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Fuzzy topologies based on fuzzy relations

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Contents

Introduction

General topology is a branch of mathematics that provides a vocabulary and a general framework for dealing with notions of limit, continuity, and neighborhood. Topology, as a well-defined mathematical discipline, originates in the early part of the twentieth century, but some isolated results can be traced back several centuries. In 1736, Euler propossed the Seven Bridges of Königsberg as one of the first practical applications of topology. Also, in 1750, Euler wrote to a friend that he had realized the importance of the edges of a polyhedron. This led to his polyhedron formula, $V - E + F = 2$ (where V, E, and F respectively indicate the number of vertices, edges, and faces of the polyhedron). Topology generated by binary relation is one of the famous classes of general topology and play a prominent role in pure and applied mathematics. They apply in different fields especially in preference representation theorems [4] and they appear to provide the notion of nearness or proximity between two elements of an arbitrary set without using any distance function on it [8].

The study of the topology generated by binary relation was initiated by Smithson (1969) [21]. Then after that, many researchers have been working in this topic, In (2009), Knoblauch [8] introduced topology induced by a binary relation, which are generated by the set of all upper and lower contours of this relation and he obtained a characterization about this class of topology. Salama (2008) [19] used binary relation to generate topological structures using the lower and the upper approximations. Campión et al. (2009) have characterized topologies induced by total preorder relations by utility functions. Recently, Induráin and Knoblauch (2013) have studied the problem of characterizing which topologies on a nonempty set are generated by a binary relations by means of their lower and upper contour sets, also they extended this characterization to the context of bitopological spaces induced by binary relations. In 1965, professor L.A.Zadeh [24] generalized the usual notion of the set by presenting "fuzzy sets". Fuzzy subsets are the classes of objects with a degree of membership ranging between 0 and 1 $(\mu : X \to [0, 1])$. Fuzzy sets allow us to represent vague concepts expressed in natural language. It also allows us to use it in a lot of concepts like fuzzy trellises, fuzzy rings,..., ect

The notion of fuzzy topology generated by a fuzzy relation is introduced by Mishra and Srivastava (2018)[12] as generalization of topology generated by a crisp relation and they studied several related results.

The main objective of this memory is to improve our knowledge about the notions of fuzzy relations based on fuzzy relation Due to the usefulness of these concepts, the first aim of this memoir is to study a special type of fuzzy topological space, which is the fuzzy topologies generated by fuzzy relations and some of the fundamental properties. The second aim is to prove the family of open fuzzy set in this topology forms a lattice.

The memoir is divided into three chapters:

- The purpose of the first chapter is to provide a basic introduction to the notion of fuzzy sets, operation of fuzzy sets, characteristics of fuzzy sets, cartesian product, T-norms and T-conorms. Finally, we recall a basic definitions and properties of fuzzy relations.
- In the second chapter, we treat the concept of fuzzy topology introduced by C.L Change as a generalization of classical topology. In this sense, we study some notions related to fuzzy topology like: nieghborhoods, closed, interior ...ect. Finally, we discuss properties in fuzzy topology.
- In the third chapter, we present some definitions, properties and theorems of fuzzy topologies based on fuzzy relations. Moreover, we study the lattice structure of the family of fuzzy open sets in this topology.

Chapter 1

Generalities on fuzzy sets and relations

Fuzzy sets were introduced by Zadeh [24] as a generalization of crisp sets. The purpose of this chapter is to provide a basic introduction to the notion of fuzzy sets, operation of fuzzy sets, characteristics of fuzzy sets, cartesian product, T-norms and T-conorms. Finally, we recall a basic definitions and properties of fuzzy relations. Many of the properties of these concepts will be used in the next chapter. For more details see [3, 10, 14, 17, 24, 25].

1.1 Generalities on fuzzy sets

In this section, we give a basic introduction to the notion of fuzzy set, operation of fuzzy sets, characteristics of fuzzy sets, cartesian product, T-norms and T-conorms.

1.1.1 Crisp sets

The concept of a set is fundamental in mathematics and intuitively can be described as a collection of objects possibly linked through some properties. The crisp set has clear boundaries, i.e., $x \in A$ or $x \notin A$ exclude any other possibility.

Definition 1.1. [3] *The set can be defined by:*

(i) Writing of all its elements, whose elements are $\alpha_1, \alpha_2, ..., \alpha_n$*, and we write,* $A = \{\alpha_1, \alpha_2, ..., \alpha_n\}$.

(ii) A property or properties are satisfied by its elements, and we write, $A = \{x \mid P(x)\}$. *Where the symbol "*$|$*" means the sentence "such that" and* $P(x)$ *a proposal of the form "*x *a* P *property".*

(iii) *Let X be a set and A be a subset on X $\;(A \subseteq X)$. Then the function*

$$\begin{array}{rl} \chi_A : X \longrightarrow \{0,1\} \\ x \longmapsto & \begin{cases} 0 \;\; if \;\; x \;\notin\; A; \\ 1 \;\; if \;\; x \;\in\; A, \end{cases} \end{array}$$

$$\begin{array}{rcl} \chi_A : X & \longrightarrow & \{0,1\} \\ x & \longmapsto & \begin{cases} 0 \;\; if \;\; x \;\notin\; A; \\ 1 \;\; if \;\; x \;\in\; A, \end{cases} \end{array}$$

is called the characteristic function of the set $A \in X$.

Classical sets and their operations can be represented by their characteristic function.

Definition 1.2 (Operations on crisp sets)**.** *Let X be a universal set, let A and B are a subsets on X.*

(i) *Inclusion: $A \subset B$ if and only if any $x \in X$, $(x \in A) \Rightarrow (x \in B)$,i,e.,$\chi_A(x) \leq \chi_B(x)$.*

(ii) *Equality: $A = B$ if and only if any $A \subseteq B$ and $B \subseteq A$ i.e., $(\chi_A(x) = \chi_B(x))$.*

(iii) *Complement: $A^c = \{x \in X \mid x \notin X\}$ i.e., $(\chi_{A^c}(x) = 1 - \chi_A(x))$.*

(iv) *Intersection: $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$i.e., $\chi_{A\cap B}(x) = min(\chi_A(x), \chi_B(x))$.*

(v) *Union : $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$i.e., $\chi_{A\cup B}(x) = max(\chi_A(x), \chi_B(x))$.*

(vi) *Relative complement: $A \setminus B = A - B = A \cap B^c = \{x \in X \mid x \in A \text{ and } x \notin B\}$i.e.,*

$$\chi_{A-B}(x) = \chi_{A\cap B^c}(x) = min(\chi_A(x), \chi_{B^c}(x)).$$

Example 1.1. *Let $X = \{x, y, z, t, u\}$ be a set, let A and B be two subsets on X such that $A = \{x, y, t, u\}$ and $B = \{x, y, z\}$. Then*

$$A^c = \{z\};$$

$$B^c = \{t, u\};$$

$$A \cap B = \{x, y\};$$

$$A \cup B = \{x, y, z, t, u\};$$

$$A \setminus B = \{t, u\};$$

$$B \setminus A = \{z\}.$$

1.1.2 Fuzzy sets

This subsection contains the basic definitions and properties of fuzzy sets, and several operations on fuzzy sets. The notion of fuzzy set was introduced in 1965 by L.A. Zadeh [24].

Definition 1.3. [24] *Let X be a non empty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.*

Example 1.2. *(1) Let $X = \{a, b, c\}$. We define the fuzzy set A on X by:*
$A = \{\langle a, 0.1\rangle, \langle b, 0.4\rangle, \langle c, 1\rangle\}$.

(2) Let $X = [0, 10]$, and $A : [-5, 5] \to [0, 1]$ is a fuzzy subset on X, defined by :

$$A(x) = \frac{1}{1+x^2}.$$

This fuzzy set can model the linguistic expression "real number near 0".

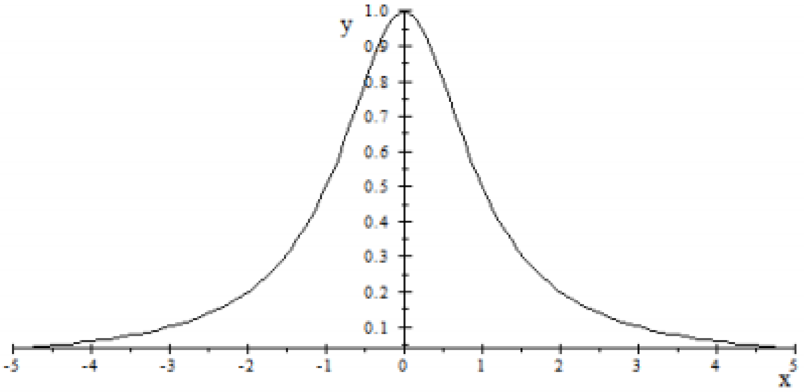
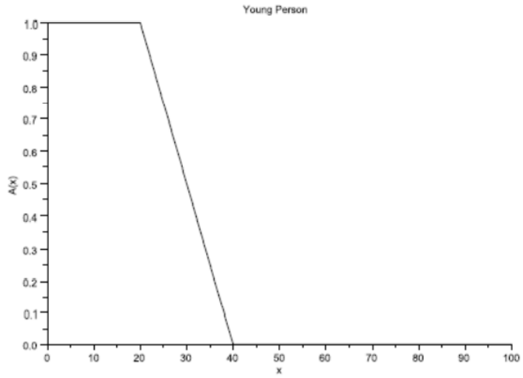


Figure 1.1: The membership function of the fuzzy set A.

(3) In this example, we consider the expression "young"in the context "a young person" in order to exemplify how linguistic expression can be modeled using fuzzy sets. The fuzzy set $A : [0, 100] \to [0, 1]$

$$A(x) = \begin{cases} 1 \ , \ if \ 0 \leqslant x \leqslant 20, \\ \dfrac{40-x}{20} \ , \ if \ \ 20 \leqslant x \leqslant 40, \\ 0, \ if \qquad otherwise. \end{cases}$$



Graph of fuzzy set for modeling the expression young person.

1.1.3 Operations of fuzzy sets

Definition 1.4 (**Inclusion**). [24] *Let* X *be a non empty set and let* A *and* B *be two fuzzy subsets, we say that* $A \subseteq B$, *if and only if* $\mu_A(x) \leq \mu_B(x)$, *for any* $x \in X$.

Definition 1.5 (**Equality**). [24] *Let* X *be a non empty set and let* A *and* B *be two fuzzy subsets, we say that* $A = B$, *if and only if* $\mu_A(x) = \mu_B(x)$, *for any* $x \in X$.

Definition 1.6 (**Intersection**). [24] *Let* X *be a non empty set and let* A *and* B *be two fuzzy subsets, the intersection defined by for any* $x \in X$

$$\mu_{A \cap B}(x) = min\left\{\mu_A(x), \mu_B(x)\right\} = \mu_A(x) \wedge \mu_B(x).$$

Definition 1.7 (**Union**). [24] *Let* X *be non empty set and let* A *and* B *be two fuzzy subsets, the union defined by for any* $x \in X$

$$\mu_{A \cup B}(x) = max\left\{\mu_A(x), \mu_B(x)\right\} = \mu_A(x) \vee \mu_B(x).$$

Definition 1.8 (**Complement**). [24] *The complement of a fuzzy set* A *denoted by* $C(A)$ *and is defined by for any* $x \in X$

$$\mu_{C(A)}(x) = 1 - \mu_A(x).$$

Definition 1.9 (**Sum**). [24] *Let* X *be non empty set and let* A *and* B *be two fuzzy subsets, the sum defined by for any* $x \in X$

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x).$$

Definition 1.10 (Product). [24] *Let X be non empty set and let A and B be two fuzzy subsets, the product defined by for any $x \in X$*

$$\mu_{A\times B}(x) = \mu_A(x)\mu_B(x).$$

Example 1.3. *Let $X = \{a, b, c\}$, let $A = \{\langle a, 0.1\rangle; \langle b, 0.4\rangle; \langle c, 0.3\rangle\}$, and* $B = \{\langle a, 0.3\rangle; \langle b, 0.2\rangle; \langle c, 1\rangle\}$. *Then*

1. $A \cap B = \{\langle a, 0.1\rangle; \langle b, 0.2\rangle; \langle c, 0.3\rangle\}$;
2. $A \cup B = \{\langle a, 0.3\rangle; \langle b, 0.4\rangle; \langle c, 1\rangle\}$;
3. $A \times B = \{\langle a, 0.03\rangle; \langle b, 0.08\rangle; \langle c, 0.3\rangle\}$;
4. $A + B = \{\langle a, 0.37\rangle; \langle b, 0.52\rangle; \langle c, 1\rangle\}$;
5. $C(A) = \{\langle a, 0.9\rangle; \langle b, 0.6\rangle; \langle c, 0.7\rangle\}$.

1.1.4 Characteristics of fuzzy sets

Definition 1.11 (Support). [24, 25] *Let A be a fuzzy set on a set X. The support of A is the crisp subset on X given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 1.12 (Kernel). [24, 25] *Let A be a fuzzy set on a set X. The kernel of A is the crisp subset on X given by*

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Definition 1.13 (Height). [24, 25] *Let A be a fuzzy set on a set X. The height of A is the highest value taken by its membership function given by*

$$H(A) = sup\,\{\mu_A \mid x \in X\}.$$

Definition 1.14 (Cardinality). [24, 25] *The cardinality of a finite fuzzy set A, denoted $\mid A \mid$ is defined as*

$$\mid A \mid = \sum_{x\in X} \mu_A(x).$$

Definition 1.15 (**α-cuts**)**.** [24, 25] *Let A be a fuzzy set on a set X. The α-cut of Ais a crisp subset, denoted A_α.*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Where $\alpha \in [0,1]$, $A_0 = X$ and $A_1 = Ker(A)$.

Example 1.4. *(1) Let $X = \{a, b, c, d\}$, and $A = \{\langle a, 0.7\rangle, \langle b, 1\rangle, \langle c, 0\rangle, \langle d, 0.1\rangle\}$*

$A_{0.6} = \{a, b\}$;

$Supp(A) = \{a, b, d\}$;

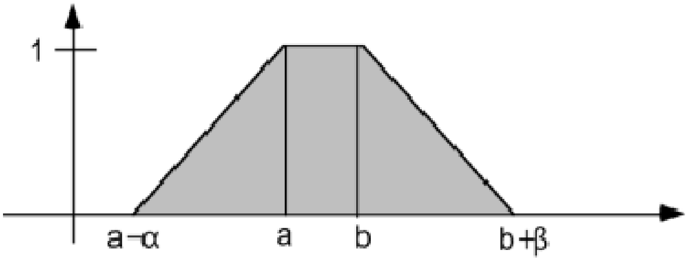
$ker(A) = \{b\}$;

$H(A) = 1$;

$|A| = 1.8$.

(2) Let $X = [0,1]$ with $\alpha, \beta \in \mathbb{R}$ and let $a, b \in \mathbb{R}$. We defined the fuzzy set A on X by:

$$\mu_A(x,y) = \begin{cases} 0\ ,\ if\ x < a-\alpha \text{ or } b+\beta < x, \\ 1\ ,\ if\ \ a < x < b, \\ 1 + (\dfrac{x-a}{\alpha}),\ if\ \ a-\alpha < x < a, \\ 1 - (\dfrac{b-x}{\beta}),\ if\ \ b < x < b-\beta. \end{cases}$$



Graph of μ_A

Then, $kerA = [0,1]$, $supA = [a-\alpha, b+\beta]$ and $H(A) = 1$.

Proposition 1.1. *Let X be a nonempty set and let A be a fuzzy subset on X. Then*

(i) $Supp(A^c) = (Ker(A))^c = X - Ker(A)$;

(ii) $Ker(A^c) = (Supp(A))^c = X - Supp(A)$.

Proof. (i)

$$\begin{aligned} Supp(A^c) &= \{x \in X | A^c(x) \neq 0\}\,; \\ &= \{x \in X | 1 - A(x) \neq 0\}\,; \\ &= \{x \in X | x \notin ker(A)\}\,; \\ &= X - Ker(A). \end{aligned}$$

(ii)

$$\begin{aligned} Ker(A^c) &= \{x \in X | A^c(x) = 1\}\,; \\ &= \{x \in X | 1 - A(x) = 1\}\,; \\ &= \{x \in X | A(x) = 0\}\,; \\ &= \{x \in X | x \notin Supp(A)\}\,; \\ &= X - Supp(A). \end{aligned}$$

□

Theorem 1.1. [10] *Any fuzzy subset A of the reference set X is defined from its α-cute for any x on X.*

$$\mu_A(x) = \underset{x \in]0,1]}{sup}\,(\alpha.\chi_{A_\alpha}(x)).$$

Let χ_{A_α} is the characteristic function of A^α.

Proof. Let the characteristic function:

$$\chi_{A_\alpha} = \begin{cases} 1 \;\; if \; \mu_A(x) \geq \alpha \;; \\ 0 \;\; if \;\; otherwise. \end{cases}$$

By multipling each member by a real number α, we obtain

$$\alpha\chi_{A_\alpha} = \begin{cases} \alpha \;\; if \; \mu_A(x) \geq \alpha \;; \\ 0 \;\; if \;\; otherwise. \end{cases}$$

By introducing the operator "sup" in each member, we get :

$\underset{x \in]0,1]}{sup}\,(\alpha\chi_{A_\alpha}(x)) = \underset{x \in]0,1]}{sup}\,\{\mu_A(x) \geq \alpha\}$,
this implies that $\underset{x \in]0,1]}{sup}\,(\alpha\chi_{A_\alpha}(x)) = \underset{x \in]0,1]}{sup}\,\{\alpha \leq \mu_A(x) \geq \alpha\}$.
We conclude that:

$$\mu_A(x) = \underset{x \in]0,1]}{sup}\,(\alpha.\chi_{A_\alpha}(x)).$$

□

Example 1.5. *Let* $X = \{0, 1, 2, 3, ..., 9\}$ *and* $A = \{\langle 1, 0.9\rangle; \langle 2, 0.3\rangle; \langle 3, 0.5\rangle; \langle 4, 1\rangle; \langle 5, 0.8\rangle\}$. *We have for any level* α *in* $[0, 1]$.

$$A_1 = \{x \in X \mid \mu_A(x) \geq 1\} = \{4\};$$

$$A_{0.9} = \{x \in X \mid \mu_A(x) \geq 0.9\} = \{1, 4\};$$

$$A_{0.8} = \{x \in X \mid \mu_A(x) \geq 0.8\} = \{1, 4, 5\}.;$$

$$A_{0.5} = \{x \in X \mid \mu_A(x) \geq 0.5\} = \{1, 3, 4, 5\};$$

$$A_{0.3} = \{x \in X \mid \mu_A(x) \geq 0.8\} = \{1, 2, 3, 4, 5\}.$$

Also, we get

$$\mu_A(1) = max(1 \times 0, ..., 0.9 \times 1, ..., 0 \times 1) = 0.9;$$

$$\mu_A(2) = max(1 \times 0, ..., 0.3 \times 1, ..., 0 \times 1) = 0.3;$$

$$\mu_A(3) = max(1 \times 0, ..., 0.5 \times 1, ..., 0 \times 1) = 0.5;$$

$$\mu_A(4) = max(1 \times 0, ..., 0 \times 1) = 1;$$

$$\mu_A(5) = max(1 \times 0, ..., 0.8 \times 1, ..., 0 \times 1) = 0.9.$$

1.1.5 Cartesian product and projection on fuzzy sets

The cartesian product of the fuzzy subsets is the minimum of these degrees of belonging and these projection is the maximun of these cartesian product.

Definition 1.16 (**Cartesian product**). [10] *The cartesian product applied to n fuzzy sets can be defined as follows. Let* $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, *be membership functions of* $A = A_1, A_2, ..., A_n$. *Then, the membership degree of* $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ *on the fuzzy set* $A = A_1 \times A_2 \times ... \times A_n$ *is given by*

$$\mu_A(x_1, x_2, ..., x_n) = min\,\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}\,.$$

Which provides the set A.

Example 1.6. *Lets* $X_1 = \{a, b, c, d\}$, $X_2 = \{\alpha, \beta\}$, *lets* A_1, A_2 *be two fuzzy subsets respectively defined on* X_1 *and* X_2 *by:*
$A_1 = \{\langle a, 0.2\rangle; \langle b, 0.3\rangle; \langle c, 0.6\rangle; \langle d, 0.1\rangle\}$;
$A_2 = \{\langle \alpha, 0.4\rangle; \langle \beta, 0.7\rangle\}$.
So, we get:

$$A_1 \times A_2 = \{\langle (a, \alpha), 0.2\rangle; \langle (a, \beta), 0.2\rangle; \langle (b, \alpha), 0.3\rangle; \langle (b, \beta), 0.3\rangle; \langle (c, \alpha), 0.4\rangle; \langle (d, \alpha), 0.1\rangle; \langle (d, \beta), 0.1\rangle\}\,.$$

Definition 1.17 (Projection). [10] *The projection on* X_1 *of the fuzzy set* A *of* $X_1 \times X_2 \times ... \times X_n$ *is the fuzzy set* $Proj_{X_1}(A)$ *of* X_1*, whose membership function is defined by: for any* $x_1 \in X_1$*,*

$$\mu_{ProjX_{1(A)}}(x_1) = sup_{x_2 \in X_2, x_3 \in X_3, ..., x_n \in X_n}(\mu_A(x_1, x_2, ..., x_n)).$$

Example 1.7. *Let* $X = X_1 \times X_2$*, such that* X_1 *and* X_2 *two sets, we consider* $A_1 \times A_2 = B$ *given by:*

$$B = \{\langle(a,\alpha),0.2\rangle;\langle(a,\beta),0.2\rangle;\langle(b,\alpha),0.3\rangle;\langle(b,\beta),0.3\rangle;\langle(c,\alpha),0.4\rangle;\langle(c,\beta),0.6\rangle;\langle(d,\alpha),0.1\rangle;\langle(d,\beta),0.1\rangle\};$$

So, we get:

$$\begin{aligned} Proj_{X_1}(B) &= \{\langle a, max(0.2,0.2)\rangle;\langle b, max(0.3,0.3)\rangle;\langle c, max(0.4,0.6)\rangle;\langle d, max(0.1,0.1)\rangle\} \\ &= \{\langle a,0.2\rangle;\langle b,0.3\rangle;\langle c,0.6\rangle;\langle d,0.1\rangle\}; \end{aligned}$$

$$\begin{aligned} Proj_{X_2}(B) &= \{\langle \alpha, max(0.2,0.3,0.4,0.1)\rangle;\langle \beta, max(0.2,0.3,0.6,0.1)\rangle\} \\ &= \{\langle \alpha,0.4\rangle;\langle \beta,0.6\rangle\}. \end{aligned}$$

1.1.6 T-norms and T-conorms

The history of triangular norms(t-norm) started with the paper [9]. The main idea was to construct metric spaces where probability distributions rather than numbers are used in order to describe the distance between two elements. Schweizer and Abe Sklar in [20] provided the axioms of t-norm, as they are used today.

Definition 1.18. [9] *A t-norms on* $[0,1]$ *is function* $T : [0,1]^2 \longrightarrow [0,1]$ *satisfies the following four axioms:*

(T1) Commutativity: $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$.

(T2) Associativity: $(\forall x, y, z \in [0,1])(T(x,T(y,z)) = T(T(x,y),z))$.

(T3) Monotonicity: $(\forall x, y, z \in [0,1])(x \leq y \Rightarrow T(x,y) \leq T(y,z))$.

(T4) Boundary condition: $(\forall x \in [0,1])(T(x,1) = x)$.

Condition $(T3)$ *and* $(T4)$ *imply that for any t-norms* T *it hold that* $T(x,y) \leq x$, $T(x,y) \leq y$, $T(x,y) \leq Min(x,y)$ *and* $T(x,0) = x$.

Example 1.8. *The following four operations are the most common t-norms:*

(T5) Minimum: $T_M(x,y) = min\{x,y\}$.

(T6) Product: $T_P(x,y) = x.y$.

(T7) Lukasiewicz: $T_L(x,y) = min\{x+y-1,0\}$.

(T8) Drastic product:

$$T_D(x,y) = \begin{cases} x & if \quad y = 1 \\ y & if \quad x = 1 \\ 0 & if \quad x,y < 1. \end{cases}$$

Let T be a t-norm on $[0,1]$.
An element $a \in]0,1[$ is called a zero divisor of T if there exists some $b > 0$ such that $T(a,b) = 0$.
An element $a \in [0,1]$ is called an idempotent element of T if $T(a,a) = a$.
T is called Archimedean if $T(x,x) < x$, for any $x \in [0,1]$.
Each $a \in [a,b]$ is an idempotent element of the Minimum t-norm T_M (Actually T_M is the only t-norm whose set of idempotent is equal $[0,1]$), T_M has no zero divisor. Each $\alpha \in]0,1[$ is a zero divisor of the Lukasiewicz t-norm T_L as well of the Drastic product t-norm T_D.
For two t-norms T_1 and T_2 on $[0,1]$, we define:

$$T_1 \leq T_2 \Leftrightarrow (\forall x,y \in [0.1])(T_1(x,y) \leq T_2(x,y)).$$

Let be T_1 and T_2 two t-norms. If $T_1 \leq T_2$, then T_1 is called weaker than T_2 (or, equivalently, T_2 is called stronger than T_1). Note that T_D is the weakest t-norm, and T_M is the strongest t-norm, i.e., for any t-norm it holds: (T9) $T_D \leq T \leq T_M$.
Since $T_L \leq T_p$, it obviously holds: (T10) $T_D \leq T_L \leq T_P \leq T_M$.
Triangular conorms (t-conorms) are dual operations of t-norms, we recall the following definition of conorms.

Definition 1.19. [9] *A triangular conorm is a binary operation* S *on the unit interval* $[0,1]$, *i.e., it is a function* $S : [0,1]^2 \longrightarrow [0,1]$ *: the following four axioms are satisfied*

(S1) Commutativity : $S(x,y) = S(y,x)$.

(S2) Associativity : $S(x, S(y, z)) = S(S(x, y), z)$.

(S3) Monotonicity : $S(x, y) \leq S(x, z)$ *whenever* $y \leq z$.

(S4) Boundary condition : $S(x, 0) = x$.

Example 1.9.

(1) Maximun: $S_M(x, y) = max\{x, y\}$.

(2) Probabilistic sum: $S_P(x, y) = x + y - s.y$.

(4) Lukasiewicz: $S_L(x, y) = max\{x + y, 1\}$.

(5) Drastic sum:

$$S_D(x, y) = \begin{cases} 1 & if \quad (x, y) \in [0, 1]^2; \\ max\{x, y\}, & \quad otherwise. \end{cases}$$

Example 1.10. *Let* $X = \{a, b, c\}$, *lets* A *and* B *be two fuzzy subsets on* X *as* $A = \{\langle a, 0.1\rangle; \langle b, 0.2\rangle; \langle c, 0.5\rangle\}, B = \{\langle a, 0.3\rangle; \langle b, 0.1\rangle; \langle c, 0.5\rangle\}$. *Lukasiewicz operators can be used to define intersection and union by:*

(i) $\mu_{A \cap_T B}(x) = max(\mu_A(x) + \mu_B(x) - 1, 0), \forall x \in X$;

(ii) $\mu_{A \cup_S B}(x) = min(\mu_A(x) + \mu_B(x), 1), \forall x \in X$.

Then, we get

(i) $A \cap_T B = \{\langle a, 0\rangle; \langle b, 0\rangle; \langle c, 0\rangle\}$;

(ii) $A \cup_S B = \{\langle a, 0.4\rangle; \langle b, 0.3\rangle; \langle c, 1\rangle\}$.

1.2 Fuzzy relations

This section contains the basic definitions and proprieties of fuzzy relation, fuzzy relations, fuzzy order relations and several operations on fuzzy relations. The notion of fuzzy relations was first introduced by Zadeh [25] as a natural extension of fuzzy set and plays an important role in the theory of such sets and their applications.

1.2.1 Basic definition of fuzzy relation

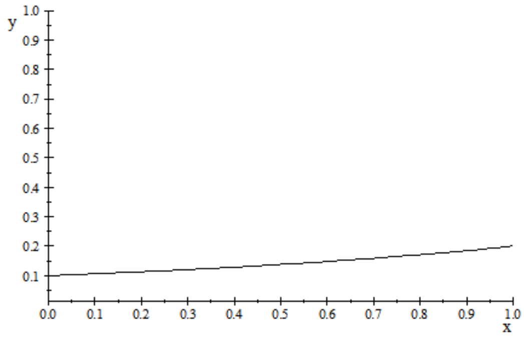
Definition 1.20. [25] *Let* X *and* Y *be two nonempty sets. A binary fuzzy relation from* X *to* Y*, is a fuzzy subset of* $X \times Y$ *characterized by a membership function* μ_R *which associates with each pair* (x, y) *its grade of membership* $R(x, y)$ *in the interval* $[0, 1]$.

Example 1.11. *Let* R *fuzzy relation on* $x = \{1, 2, 3\}$,

R	*1*	*2*	*3*
1	*1*	*0.7*	*0.2*
2	*0.7*	*1*	*0.7*
3	*0.2*	*0.7*	*1*

Example 1.12. *The fuzzy relation* R *"x approximately equal to* 3*"can be defined on* $R \times R$ *by the membership function:*

$$R(x, 3) = \frac{1}{1 + (x - 3)^2}.$$



Graph of $R(x, 3)$.

1.2.2 Operations on fuzzy relations

Let R, P be two fuzzy relations. R is said to be contained in P(or we say that P contains R), denoted by $R \subseteq P$, if for any $(x, y) \in X \times Y$ it hold that $R(x, y) \leq P(x, y)$.

The transpose (or the inverse) R^t of R is the fuzzy relation from Y to X defined by:

$$R^t = \{\langle (x, y), R^t(x, y)\rangle \mid (x, y) \in X \times Y\}.$$

The intersection of two fuzzy relations R and P is defined as

$$R \cap P = \{\langle (x, y), min(R(x, y), P(x, y))\rangle \mid (x, y) \in X \times Y\}.$$

The union of two fuzzy relations R and P is defined as

$$R \cup P = \{\langle (x, y), max(R(x, y), P(x, y))\rangle \mid (x, y) \in X \times Y\}.$$

Example 1.13. *Let R and P be two fuzzy relations on $X \times Y$, $X = \{x, y, z\}$,*

R	x	y	z
x	1	0.7	0.4
y	0.5	1	0.2
z	0.1	0.6	1

P	x	y	z
x	1	0.3	0
y	0.7	0.2	1
z	0.3	0	0.4

$R \cup P$	x	y	z
x	1	0.7	0.4
y	0.7	1	1
z	0.3	0.6	1

$R \cap P$	x	y	z
x	1	0.3	0
y	0.5	0.2	0.2
z	0.1	0	0.4

Proposition 1.2. [24, 25] *let R, P and Q be three fuzzy relations from a universe X to a universe Y*

(1) if $R \subset P$, then $R^t \subset P^t$;

(2) $(R \cup P)^t = R^t \cup P^t$;

(3) $(R \cap P)^t = R^t \cap P^t$;

(4) $(R^t)^t = R$;

(5) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ and $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(6) $R \subseteq R \cup P, P \subseteq R \cup P, R \cup P \subseteq R, R \cup P \subseteq P$;

(7) *if* $P \subseteq R$ *and* $Q \subseteq R$, *then* $P \cup Q \subseteq R$;

(8) *if* $R \subseteq P$ *and* $R \subseteq Q$, *thenR* $\subseteq P \cap Q$.

Definition 1.21. [24, 25] *Let* R *be a fuzzy relation (fuzzy relation on* X*, for short). The following properties are crucial in (see e.g* [23]*):*

(i) *Reflexivity: If* $R(x, x) = 1$, *for any* $x \in X$.

(ii) *Symmetry: If* $R(x, y) = R(y, x)$, *for all* $x, y \in X$.

(iii) *Antisymmetry: If* $x \neq y \quad R(x, y) = 0 \vee R(y, x) = 0$, *for all* $x, y \in X$.

(iv) *Transitivity: if* $R(x, z) \geq max\{min\{R(x, y), R(y, z)\}\}$, *for all* $x, y, z \in X$.

Definition 1.22. *Let* X *be a nonempty crisp set and* R *be a fuzzy relation on* X. R *is called fuzzy order or partial fuzzy order if it is reflexive, transitive and antisymmetric.*
A nonempty set X *with a fuzzy order* R *defined on it is called a fuzzy ordered set and is denoted by* (X, R). *It easily follows that each partially ordered set* $(X, \leq)$ *and each fuzzy ordered set* (X, R) *can be viewed as fuzzy ordered sets.*

Example 1.14. *Let* $X = \{a, b, c, d\}$, *then the fuzzy relation* R *defined on* X *by*

$$R = \{\langle (x, y), R(x, y) \rangle \mid (x, y) \in X\}$$

R(.,.)	*a*	*b*	*c*	*d*
a	*1*	*0.6*	*0.8*	*0.8*
b	*0*	*1*	*0*	*0.2*
c	*0*	*0.6*	*1*	*0.4*
d	*0*	*0*	*0*	*1*

Then, R *is a fuzzy order relation.*

Example 1.15. *Let* $m, n \in \mathbb{N}$. *Then the following fuzzy relation* R *on* $\mathbb{N}$ *is a fuzzy order relation, where*

$$R(x, y) = \begin{cases} 1, \ if \ m = n; \\ 1 - \frac{m}{n} \ , \quad if \ m < n; \\ 0 \ , \quad if \quad m > n. \end{cases}$$

On the basis of the above definition of antisymmetry we define a complete (or total) fuzzy order as follows.

Definition 1.23. *A fuzzy order R on a universe X is called complete (or total) if for all $x, y \in X$ it holds that*

$$[R(x, y) > 0 \quad or \quad (R(y, x) > 0)].$$

Example 1.16. *Let Ra fuzzy relation on $X = \{x, y, z\}$ given by:*

R	x	y	z
x	*1*	*0.7*	*0.7*
y	*0*	*1*	*0.5*
z	*0*	*0*	*1*

R is a fuzzy order total.

Definition 1.24. *A fuzzy ordered set (X, R) in which R is linear is called a linearly fuzzy ordered set or a fuzzy chain.*

Chapter 2

Fuzzy topological spaces

In this chapter, we study the concept of fuzzy topology introduced by C.L. Change [5] as a generalization of the classical topology. In this sense, we treat some notions related to this notion fuzzy topology like: neighborhoods, closed, interior,..., ect. Finally, we discuss some fundamental properties of fuzzy topology. For more details see [5, 8, 12, 16, 19].

2.1 Definitions and examples

In this section, we recall the notion of fuzzy topology introduced by C.L. Change [5] with related notions that will be used in this memoir.

Definition 2.1. [5] *A fuzzy topological space is a pair (X, τ) consisting of a non-empty set X and a family τ of fuzzy sets in X satisfying the following conditions:*

(1) $\emptyset, X \in \tau$.

(1) If $\{A_i : i \in \Omega\}$ *is an arbitrary family of fuzzy sets in* τ*, then* $\bigcup\limits_{i\in\Omega} A_i \in \tau$.

(3) If $A, B \in \tau$*, then* $A \cap B \in \tau$.

τ is called a fuzzy topology for X, and the pair (X, τ) is fuzzy topological space, or FTS for short. Every member of τ is called a τ-open fuzzy set (OFS, for short). A fuzzy set is τ -closed (CFS, for short) if and only if its complement is τ-open. In the sequel when no confusion is likely to arise, we shall call a τ-open (τ-closed) fuzzy set simply an open (closed) set. As (ordinary) topologies, the indiscrete fuzzy topology contains only $\emptyset$ and X, while the discrete fuzzy topology contains all fuzzy sets. A fuzzy topology T is said to be coarser than a fuzzy topology τ if and only if $T \subset \tau$.

Definition 2.2. *Let (X, τ) fuzzy topological space.*

(1) For each element of (X, τ) is called open.

(2) A fuzzy closed if and only if its complement is a fuzzy open.

Example 2.1. *(1) Let $X = \{a, b\}$ and A be a fuzzy subset on X given by $A = \{\langle a, 0.6\rangle; \langle b, 0.7\rangle\}$. Then $\tau = \{\emptyset, A, X\}$ is a fuzzy topology and (X, τ) is fuzzy topological space, and A is a fuzzy open and $\emptyset, X$ are open fuzzy sets and closed fuzzy sets.*

Example 2.2. *Let $X = [0, 1]$ and let $h \in]0, 1]$, we consider the following function:*

$$f_h(x, y) = \begin{cases} 2hx & if \quad x \in [0, \frac{1}{2}]; \\ 2h(1-x), & if \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

The family $\tau = \{f_h : 0 < h \leq 1\} \cup \{\emptyset, X\}$ is a fuzzy topology and (X, τ) is fuzzy topological space.

Definition 2.3. [6] *A fuzzy set U in a FTS (X, τ) is a neighborhood (nbhd, for short), of a fuzzy set A if and only if there exists an open fuzzy set O such that $A \subset O \subset U$.*
The above definition differs some what from the ordinary one in that we consider here a nbhd of a fuzzy set instead of a nbhd of a point.

Example 2.3. *1- Let $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$, then*
$N(1) = \{\{1\}, \{1, 2\}, \{1, 3\}, X\}$;
$N(2) = \{\{2\}, \{1, 2\}, \{2, 3\}, X\}$.

2- Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}\}$, then
$N(a) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$;
$N(b) = \{X\}$;
$N(c) = \{\{a, c\}, \{a, c, b\}, \{a, c, d\}\}$;
$N(d) = \{\{a, d\}, \{a, d, b\}, \{a, d, c\}, X\}$.

Theorem 2.1. [6] *A fuzzy set A is open if only if for each fuzzy set B contained in A, A is a nbdh of B.*

Proof. $(\Rightarrow)$ Obvious.

($\Leftarrow$) Since $A \subset A$, there exists an open fuzzy set O such that $A \subset O \subset A$.

Hence, $A = O$ and A is open.

□

The nbhd system of fuzzy set is the family of all nbhd's of the fuzzy set.

Theorem 2.2. [6] *If U is the nbhd system of a fuzzy set, then the finite intersections of members of U belong to U, and each fuzzy set which contains a member of U belongs to U.*

Proof. If R and S are nbhd's of a fuzzy set A, there are open nbhd's R_0 and S_0 contained on R and S, respectively. Then $R \cap S$ contains the open nbhd $R_0 \cap S_0$ and is hence a nbhd on A . Thus the intersection of two (and hence of any finite number of) members of U is a member on U. Hence, if a fuzzy set R contains a nbhd of A it contains an open nbhd of A and consequently is itself a nbhd. □

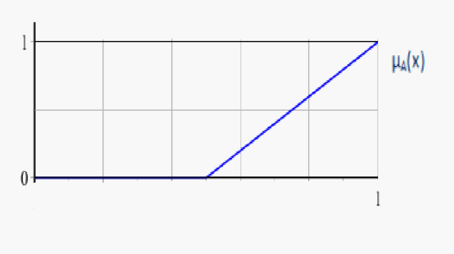
Definition 2.4. [6] *Let A and B are a fuzzy sets in a FTS (X, τ), and let $B \subset A$. Then B is called an interior fuzzy set of A if and only if A is a nbhd of B. The union of all interior fuzzy sets of A is called the interior of A and is denoted by A^0.*

Definition 2.5. [15] *Let (X, τ) be fuzzy topological space. A is a fuzzy subset on X. Closing A is a fuzzy set $\overline{A}$ defined by:*

$$\overline{A} = \{< x, \max_{x \in X} \mu_A(x) >| \; x \in X\}.$$

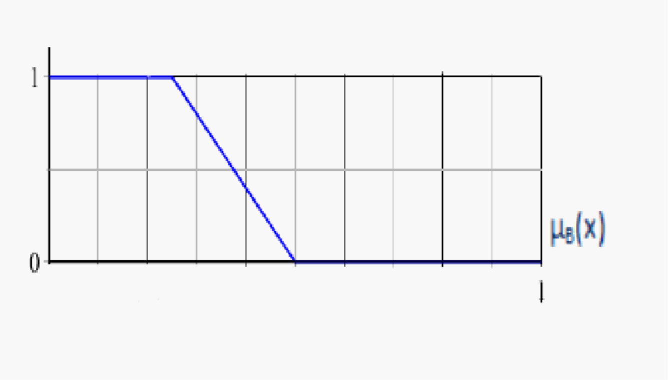
Example 2.4. [15] *Let A and B two fuzzy sets on $X = \mathbb{R}$*

$$A(x) = \begin{cases} 0 & if \quad 0 \leq x \leq \frac{1}{2}; \\ 2x - 1 & if \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$



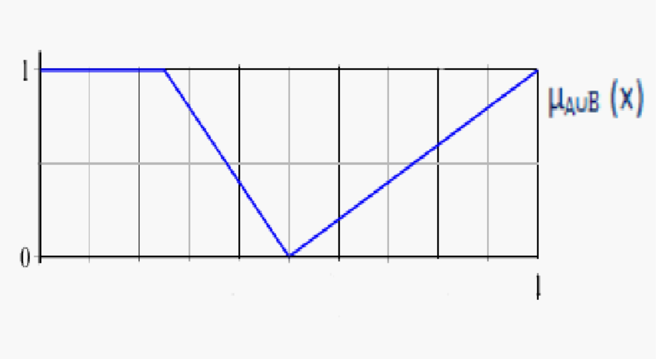
Graph of μ_A.

$$B(x) = \begin{cases} 1 & if \quad 0 \le x \le \frac{1}{4}; \\ -4x+2 & if \quad \frac{1}{4} \le x \le \frac{1}{2}; \\ 0 & if \quad \frac{1}{2} \le x \le 1. \end{cases}$$



Graph of μ_B.

And the curve of $A \cup B$ given by:



Graph of $\mu_{A \cup B}$.

Then, $\tau = \{\emptyset, A, B, A \cup B, X\}$ is a fuzzy topology on X, and with $\overline{A} = B^c$, $\overline{B} = A^c$, $(A^c)^0 = B$, $(B^c)^0 = A$, and $((A \cup B)^c)^0 = \emptyset$.

Theorem 2.3. [6] *Let A be a fuzzy set in a FTS (X, τ). Then A^0 is open and is the largest open fuzzy set contained on A. The fuzzy set A is open if and only if $A = A^0$.*

Proof. By Definition 2.4, A^0 is itself an interior fuzzy set of A. Hence, there exists an open fuzzy set O such that $A^0 \subset O \subset A$. But O is an interior fuzzy set of A. Hence, $O \subset A^0$. Therefore $A^0 = O$. Thus, A^0 is open and is the largest open fuzzy set contained on A. If A is open, then $A \subset A^0$, for A is an interior fuzzy set of A. Hence, $A = A^0$. The converse is obviously true. □

Definition 2.6. [16] *Let (X, τ) be a fuzzy topological space. Then a subfamily $\mathcal{B}$ of τ is called a base for τ if every member of τ can be written as a union of members of $\mathcal{B}$.*

Definition 2.7. [16] *Let (X, τ) be a fuzzy topological space. Then a subfamily $\mathcal{S}$ of τ is called a subbase for τ if the family of finite intersection of its members from a base for τ.*

Definition 2.8. [22] *A fuzzy topology τ is said to be generated by a subfamily $\mathcal{S}$ of fuzzy set on X if every member of τ is a union of finite intersection of members of $\mathcal{S}$.*

Example 2.5.

(1) Let $X = \mathbb{R}$, $(X, \tau)$is fuzzy topological space, and $\mathcal{B} = \{[a, b] : a, b \in \mathbb{R} \quad \text{and} \quad a < b\}$, therefore exists a unique fuzzy topology τ for which $\mathcal{B}$ is a base.

(2) Let $X = \mathbb{R}$, have the usual fuzzy topology then $\mathcal{S} = \{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ is a subbase but not a base.

2.2 Fundamental properties of fuzzy topological spaces

In this section, we study some fundamental properties of fuzzy topological spaces. For more details see [5, 6].

Definition 2.9. [6] *Let f be a function from X to Y. Let B be a fuzzy set on Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}[B]$, is a fuzzy set on X whose membership function is defined by*

$$\mu_{f^{-1}[B]} = \mu_B(f(x)) \qquad for \quad all \quad x \quad on \quad X.$$

Conversely, let A be a fuzzy set in X with membership function μ_A. The image of A, written as $f[A]$, is a fuzzy set on Y whose membership function is given by

$$\begin{aligned} \mu_{f[A]} &= \{sup\{\mu_A(z)\}\} \qquad if \quad f^{-1}[y] \text{is not empty}, \\ &= 0 \qquad\qquad\qquad otherwise, \end{aligned}$$

for all y on Y, where $f^{-1}[y] = \{x \mid f(x) = y\}$.

Theorem 2.4. [6] *Let X, Y be two fuzzy topological spaces and f be a function from X to Y and let A, A' are two fuzzy subsets on X and B, B' are two fuzzy subsets on Y. Then*

(a) $f^{-1}[B'] = \{f^{-1}[B]\}'$ *for any fuzzy set B on Y.*

(b) $\{f[A]\}' \subset f[A']$ *for any fuzzy set A on X.*

(c) $B_1 \subset B_2, \Rightarrow f^{-1}[B_1] \subset f^{-1}[B_2]$, *where B_1, B_2 are fuzzy sets on Y.*

(d) $A_1 \subset A_2, \Rightarrow f[A_1] \subset f[A_2]$, *where A_1, A_2 are fuzzy sets on X.*

(e) $f[f^{-1}[B]] \subset B$ *for any fuzzy set B on Y.*

(f) $A \subset f^{-1}[f[A]]$ *for any fuzzy set A on X.*

(g) Let f be a function from X to Y and g be a function from Y to Z. Then $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$ for any fuzzy set C on Z, where $g \circ f$ is the composition of g and f.

Proof. (a) For any x on X,

$$\begin{aligned} \mu_{f^{-1}[B']}(x) &= \mu_{B'}[f(x)] = 1 - \mu_B[f(x)] = 1 - \mu_{f^{-1}[B]}(x) \\ &= \mu_{\{f^{-1}[B]\}'(x)}. \end{aligned}$$

(b) For any $y \in Y$, if $f^{-1}[y]$ is not empty, then

$$\begin{aligned}\mu_{f[A']}(y) = \underset{z\in f^{-1}[y]}{sup}\{\mu_{A'}(z)\} &= \underset{z\in f^{-1}[y]}{sup}\{1-\mu_A(z)\}\\ &= 1-\underset{z\in f^{-1}[y]}{inf}\{\mu_A(z)\},\end{aligned}$$

and

$$\mu_{\{f[A]\}'}(y) = 1-\mu_{f[A]}(y) = 1-\underset{z\in f^{-1}[y]}{sup}\{\mu_A(z)\}.$$

Hence,

$$\mu_{f[A']}(y) \geq \mu_{\{f[A]\}'}(y).$$

(c) For any x on X,

$$\mu_{f^{-1}[B_1]}(x) = \mu_{B_1}[f(x)]$$

and

$$\mu_{f^{-1}[B_2]}(x) = \mu_{B_2}[f(x)] \quad for \;\; any \;\; x \in X.$$

Since

$$B_1 \subset B_2, \qquad \mu_{f^{-1}[B_1]}(x) \leq \mu_{f^{-1}[B_2]}(x) \qquad for \;\; any \;\; x \in X.$$

Hence,

$$f^{-1}[B_1] \subset f^{-1}[B_2]\ .$$

(d) $\mu_{f[A_1]} = \underset{z\in f^{-1}[y]}{sup}\{\mu_{A_1}(z)\} \qquad and \qquad \mu_{f[A_2]} = \underset{z\in f^{-1}[y]}{sup}\{\mu_{A_2}(z)\}.$
Therefore, $A_1 \subset A_2$

$$\mu_{f[A_1]}(y) \leq \mu_{f[A_2]}(y) \qquad for \;\; any \;\; y \in Y.$$

Since

$$f[A_1] \subset f[A_2].$$

(e) If $f^{-1}[y]$ is not empty,

$$\mu_{f[f^{-1}[B]]}(y) = \underset{z\in f^{-1}[y]}{sup}\{\mu_{f^{-1}[B]}(z)\} = \underset{z\in f^{-1}[y]}{sup}\{\mu_B(f(z))\} = \mu_B(y).$$

If $f^{-1}(y)$ is empty

$$\mu_{f[f^{-1}[B]]}(y) = 0.$$

Therefore,

$$\mu_{f[f^{-1}[B]]}(y) \leq \mu_B(y) \qquad for \;\; any \quad y \in Y.$$

(f) for any $x \in X$.

$$\mu_{f^{-1}[f[A]]}(x) = \mu_{f[A]}[f(x)] = \underset{z \in f^{-1}[f(x)]}{sup} \{\mu_A(z)\} \geq \mu_A(x).$$

(g) For any $x \in X$,

$$\begin{aligned} \mu_{(g \circ f)^{-1}[C]}(x) &= \mu_C[g \circ f(x)] = \mu_C[g[f(x)]] \\ &= \mu_{g^{-1}[C]}[f(x)] = \mu_{f^{-1}[g^{-1}[C]]}(x). \end{aligned}$$

□

Definition 2.10. [6] *A function f from a FTS (X, τ_1) to a FTS (Y, τ_2) is fuzzy continuous if and only if the inverse of each τ_2-open fuzzy set is τ_1-open set.*
Clearly, if f is an fuzzy continuous function on X to Y and g is an fuzzy continuous function on Y to Z, then the composition $g \circ f$ is an fuzzy continuous function on X to Z, for $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$ for each fuzzy set V on Z, and using the fuzzy continuity of g and f it follows that if V is open so is $(g \circ f)^{-1}[V]$.

Theorem 2.5. [6] *If X and Y are two fuzzy topological spaces, and f is a function on X to Y, then the conditions below are related as follows: (a) and (b) are equivalent;(c) and (d) are equivalent; (a) implies(c), and (d) implies(e).*

(a) The function f is fuzzy continuous.

(b) The inverse of every closed fuzzy set is closed.

(c) For each fuzzy set A on X, the inverse of every nbhd of $f[A]$ is a nbhd of A.

(d) For each fuzzy set A on X and each nbhd V of $f[A]$, there is a nbhd W of A such that $f[W] \subset V$.

(e) For each sequence of fuzzy sets $\{A_n, n = 1, 2, ...\}$ on X which converges to a fuzzy set A in X, the sequence $\{f[A_n], n = 1, 2, ...\}$ converges to $f[A]$.

Proof. (a)⇔ (b) This is an immediate consequence of the fact that $f^{-1}[B'] = \{f^{-1}[B]\}'$ for every fuzzy set B on Y.
(a) ⇒(c) If f is fuzzy continuous, A is a fuzzy set on X, and V is a nbhd of $f[A]$, therefore V contains an open nbhd W of $f[A]$. Since $f[A] \subset W \subset V, f^{-1}[f[A]] \subset f^{-1}[W] \subset f^{-1}[V]$. But $A \subset f^{-1}[f[A]]$ and $f^{-1}[W]$ is open. Thus, $f^{-1}[V]$ is a nbhd of A.

(c) $\Rightarrow$ (d) Since $f^{-1}[V]$ is a nbhd of A, we have $f[W] = f[f^{-1}[V]] \subset V$, wherever $W = f^{-1}[V]$.
(d) $\Rightarrow$(c) Suppose V be a nbhd of $f[A]$. Then there is a nbhd W of A such that $f[W] \subset V$. Hence, $f^{-1}[f[W]] \subset f^{-1}[V]$. Moreover, since $W \subset f^{-1}[f[W]]$, $f^{-1}[V]$ is a nbhd of A.
(d) $\Rightarrow$ (e) If V is a nbhd of $f[A]$, there is a nbhd W of A such that $f[W] \subset V$. Since$\{A_n, n = 1, 2, ...\}$ is eventually contained in W, i.e., there is an m such that for $n \geq m, A_n \subset W$, we have $f[A_n] \subset f[W] \subset V$ for $n \geq m$.
Therefore $\{f[A_n], n = 1, 2, ...\}$ converges to $f[A]$. □

A fuzzy homeomorphism is an fuzzy continuous one-to-one map of a FTS X onto a FTS Y such that the inverse of the map is also fuzzy continuous. If there exists a fuzzy homeomorphism of one fuzzy space onto another, the two fuzzy spaces are said to be F-homeomorphic and each is a fuzzy homeomorph of the other. Two FTS's are topologically fuzzy equivalent if and only if they are F-homeomorphic.

2.3 Compact fuzzy topological space

In this section, we study the Compactness properties for fuzzy topological spaces. For more details[6].

Definition 2.11. [5, 6] *A family $\mathcal{A}$ of fuzzy set is a cover of a fuzzy set B if and only if $B \subset \cup\{A \mid A \in \mathcal{A}\}$. It is an open cover if and only if each member of $\mathcal{A}$ is an open fuzzy set. A subcover of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover.*

Example 2.6. *Let $C = \{(-n, n) \mid n \in \mathbb{N}\}$ and $C' = \{(-3n, 3n) \mid n \in \mathbb{N}\}$.
Both are open cover, but C' is subcover of C.*

Definition 2.12. [6] *A FTS (X, τ) is compact if and only if each open cover has a finite subcover.*

Example 2.7. *(i) Let B_x be a fuzzy set with continuous membership function μ_{B_x} such that*

$$\mu_{B_x}(y) = \begin{cases} 1 & if \quad y = x \\ 0 & if \quad otherwise. \end{cases}$$

Hence, the family $\{B_x\}$, $x \in X$ forms an open cover of (X, τ). Clearly, it contains no finite subcover if X is not a finite set, hence (X, τ) is not compact.

(ii) Let $\{V_n\}, n = 1, 2, ...,$ be a countable family of disjoint subsets on X such that their union is X. Consider the family of fuzzy sets $\{B_n\}, n = 1, 2, ...,$ with continuous membership functions such that

$$\mu_{B_x}(y) = \begin{cases} 1 & if \quad y \in V_n \\ 0 & if \quad otherwise. \end{cases}$$

Clearly,$\{B_n\}, n = 1, 2, ...,$ is a countable open cover of (X, τ). If all V_n's are nonempty, then $\{B_n\}$ has no finite subcover.

Definition 2.13. [6] *A family $\mathcal{A}$ of fuzzy sets has the finite intersection property if and only if the intersection of the members of each finite subfamily of $\mathcal{A}$ is nonempty.*

Theorem 2.6. [6] *A FTS is compact if and only if each family of closed fuzzy sets which has the finite intersection property has a nonempty intersection.*

Proof. If $\mathcal{A}$ is a family of fuzzy sets in a FTS (X, τ), thereafter $\mathcal{A}$ is a cover of X if and only if $\cup\{A \mid A \in \mathcal{A}\} = X$, or if and only if $\{\cup[A \mid A \in \mathcal{A}]\}' = X' = \emptyset$, or if and only if

$\cap\{A' \mid A \in \mathcal{A}\} = \emptyset$ by the De Morgan's laws. Therefore, the fuzzy space X is compact if and only if each family of open fuzzy sets on X such that no finite subfamily covers X, fails to be a cover, and this is true if and only if each family of closed fuzzy sets which possesses the finite intersection property has a nonempty intersection. $\square$

Theorem 2.7. [6] *Let f be an fuzzy continuous function carrying the compact FTS (X) on to the FTS (Y). Then Y is compact.*

Proof. Let $\mathcal{B}$ be an open cover on Y. Then,

$$\mu_{\underset{B\in\mathcal{B}}{\cup} f^{-1}[B]}(x) = \underset{B\in\mathcal{B}}{sup}\{\mu_{f^{-1}[B]}(x)\} = \underset{B\in\mathcal{B}}{sup}\{\mu_B(f(x))\} = 1 \quad for \;\; any \;\; x \in X,$$

the family of all fuzzy sets of the form $f^{-1}[B]$, for B in $\mathcal{B}$, is an open cover of X which has a finite subcover. However, if f is onto, then it is easily seen that $f[f^{-1}[B]] = B$ for any fuzzy set B on Y. Thus, the family of images of members of the subcover is a finite subfamily of $\mathcal{B}$ which covers Y and consequently Y is compact. $\square$

Chapter 3

Fuzzy topologies based on fuzzy relations

In 2009, Knoblauch [8] introduced the notion of topology based on binary relations. Recently Mishra and srivastava [12] generalized this notion in the case fuzzy sets. In this chapter, we present some definitions, properties and theorems of fuzzy topologies based on fuzzy relations. Moreover, we study the lattice structure of the family of fuzzy open sets in this topology.

3.1 Fuzzy topologies based on fuzzy relations

In this section, we recall the notion of fuzzy topologies based on fuzzy relations given by Mishra and srivastava [12].

Definition 3.1. [12] *Let $\mathcal{R}$ be a fuzzy relation on a set X. Then for $x \in X$, the fuzzy sets L_x and R_x, which are defined as*

$$L_x(y) = \mathcal{R}(y, x) \ for \ all \ y \in X,$$

$$R_x(y) = \mathcal{R}(x, y), \ for \ all \ y \in X,$$

are called lower and upper contour, respectively, of the element $x \in X$.
The fuzzy topology generated by the collection $\mathcal{S}_1$ of all lower contours(i,e., $\mathcal{S}_1 = \{L_x : x \in X\}$) will be denoted by τ_1, and the fuzzy topology generated by the collection $\mathcal{S}_2$ of all upper contours(i,e., $\mathcal{S}_2 = \{R_x : x \in X\}$ will be denoted by τ_2.

Definition 3.2. [12] *The fuzzy topology which is generated by the subbase*

$$\mathcal{S} = \{L_x\}_{x \in X} \cup \{R_x\}_{x \in X}.$$

Is called the fuzzy topology generated by $\mathcal{R}$ and is denoted by $\tau_{\mathcal{R}}$.

Example 3.1. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{x, y\}$ given by:*

$\mathcal{R}$	x	y
x	0.5	0.6
y	0.2	0.4

Then L_x, L_y, R_x, R_y are the fuzzy sets on X given by:

$$L_x = \{\langle x, 0.5\rangle; \langle y, 0.2\rangle\};$$

$$L_y = \{\langle x, 0.6\rangle; \langle y, 0.4\rangle\};$$

$$R_x = \{\langle x, 0.5\rangle; \langle y, 0.6\rangle\};$$

$$R_y = \{\langle x, 0.2\rangle; \langle y, 0.4\rangle\}.$$

Therefore,

$$\tau_1 = \{\emptyset, X, L_x, L_y\};$$

$$\tau_2 = \{\emptyset, X, R_x, R_y\}.$$

And

$$\begin{aligned} \mathcal{S} &= \{\{\langle x, 0.5\rangle; \langle y, 0.2\rangle\}; \{\langle x, 0.6\rangle; \langle y, 0.4\rangle\}; \{\langle x, 0.5\rangle; \langle y, 0.6\rangle\}; \{\langle x, 0.2\rangle; \langle y, 0.4\rangle\}\}; \\ &= \{L_x, L_y\} \cup \{R_x, R_y\}. \end{aligned}$$

Therefore,

$$\tau_{\mathcal{R}} = \{\emptyset, X, L_x, L_y, R_x, R_y, \{\langle x, 0.2\rangle; \langle y, 0.2\rangle\}, \{\langle x, 0.5\rangle; \langle y, 0.4\rangle\}, \{\langle x, 0.6\rangle; \langle y, 0.6\rangle\}\}.$$

Example 3.2. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{x, y, z\}$ given by*

$\mathcal{R}$	x	y	z
x	1	0.4	0
y	0	1	0.7
z	0.6	0	1

Then $L_x,\ L_y,\ L_z, R_x, R_y R_z$ are the fuzzy sets on X given by:

$$L_x = \{\langle x,1\rangle; \langle y,0\rangle; \langle z,0.6\rangle\};$$

$$L_y = \{\langle x,0.4\rangle; \langle y,1\rangle; \langle z,0\rangle\};$$

$$L_z = \{\langle x,0\rangle; \langle y,0.7\rangle; \langle z,1\rangle\};$$

$$R_x = \{\langle x,1\rangle; \langle y,0.4\rangle; \langle z,0\rangle\};$$

$$R_y = \{\langle x,0\rangle; \langle y,1\rangle; \langle z,0.7\rangle\};$$

$$R_z = \{\langle x,0.6\rangle; \langle y,0\rangle; \langle z,1\rangle\}.$$

Therefore,

$$\tau_1 = \{\emptyset, X, L_x, L_y, L_z\};$$

$$\tau_2 = \{\emptyset, X, R_x, R_y, R_z\}.$$

And

$$\begin{aligned}\mathcal{S} &= \Big\{\{\langle x,1\rangle; \langle z,0.6\rangle\}, \{\langle x,0.4\rangle; \langle y,1\rangle\}, \{\langle y,0.7\rangle; \langle z,1\rangle\}, \{\langle x,1\rangle; \langle y,0.4\rangle\},\\ &\quad \{\langle y,1\rangle; \langle z,0.7\rangle\}, \{\langle x,0.6\rangle; \langle z,1\rangle\}\Big\}\\ &= \{L_x, L_y, L_z\} \cup \{R_x, R_y, R_z\}.\end{aligned}$$

Hence,

$$\begin{aligned}\tau_{\mathcal{R}} &= \Big\{\{\emptyset, X, L_x, L_y, L_z, R_x, R_y, R_z, \{\langle x,1\rangle; \langle y,1\rangle; \langle z,0.6\rangle\}, \{\langle x,0.4\rangle\},\\ &\quad \{\langle z,0.6\rangle\}, \{\langle x,1\rangle; \langle y,0.4\rangle; \langle z,0.6\rangle\}, \{\langle x,1\rangle\}, \{\langle x,1\rangle; \langle y,1\rangle; \langle z,0.7\rangle\},\\ &\quad \{\langle x,1\rangle; \langle z,1\rangle\}, \{\langle x,0.6\rangle; \langle z,0.6\rangle\}, \{\langle x,0.4\rangle; \langle y,1\rangle; \langle z,1\rangle\}, \{\langle y,0.7\rangle\},\\ &\quad \{\langle x,1\rangle; \langle y,1\rangle\}, \{\langle x,0.4\rangle; \langle y,0.4\rangle\}, \{\langle x,0.4\rangle; \langle y,1\rangle; \langle z,0.7\rangle\}, \{\langle y,1\rangle\},\\ &\quad \{\langle x,0.6\rangle; \langle y,1\rangle; \langle z,1\rangle\}, \{\langle y,0.4\rangle\}, \{\langle y,1\rangle; \langle z,1\rangle\}, \{\langle y,0.7\rangle; \langle z,0.7\rangle\},\\ &\quad \{\langle x,0.6\rangle; \langle y,0.7\rangle; \langle z,1\rangle\}, \{\langle z,1\rangle\}, \{\langle x,1\rangle; \langle y,0.4\rangle; \langle z,1\rangle\}, \{\langle x,0.6\rangle\},\\ &\quad \{\langle z,0.7\rangle\}, \{\langle x,1\rangle; \langle y,0.7\rangle; \langle z,1\rangle\}\Big\}.\end{aligned}$$

Proposition 3.1. *If $\mathcal{R}$ is a symmetric fuzzy relation, then $\tau_1 = \tau_2$.*

Proof. Since $\mathcal{R}$ is symmetric fuzzy relation, it follows that $\mathcal{R}(x,y) = \mathcal{R}(y,x)$, for all $x, y \in X$. This implies that $R_x(y) = L_x(y)$, for any $x \in X$. Thus the topologies τ_1 and τ_2, which are generated by $\{L_x : x \in X\}$ and $\{R_x : x \in X\}$, respectively, are the same. ☐

Proposition 3.2. *If $\mathcal{R}$ is a preorder relation. Then*

1. *If $A \in \tau_1$, then $A \supseteq \bigcup\limits_{x:A(x)=1} R_x$.*
2. *If $A \in \tau_2$, then $A \supseteq \bigcup\limits_{x:A(x)=1} L_x$.*

Proof. 1. To show that $A \supseteq \bigcup\limits_{x:A(x)=1} L_x$, suppose $y_r \in A \supseteq \bigcup\limits_{x:A(x)=1} L_x$. This implies that there exists some x where $A(x) = 1$ and $y_r \in L_x$. It follows that $r < \mathcal{R}(y, x)$. Now that, A is open and $A(x) = 1$, so $x_r \in A$ and there exists a basic fuzzy open set $\bigcap\limits_{i=1}^{n} L_{x_i}$ such that

$$x_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A$$

$$\begin{aligned}
&\Rightarrow \quad r < \mathcal{R}(x, x_i),\ for\ each\ i = 1, 2, ..., n;\\
&\Rightarrow \quad r < min\{\mathcal{R}(y, x), \mathcal{R}(x, x_i)\} \leq \mathcal{R}(y, x_i),\ for\ each\ i = 1, 2, ..., n;\\
&\Rightarrow \quad y_r \in L_{x_i},\ for\ each\ i = 1, 2, ..., n;\\
&\Rightarrow \quad y_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A;\\
&\Rightarrow \quad y_r \in A;\\
&\Rightarrow \quad A \in \tau_2,\ then\ A \supseteq \bigcup_{x:A(x)=1} L_x\ .
\end{aligned}$$

(2) The proof is similar to that of part 1.

□

3.2 The lattice of a family of fuzzy open sets

In this section, we introduce the notion of the lattice structure of a family of fuzzy open sets. First, we introduce the notion of inclusion, union, intersection of open sets in the fuzzy topologies based on fuzzy relations.

Definition 3.3. *Let* τ_R *be the fuzzy topology based on fuzzy relation* R *and let* U_1, U_2 *be two fuzzy open sets. Then,* U_1 *is said to be contained in* U_2 *(in symbols,* $U_1 \sqsubseteq U_2$*) if* $\mu_{U_1}(x) \leq \mu_{U_2}(x)$*, for any* $x \in X$*.*

Definition 3.4. *Let* τ_R *be the fuzzy topology based on fuzzy relation* R *and let* U_1, U_2 *be two fuzzy open sets. Then, the intersection of* U_1 *and* U_2 *(in symbols,* $U_1 \sqcap U_2$*) is a fuzzy open set* U *such that* $\mu_U(x) = min\{\mu_{U_1}(x), \mu_{U_2}(x)\}$ *for any* $x \in X$*.*

Definition 3.5. *Let* τ_R *be the fuzzy topology based on fuzzy relation* R *and let* U_1, U_2 *be two fuzzy open sets. Then, the union of* U_1 *and* U_2 *(in symbols,* $U_1 \sqcup U_2$*) is a fuzzy open set* U *such that* $\mu_U(x) = max\{\mu_{U_1}(x), \mu_{U_2}(x)\}$ *for any* $x \in X$*.*

Theorem 3.1. *Let* τ_R *be the fuzzy topology based on fuzzy relation* R*,* U_i *are a fuzzy open sets. Then the family* $\mathcal{L} = \{\cup_i \mid \cup_i \in \ FOS\ (\tau_R)\}$ *is a fuzzy lattice.*

Proof. Suppose that U_i is a set of fuzzy open sets. Definition of fuzzy topology guarantees that U_i is nonempty set. Now , let U_1 and U_2 be two fuzzy open sets. It is easy to cheek that $U_1 \sqsubseteq U_1$, i.e., the fuzzy reflexivity and if we suppose that $U_1 \sqsubseteq U_2$ and $U_2 \sqsubseteq U_1$, it follows that $U_1 = U_2$ i.e., the fuzzy antisymmetry. In the order to verify the transitivity, we suppose that $U_1 \sqsubseteq U_2$ and $U_2 \sqsubseteq U_3$, it follows that $U_1 \sqsubseteq U_3$, i.e., the fuzzy transitivity. Hence $(\mathcal{L}, \sqsubseteq)$ is a fuzzy poset on X. Moreover, the least upper bound (resp. the greatest lower bound) of U_1 and U_2 is coincides with the intersection of fuzzy open sets (resp. the union of fuzzy open sets)i.e., $U_1 \wedge U_2 = U_1 \sqcap U_2$(resp. $U_1 \vee U_2 = U_1 \sqcup U_2$). Thus, we can conclude that $(\mathcal{L}, \sqsubseteq)$ is a lattice on X. □

Conclusion

In this memoir, we have studied a special type of fuzzy topological space, which is the fuzzy topologies based on fuzzy relations with their fundamental properties. Also, we have proved that the family of fuzzy open sets in this topology forms the structure of a lattice.
In future work, we can treat some other topological properties on this topology.

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ملخص

في هذه المذكرة قمنا بدراسة نوع خاص من الفضاءات الطبولوجية الفازية وهي الطبولوجية الفازية المولدة بعلاقة فازية ، مع بعض الخواص الأساسية .

أيضا قمنا بإثبات أن عائلة المفتوحات الفازية في هذه الطبولوجية تشكل بنية شبكة .

الكلمات المفتاحية: مجموعة فازية، طبولوجية، طبولوجية مولدة بعلاقة ، شبكة.

Abstract

In this memoir, we have studied a special type of fuzzy topological spaces, which is the fuzzy topologies based on fuzzy relations with their fundamental properties. Also, we have proved that the family of fuzzy open sets in this topology forms the lattice structure.

Key words : Fuzzy set , Topology, Topology based on relation, Lattice.

Résumé

Dans ce mémoire, nous avons étudié un type spécial d'espaces topologiques floues, qui est la topologie floue basée sur une relation floue avec quelques propriétés de base.

De plus, nous avons montré que la famille des ouverts flous dans cette topologie forme une structure d'une treillis.

Mots clés : Ensemble floue, Topologie, Topologie basée sur une relation, Treillis.