(x)$ and $\varphi(x)$ to (1.1.2). We define a norm $\| \, . \, \|_\alpha$in the space $C([x_0, x_0 + a_1], Y)$ by the formula

$$\| \varphi \|_\alpha := \sup\{e^{-\alpha(x-x_0)} \| \varphi(x) \| : x \in [x_0, x_0 + a_1]\},$$

where $\alpha > 0$ and $\| \, . \, \|$ is any norm in $\mathbb{R}^m$.Subtracting the integral equations for $\psi(x)$ and $\varphi(x)$ equivalent to (2.2.1), we obtain

$$\psi(x) - \varphi(x) = \int_{x_0}^{x} (f(s, \psi(s)) - f(s, \varphi(s)))ds.$$

and it follows that

$$\begin{aligned}
\| \quad \psi(x) - \varphi(x) \| &\leq L \int_{x_0}^{x} e^{\alpha(s-x_0)} e^{-\alpha(s-x_0)} \| \psi(s) - \varphi(s) \| \, ds. \\
\leq \quad L \| \psi - \varphi \|_\alpha & \int_{x_0}^{x} e^{\alpha(s-x_0)} ds = L \| \psi - \varphi \|_\alpha \frac{e^{\alpha(x-x_0)} - 1}{\alpha} \\
\leq \quad \frac{L}{\alpha} \| \psi - \varphi \|_\alpha & e^{\alpha(x-x_0)}.
\end{aligned}$$

this leads to

$$e^{-\alpha(x-x_0)} \| \psi(x) - \varphi(x) \| \leq \frac{L}{\alpha} \| \psi - \varphi \|_\alpha .$$

and since the right-hand side of this inequality is independent of x, we obtain

$$\| \psi - \varphi \|_\alpha \leq \frac{L}{\alpha} \| \psi - \varphi \|_\alpha .$$

choosing α such that $\frac{L}{\alpha} < 1$,we obtain

$\| \psi - \varphi \|_\alpha = 0$ or $\psi(x) = \varphi(x), x \in [x_0, x_0 + a_1]$. ■

2.3 Numerical Methods of IVP for a first-order:

(see [4])

Analysis the initial value problem to find the solution of ordinary differential equation

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0 \end{aligned}$$

over the interval,$x \in [a, b]$.

we divide the interval $a \leq x \leq b$ into N segments of length h, called the step size .thus the step size is $h = \frac{b-a}{N}$. This defines a set of equally spaced discrete points $a = x_0 \leq x_1 \leq x_2 \leq ... \leq x_n = b$, where $x_{i+1} = x_i + h$,$i = 1, 2, ..., N$.

2.4 The Runge-Kutta methods

"(see [5])"

the general form of Runge-kutta method is

$$\begin{aligned} y_{i+1} &= y_i + hF(x_i, y_i; h), \quad i = 0, ..., i-1 \\ &= y_i + h \sum_{i=1}^{s} k_i \end{aligned}$$

where $F(x_i, y_i; h)$ is called an increment function on the interval $[x_i, x_{i+1}]$ .It can be defined in general form as

$$F(x_i, y_i; h) = k_1 + k_2 + ... + k_i$$

where k_n is

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f(x_i + h, y_i + k_1 h)\\
k_3 &= f(x_i + h, y_i + (k_1 + k_2)h)\\
&.\\
&.\\
&.\\
k_s &= f(x_i + h, y_i + (k_1 + k_2 + ... + k_{s-1})h)
\end{aligned}
$$

2.4.1 Fourth order Runge-Kutta method

(see [6])

the most popular method of the fourth order Runge-Kutta method is obtained as

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f(x_i + h, y_i + k_1 h)\\
k_3 &= f(x_i + h, y_i + k_2 h)\\
k_4 &= f(x_i + h, y_i + k_3 h)
\end{aligned}
$$

Fourth order Runge-Kutta method is the most well-known sample of all Runge-Kutta methods.

Example 2.4.1 *solve the following equation:*

$$\begin{cases} y'(x) = \exp(-2x) - (y + y^2); \\ \qquad y(0) = 1 \end{cases}$$

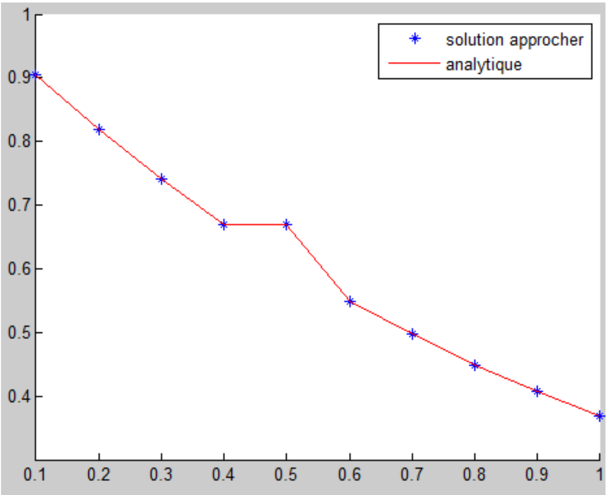
knowing that the exact solution is:

$phi = \exp(-x);$

we have $N = 10, a = 0, b = 1, t = a : h : b - h$

such as $h = (b - a)/N + 1$.

values of X	exact solution	approx solution	error
1.000000e-001	9.048374e-001	9.048390e-001	1.601954e-006
2.000000e-001	8.187308e-001	8.187331e-001	2.392933e-006
3.000000e-001	7.408182e-001	7.408209e-001	2.718255e-006
4.000000e-001	6.703200e-001	6.703228e-001	2.780293e-006
5.000000e-001	6.703200e-001	6.065334e-001	2.698167e-006
6.000000e-001	5.488116e-001	5.488142e-001	2.541963e-006
7.000000e-001	4.965853e-001	4.965877e-001	2.352640e-006
8.000000e-001	4.493290e-001	4.493311e-001	2.153763e-006
9.000000e-001	4.065697e-001	4.065716e-001	1.958489e-006
1.000000e-000	3.678794e-001	3.678812e-001	1.773763e-006



$approximation\ of\ method\ Runge - Kutta$

Example 2.4.2 *solve the following equation :*

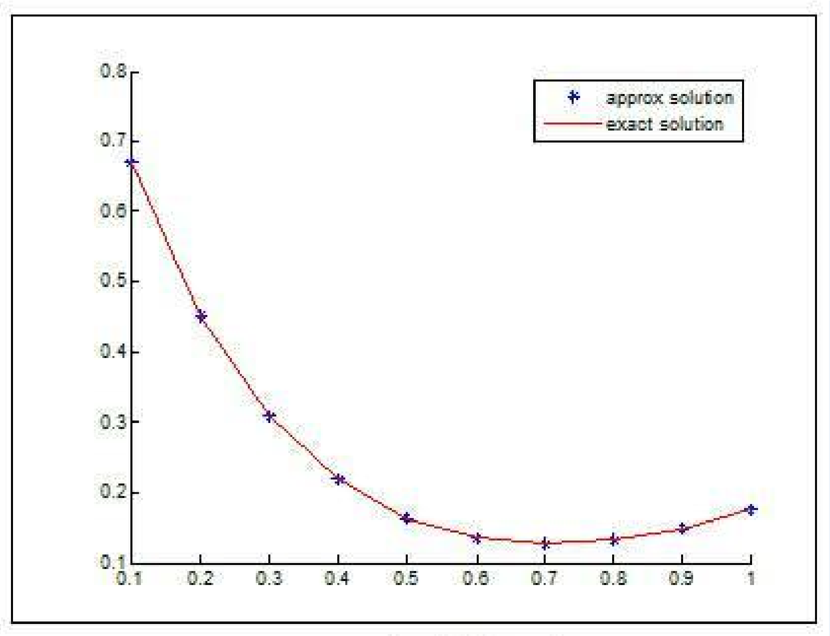
$$\begin{cases} y'(x) = x^2 - 4y \\ y(0) = 1 \end{cases}$$

the exact solution of initial problem is known by

$$phi = \frac{31}{32}\exp(-4x) + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32};$$

we have $N = 10, a = 0, b = 1$. such that $h = (b - a)/N + 1$.

values of X	exact solution	approx solution	error
1.000000e-001	6.706225e-001	6.707033e-001	8.078874e-005
2.000000e-001	4.515374e-001	4.516468e-001	1.094140e-004
3.000000e-001	3.080319e-001	3.081434e-001	1.114874e-004
4.000000e-001	2.168373e-001	2.169387e-001	1.014036e-004
5.000000e-001	1.623561e-001	1.624430e-001	8.695231e-005
6.000000e-001	1.341330e-001	1.342051e-001	7.210861e-005
7.000000e-001	1.251597e-001	1.252184e-001	5.870154e-005
8.000000e-001	1.307384e-001	1.307858e-001	4.739691e-005
9.000000e-001	1.477199e-001	1.477581e-001	3.826548e-005
1.000000e-000	1.739933e-001	1.740244e-001	3.110288e-005



$approximation\ of\ method\ Runge-Kutta$

Chapter 3

Numerical Solution of Volterra Integral Equations

3.1 Volterra Integral Equations

(see [7])

volterra integral equations are in the following form :

$$\varphi(x) = f(x) + \int_{x_0}^{x} k(x,y)\varphi(y)dy \qquad a \leq x \leq b \tag{3.1.1}$$

where the function $f(x)$ and the regular kernel$k(x,y)$ are given, and $\varphi(x)$ is the unknown function to be determined.

3.2 Transformation of the initial value problem into Volterra integral equation first kind

First order : we transform differential equation to volterra integral equation, It becomes as follow:

$$\begin{cases} y' = f(x,y) \\ y(x_0) = \alpha \end{cases}$$

$$\begin{aligned}\int_{x_0}^{x} y'(x)dx &= \int_{x_0}^{x} f(x,y)dx \\ y(x) - y(x_0) &= \int_{x_0}^{x} f(x,y)dx \\ y(x) &= \alpha + \int_{x_0}^{x} f(x,y)dx\end{aligned}$$

3.3 Development of the Method

(see [8])

considering a volterra integral equation given by

$$\varphi(x) = f(x) + \int_{x_0}^{x} k(x,y)\varphi(y)dy \qquad a \leq x \leq b \tag{3.3.1}$$

Let the interval $[a,b]$ be finite and partitioned by $2n$ equally spaced points

$$x_0 = a < x_1 < ... < x_{2j-1} < x_{2j} < ... < x_{2n}.$$

we solve the problem by marching in time. The approximation of (3.1.1) in even nodes (x_{2j}) is given by

$$\begin{aligned}\varphi(x_{2j}) &= f(x_{2j}) + \int_{a}^{x_{2j}} k(x_{2j},y)\varphi(y)dy \\ &= f(x_{2j}) + \sum_{i=0}^{j-1} \int_{a}^{t_{2i+2}} k(x_{2j},y)\varphi(y)dy.\end{aligned}$$

which can be rewritten as

$$\varphi_{2j} = f(x_{2j}) + \sum_{i=0}^{j-1} \int_{a}^{t_{2i+2}} k(x_{2j},y)\varphi(y)dy.$$

Using Simpson's quadratic formula, the above discrete equation becomes

$$\varphi_{2j} = f(x_{2j}) + \sum_{i=0}^{j-1} \frac{h}{3}(k_{2j,2i}\varphi_{2i} + 4k_{2j,2i}\varphi_{2i+1} + k_{2j,2i}\varphi_{2i+2}).$$

For a smaller step h, an approximation toφ_{2j} can then be computed by replacingφ_{2i+1} by the average $\frac{\varphi_{2i}+\varphi_{2i+2}}{2}$,

$$\begin{aligned}
\varphi_{2j} &= f_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}[(k_{2j,2i}\varphi_{2i} + 4k_{2j,2i}\frac{\varphi_{2i}+\varphi_{2i+2}}{2} + k_{2j,2i}\varphi_{2i+2})] \\
&= f_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}[(k_{2j,2i} + 2k_{2j,2i+1})\varphi_{2i} + (2k_{2j,2i+1} + k_{2j,2i+2})\varphi_{2i+2}] \\
&= f_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}(k_{2j,2i} + 2k_{2j,2i+1})\varphi_{2i} + \sum_{i=0}^{j-1}(k_{2j,2i+1} + 2k_{2j,2i+2})\varphi_{2i+2} \\
&= f_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}(k_{2j,2i} + 2k_{2j,2i+1})\varphi_{2i} + \sum_{i=0}^{j-1} \frac{h}{3}(2k_{2j,2i+1} + k_{2j,2i})\varphi_{2i},
\end{aligned}$$

$$\varphi_{2j} = f_{2j} + \frac{h}{3}(k_{2j,0} + 2k_{2j,1})\varphi_0 + \frac{h}{3}(2k_{2j,2j-1} + k_{2j,2j})\varphi_{2i} + \frac{2h}{3}\sum_{i=1}^{j-1}(k_{2j,2i-1} + k_{2j,2i} + k_{2j,2i+1})\varphi_{2i}$$

In general, for j=1,2,...,n

$$\varphi_{2j}(1 - \frac{h}{3}(2k_{2j,2j-1} + k_{2j,2j})) = f_{2j} + \frac{h}{3}(k_{2j,0} + 2k_{2j,1})\varphi_0 + \frac{2h}{3}\sum_{i=1}^{j-1}(k_{2j,2i-1} + k_{2j,2i} + k_{2j,2i+1})\varphi_{2i}.$$

clearly from (3.1.1)the value of $\varphi(0)$ is $f(0)$, $\varphi_0 = f_0$.

3.3.1 Numerical Experiments

volterra integral equation IVP for a first-order

Example 3.3.1 *solve the initial value problem*

$$y'(x) = \exp(-2x) - (y + y^2)\ ; \quad y(0) = 1$$

On the other hand, the transformation of the initial value problem into Volterra integral equation gives the best approximation solution using the modified Simpson method

$$\begin{aligned}
\int_0^x y'(x)dx &= \int_0^x \exp(-2x) - (y+y^2)dx; \\
y(x) - y(0) &= \int_0^x \exp(-2x)dx - \int_0^x (y+y^2)dx \\
y(x) &= 1 + \frac{1}{2} - \frac{1}{2}\exp(-2x) - \int_0^x (y+y^2)dx \\
y(x) &= \frac{3}{2} - \frac{1}{2}\exp(-2x) - \int_0^x (y+y^2)dx
\end{aligned}$$

Example 3.3.2 *solve the initial value problem*

$$y'(x) = x^2 - 4y \ ; \qquad y(0) = 1$$

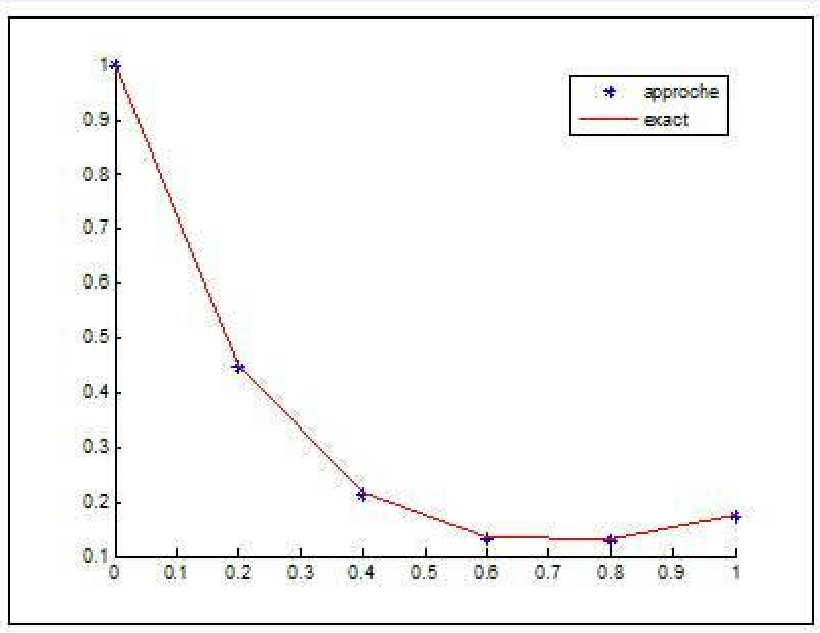
On the orher hand, the transformation of the initial value problem into Volterra integral equation gives the best approximation solution using the modified Simpson method

$$\begin{aligned}
\int_0^x y'(x)dx &= \int_0^x (x^2 - 4y)dx \\
y(x) + y(0) &= \int_0^x x^2 dx - \int_0^x 4y(x)dx \\
y(x) &= \int_0^x -4y(x)dx + \frac{1}{3}x^3 + 1.
\end{aligned}$$

wich is computed

valuesof x	Exact solution	Approx solution	Error
0.000e+000	1.000000e+000	1.000000e+000	0.000000e+000
2.000e-001	4.515374e-001	4.465741e-001	4.963360e-003
4.000e-001	2.168373e-001	2.122737e-001	4.563589e-003
6.000e-001	1.341330e-001	1.309179e-001	3.215093e-003
8.000e-001	1.307384e-001	1.286487e-001	2.089678e-003
1.000e+000	1.739933e-001	1.726402e-001	1.353109e-003

Table of aproximation by modied Simpson method



The solution with the modied Simpson method

Conclution

The current study was intended to compare between the solution differential equation and the Integral solution, accordingly we remark that all initial value problem can be transformed into first order Volterra integral equation, where the numerical solution of those equations using the modified Simpson method is better than the Runge-Kutta method, moreover it is concluded that both methods used in the research are reliable, and reveal the same results.

Bibliography

[1] Mostefa Nadir (cours d'analyse fonctionnelle université de Msila 2004)

[2] Mostefa Nadir .cours sur théoréme d'existence et l'unicité university of Msila ,Algeria ,2008.

[3] John Wiley and sons,Inc (General linear methods for ordinary differential equations), New Jersey .simultaneously in canada ,2009.

[4] Mostefa.Nadir,Rahima Frahtia (comparaison entre taylor et méthode de simpson modifie pour un problème initial). Universite de M'sila, Master, 2018/2019."

[5] E.Hairer ,S.P.Norsett ,G.Wanner (solving ordinary differential equation).Université de Genève .

[6] Simruy Hurol ,Numerical Methods for Solving Systems of Ordinary Differential Equations ,Master of Science in Applied Mathematics and Computer Science ,Eastern Mediterranean University January 2013 Gazimağusa, North Cyprus .

[7] Abdul-Majid Wazwaz ,(Linear and Nonlinear Integral Equations Methods and Application),London New York.

[8] Azedine RAHMOUNE ,(Président : Madani. MOUSSAI),RESOLUTION NUMERIQUE DES EQUATIONS INTEGRALES ,MAGISTER ,Msila 2004.

[9] Mostefa.Nadir,B FREIDJA (comparaison entre euler et méthode de trapèze pour un problème initial),Universite de M'sila ,Master ,2018/2019."

Résume :

Dans ce travail, Nous avons résolu l'équation differentielle du premier ordre avec des valeurs initiales en utilisant la méthode Runge-Kutta, puis nous convertissons l'équation differentielle en équations intégrales de volterra pour la résoudre dans une méthode du Simpson Modifiée premier degré pour trouver que les deux méthodes sont améliorées, en d'autre termes elles donnent les meme résults.

Mots clé :

Equation differentielle lineaire du premier ordre, méthode de Runge-Kutta d'ordre quatre, Equation integrale lineaire de volterra, méthode de Simpson Modifiée.

Abstract :

In this study, I have solved the first ordre differential equation using Runge-kutta method. Moreover the differential equation was transformed to an integral equation of volterra, to be solved by the Simpson Modifiée methode of the first order. therefore, the findings revealed that both methods used in the research lead to the same results.

Key Words :

First-ordre linear differential equation, Runge-Kutta method the fourth order, Linear integral equation of Volterra, Simpson Modifiée method.

ملخص:

في هذا العمل ,قمنا بحل المعادلة التفاضلية من الدرجة الاولى ذات القيم الابتدائية باستعمال طريقة رانج كيتا . ثم نقوم بتحويل المعادلة التفاضلية الى معادلة تكاملية من نوع فولتيرا لنقوم بحلها بطريقة سيمسون المعدلة لنجد ان الطريقتين تؤديان الى نفس النتائج

الكلمات المفتاحية

المعادلة التفاضلية الخطية من الدرجة الاولى , طريقة رونج كوتا من الرتبة الرابعة , المعادلة التكاملية الخطية من النوع فولتيرا , طريقة سيمبسون المعدلة.