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Contents

Notations

$\leq$	Partial order.
$\preccurlyeq$	Partial order.
$\leq_{L^*}$	Partial order in L^*.
τ	Topology.
$T(A)$	Dual space of the distributive lattice A.
$L(\delta)$	Set of increasing and $\tau-$clopen sets of X.
$L(T(A))$	Bidual of the distributive lattice A.
$T(L(\delta))$	Bidual of the Priestley space δ.
$\mu_A(x)$	Degree of membership of the element x in A.
$\vartheta_A(x)$	Degree of nonmembership of the element x in A.
$h(R)$	Height of the relation R.
$\uparrow x$	Principal filter generated by x.
$\downarrow x$	Principal ideal generated by x.

Introduction

The study of fuzzy relations was started by Zadeh [39] in 1971. In that celebrated paper the author introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy orderings. The concept of fuzzy order was introduced by generalizing the notion of reflexivity, antisymmetry and transitivity, there by facilitating the derivation of known results in various areas and stimulating the discovery of new ones. Fuzzy orderings have broad utility. They can be applied, for example, when expressing our preferences with a set of alternatives. Since then many notions and results from the theory of ordered sets have been extended to the fuzzy ordered sets. In [36], Venugopalan introduced a definition of fuzzy ordered set (foset) (P, μ) and presented an example on the set of positive integers. He extended this concept to obtain a fuzzy lattice in which he defined a (fuzzy) relation as a generalization of equivalence. Another approach was proposed by Chon [12]. Hence, a fuzzy lattice is defined to be just a set equipped with a fuzzy ordering relation, albeit the simple definition, many interesting properties of these lattices were deduced [12].

In June 1983 Atanassov introduced the concept of intuitionistic fuzzy set as a generalization of notion of fuzzy set. They differ from fuzzy both by their motivations and their underlying mathematical structure. Mathematical objects introduced by Atanassov [6, 7] and studied under the name "intuitionistic fuzzy sets" (IFS) have become a popular topic of investigation in the fuzzy set community. The first widely accessible reference was published in 1986 [7]. An intuitionistic fuzzy set in the sense of Atanassov is defined by a pair of membership functions (denoted by μ, ϑ where μ is the degree of membership and ϑ is the degree of non-membership with $\mu(x) + \vartheta(x) \leq 1$). The representation theorems appeared in the thirties of the last century; M. Stone [35] proved that every Boolean algebra is isomorphic to a set of $\{I_a / a \in A\}$ (where I_a denotes the set of prime ideals of A not containing a). The representation theorem for distributive lattices proved by Birkhoff [8]; asserts that any finite distributive lattice L is isomorphic to the lattice of the ideals of the partial order of the join-irreducible elements of L. H. Priestley developed another kind of duality for bounded distributive lattices see [29, 30]. Such representation theorems enable a deep and a concrete comprehension of the lattices as well their structures. The duality is central in making the link between syntactical and semantic approaches to logic, also in theoretical computer science this link is central as the two sides correspond to specification languages and the space of computational states. This ability to translate faithfully between algebraic specification and spatial dynamics has often proved itself to be a powerful theoretical tool as well as a handle for making practical problems decidable. Topological duality for Boolean algebras [34] and distributive lattices [35] is a useful tool for studying relational semantics for propositional logics. Canonical extensions [16, 17, 18, 19], provide a way of looking at these semantics algebraically. Priestley's duality for bounded distributive lattices has enjoyed growing attention and has been variously applied in the international literature since its inception in 1970 [29, 30].

In the first Chapter, we give some basic notions and definitions of topology, ordered sets, and fuzzy ordered sets needed in the sequel and we present the duality for bounded distributive lattices in the crisp case using the Priestley approach [29, 30].

The second Chapter, we focus on some results concerning the Priestley duality for finite fuzzy distributive lattices.

In the third Chapter, we extend some results in [2, 29, 30], where a representation theorem of fuzzy distributive lattices in the infinite case is presented. In this case, we show that the category of infinite fuzzy Priestley spaces is equivalent to the dual of the category of infinite fuzzy distributive lattices (see[4]).

In Chapter 4, we focus on the basic notions and definitions concerning intuitionistic fuzzy perfect lattices and we introduce the notion of intuitionistic fuzzy Priestley spaces on one hand. On the other hand, we extend some results of [2, 29, 30], more precisely, we give a representation theory of perfect intuitionistic fuzzy distributive lattices in the finite case. The finding result obtained in this chapter, is that the category of finite intuitionistic fuzzy perfect Priestley spaces is equivalent to the dual of the category of finite intuitionistic fuzzy perfect distributive lattices (see[5]).

Chapter 1

Preliminaries

In this chapter, we recall some useful notions on topological spaces, lattices, lattices isomorphisms, Priestley duality in the crisp case, fuzzy lattices, filters and ideals of fuzzy lattices, fuzzy Priestley spaces and fuzzy Priestley spaces isomorphisms.

1.1 Topological spaces

Let X be a set and τ be a family of subsets of X. If τ
satisfies the following conditions

(O1) $\phi \in \tau$ and $X \in \tau$.

(O2) Any intersection of finitely many elements of τ is an element of τ.

(O3) Any union of elements of τ is an element of τ.

Then τ is called a topology on X and the pair (X, τ) is called a topological space. Every element of (X, τ) is called a point. Every member of τ is called an open set of X or open in X. If $\{x\} \in \tau$, then the point x is called an isolated point of X. The complement of an open set is called a closed set of X or closed in X.

If a set is open and closed in a topological space, then it is called open-and-closed or closed-and-open (or clopen for short).

Let $\mathcal{C}$ be the family of closed sets of the topological space (X, τ). Then $\mathcal{C}$ satisfies the following conditions

(C1) $\phi \in \mathcal{C}$ and $X \in \mathcal{C}$.

(C2) Any union of finitely many elements of $\mathcal{C}$ is an element of $\mathcal{C}$.

(C3) Any intersection of elements of $\mathcal{C}$ is an element of $\mathcal{C}$.

1.1.1 Basis for a topological space

Definition 1.1. *If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis element) such that:*

1. *For each $x \in X$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2,then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

If $\mathcal{B}$ satisfies these two conditions, then we define the topology τ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of τ) if for each element $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of τ.

1.1.2 Subspace topology

Definition 1.2. *Let X be a topological space with topology τ. If Y is a subset of X, the collection*

$$\tau_Y = \{Y \cap U \diagup U \in \tau\}$$

is a topology on Y. It is called the induced topology, the subspace topology, or the relative topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of opens sets of X with Y.

Lemma 1.3. *If $\mathcal{B}$ is a basis for the topology X, then the collection*

$$\mathcal{B}_Y = \{B \cap Y \diagup B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

1.1.3 Compact spaces

Definition 1.4. *Let A be a subset of the topological space X. An open cover for A is a collection $\mathcal{O}$of open sets whose union contains A. A subcover derived from the open cover $\mathcal{O}$ is a subcollection $\mathcal{O}'$of $\mathcal{O}$ whose union contains A.*

Definition 1.5. *A topological space X is compact provided that every open cover of X has a finite subcover.*

This says that however we write X as a union of open sets, there is always a finite sub collection $\{\mathcal{O}_i\}_{i=1}^{i=n}$ of these sets whose union is X. A subspace A of X is compact if A is a compact space in its subspace topology. Since relatively open sets in the subspace topology are the intersections of open sets in X with the subspace A, the definition of compactness for subspaces can be restated as follows.

Definition 1.6. *A subspace A of X is compact if and only if every open cover of A by open sets in X has a finite subcover.*

Theorem 1.7. *Each closed subset of a compact space is compact.*

Finite products of topological spaces

Let X and Y be two topological spaces, and consider the Cartesian product $X \times Y$.

Definition 1.8. *The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open set of X and V is an open set of Y.*

Theorem 1.9. *If $\mathcal{B}$ is a basis for the topology X and $\mathcal{C}$ is a basis for the topology Y, then the collection*

$$\mathcal{D} = \{B \times C/\ B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

1.2 Lattices

1.2.1 Partial ordered sets

A partial order (order for short) is a binary relation $\leq$ over a set X which is reflexive ($a \leq a$ for any $a \in X$), antisymmetric ($a \leq b$ and $b \leq a$ implies $a = b$ for any $a, b \in X$) and transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$ for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset).

Zorn's lemma

Lemma 1.10. *(Zorn's lemma). Let S be a partially ordered set. If every totally ordered subset of S has an upper bound, then S contains a maximal element*

1.2.2 Lattices

We shall be particularly interested in ordered sets $(X, \leq)$ in which $sup\{x, y\}$, $inf\{x, y\}$ exist for all $x, y \in X$.

Notation 1.11. *We shall adopt the following neater notation: we write $x \vee y$ (read as x **join** y) in place of $sup\{x, y\}$ when it exists and $x \wedge y$ (read as x **meet** y) in place of $\inf\{x, y\}$ when it exists. Similarly we write $\vee S$ (the **join of** S) and $\wedge S$ (the **meet of** S) instead of $\sup S$ and $\inf S$ when these exist.*

Definition 1.12. *([14])Let $(X, \leq)$ be an ordered set.*

(i) *If $x \vee y$, $x \wedge y$ exist for all $x, y \in X$, then $(X, \leq)$ is called a lattice.*

(ii) *If $\vee S$, $\wedge S$ exist for all $S \subseteq X$, then $(X, \leq)$ is called a complete lattice.*

1.2.3 Distributive lattices

Lemma 1.13. *([14])Let L be a lattice. Then the following are equivalent*

$(D1)$ $(\forall a, b, c \in L)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

$(D2)$ $(\forall a, b, c \in L)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition 1.14. *([14]) Let L be a lattice. L is said to be distributive if it satisfies the distributive law, $(\forall a, b, c \in L)\, a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.*

1.2.4 Filters and ideals

Definition 1.15. *([14]) A nonempty subset F of a lattice L is called a filter of L if for all $x, y \in L$*

1. *if $y \in F$ with $y \leq x$, then $x \in F$,*
2. *if $x, y \in F$ implies $x \wedge y \in F$.*

Definition 1.16. *([10]) A filter F of L is a prime filter if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for any $x, y \in X$.*

Definition 1.17. *([14]) A nonempty subset I of a lattice L is called a ideal of L if for all $x, y \in L$*

1. *if $y \in I$ with $x \leq y$, then $x \in I$,*
2. *if $x, y \in I$ implies $x \vee y \in I$.*

Definition 1.18. *([10]) An ideal I of L is a prime ideal if $x \wedge y \in I$ implies $x \in I$ or $y \in I$ for any $x, y \in X$.*

1.2.5 Lattices isomorphisms

Definition 1.19. *([14]) Let L and L'be two lattices. A mapping $f : L \to L'$ is said to be an homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$. If f is a bijection, then f is said to be lattices isomorphism.*

Proposition 1.20. *([10, Proposition 1.3.9]) Let L be a lattice and F be a subset of L. The following conditions are equivalents,*

1. *F is a prime filter;*

2. *$L - F$ is a prime ideal.*

3. *There is a surjective lattice homomorphism $f : X \to \{0, 1\}$ such that $F = f^{-1}(\{1\})$.*

Corollary 1.21. *([10, Corollary 1.3.13]) Let L be a distributive lattice. If $a, b \in X$ are such that $a \nleq b$ there is a prime filter F such that $a \in F$ and $b \notin F$*

1.3 Priestley duality for bounded distributive lattices

Most of the definitions and results presented here are classical and can be found in [10],[14].

Definition 1.22. *Let $(X, \leq)$ be a poset. A subset $E \subseteq X$ is said to be increasing (decreasing) if $\forall x, y \in X$: $x \in E$ and $x \leq y$ $(y \leq x)$ imply $y \in E$.*

Definition 1.23. *An ordered topological space is a triple $(X, \tau, \leq)$ such that (X, τ) is a topological space and $(X, \leq)$ is a poset. A $\tau-$clopen set in a topological space is a set which is both open and closed. The ordered topological space is said to be totally disconnected for the order $(\leq)$ if for every $x, y \in X$ such that $x \nleq y$ there exist an increasing $\tau-$clopen U and a decreasing $\tau-$clopen V such that $U \cap V = \emptyset$ with $x \in U$ and $y \in V$.*

Definition 1.24. *A Priestley space is a compact totally disconnected ordered topological space.*

If A is a distributive lattice then its dual space is defined to be $T(A) = (X, \tau, \leq)$, where X is the set of homomorphisms from A onto $\{0, 1\}$, preserving 0 and 1, τ be the topology induced by the product topology of $\{0, 1\}^A$, and $\leq$ is the partial order: $f \leq g$ in X if and only if $f(a) \leq g(a)$ for all $a \in A$.

$T(A) = (X, \tau, \leq)$ is compact totally order disconnected, i.e., a Priestley space.

1.3.1 Priestley spaces isomorphisms

Definition 1.25. *Let $(X, \tau, \leq)$, and $(X', \tau', \preceq)$ be two Priestley spaces.*

1. *A function $f : X \to X'$ is called increasing if for all $x, y \in X$, $x \leq y \Rightarrow f(x) \preceq f(y)$.*

2. *Let $f : X \to X'$ be a function between Priestley spaces. Then f is called a Priestley spaces homomorphism if is increasing and continuous.*

If f is a bijection, then f is said to be Priestley spaces isomorphism.

Lemma 1.26. *If $\delta = (X, \tau, \leq)$ is a Priestley space. Then $(L(\delta), \cap, \cup, \phi, X)$ is a bounded distributive lattice. Where $L(\delta) = \{Y \subseteq X | Y$ is increasing and $\tau-$clopen$\}$.*

Lemma 1.27. *Let A be a bounded distributive lattice. The map $F_A : A \mapsto L(T(A))$ defined by*

$$F_A\,(a) = \{f \in X/f(a) = 1\}$$

is a lattice isomorphism.

Lemma 1.28. *If $f : A_1 \mapsto A_2$ is a lattice homomorphism, then the map $T(f) : T(A_1) \mapsto T(A_2)$ defined by $T(f)(g) = g \circ f$ is an homomorphism of Priestley space, i.e., a continuous and increasing map.*

Lemma 1.29. *If $\delta = (X, \tau, r)$ is a Priestley space, then the map $G_\delta : \delta \mapsto T(L(\delta))$ defined by*

$$G_\delta(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

for all $Y \in L(\delta)$ is an isomorphism of Priestley space, i.e., a bijection, continuous and increasing map.

Lemma 1.30. *If $h : \delta_1 \mapsto \delta_2$ is an homomorphism of Priestley space, then the map $L(h) : L(\delta_2) \mapsto L(\delta_1)$ defined by $L(h)(y) = h^{-1}(y)$ for every $y \in L(\delta_2)$ is a lattices homomorphism.*

Theorem 1.31. *If $f : A_1 \mapsto A_2$ is a lattice homomorphism, then*

$$L\,(T(f)) \circ F_{A_1} = F_{A_2} \circ f.$$

$$\begin{array}{ccccc} & & f & & \\ & A_1 - -- & & ----\;\; \rightarrow & A_2 \\ & | & & & | \\ F_{A_1} & | & & & |\; F_{A_2} \\ & \downarrow & & & \downarrow \\ & L\,(T\,(A_1)) & & ----\;\; \rightarrow & L\,(T\,(A_2)) \\ & & L\,(T\,(f)) & & \end{array}$$

Theorem 1.32. *If $h : \delta_1 \mapsto \delta_2$ is an homomorphism of Priestley space, then*

T

$$(L\,(h)) \circ G_{\delta_1} = G_{\delta_2} \circ h.$$

$$\begin{array}{ccccc} & & h & & \\ & \delta_1 & ----\rightarrow & \delta_2 & \\ & | & & | & \\ G_{\delta_1} & | & & | & G_{\delta_2} \\ & \downarrow & & \downarrow & \\ & T\left(L\left(\delta_1\right)\right) & ----\rightarrow & T\left(L\left(\delta_2\right)\right) & \\ & & T\left(L\left(h\right)\right) & & \end{array} .$$

Theorem 1.33. *The dual of the category of distributive lattices is equivalent to the category of Priestley spaces.*

1.4 Fuzzy lattices

In this section, we recall some definitions and concepts needed in the sequel.

1.4.1 Fuzzy set

Let X be a non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy set on X.

The following expressions are defined in X for all fuzzy sets A, B in X;

1. $A \subseteq B$ if and only if $\mu_A \leq \mu_B$,
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
3. $A^c = 1 - \mu_A$,
4. $A \cap B = \mu_A \wedge \mu_B$,
5. $A \cup B = \mu_A \vee \mu_B$.

1.4.2 Fuzzy ordered sets

Let X be a non-empty set. A mapping $R : X \times X \to [0,1]$ is called a fuzzy binary relation on X.

A fuzzy binary relation R on X is called

1. Reflexive, if $R(x,x) = 1$, for all $x \in X$.

2. Antisymmetric, if $R(x, y) \wedge R(y, x) = 0$ whenever $x \neq y$, for all $x, y \in X$.

3. Transitive, if $R(x, y) \wedge R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

A reflexive, antisymmetric and transitive fuzzy relation is called a fuzzy partial ordering relation. A set equipped with a fuzzy order relation is called a fuzzy ordered set (foset). The height of R, denoted by $h(R)$, is defined by $h(R) = \vee_{\{(x,y) \in X^2 : x \neq y\}} R(x, y)$.

1.4.3 Fuzzy lattices

The terminology used in this section is the same as in [12, 36], where all the necessary definitions and results may be found.

Definition 1.34. *Let (X, R) be a fuzzy poset and let A be a subset of X. An element $u \in X$ is said to be an upper bound of A if and only if $R(a, u) > 0$ for all $a \in A$. An upper bound u_0 of A is the least upper bound of A if and only if $R(u_0, u) > 0$ for every upper bound u of A. An element $l \in X$ is said to be a lower bound of A if and only if $R(l, a) > 0$ for all $a \in A$. A lower bound l_0 of A is the greatest lower bound of A if and only if $R(l, l_0) > 0$ for every lower bound l of A.*

The least upper bound and the greatest lower bound of the set $\{x, y\}$ are denoted by $x \vee y$ and $x \wedge y$ respectively.

Proposition 1.35. *[12]Let (X, R) be a fuzzy lattice. For any $x, y, z \in X$, it follows that,*

1. $R(x, x \vee y) > 0$, $R(y, x \vee y) > 0$, $R(x \wedge y, x) > 0$ *and* $R(x \wedge y, y) > 0$,

2. $R(x, z) > 0$ *and* $R(y, z) > 0$ *implies* $R(x \vee y, z) > 0$,

3. $R(z, x) > 0$ *and* $R(z, y) > 0$ *implies* $R(z, x \wedge y) > 0$,

4. $R(x, y) > 0$ *if and only if* $x \vee y = y$,

5. $R(x, y) > 0$ *if and only if* $x \wedge y = x$,

6. *If* $R(y, z) > 0$, *then* $R(x \wedge y, x \wedge z) > 0$ *and* $R(x \vee y, x \vee z) > 0$.

Proposition 1.36. *[12]Let (X, R) be a fuzzy lattice. For any $x, y, z \in X$, it follows that,* *a)* $x \vee x = x$, $x \wedge x = x$,

b) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$,

c) $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,

d) $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$.

The following definition give a characterizations of fuzzy distributive lattices.

Definition 1.37. *[12]Let (X, R) be a fuzzy lattice. (X, R) is distributive if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.*

1.4.4 Filters and ideals of a fuzzy lattice

Definition 1.38. *[26, 27]Let (X, R) be a fuzzy lattice and F be a nonempty crisp subset of X. F is said to be a filter of (X, R) if for all $x, y \in X$, it holds that*

(i) *If $y \in F$ and $R(y, x) > 0$, then $x \in F$,*

(ii) *If $x, y \in F$, then $x \wedge y \in F$.*

Definition 1.39. *Let (X, R) be a fuzzy lattice and F be a filter of (X, R). Then F is called a prime filter if F is proper $(F \neq X)$ and for all $x, y \in X$, $x \vee_R y \in F$ imply $x \in F$ or $y \in F$.*

Definition 1.40. *[26, 27]Let (X, R) be a fuzzy lattice, and I be a nonempty crisp subset of X. I is said to be an ideal of (X, R) if for all $x, y \in X$, it holds that*

(i) *If $y \in I$ and $R(x, y) > 0$, then $x \in I$,*

(ii) *If $x, y \in I$, then $x \vee y \in I$.*

Definition 1.41. *Let (X, R) be a fuzzy lattice and I be an ideal of (X, R). Then I is called a prime ideal if I is proper $(I \neq X)$ and for all $x, y \in X$, $x \wedge_R y \in I$ imply $x \in I$ or $y \in I$.*

For every subset δ of X, the smallest filter of X (with respect to the inclusion) which contains δ is said to be the filter generated by δ and will be denoted by $\langle \delta \rangle$.

Proposition 1.42. *If δ is a non-empty subset of a fuzzy lattice (X, R), then*

$$\langle \delta \rangle = \{x \in X / R\left(a_1 \wedge ... \wedge a_n, x\right) > 0, \text{ for some } a_1, .., a_n \in \delta\}.$$

Proof. Let $\langle \delta \rangle = \{x/R(\wedge_{i=1}^{n} a_i, x) > 0,\ a_1, a_2, .., a_n \in \delta\}$.

First, we prove that $\langle \delta \rangle$ is non-empty. Let $a \in \delta$, since $R(a,a) > 0$, then $a \in \langle \delta \rangle$, hence $\langle \delta \rangle \neq \emptyset$. To proof that $\langle \delta \rangle$ is a filter. Let $x \in \langle \delta \rangle, y \in X$ such that $R(x,y) > 0$, there exist $a_1, a_2, .., a_n$ such that $R(\wedge_{i=1}^{n} a_i, x) > 0$. Then, $R(\wedge_{i=1}^{n} a_i, y) > 0$, then $y \in F$.

On the other hand, let $x, y \in \langle \delta \rangle$, there exist $a_1, a_2, .., a_n, b_1, b_2, .., b_m$ such that $R(\wedge_{i=1}^{n} a_i, x) > 0$ and $R\left(\wedge_{j=1}^{m} b_j, y\right) > 0$.

Then, $R\left((\wedge_{i=1}^{n} a_i) \wedge (\wedge_{j=1}^{m} b_j), x \wedge y\right) > 0$. Therefore $x \wedge y \in \langle \delta \rangle$.

Next, let $a \in \delta$, since $R(a,a) > 0$, we have $a \in \langle \delta \rangle$. Then $\delta \subseteq \langle \delta \rangle$.

Finally, suppose that F is a filter with $\delta \subseteq F$. Then for any $x \in \langle \delta \rangle$, then there exist $a_1, a_2, .., a_n$ such that

$R(\wedge_{i=1}^{n} a_i, x) > 0$, then $x \in F$. Therefore $\langle \delta \rangle \subseteq F$. □

If x is an element of the fuzzy lattice X, then $\uparrow x = \{y \in X : R(x,y) > 0\}$ is the principal filter generated by x and $\downarrow x = \{y \in X : R(y,x) > 0\}$ is the principal ideal generated by x. We denote the set of all principal filters (principal ideal) generated by elements of X by $PF(X)$ $(PI(X))$.

Proposition 1.43. *$(PF(X), \subseteq, \wedge, \vee)$ is a lattice with $(\uparrow x) \wedge (\uparrow y) = (\uparrow x) \cap (\uparrow y) = \uparrow (x \vee y)$ and $(\uparrow x) \vee (\uparrow y) = \uparrow (x \wedge y)$, for any $\uparrow x, \uparrow y \in PF(X)$.*

Proof. Let $a \in X$,

$$\begin{aligned} a \in \uparrow (x \vee y) &\Leftrightarrow R(x \vee y, a) > 0 \\ &\Leftrightarrow R(x,a) > 0 \text{ and } R(y,a) > 0 \\ &\Leftrightarrow a \in (\uparrow x) \cap (\uparrow y) \\ &\Leftrightarrow a \in (\uparrow x) \wedge (\uparrow y). \end{aligned}$$

and

$$\begin{aligned} \begin{cases} R(x \wedge y, x) > 0 \\ R(x \wedge y, y) > 0 \end{cases} &\Rightarrow \begin{cases} \uparrow x \subseteq \uparrow (x \wedge y) \\ \uparrow y \subseteq \uparrow (x \wedge y) \end{cases} \\ &\Rightarrow (\uparrow x) \cup (\uparrow y) \subseteq \uparrow (x \wedge y) \\ &\Rightarrow \langle (\uparrow x) \cup (\uparrow y) \rangle \subseteq \uparrow (x \wedge y). \end{aligned}$$

Let $a \in \uparrow (x \wedge y)$, hence $R(x \wedge y, a) > 0$, it follows that $R(x,a) > 0$ and $R(y,a) > 0$, therefore $a \in \uparrow x$ and $a \in \uparrow y$, it follows that $a \in (\uparrow x) \cup (\uparrow y)$, then $a \in \langle (\uparrow x) \cup (\uparrow y) \rangle$. Then, $\uparrow (x \wedge y) \subseteq \langle (\uparrow x) \cup (\uparrow y) \rangle$. □

Theorem 1.44. *Let X be a fuzzy distributive lattice, F a filter and I an ideal of X. If $F \cap I = \phi$, then there is a prime filter P such that $F \subseteq P$ and $P \cap I = \phi$.*

Proof. Let G be the family of those filters F' which satisfy $F \subseteq F'$ and $F' \cap I = \phi$. It follows from the Zorn's lemma that G has a maximal element P. Since $P \in G$, it remains to prove that the filter P is prime. P is proper because $P \cap I = \phi$. Suppose P is not prime. Then there exist $a, b \in X$ such that $a \vee b \in P$, $a \notin P$ and $b \notin P$. Let $\delta = P \cup \{a\}$. Then $\langle \delta \rangle \cap I \neq \phi$, otherwise

$P \subseteq \langle \delta \rangle \in G$ contradicting the maximality of P. Take $x \in \langle \delta \rangle \cap I$. Then Proposition 1.42, implies easily the existence of $p \in P$ such that $R(p \wedge a, x) > 0$ and since $x \in I$ it follows that $p \wedge a \in I$. Similarly there is $q \in P$ such that $q \wedge b \in I$. Then $(p \wedge a) \vee (q \wedge b) \in I$ and on the other hand $(p \wedge a) \vee (q \wedge b) = (p \vee q) \wedge (p \vee b) \wedge (a \vee q) \wedge (a \vee b) \in P$, therefore $I \cap P \neq \phi$, a contradiction. □

Corollary 1.45. *Let X be a fuzzy distributive lattice. If I is an ideal and $a \in X - I$ there is a prime filter P such that $a \in P$ and $P \cap I = \phi$.*

Proof. Let I be an ideal and $a \in X - I$. Take $F = \langle a \rangle$ it follows $F \cap I = \phi$. By Theorem there is a prime filter P such that $F \subseteq P$ and $P \cap I = \phi$. Since F is maximal then $P = F$, i.e., F is a prime filter. □

1.4.5 Fuzzy lattices isomorphisms

We recall the following definition, see [36]

Definition 1.46. *Let $(L, r, \wedge, \vee)$, and $(M, R, \wedge, \vee)$ be two fuzzy lattices.*

1. *A function $f : L \to M$ is called increasing if for all $x, y \in L$, $r(x, y) \leq R(f(x), f(y))$.*

2. *Let $f : L \to M$ be increasing function between fuzzy lattices. Then f is called a lattice homomorphism if for any $x, y \in L$, $f(x \wedge y) = f(x) \wedge f(y)$, and $f(x \vee y) = f(x) \vee f(y)$. If f is a bijection, then f is said to be fuzzy lattices isomorphism.*

This Proposition is a fuzzy version of [10, Proposition 1.3.9].

Proposition 1.47. *Let $(X, R, \wedge, \vee)$ be a fuzzy lattice and F be a subset of $(X, R, \wedge, \vee)$. The following conditions are equivalents,*

1. *F is a prime filter;*

2. *F is a proper filter and for every $x, y \in X$, $x \vee y \in F \Leftrightarrow x \in F$ or $y \in F$;*

3. *$L - F$ is a prime ideal;*

4. *There is a surjective lattice homomorphism $f : X \to \{0, 1\}$ such that $F = f^{-1}(\{1\})$.*

Proof. (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (4): Set f defined by $f(x) = \begin{cases} 1 \text{ for } x \in F; \\ 0 \text{ for } x \in X - F. \end{cases}$. Then f is surjective. Since $\phi \neq F \neq X$. Since $x \wedge y \in F \Leftrightarrow x \in F$ and $y \in F$ it follows that $f(x \wedge y) = 1 \Leftrightarrow f(x) = 1$ and $f(y) = 1 \Longleftrightarrow f(x) \wedge f(y) = 1$, therefore $f(x \wedge y) = f(x) \wedge f(y)$. Similarly we can show that $f(x \vee y) = f(x) \vee f(y)$.

(4) $\Rightarrow$ (2): $\phi \neq F \neq X$ because f is surjective. Then $x \wedge y \in F \Leftrightarrow f(x) \wedge f(y) = f(x \wedge y) = 1 \Leftrightarrow x \in F$ and $y \in F$, therefore F is a proper filter by

$x \vee y \in F \Leftrightarrow x \in F$ or $y \in F$,we have

$x \vee y \in F \Leftrightarrow f(x \vee y) = 1 \Leftrightarrow f(x) \vee f(y) = 1 \Leftrightarrow x \in F$ or $y \in F$.

(2) $\Rightarrow$ (1): Trivial.

Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) and it follows by duality that (3) is equivalent to the existence of a surjective homomorphism $f : X \to \{0, 1\}$ such that $X - F = f^{-1}(\{0\})$, but the latter property is equivalent to (4). □

This Corollary is a fuzzy version of [10, Corollary1.3.13]

Corollary 1.48. *Let $(X, R, \wedge, \vee)$ be a fuzzy distributive lattice. If $a, b \in X$ are such that $R(a, b) = 0$, then there is a prime filter F such that $a \in F$ and $b \notin F$.*

Proof. Take $I = \downarrow b$ in Corollary 1.45. □

1.5 Fuzzy Priestley spaces

Definition 1.49. *[2]Let (X, R) be a foset. A subset E of X is called increasing, if for all x belongs to E and $R(x, y) > 0$ (y is an upper bound of x), then y belongs to E. A decreasing set is defined in a dually.*

Definition 1.50. *[2]A fuzzy ordered space is a triplet (X, τ, R), where X is a nonempty set, τ is a topology on X and R is a fuzzy order on X. A fuzzy ordered space (X, τ, R) is called* *totally order disconnected if for $x, y \in X$ and $R(x, y) = 0$, there exist an increasing $\tau-$clopen U and a decreasing $\tau-$clopen V, such that $U \cap V = \phi$, with $x \in U$ and $y \in V$.*

Recall that a $\tau-$clopen set in a topological space is a set which is both open and closed.

Definition 1.51. *[2]A fuzzy ordered space (X, τ, R) is called a fuzzy Priestley space, if it is compact and totally order disconnected.*

1.5.1 Fuzzy Priestley spaces isomorphisms

Definition 1.52. *Let (X, τ, r), and (X', τ', r') be two fuzzy Priestley spaces.*

1. *A function $f : X \to X'$ is called increasing if for all $x, y \in X$, $r(x, y) \leq r'(f(x), f(y))$.*

2. *Let $f : X \to X'$ be a function between fuzzy Priestley spaces. Then f is called a Priestley spaces homomorphism if is increasing and continuous. If f is a bijection, then f is said to be fuzzy Priestley spaces isomorphism.*

Chapter 2

Priestley duality for finite fuzzy distributive lattices

The terminology used in this chapter is the same as in [2]. And using the terminology in [10] we redemonstrate the results in [2]. Throughout this chapter, all fuzzy lattices are finite distributive lattices, and homomorphisms preserve the smallest element (denoted 0) and the greatest element (denoted 1). If $(A, \wedge, \vee, R)$ is a fuzzy distributive lattice, then its dual is a fuzzy priestley space defined by $T(A) = (X, \tau, R_1)$, where X is the set of $0-1$-homomorphisms from A onto $\{0,1\}$ τ be the topology induced by the product topology of $\{0,1\}^A$ and R_1 is a fuzzy order adequately chosen on X.

If $\delta = (X, \tau, r)$ is a fuzzy Priestley space, then its dual is a fuzzy distributive lattice a defined by $(L(\delta), \vee, \wedge, r_1)$, where $L(\delta) = \{Y \subseteq X/ \ Y \text{ is increasing and } \tau - \text{clopen}\}$ and r_1 is a fuzzy order adequately chosen.

2.1 Dual and bidual of a finite fuzzy distributive lattice

Lemma 2.1. *[2]If $(A, \vee, \wedge, R)$ is a fuzzy finite distributive lattice, then there exist two fuzzy orders R_1, R_2 such that*

1. *$T(A) = (X, \tau, R_1)$ is a fuzzy Priestley space,*

2. *$(L(T(A)), \vee, \wedge, R_2)$ is a fuzzy distributive lattice.*

Proof. (1) Let R_1 be the relation defined by

$$R_1(f,g)=\begin{cases} R\left(\wedge g^{-1}(1), \wedge f^{-1}(1)\right) & \text{if } f^{-1}(1)\subseteq g^{-1}(1); \\ 0 & \text{otherwise.}\end{cases}$$

where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation R. We show that R_1 is a fuzzy order. We have $R_1(f,f) = R\left(\wedge f^{-1}(1), \wedge f^{-1}(1)\right) = R(a,a) = 1$ for all $f \in X$, then R_1 is reflexive.

On the other hand for all $f, g \in X$ such that $f \neq g$, if $R_1(f,g) > 0$, then $f^{-1}(1) \subseteq g^{-1}(1)$, which imply $g^{-1}(1) \nsubseteq f^{-1}(1)$, it follows that $R_1(g,f) = 0$. Hence, R_1 is antisymmetric relation.

In order to verify the transitivity of R_1, let $f, g, h \in X$, we show that $R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h)$. We use the following truth table, where the proposition (P) is $R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h)$

$f^{-1}(1) \subset g^{-1}(1)$	$g^{-1}(1) \subset h^{-1}(1)$	$f^{-1}(1) \subset h^{-1}(1)$	(P)
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

The only case for investigating is $f^{-1}(1) \subset g^{-1}(1)$ and $g^{-1}(1) \subset h^{-1}(1)$. By the transitivity of R, for every a, b, c in A, we have $R(a,b) \wedge R(b,c) \leq R(a,c)$. This yields

$$R\left(\wedge g^{-1}(1), \wedge f^{-1}(1)\right) \wedge R\left(\wedge h^{-1}(1), \wedge g^{-1}(1)\right) \leq R\left(\wedge h^{-1}(1), \wedge f^{-1}(1)\right)$$

Then for all $f, g, h \in X$, $R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h)$ is hold, i.e. R_1 is transitive. Hence, R_1 is a fuzzy order and by [29, 30] $T(A) = (X, \tau, R_1)$ is a fuzzy Priestley space.

(2) Let $M_0 = \wedge\{R(x,y)\,/x, y \in X,\ x \neq y \text{ and } R(x,y) \neq 0\}$. We define R_2 by

$$R_2(H,D)=\begin{cases} 1 & \text{if } H=D,\\ R\left(\wedge\cap_{f\in H} f^{-1}(1),\wedge\cap_{g\in D} g^{-1}(1)\right) & \text{if } H\subset D \text{ and } H\neq\phi,\\ M_0 & \text{if } H=\phi,\\ 0 & \text{otherwise.}\end{cases}$$

for all $H, D \in L(T(A))$, where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation R. To show the reflexivity we have $R_2(H,H)=1$. To proof the antisymmetrical of R_2 let $H, D \in L(T(A))$ such that $H \neq D$, if $R_2(H,D)>0$, then $D \not\subseteq H$, it follows that $R_2(H,D)=0$, hence R_2 is antisymmetric. In order to verify the transitivity, we use the following truth table, where the proposition (P) is $(R_2(H,D) \wedge R_2(D,E) \leq R_2(H,E))$.

$H \subseteq D$	$D \subseteq E$	$H \subseteq E$	(P)
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

First, if one of the three elements H, D, E is empty, then the transitivity is a trivial fact.

If $H \neq \phi$ and $D \neq \phi$ and $E \neq \phi$, the only case that need investigation is when $H \subset D$ and $D \subset E$. This yields to

$R\left(\wedge\cap_{f\in H} f^{-1}(1),\wedge\cap_{g\in D} g^{-1}(1)\right) \wedge R\left(\wedge\cap_{g\in D} g^{-1}(1),\wedge\cap_{h\in E} h^{-1}(1)\right) \leq$
$R\left(\wedge\cap_{f\in H} f^{-1}(1),\wedge\cap_{h\in E} h^{-1}(1)\right)$.

Hence, R_2 is transitive.

Finally, the least upper and greatest lower bounds of H and D (with respect of the fuzzy ordering relation R_2) are denoted by $H \vee_{R_2} D$ and $H \wedge_{R_2} D$, respectively, to show that $H \vee_{R_2} D = H \cup D$ and $H \wedge_{R_2} D = H \cap D$. We have that $H \cup D$ is upper bound of $\{H, D\}$ because $\mu_{R_2}(H, H \cup D) > 0$ and $\mu_{R_2}(D, H \cup D) > 0$, if C is the least upper bound of $\{H, D\}$ we have fore cases:

1) if $H=\phi$ and $D=\phi$, it follows that $R_2(C, H\cup D) = R_2(C,\phi)$, which imply that $R_2(C, H\cup D)$ deferent to 0 if and only if $C=\phi$, hence $C = H \cup D$.

2) if $H = \phi$ and $D \neq \phi$ and $D \subseteq C$ (because $R_2(D, C) > 0$), it follows that $R_2(C, H \cup D) = R_2(C, D)$, which imply that $R_2(C, H \cup D)$, deferent to 0 if and only if $C \subseteq D$, hence $C = D = H \cup D$.

3) if $H \neq \phi$ and $D = \phi$ and $H \subseteq C$ (because $\mu_R(H, C) > 0$), it follows that $R_2(C, H \cup D) = R_2(C, H)$, which imply that $R_2(C, H \cup D)$ deferent to 0 if and only if $C \subseteq H$, hence $C = H = H \cup D$.

4) if $H \neq \phi$ and $D \neq \phi$ and $H \subseteq C$ and $D \subseteq C$, which imply $H \cup D \subseteq C$, it follows that $R_2(C, H \cup D)$ deferent to 0 if and only if $C \subseteq H \cup D$, hence $C = H \cup D$.

Similarly, we proof that $H \wedge_{R_2} D = H \cap D$, its known that $H \cup D$ and $H \cap D$ are increasing and $\tau-$clopens. This shows that $(L(T(A)), \vee, \wedge, R_2)$ is a fuzzy distributive lattice. □

2.2 Dual and bidual of a finite fuzzy Priestley space

Lemma 2.2. *[2]If $\delta = (X, \tau, r)$ is a fuzzy finite Priestley space, then there exist two fuzzy orders r_1 and r_2 such that*

1. *$(L(\delta), \vee, \wedge, r_1)$ is a fuzzy distributive lattice,*

2. *$(T(L(\delta)), \tau, r_2)$ is a fuzzy Priestley space.*

Proof. (1) (i) If $h(r) = 0$, then X is an antichain and we can write r_1 as follows

$$r_1(A, B) = \begin{cases} 1 & \text{if } A = B, \\ 1 - \frac{cardA}{cardB} & \text{if } A \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

r_1 is a fuzzy relation. It is easy to show that r_1 is a fuzzy order, and $A \vee_{r_1} B = A \cup B$, $A \wedge_{r_1} B = A \cap B$ exist for every A and B from $L(\delta)$, and $A \cup B$, $A \cap B$ are increasing and τ-clopens sets of $L(\delta)$, where $(L(\delta), \vee, \wedge, r_1)$ is a fuzzy distributive lattice.

If $h(r) \neq 0$, then X is not an antichain, setting $M_1 = \wedge\{r(x, y) / x, y \in X, x \neq y$ and $r(x, y) \neq 0\}$. Then, $M_1 \neq 0$ and we can write r_1 as follows

$$r_1(A, B) = \begin{cases} 1 & \text{if } A = B, \\ Max\left(M_1, \vee_{a \in A, b \in B_{a \neq b}} r(a, b)\right) & \text{if } A \subset B \text{ and } A \neq \phi, \\ M_1 & \text{if } A = \phi, \\ 0 & \text{othewise.} \end{cases}$$

Similar to the previous lemma, r_1 is a fuzzy order and we can assume that $A\vee_{r_1}B = A\cup B$ and $A\wedge_{r_1}B = A\cap B$, for every A and B from $L(\delta)$, where $(L(\delta),\vee,\wedge,r_1)$ is a fuzzy distributive lattice.

(2) To proof the second assertion, we define r_2 by

$$r_2(f,g) = \begin{cases} 1 & \text{if } f = g, \\ r(\wedge\wedge_{A\in f^{-1}(1)}A, \wedge\wedge_{B\in g^{-1}(1)}B) & \text{if } f^{-1}(1)\subset g^{-1}(1), \\ 0 & \text{otherwise.} \end{cases}$$

where the first infimum $\wedge$ is in the sense of the fuzzy ordering relation r and the second infimum $\wedge$ is in the sense of the fuzzy relation r_1. Note that r_2 is well defined: $A_1 = \wedge_{A\in f^{-1}(1)}A$, where the symbol $\wedge$ stands for an infimum with respect to the fuzzy ordering relation r_1, it exists because $L(\delta)$ is a lattice and $a = \wedge A_1$, where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation r. Then, a exists because A_1 is a finite increasing τ-clopen, if A_1 has two minimal elements x, y, then $r(x,y) = 0$, there exist an increasing $\tau-$clopen U and a decreasing $\tau-$clopen V such that $U\cap V = \emptyset$ with $x\in U$ and $y\in V$. It is easy to see that $A_1\subset U$, then $U\cap V\neq\emptyset$ contradiction. By definition r_2 is a fuzzy relation. It is easy to show that r_2 is a fuzzy ordering relation. Furthermore, By [29, 30] $(T(L(\delta)),\tau,r_2)$ is a fuzzy Priestley space. □

The following theorem, shows that the category of finite fuzzy Priestley spaces is equivalent to the dual of the category of finite fuzzy distributive lattices.

In [29, 30], Priestley remarked the basis can be characterized by the fact that they are increasing according to inclusion of prime filters from A by taking the sets $\{F_a/a\in A\}$ as basis, where F_A is the set of all lattice homomorphisms from A onto the chain $\{0,1\}$, non-identical nulls (taking 1 in a).

2.3 Bidual of a finite fuzzy distributive lattice and isomorphisms

Lemma 2.3. *[2]Let A be a fuzzy distributive lattice. The map $F_A : A\mapsto L(T(A))$ defined by*

$$F_A(a) = \{f\in X/f(a) = 1\}$$

is a fuzzy lattice isomorphism.

Proof. For all $a, b \in A$, it is not difficult to see that $F_A(a \wedge b) = F_A(a) \wedge_{R_2} F_A(b)$ and $F_A(a \vee b) = F_A(a) \vee_{R_2} F_A(b)$. Let us show that the map $F_A(a) = \{f \in X / f(a) = 1\}$ is a fuzzy lattice isomorphism.

Suppose that $a \neq b$, it follows $R(a, b) = 0$ or $R(b, a) = 0$.

If $R(a, b) = 0$ then, there exist a prime filter F such that $a \in F$ and $b \notin F$ (by Corollary 1.48), by proposition 1.47 it follows that there exist a surjection $f : A \to \{0, 1\}$ such that $a \in f^{-1}(\{1\})$ and $b \notin f^{-1}(\{1\})$, hence $f(a) = 1$ and $f(b) = 0$ i.e., $R_2(F_A(a), F_A(b)) = 0$.

Similarly if $R(b, a) = 0$ we have $R_2(F_A(b), F_A(a)) = 0$. Hence, $a \neq b$ imply $F_A(a) \neq F_A(b)$ i.e., F_A is injective.

To prove that F_A is surjective let $U \in L(T(A))$, for all $f \in U$ and $g \in L(T(A)) - U$ we have $g < f$, it follows that $\exists a_{fg} \in A$ such that $f(a_{fg}) = 1$ and $g(a_{fg}) = 0$. Then, $f \in F_A(a_{fg})$ and $g \in L(T(A)) - F_A(a_{fg})$. For each fixed $f \in U$ we have $g \in L(T(A)) - U \subseteq \bigcup_{i=1}^{n} (L(T(A)) - F_A(a_{fgi})) = L(T(A)) - F_A(\bigwedge_{i=1}^{n} a_{fgi})$. Setting $\bigwedge_{i=1}^{n} a_{fgi} = a_f$, it follows $F_A(a_f) = F_A(\bigwedge_{i=1}^{n} a_{fgi}) \subset U$, on the other hand $f(a_f) = 1$, then $f \in F_A(a_f)$. Setting $U = \cup_{f \in U} F_A(a_f)$, it follows $U = \bigcup_{j=1}^{n} F_A(a_{fj}) = F_A(\bigvee_{j=1}^{n} a_{fj}) \in L(T(A))$, hence $\exists a = \bigvee_{j=1}^{n} a_{fj}$ such that $U = F_A(a)$ i.e., F_A is surjective.

Let us show that the map $F_A(a) = \{f \in X / f(a) = 1\}$ is increasing i.e., $R(x, y) \leq R_2(F_A(x), F_A(y))$ for all $x, y \in A$, where

$$R_2(F_A(x), F_A(y)) = \begin{cases} 1 & \text{if } F_A(x) = F_A(y), \\ R\left(\wedge \cap_{f \in F_A(x)} f^{-1}(1), \wedge \cap_{g \in F_A(y)} g^{-1}(1)\right) & \text{if } F_A(x) \subset F_A(y), \\ 0 & \text{otherwise.} \end{cases}$$

and the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation R.

Note that $F_A(x) \neq \phi$ for all $x \in A$.

If $x = y$, it follows that $R(x, y) = R_2(F_A(x), F_A(y))$.

If $x \neq y$, we consider two cases

1. If $R(x, y) = 0$, then we have $R(x, y) \leq R_2(F_A(x), F_A(y))$.

2. If $R(x, y) > 0$, it follows that $F_A(x) \subset F_A(y)$, which implies that $R(x, y) = R_2(F_A(x), F_A(y))$ (Since, $\wedge \cap_{f \in F_A(x)} f^{-1}(1) = x$ and $\wedge \cap_{g \in F_A(y)} g^{-1}(1) = y$, because if $\wedge \cap_{f \in F_A(x)} f^{-1}(1) \neq x$, it follows that $\cap_{f \in F_A(x)} f^{-1}(1)$ has minimal elements $z \neq x$, then $r(x, z) = 0$, there exist an increasing $\tau-$clopen U and a decreasing $\tau-$clopen V such that $U \cap V = \emptyset$ with $x \in U$ and

$z \in V$. It is easy to see that $\cap_{f\in F_A(x)} f^{-1}(1) \subset U$, then $U \cap V \neq \emptyset$, contradiction). It follows that the map F_A is a fuzzy lattice isomorphism. □

Lemma 2.4. *[2]If $f : A_1 \mapsto A_2$ is a fuzzy lattice homomorphism, then the map $T(f) : T(A_1) \mapsto T(A_2)$ defined by $T(f)(g) = g \circ f$ is an homomorphism of fuzzy Priestley space, i.e., a continuous and increasing map .*

Proof. For all $g_1, g_2 \in T(A_1)$ $g_1 \leq g_2 \Rightarrow g_1 \circ f \leq g_2 \circ f$, hence $T(f)$ is increasing.

The continuity of $T(f)$ follows from the fact that for every $a \in A_1$,

$$\begin{aligned} T(f)^{-1}(F_{A_1}(a)) &= \{g \in T(A_2) / T(f)(g) \in F_{A_1}(a)\} \\ &= \{g \in T(A_2) / g \circ f(a) = 1\} \\ &= \{g \in T(A_2) / g(f(a)) = 1\} \\ &= F_{A_2}(f(a)). \end{aligned}$$

Hence, $T(f)$ is continuous. □

2.4 Bidual of a finite fuzzy Priestley space and isomorphisms

Lemma 2.5. *[2]If $\delta = (X, \tau, r)$ is a fuzzy Priestley space, then the map $G_\delta : \delta \mapsto T(L(\delta))$ defined by*

$$G_\delta(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

for all $Y \in L(\delta)$ is an isomorphism of fuzzy Priestley space, i.e., a bijection, continuous and increasing map.

Proof. Let $G_\delta : \delta \mapsto T(L(\delta))$ defined by

$$G_\delta(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

To prove the surjections take $f \in T(L(\delta))$ and set $U = \{Y \in L(\delta) : f(Y) = 1\}$, $V = \{Z \in L(\delta) : f(Z) = 0\}$, $A = \cap_{Y\in U} Y$ and $B = \cup_{Z\in V} Z$. To show that $A - B \neq \phi$, suppose that $A - B = \phi$, it follows that $(\cap_{Y\in U} Y) \cap (\cup_{Z\in V} Z)^c = \phi$, then $(\cap_{Y\in U} Y) \cap (\cap_{Z\in V} Z^C) = \phi$, since X is compact we have $(\bigcap_{i=1}^{n} Y_i) \cap (\bigcap_{j=1}^{m} Z_j^C) = \phi$, it follows that $\bigcap_{i=1}^{n} Y_i \subseteq \bigcup_{j=1}^{m} Z_j$, then $f(\bigcup_{j=1}^{m} Z_j) = 1$, contraduction since $f(\overset{m}{\underset{j=1}{\cup}} Z_j) = \overset{m}{\underset{j=i}{\vee}} f(Z_j) = 0$, hence $A - B \neq \phi$. Then, there exists $x \in A - B$

such that $G_\delta(x) = f$. Therefore,

$G_\delta(x)(Y) = 1 \Leftrightarrow x \in Y \Leftrightarrow Y \in U \Leftrightarrow f(Y) = 1$

hence G_δ is surjective.

Let $x_1, x_2 \in \delta$, $x_1 \neq x_2 \Rightarrow r(x_1, x_2) = 0$ or $r(x_2, x_1) = 0$.

If $r(x_1, x_2) = 0$, then there exists $Y_0 \in L(\delta)$ such that $x_1 \in Y_0$ and $x_2 \notin Y_0$, hence $G_\delta(x_1)(Y_0) \neq G_\delta(x_2)(Y_0)$.

If $r(x_2, x_1) = 0$, then, there exists $Y_1 \in L(\delta)$ such that $x_2 \in Y_1$ and $x_1 \notin Y_1$, hence $G_\delta(x_2)(Y_1) \neq G_\delta(x_1)(Y_1)$. It follows that $x_1 \neq x_2 \Rightarrow G_\delta(x_1)(Y) \neq G_\delta(x_2)(Y)$, hence G_δ is injective.

To prove that G_δ is continuous let Z a $\tau-$clopen of $T(L(\delta))$. Then, there exists $y \in L(\delta)$ such that $Y = F_{L(\delta)}(y)$.

$$\begin{aligned} G_\delta^{-1}(Y) &= G_\delta^{-1}(F_{L(\delta)}(y)) \\ &= \left\{x \in X / G_\delta(x) \in F_{L(\delta)}(y)\right\} \\ &= \{x \in X : G_\delta(x)(y) = 1\} \\ &= \{x \in X : x \in y\} \\ &= X \cap y \\ &= y \ (\tau - \text{clopen}) \end{aligned}.$$

Hence, G_δ is continuous.

To prove that G_δ is increasing it suffices to show that $r(x, y) \leq r_2(G_\delta(x), G_\delta(y))$. then it suffices to show that $r(x, y) \leq r_2(G_\delta(x), G_\delta(y))$ for all $x, y \in \delta$.

$$r_2(G_\delta(x), G_\delta(y)) = \begin{cases} 1 & \text{if } G_\delta^{-1}(x) = G_\delta^{-1}(y), \\ r\left(\wedge\wedge_{A \in G_\delta(x)^{-1}(1)} A, \wedge\wedge_{B \in G_\delta(y)^{-1}(1)} B\right) & \text{if } G_\delta^{-1}(x)(1) \subset G_\delta^{-1}(y)(1), \\ 0 & \text{otherwise.} \end{cases}$$

If $x = y$, then $r_2(G_\delta(x), G_\delta(y)) = r(x, y) = 1$.

If $x \neq y$, then there are two cases to investigate

Case 1: if $r(x, y) = 0$, then we have $r(x, y) \leq r_2(G_\delta(x), G_\delta(y))$.

Case 2: if $r(x, y) > 0$, then y belongs to each τ-clopen which contains x, so,

$G_\delta^{-1}(x)(1) \subset G_\delta^{-1}(y)(1)$. It follows that $\wedge\wedge_{A \in G_\delta^{-1}(x)(1)} A = x$ and $\wedge\wedge_{B \in G_\delta^{-1}(y)(1)} B = y$. Then, $r_2(G_\delta(x), G_\delta(y)) = r\left(\wedge\wedge_{A \in G_\delta^{-1}(x)(1)} A, \wedge\wedge_{B \in G_\delta^{-1}(y)(1)} B\right) = r(x, y)$,

hence $r_2(G_\delta(x), G_\delta(y)) = r(x, y)$. □

Lemma 2.6. *[2]If $h : \delta_1 \mapsto \delta_2$ is an homomorphism of fuzzy Priestley space, then the map $L(h) : L(\delta_2) \mapsto L(\delta_1)$ defined by $L(h)(y) = h^{-1}(y)$ for every $y \in L(\delta_2)$ is a fuzzy lattices homomorphism.*

Proof. For all $y \in L(\delta_2)$ we have $L(h)(y) \in L(\delta_1)$.

For all $y, z \in L(\delta_2)$ since h^{-1} commutes with set-theoretical operations we have,

$$\begin{aligned} L(h)(y \cup z) &= h^{-1}(y \cup z) \\ &= h^{-1}(y) \cup h^{-1}(z) \\ &= L(h)(y) \cup L(h)(z). \end{aligned}$$

and

$$\begin{aligned} L(h)(y \cap z) &= h^{-1}(y \cap z) \\ &= h^{-1}(y) \cap h^{-1}(z) \\ &= L(h)(y) \cap L(h)(z). \end{aligned}$$

and for all $y, z \in L(\delta_2)$

$$\begin{aligned} y \subseteq z &\Rightarrow h^{-1}(y) \subseteq h^{-1}(z) \\ &\Rightarrow L(h)(y) \subseteq L(h)(z). \end{aligned}$$

Hence, $L(h)$ is a fuzzy lattice homomorphism. □

Theorem 2.7. *[2]If $f : A_1 \mapsto A_2$ is a fuzzy lattice homomorphism, then*

$$L(T(f)) \circ F_{A_1} = F_{A_2} \circ f.$$

$$\begin{array}{ccccc} & & f & & \\ & A_1 & \dashrightarrow & A_2 & \\ & | & & | & \\ F_{A_1} & | & & | & F_{A_2} \\ & \downarrow & & \downarrow & \\ & L(T(A_1)) & \dashrightarrow & L(T(A_2)) & \\ & & L(T(f)) & & \end{array}.$$

Proof. For all $a \in A_1$,

$$
\begin{aligned}
(L(T(f)) \circ F_{A_1})(a) &= L(T(f))(F_{A_1}(a)) \\
&= T^{-1}(f)(F_{A_1}(a)) \\
&= \{g \in T(A_2) : T(f)(g) \in F_{A_1}(a)\} \\
&= \{g \in T(A_2) : g \circ f \in F_{A_1}(a)\} \\
&= \{g \in T(A_2) : g(f(a)) = 1\} \\
&= F_{A_2}(f(a)) \\
&= F_{A_2} \circ f(a).
\end{aligned}
$$

□

Theorem 2.8. *[2]If $h : \delta_1 \mapsto \delta_2$ is an homomorphism of fuzzy Priestley space, then*

$$T(L(h)) \circ G_{\delta_1} = G_{\delta_2} \circ h.$$

$$
\begin{array}{ccccc}
 & & h & & \\
 & \delta_1 & ----\rightarrow & \delta_2 & \\
 & | & & | & \\
G_{\delta_1} & | & & | & G_{\delta_2} \\
 & \downarrow & & \downarrow & \\
 & T(L(\delta_1)) & ----\rightarrow & T(L(\delta_2)) & \\
 & & T(L(h)) & &
\end{array}.
$$

Proof. For all $f \in \delta_1$,

$$
\begin{aligned}
(T(L(h)) \circ G_{\delta_1})(f) &= T(L(h))(G_{\delta_1}(f)) \\
&= G_{\delta_1}(f) \circ L(h) \,(\text{since } T(f)(g) = g \circ f)
\end{aligned}
$$

hence for all $y \in L(\delta_2)$,

$$
\begin{aligned}
(T(L(h)) \circ G_{\delta_1})(f)(y) &= (G_{\delta_1}(f) \circ L(h))(y) \\
&= G_{\delta_1}(f)(h^{-1}(y)) \\
&= \begin{cases} 1 \text{ if } f \in h^{-1}(y) \\ 0 \text{ if } f \notin h^{-1}(y) \end{cases} \\
&= \begin{cases} 1 \text{ if } h(f) \in y \\ 0 \text{ if } h(f) \notin y \end{cases} \\
&= G_{\delta_2}(h(f))(y) \\
&= (G_{\delta_2} \circ h)(f)(y).
\end{aligned}
$$

□

Theorem 2.9. *[2]The dual of the category of fuzzy finite distributive lattices is equivalent to the category of fuzzy finite Priestley spaces.*

Proof. Lemmas 2.3, 2.5, Theorems 2.7, 2.8 establish the functorial isomorphisms. □

Example 2.10. *Let* $(A, \vee, \wedge, R)$ *be a fuzzy distributive lattice, where* $A = \{a, b, c, d, e\}$ *and* R *is a fuzzy relation defined by*

R	a	b	c	d	e
a	1	0.1	0.3	0.3	0.8
b	0	1	0	0.4	0.6
c	0	0	1	0.4	0.4
d	0	0	0	1	0.3
e	0	0	0	0	1

Then its dual is

$$T(A) = \textit{The set of } 0-1 \textit{ homomorphisms from } A \textit{ onto } \{0,1\} = \{f_1, f_2, f_3\}$$

A	f_1	f_2	f_3
a	0	0	0
b	0	0	1
c	0	1	0
d	0	1	1
e	1	1	1

and its bidual is $L(T(A)) = \{\phi, \{f_2\}, \{f_3\}, \{f_2, f_3\}, X\}$, *where* R_2 *is given by*

μ_{R_2}	ϕ	$\{f_2\}$	$\{f_3\}$	$\{f_2, f_3\}$	X
ϕ	1	0.1	0.1	0.1	0.1
$\{f_2\}$	0	1	0	0.4	0.4
$\{f_3\}$	0	0	1	0.4	0.4
$\{f_2, f_3\}$	0	0	0	1	1
X	0	0	0	0	1

Finally, $F_A : A \mapsto L(T(A))$ *is given by*

A	$F_A(a_i)/i = 1\ to\ 5$
a	ϕ
b	$\{f_2\}$
c	$\{f_3\}$
d	$\{f_2, f_3\}$
e	X

Example 2.11. *Let (X, τ, r) be a Priestley space, where $X = \{x, y, z\}$ and r is given by*

r	x	y	z
x	1	0	0
y	0	1	0
z	0	0	1

Then $L(X) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}$ and r_1 can be given by

$$r_1(A, B) = \begin{cases} 1 & \textit{if } A = B, \\ 1 - \frac{cardA}{cardB} & \text{if } A \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

and r_1 given by

r_1	ϕ	$\{x\}$	$\{y\}$	$\{z\}$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	X
ϕ	1	1	1	1	1	1	1	1
$\{x\}$	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$
$\{y\}$	0	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$
$\{z\}$	0	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$
$\{x, y\}$	0	0	0	0	1	0	0	$\frac{2}{3}$
$\{x, z\}$	0	0	0	0	0	1	0	$\frac{2}{3}$
$\{y, z\}$	0	0	0	0	0	0	1	$\frac{2}{3}$
X	0	0	0	0	0	0	0	1

and the set of $0-1$ *homomorphisms from* $L(X)$ *onto* $\{0,1\}$, *i.e.,* $T(L(X))$ *is equal to* $\{f_1, f_2, f_3\}$

$L(X)$	$f_1(X_i)$	$f_2(X_i)$	$f_3(X_i)$
ϕ	0	0	0
$\{x\}$	1	0	0
$\{y\}$	0	1	0
$\{z\}$	0	0	1
$\{x,y\}$	1	1	0
$\{x,z\}$	1	0	1
$\{y,z\}$	0	1	1
X	1	1	1

And r_2 *will be given by*

r_2	f_1	f_2	f_3
f_1	1	0	0
f_2	0	1	0
f_3	0	0	1

and the isomorphism G_X *is defined by* $G_X : X \to T(L(X))$, *where*

X	$G_X(X_i)/X_i \in X$
x	f_1
y	f_2
z	f_3

.

Example 2.12. *Let* (X, τ, r) *be a Priestley space, where* $X = \{x, y, z, t\}$ *where* r *is given by*

r	x	y	z	t
x	1	0	0.5	0.6
y	0	1	0.4	0.5
z	0	0	1	0.4
t	0	0	0	1

and $L(X) = \{\phi, \{t\}, \{z,t\}, \{x,z,t\}, \{y,z,t\}, X\}$ *, where* r_1 *can be given by*

r_1	ϕ	$\{t\}$	$\{z,t\}$	$\{x,z,t\}$	$\{y,z,t\},$	X
ϕ	1	0.2	0.2	0.2	0.2	0.2
$\{t\}$	0	1	0	0.2	0.2	0.2
$\{z,t\}$	0	0	1	0.2	0.2	0.2
$\{x,z,t\}$	0	0	0	1	0.2	0.2
$\{y,z,t\}$	0	0	0	0	1	0.4
X	0	0	0	0	0	1

and $T(L(X)) = \{f_1, f_2, f_3, f_4\}$ *such that*

$L(X)$	$f_1(X_i)$	$f_2(X_i)$	$f_3(X_i)$	$f_4(X_i)$
ϕ	0	0	0	0
$\{t\}$	0	0	0	1
$\{z,t\}$	0	0	1	1
$\{x,z,t\}$	0	1	1	1
$\{y,z,t\}$	1	0	1	1
X	1	1	1	1

.

The isomorphism G_X *is defined as follows* $G_X : X \mapsto T(L(X))$

X	$G_X(X_i)\ X_i \in X$
x	f_1
y	f_2
z	f_3
t	f_4

Chapter 3

Priestley duality for infinite fuzzy bounded distributive lattices

The results obtained in this chapter have been published in the "Journal of Intelligent and Fuzzy Systems"[4].

In this chapter we extend the result obtained by A. Amroune and B. Davvaz in [2] to the case of infinite fuzzy bounded distributive lattices.

Throughout this chapter , all fuzzy lattices are distributive and bounded (fuzzy closed distributive lattices), and homomorphisms preserve the smallest element (denoted 0) and the greatest element (denoted 1). If $(A, \wedge, \vee, R)$ is a fuzzy bounded distributive lattice, then its dual is a fuzzy priestley space defined by $T(A) = (X, \tau, R_1)$, where X is the set of $0-1$ homomorphisms from A onto $\{0, 1\}$ τ be the topology induced by the product topology of $\{0, 1\}^A$ and R_1 is a fuzzy order on X adequately chosen.

If $\delta = (X, \tau, r)$ is a fuzzy Priestley space, then its dual is a fuzzy distributive lattice a defined by $(L(\delta), \vee, \wedge, r_1)$, where $L(\delta) = \{Y \subseteq X/\ Y$ is increasing and $\tau-$ clopen$\}$ and r_1 is a fuzzy order adequately chosen.

3.1 Dual and bidual of infinite fuzzy bounded distributive lattice

Lemma 3.1. *If $(A, \vee, \wedge, R)$ is a fuzzy bounded distributive lattice, then there exist two fuzzy orders R_1, R_2 such that*

(1) $T(A) = (X, \tau, R_1)$ *is a fuzzy Priestley space,*

(2) $(L(T(A)), \vee_{R_2}, \wedge_{R_2}, R_2)$ *is a fuzzy distributive lattice.*

Proof. Let R_1 be the relation defined by

$$R_1(f,g) = \begin{cases} 1 & f = g; \\ \vee_{\substack{a\in g^{-1}(1), b\in f^{-1}(1) \\ a\neq b}} R(a,b) & f^{-1}(1) \subset g^{-1}(1); \\ 0 & \text{otherwise.} \end{cases}$$

We show that R_1 is a fuzzy order. Since $R_1(f,f) = 1$ for all $f \in X$, it follows that R_1 is reflexive. For all $f, g \in X$ such that $f \neq g$, we have $R_1(f,g) \wedge R_1(g,f) = 0$ (it suffices to apply the definition of R_1). Hence, R_1 is antisymmetric. Now, for all $f, g, h \in X$ we show that $R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h)$. Indeed, there are eight possible relations between $f^{-1}(1)$, $g^{-1}(1)$, $h^{-1}(1)$ in term of inclusion. Let (P) be the proposition : $(R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h))$, and put $A = (f^{-1}(1) \subseteq g^{-1}(1))$, $B = (g^{-1}(1) \subseteq h^{-1}(1))$ and $C = (f^{-1}(1) \subseteq h^{-1}(1))$.

The only case that requires an investigation is $f^{-1}(1) \subseteq g^{-1}(1)$ and $g^{-1}(1) \subseteq h^{-1}(1)$. Since $\vee_{\substack{a\in g^{-1}(1), b\in f^{-1}(1) \\ a\neq b}} R(a,b) \leq \vee_{\substack{c\in h^{-1}(1), b\in f^{-1}(1) \\ c\neq b}} R(c,b)$, where $g^{-1}(1) \subseteq h^{-1}(1)$,

A	B	C	(P)
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

then it follows that $\vee_{\substack{a\in g^{-1}(1), b\in f^{-1}(1) \\ a\neq b}} R(a,b) \wedge \vee_{\substack{c\in h^{-1}(1), a\in g^{-1}(1) \\ c\neq a}} R(a,c) \leq \vee_{\substack{c\in h^{-1}(1), b\in f^{-1}(1) \\ c\neq b}} R(c,b)$. Hence for all $f, g, h \in X$, the inequality $R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h)$ holds, and R_1 is transitive. It follows that R_1 is a fuzzy order and according to [29], $T(A) = (X, \tau, R_1)$ is a fuzzy Priestley space. This completes the proof of (1) For the second case (2), let

$M_0 = \wedge_{x \in A} \wedge_{y \in A} \{R(x,y)/x \neq y \text{ and } R(x,y) > 0\}$ and let R_2 be the relation defined by

$$R_2(H,D) = \begin{cases} 1 & \text{if } H = D; \\ Max\,(M_0, \delta) & \text{if } H \subset D \text{ and } H \neq \phi; \\ M_0 & \text{if } H = \phi; \\ 0 & \text{otherwise.} \end{cases}$$

where $\delta = \vee_{\substack{a \in \cup_{f \in H} f^{-1}(1), b \in \cup_{g \in D} g^{-1}(1) \\ (a \neq b)}} R(a,b)$.

Note that both $\cup_{f \in H} f^{-1}(1)$ and $\vee_{(a \neq b)} R(a,b)$ exist. First, we show that R_2 is a fuzzy order. For the reflexivity, we have $R_2(H,H) = 1$. To prove the antisymmetric property of R_2, let $H, D \in L(T(A))$. Clearly, $R_2(H,D) \wedge R_2(D,H)) = 0$ whenever $H = D$, i.e., R_2 is antisymmetric. To verify the transitivity, we will use the following truth table, where P is the proposition $(R_2(H,D) \wedge R_2(D,E)) \leq R_2(H,E))$. First, if one of the three elements H, D, E is empty, then the transitivity holds trivially.

If $H \neq \phi$ and $D \neq \phi$ and $E \neq \phi$, the only case that needs investigation is when $H \subset D$ and $D \subset E$. This yields to $Max(M_0, \alpha) \wedge Max(M_0, \beta) = Max(M_0, \alpha \wedge \beta) \leq Max(M_0, \delta)$.

$H \subseteq D$	$D \subseteq E$	$H \subseteq E$	(P)
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

where, $\alpha = \vee_{\substack{a \in \cup_{f \in H} f^{-1}(1), b \in \cup_{g \in D} g^{-1}(1) \\ (a \neq b)}} R(a,b)$, $\beta = \vee_{\substack{b \in \cup_{g \in D} g^{-1}(1),\, c \in \cup_{h \in E} h^{-1}(1), \\ (b \neq c)}} R(b,c)$ and $\delta = \vee_{\substack{a \in \cup_{f \in H} f^{-1}(1), c \in \cup_{h \in E} h^{-1}(1) \\ (a \neq c)}} R(a,c)$. This inequality comes from $\vee_{\substack{a \in \cup_{f \in H} f^{-1}(1), b \in \cup_{g \in D} g^{-1}(1) \\ (a \neq b)}} R(a,b) \leq \vee_{\substack{a \in \cup_{f \in H} f^{-1}(1), c \in \cup_{h \in E} h^{-1}(1) \\ (a \neq c)}} R(a,c)$, where $\cup_{g \in D} g^{-1}(1) \subset \cup_{h \in E} h^{-1}(1)$. Hence R_2 is transitive.

Finally, the least upper and greatest lower bounds of H and D (with respect to the relation R_2) are denoted by $H \vee_{R_2} D$ and $H \wedge_{R_2} D$, respectively. We prove that $H \vee_{R_2} D = H \cup D$

and $H \wedge_{R_2} D = H \cap D$. It is not difficult to see that $H \cup D$ is an upper bound of $\{H, D\}$ since $R_2(H, H \cup D) > 0$ and $R_2(D, H \cup D) > 0$. If C is the least upper bound of $\{H, D\}$, then we have four cases

1. if $H = \phi$ and $D = \phi$, then $R_2(C, H \cup D) = R_2(C, \phi)$, which is different from 0 if and only if $C = \phi$. Hence $C = H \cup D$.

2. if $H = \phi$, $D \neq \phi$ and $D \subset C$. Since $R_2(D, C) > 0$, we have $R_2(C, H \cup D) = R_2(C, D)$, which is different from 0 if and only if $C \subset D$, hence $C = D = H \cup D$.

3. if $H \neq \phi$, $D = \phi$ and $H \subset C$. Since $R(H, C) > 0$, it follows that $R_2(C, H \cup D) = R(C, H)$, which is equivalent to $C \subset H$. Hence $C = H = H \cup D$.

4. if $H \neq \phi$ and $D \neq \phi$, then we have $H \subset C$ and $D \subset C$; this implies that $H \cup D \subset C$, it follows that $R_2(C, H \cup D)$ is different from 0 if and only if $C \subset H \cup D$. Hence $C = H \cup D$.

Similarly, we proof that $H \wedge_{R_2} D = H \cap D$. It is known that $H \cap D$ and $H \cup D$ are increasing and $\tau-$clopens. This shows that $(L(T(A)), \vee_{R_2}, \wedge_{R_2}, R_2)$ is a fuzzy distributive lattice. □

3.2 Dual and bidual of infinite fuzzy Priestley space

Lemma 3.2. *If $\delta = (X, \tau, r)$ is a fuzzy Priestley space, then there exist two fuzzy orders r_1 and r_2 such that*

(1) $(L(\delta), \vee_{r_1}, \wedge_{r_1}, r_1)$ *is a fuzzy distributive lattice.*

(2) $(T(L(\delta)), \tau, r_2)$ *is a fuzzy Priestley space.*

Proof. If $h(r) = 0$, then X is an antichain and we can write r_1 as follows

$$r_1(A, B) = \begin{cases} 1 & \text{if } A = B, \\ \alpha(\alpha \in]0, 1[) & \text{if } A \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to see that r_1 is a fuzzy order, $A \vee_{r_1} B = A \cup B$ and $A \wedge_{r_1} B = A \cap B$ exist for every A and B in $L(\delta)$. Therefore, $A \cup B$ and $A \cap B$ are increasing, and τ-clopen sets of $L(\delta)$, where $(L(\delta), \vee_{r_1}, \wedge_{r_1}, r_1)$ is a fuzzy distributive lattice.

If $h(r) \neq 0$, then X is not an antichain and we can choose
$m_0 = \wedge_{x \in X} \wedge_{y \in X} \{r(x,y)/\ x \neq y \text{ and } r(x,y) > 0\}$, clearly $m_0 \neq 0$, we can set

$$r_1(A,B) = \begin{cases} 1 & \text{if } A = B, \\ Max(m_0, \vee_{\substack{a \in A, b \in B \\ (a \neq b)}} r(a,b)) & \text{if } A \subset B \text{ and } A \neq \phi, \\ m_0 & \text{if } A \neq \phi, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to previous Lemma, r_1 is a fuzzy order and we can assume that $A \vee_{r_1} B = A \cup B$ and $A \wedge_{r_1} B = A \cap B$ for every A and B in $L(\delta)$,where $(L(\delta), \vee_{r_1}, \wedge_{r_1}, r_1)$ is a fuzzy distributive lattice. This completes the proof of (1).

To prove the second assertion, let $\alpha = \bigvee_{\substack{a \in \cap_{A \in f^{-1}(1)} A,\ b \in \cap_{B \in g^{-1}(1)} B \\ a \neq b}} r(a,b)$ for all f and g in $T(L(\delta))$.

$$r_2(f,g) = \begin{cases} 1 & \text{if } f = g, \\ \alpha & \text{if } f^{-1}(1) \subset g^{-1}(1) \text{ and } \cap_{B \in g^{-1}(1)} B \neq \phi, \\ m_0 & \text{if } \cap_{B \in g^{-1}(1)} B = \phi, \\ 0 & \text{otherwise.} \end{cases}$$

We will show that the choice of $\vee_{\substack{a \in \cap_{A \in f^{-1}(1)} A,\ b \in \cap_{B \in g^{-1}(1)} B \\ a \neq b}} r(a,b)$ is possible.

Firstly, it is not difficult to see that r_2 is well defined since the closed set $\cap_{A \in f^{-1}(1)} A$ exists in τ.

Secondly,

1. $\cap_{A \in f^{-1}(1)} A \neq \phi$, since $f^{-1}(1) \subset g^{-1}(1)$ which implies that $\cap_{B \in g^{-1}(1)} B \subset \cap_{A \in f^{-1}(1)} A$

   with $\cap_{B \in g^{-1}(1)} B \neq \phi$, i.e. $\cap_{A \in f^{-1}(1)} A \neq \phi$.

2. We show that $\cap_{A \in f^{-1}(1)} A \neq \cap_{B \in g^{-1}(1)} B$. Suppose that $\cap_{A \in f^{-1}(1)} A = \cap_{B \in g^{-1}(1)} B$. If $f^{-1}(1) = \{Y_1 : i \in I\}$ and $g^{-1}(1) = \{Y_1 : i \in I\} \cup \{Zi : j \in J\}$, where $Z_j \neq Y_i$ for all $i \in I$ and $j \in J$, then $\cap_{i \in I} Y_i = \cap_{i \in I} Y_i \cap \cap_{j \in J} Z_j$, which implies that $\cap_{i \in I} Y_i \subset Z_j$ for all $j \in J$. Let j_0 be a fixed element of J. Then $\cap_{i \in I} Y_i \cap Z_{j_0} = \cap_{i \in I} Y_i$. On the other

hand we have $\cap_{i\in I} Y_i \subset Z_{j_0}$ where Z_{j_0} is increasing and τ-clopen, which implies that $Z_{j_0}^c \subset \cup_{i\in I} Y_i^c$. Since $Z_{j_0}^c$ is $\tau-$clopen, it is compact (see [28, Page 76]). Therefore $Z_{j_0}^c \subset \bigcup_{i=1}^{n} Y_i^c$, this implies that $\bigcap_{i=1}^{n} Y_i \subset Z_{j_0}$, hence $f(Z_{j_0}) \geq f(\bigcap_{i=1}^{n} Y_i) = \bigwedge_{i=1}^{n} f(Y_i) = 1$, which yields $Z_{j_0} \in f^{-1}(1)$. This is impossible. Hence, $\cap_{A\in f^{-1}(1)} A \neq \cap_{B\in g^{-1}(1)} B$.

To show that r_2 is a fuzzy order, we have $r_2(f,f) = 1$ for all $f \in X$, then r_2 is reflexive. For all $f, g \in T(L(\delta))$, clearly $r_2(f,g) \wedge r_2(g,f) = 0$, whenever $f = g$, then r_2 is antisymmetric.

To verify the transitivity, we use the following truth table, where the proposition P is $(r_2(f,g) \wedge r_2(g,h) \leq r_2(f,h))$, for all $f, g, h \in T(L(\delta))$; also $*$ means can not occur and $\varphi_1^{-1} \subset \psi_1^{-1}$ is $\varphi^{-1}(1) \subset \psi^{-1}(1)$.

$f_1^{-1} \subset g_1^{-1}$	$g_1^{-1} \subset h_1^{-1}$	$f_1^{-1} \subset h_1^{-1}$	(p)
1	1	1	1
1	1	0	$*$
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

As shown above, the only case who merits to be checked is $f^{-1}(1) \subset g^{-1}(1)$ and $g^{-1}(1) \subset h^{-1}(1)$.

1. If $\cap_{A\in f^{-1}(1)} A = \phi$ or $\cap_{B\in g^{-1}(1)} B = \phi$ or $\cap_{C\in h^{-1}(1)} C = \phi$, the transitivity is trivial.

2. If $\cap_{A\in f^{-1}(1)} A \neq \phi$ and $\cap_{B\in g^{-1}(1)} B \neq \phi$ and $\cap_{C\in h^{-1}(1)} C \neq \phi$,

Let $\alpha = \vee_{\substack{a\in\cap_{A\in f^{-1}(1)} A,\, b\in\cap_{B\in g^{-1}(1)} B \\ (a\neq b)}} r(a,b)$, $\beta = \vee_{\substack{b\in\cap_{B\in g^{-1}(1)} B,\, c\in\cap_{C\in h^{-1}(1)} C \\ (b\neq c)}} r(b,c)$ and $\gamma = \vee_{\substack{a\in\cap_{A\in f^{-1}(1)} A,\, c\in\cap_{C\in h^{-1}(1)} C \\ (a\neq c)}} r(a,c)$, we have $\alpha \wedge \beta \leq \gamma$, since $\beta \leq \gamma$, where $\cap_{B\in g^{-1}(1)} B \subset \cap_{A\in f^{-1}(1)} A$. Hence, r_2 is transitive. Furthermore, $(T(L(\delta)), \tau, r_2)$ is a fuzzy Priestley space. □

In the following we extend the main result of [2] in the infinite case i.e., we show that the category of fuzzy Priestley spaces is equivalent to the dual of the category of fuzzy bounded distributive lattices.

3.3 Bidual of infinite fuzzy bounded distributive lattice and isomorphisms

Lemma 3.3. *Let A be a fuzzy distributive lattice. The map $F_A : A \mapsto L(T(A))$ defined by*

$$F_A\left(a\right) = \{f \in X / f(a) = 1\}$$

is a fuzzy lattice isomorphism.

Proof. Let us show that the map $F_A(a) = \{f \in X/f(a) = 1\}$ is a fuzzy lattice isomorphism.

For all $a, b \in A$, it is not difficult to see that $F_A\left(a \wedge b\right) = F_A\left(a\right) \wedge_{R_2} F_A\left(b\right)$ and $F_A\left(a \vee b\right) = F_A\left(a\right) \vee_{R_2} F_A\left(b\right)$.

Suppose that $a \neq b$, it follows $R\left(a, b\right) = 0$ or $R\left(b, a\right) = 0$.

If $R\left(a, b\right) = 0$ then, there exist a prime filter F such that $a \in F$ and $b \notin F$ (Corollary 1.48), by proposition 1.47 it follows that there exists a surjection $f : A \to \{0, 1\}$ such that $a \in f^{-1}\left(\{1\}\right)$ and $b \notin f^{-1}\left(\{1\}\right)$, hence $f\left(a\right) = 1$ and $f\left(b\right) = 0$ i.e., $R_2\left(F_A\left(a\right), F_A\left(b\right)\right) = 0$.

Similarly if $R\left(b, a\right) = 0$ we have $R_2\left(F_A\left(b\right), F_A\left(a\right)\right) = 0$. Hence, $a \neq b$ imply $F_A\left(a\right) \neq F_A\left(b\right)$ i.e., F_A is injective.

To prove that F_A is surjective let $U \in L\left(T\left(A\right)\right)$, for all $f \in U$ and $g \in L\left(T\left(A\right)\right) - U$ we have $g < f$, it follows that $\exists a_{fg} \in A$ such that $f\left(a_{fg}\right) = 1$ and $g\left(a_{fg}\right) = 0$. Then, $f \in F_A\left(a_{fg}\right)$ and $g \in L\left(T\left(A\right)\right) - F_A\left(a_{fg}\right)$. For fixed $f \in U$ we have $g \in L\left(T\left(A\right)\right) - U \subseteq \overset{n}{\underset{i=1}{\cup}}\left(L\left(T\left(A\right)\right) - F_A\left(a_{fgi}\right)\right) = L\left(T\left(A\right)\right) - F_A(\overset{n}{\underset{i=1}{\wedge}} a_{fgi})$. Setting $\overset{n}{\underset{i=1}{\wedge}} a_{fgi} = a_f$, it follows $F_A\left(a_f\right) = F_A(\overset{n}{\underset{i=1}{\wedge}} a_{fgi}) \subset U$, on the other hand $f\left(a_f\right) = 1$ then $f \in F_A\left(a_f\right)$. Setting $U = \cup_{f \in U} F_A\left(a_f\right)$, it follows $U = \overset{n}{\underset{j=1}{\cup}} F_A\left(a_{fj}\right) = F_A(\overset{n}{\underset{j=i}{\vee}} a_{fj}) \in L\left(T\left(A\right)\right)$, hence $\exists a = \overset{n}{\underset{j=i}{\vee}} a_{fj}$ such that $U = F_A\left(a\right)$ i.e., F_A is surjective.

Let us show that the map $F_A(a) = \{f \in X/f(a) = 1\}$ is increasing.

We show that $R(x, y) \leq R_2(F_A(x), F_A(y))$ for all $x, y \in A$, setting

$\alpha = \max(M_0, \vee_{\substack{a \in \cup_{f \in F_A(x)} f^{-1}(1), b \in \cup_{g \in F_A(y)} g^{-1}(1) \\ (a \neq b)}} R(a, b))$, we have

$$R_2(F_A(x), F_A(y)) = \begin{cases} 1 & \text{if } F_A(x) = F_A(y), \\ \alpha & \text{if } F_A(x) \subset F_A(y), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $F_A(x) \neq \phi$ for all $x \in A$.

If $x = y$, then $R(x, y) = R_2(F_A(x), F_A(y))$.

If $x \neq y$,we consider two cases

1. If $R(x, y) = 0$, then we have $R(x, y) \leq R_2(F_A(x), F_A(y))$.

2. If $R(x, y) > 0$, then $F_A(x) \subset F_A(y)$, which implies that $R(x, y) \leq R_2(F_A(x), F_A(y))$.

   It follows that the map F_A is a fuzzy lattice isomorphism.

□

Lemma 3.4. *If $f : A_1 \mapsto A_2$ is a fuzzy lattice homomorphism, then the map $T(f) : T(A_1) \mapsto T(A_2)$ defined by $T(f)(g) = g \circ f$ is an homomorphism of fuzzy Priestley space, i.e., a continuous and increasing map.*

Proof. For all $g_1, g_2 \in T(A_1)$ $g_1 \leq g_2 \Rightarrow g_1 \circ f \leq g_2 \circ f$, hence $T(f)$ is increasing.

The continuity of $T(f)$ follows from the fact that for every $a \in A_1$,

$$\begin{aligned} T(f)^{-1}(F_{A_1}(a)) &= \{g \in T(A_2) / T(f)(g) \in F_{A_1}(a)\} \\ &= \{g \in T(A_2) / g \circ f(a) = 1\} \\ &= \{g \in T(A_2) / g(f(a)) = 1\} \\ &= F_{A_2}(f(a)). \end{aligned}$$

Hence, $T(f)$ is continuous. □

3.4 Bidual of infinite fuzzy Priestley space and isomorphisms

Lemma 3.5. *If $\delta = (X, \tau, r)$ is a fuzzy Priestley space, then the map $G_\delta : \delta \mapsto T(L(\delta))$ defined by*

$$G_\delta(x)(Y) = \begin{cases} 1 & \textit{if } x \in Y, \\ 0 & \textit{if } x \notin Y. \end{cases}$$

for all $Y \in L(\delta)$ is an isomorphism of fuzzy Priestley space, i.e., a bijection, continuous and increasing map.

Proof. Let $G_\delta : \delta \mapsto T(L(\delta))$ defined by

$$G_\delta(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

For prove the surjections take $f \in T(L(\delta))$ and setting $U = \{Y \in L(\delta) : f(Y) = 1\}$, $V = \{Z \in L(\delta) : f(Z) = 0\}$, $A = \cap_{Y \in U} Y$ and $B = \cup_{Z \in V} Z$. To show that $A - B \neq \phi$, suppose that $A - B = \phi$, it follows that $(\cap_{Y \in U} Y) \cap (\cup_{Z \in V} Z)^c = \phi$, then $(\cap_{Y \in U} Y) \cap (\cap_{Z \in V} Z^C) = \phi$, since X is compact we have $(\overset{n}{\underset{i=1}{\cap}} Y_i) \cap (\overset{m}{\underset{j=1}{\cap}} Z_j^C) = \phi$, it follows that $\overset{n}{\underset{i=1}{\cap}} Y_i \subseteq \overset{m}{\underset{j=1}{\cup}} Z_j$, then $f(\overset{m}{\underset{j=1}{\cup}} Z_j) = 1$,

conduction since $f(\overset{m}{\underset{j=1}{\cup}} Z_j) = \overset{m}{\underset{j=i}{\vee}} f(Z_j) = 0$, hence $A - B \neq \phi$. Then, there exists $x \in A - B$ such that $G_\delta(x) = f$. Therefore,

$G_\delta(x)(Y) = 1 \Leftrightarrow x \in Y \Leftrightarrow Y \in U \Leftrightarrow f(Y) = 1.$

Hence, G_δ is surjective.

Let $x_1, x_2 \in \delta$, $x_1 \neq x_2 \Rightarrow r(x_1, x_2) = 0$ or $r(x_2, x_1) = 0$.

If $r(x_1, x_2) = 0$, then there exists $Y_0 \in L(\delta)$ such that $x_1 \in Y_0$ and $x_2 \notin Y_0$, hence $G_\delta(x_1)(Y_0) \neq G_\delta(x_2)(Y_0)$.

If $r(x_2, x_1) = 0$, then, there exists $Y_1 \in L(\delta)$ such that $x_2 \in Y_1$ and $x_1 \notin Y_1$, hence $G_\delta(x_2)(Y_1) \neq G_\delta(x_1)(Y_1)$. It follows that $x_1 \neq x_2 \Rightarrow G_\delta(x_1)(Y) \neq G_\delta(x_2)(Y)$. Hence, G_δ is injective.

To prove that G_δ is continuous let Y a $\tau-$clopen of $T(L(\delta))$. Then, there exists $y \in L(\delta)$ such that $Y = F_{L(\delta)}(y)$.

$$\begin{aligned} G_\delta^{-1}(Y) &= G_\delta^{-1}(F_{L(\delta)}(y)) \\ &= \left\{x \in X / G_\delta(x) \in F_{L(\delta)}(y)\right\} \\ &= \{x \in X : G_\delta(x)(y) = 1\} \\ &= \{x \in X : x \in y\} \\ &= X \cap y \\ &= y \ (\tau - \text{clopen}). \end{aligned}$$

Hence, G_δ is continuous.

To prove that G_δ is increasing it suffices to show that $r(x,y) \leq r_2(G_\delta(x), G_\delta(y))$. then it suffices to show that $r(x,y) \leq r_2(G_\delta(x), G_\delta(y))$ for all $x, y \in \delta$.

$$\mu_{r_2}\left(G_\delta(x), G_\delta(y)\right) = \begin{cases} 1 & \text{if } G_\delta(x)^{-1} = G_\delta(y)^{-1}, \\ \gamma & \text{if } G_\delta(x)^{-1}(1) \subset G_\delta(y)^{-1}(1), \\ 0 & \text{otherwise.} \end{cases}$$

where

$\gamma = \vee_{\substack{a \in \cap_{A \in G_\delta^{-1}(x)(1)} A,\ b \in \cap_{B \in G_\delta^{-1}(y)(1)} B \\ a \neq b}} r\left(a, b\right)$

If $x = y$, then $r_2(G_\delta(x), G_\delta(y)) = r\left(x, y\right) = 1$.

If $x \neq y$ else we consider two cases

1. If $r(x,y) = 0$, then we have $r(x,y) \leq r_2(G_\delta(x), G_\delta(y))$.

2. If $r(x,y) > 0$, then y belongs to each $\tau-$clopen which contains x, then $G_\delta^{-1}(x) \subset G_\delta^{-1}(y)$, which implies that $r_2(G_\delta(x), G_\delta(y)) = \vee_{\substack{a \in \cap_{A \in G_\delta^{-1}(x)(1)} A,\ b \in \cap_{B \in G_\delta^{-1}(y)(1)} B \\ a \neq b}} r\left(a, b\right) \geq r(x,y)$. Hence, $r(x,y) \leq r_2(G_\delta(x), G_\delta(y))$.

□

Lemma 3.6. *If $h : \delta_1 \mapsto \delta_2$ is an homomorphism of fuzzy Priestley space, then the map $L(h) : L(\delta_2) \mapsto L(\delta_1)$ defined by $L(h)(y) = h^{-1}(y)$ for every $y \in L(\delta_2)$ is a fuzzy lattices homomorphism.*

Proof. For all $y \in L(\delta_2)$ we have $L(h)(y) \in L(\delta_1)$.

For all $y, z \in L(\delta_2)$ since h^{-1} commutes with set-theoretical operations we have,

$$\begin{aligned} L(h)(y \cup z) &= h^{-1}(y \cup z) \\ &= h^{-1}(y) \cup h^{-1}(z) \\ &= L(h)(y) \cup L(h)(z). \end{aligned}$$

and

$$\begin{aligned} L(h)(y \cap z) &= h^{-1}(y \cap z) \\ &= h^{-1}(y) \cap h^{-1}(z) \\ &= L(h)(y) \cap L(h)(z). \end{aligned}$$

and for all $y, z \in L\left(\delta_2\right)$

$$\begin{aligned} y \subseteq z &\Rightarrow h^{-1}\left(y\right) \subseteq h^{-1}\left(z\right) \\ &\Rightarrow L\left(h\right)\left(y\right) \subseteq L\left(h\right)\left(z\right). \end{aligned}$$

Hence, $L(h)$ is fuzzy lattices homomorphism. □

Theorem 3.7. *If $f : A_1 \mapsto A_2$ is a fuzzy lattice homomorphism, then*

$$L\left(T(f)\right) \circ F_{A_1} = F_{A_2} \circ f.$$

$$\begin{array}{ccccc}
 & & f & & \\
 & A_1 --- & ---- & \rightarrow & A_2 \\
 & | & & & | \\
F_{A_1} & | & & & | \; F_{A_2} \\
 & \downarrow & & & \downarrow \\
 & L\left(T\left(A_1\right)\right) & ---- & \rightarrow & L\left(T\left(A_2\right)\right) \\
 & & L\left(T\left(f\right)\right) & &
\end{array} .$$

Proof. For all $a \in A_1$,

$$\begin{aligned}
\left(L\left(T\left(f\right)\right) \circ F_{A_1}\right)(a) &= L\left(T\left(f\right)\right)\left(F_{A_1}\left(a\right)\right) \\
&= T^{-1}\left(f\right)\left(F_{A_1}\left(a\right)\right) \\
&= \left\{g \in T\left(A_2\right) : T\left(f\right)\left(g\right) \in F_{A_1}\left(a\right)\right\} \\
&= \left\{g \in T\left(A_2\right) : g \circ f \in F_{A_1}\left(a\right)\right\} \\
&= \left\{g \in T\left(A_2\right) : g\left(f\left(a\right)\right) = 1\right\} \\
&= F_{A_2}\left(f\left(a\right)\right) \\
&= F_{A_2} \circ f\left(a\right).
\end{aligned}$$

□

Theorem 3.8. *If $h : \delta_1 \mapsto \delta_2$ is an homomorphism of fuzzy Priestley space, then*

$$T\left(L\left(h\right)\right) \circ G_{\delta_1} = G_{\delta_2} \circ h.$$

$$\begin{array}{cccccc}
 & & h & & & \\
 & \delta_1 & ---- & \rightarrow & \delta_2 & \\
 & | & & & | & \\
G_{\delta_1} & | & & & | & G_{\delta_2} \\
 & \downarrow & & & \downarrow & \\
 & T\left(L\left(\delta_1\right)\right) & ---- & \rightarrow & T\left(L\left(\delta_2\right)\right) & \\
 & & T\left(L\left(h\right)\right) & & &
\end{array} .$$

Proof. For all $f \in \delta_1$,

$$\begin{aligned}(T(L(h)) \circ G_{\delta_1})(f) &= T(L(h))(G_{\delta_1}(f)) \\ &= G_{\delta_1}(f) \circ L(h) \text{ (since } T(f)(g) = g \circ f)\end{aligned}$$

hence for all $y \in L(\delta_2)$,

$$\begin{aligned}(T(L(h)) \circ G_{\delta_1})(f)(y) &= (G_{\delta_1}(f) \circ L(h))(y) \\ &= G_{\delta_1}(f)(h^{-1}(y)) \\ &= \begin{cases} 1 \text{ if } f \in h^{-1}(y) \\ 0 \text{ if } f \notin h^{-1}(y) \end{cases} \\ &= \begin{cases} 1 \text{ if } h(f) \in y \\ 0 \text{ if } h(f) \notin y \end{cases} \\ &= G_{\delta_2}(h(f))(y) \\ &= (G_{\delta_2} \circ h)(f)(y).\end{aligned}$$

□

Theorem 3.9. *The dual of the category of fuzzy infinite bounded distributive lattices is equivalent to the category of fuzzy infinite Priestley spaces.*

Proof. Lemmas 3.3, 3.5, Theorems 3.7, 3.8 establish the functorial isomorphisms. □

Example 3.10. *Let $(A, \vee, \wedge, R)$ be a fuzzy distributive lattice,where $A = \{a, b, c, d, e, f\}$ and R be a fuzzy relation defined by*

R	a	b	c	d	e	f
a	1	0.2	0.2	0.3	0.4	0.8
b	0	1	0.2	0.3	0.3	0.5
c	0	0	1	0	0.4	0.5
d	0	0	0	1	0.3	0.4
e	0	0	0	0	1	0.3
f	0	0	0	0	0	1

.

Then, its dual is

$$\begin{aligned}T(A) &= \textit{The set of } 0-1 \textit{ homomorphisms from } A \textit{ onto } \{0,1\} \\ &= \{f_1, f_2, f_3, f_4\}\end{aligned}$$

where f_1, f_2, f_3, f_4 are given by

	f_1	f_2	f_3	f_4
a	0	0	0	0
b	0	0	1	0
c	0	1	1	0
d	1	0	1	0
e	1	1	1	0
f	1	1	1	1

and R_1 is given by

R_1	f_1	f_2	f_3	f_4
f_1	1	0	0.5	0
f_2	0	1	0.5	0
f_3	0	0	1	0
f_4	0.5	0.5	0.5	1

which have the bidual as follows

$L(T(A)) = \{\phi, \{f_3\}, \{f_1, f_3\}, \{f_2, f_3\}, \{f_1, f_2, f_3\}, X\}$ *where R_2 is given by*

R_2	ϕ	$\{f_3\}$	$\{f_1, f_3\}$	$\{f_2, f_3\}$	$\{f_1, f_2, f_3\}$	X
ϕ	1	0.2	0.2	0.2	0.2	0.2
$\{f_3\}$	0	1	0.5	0.5	0.5	0.5
$\{f_1, f_3\}$	0	0	1	0	0.5	0.5
$\{f_2, f_3\}$	0	0	0	1	0.5	0.5
$\{f_1, f_2, f_3\}$	0	0	0	0	1	0.5
X	0	0	0	0	0	1

.

Finally, $F_A : A \to L(T(A))$ is given by

A	$F_A(a_1) : i = 1 \ to \ 6$
a	ϕ
b	$\{f_3\}$
c	$\{f_1, f_3\}$
d	$\{f_2, f_3\}$
e	$\{f_1, f_2, f_3\}$
f	X

.

Example 3.11. *Let (X, τ, r) be a Priestley space, where $X = \{x, y, z\}$ and r is given by*

r	x	y	z
x	1	0	0
y	0	1	0
z	0	0	1

.

Then $L(X) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}$ and r_1 is given by

$$r_1(A, B) = \begin{cases} 1 & \text{if } A = B; \\ \alpha\,(\alpha \in \,]0, 1[) & \text{if } A \subset B; \\ 0 & \text{otherwise.} \end{cases} .$$

Then the table of r_1 is given by

r_1	ϕ	$\{x\}$	$\{y\}$	$\{z\}$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	X
ϕ	1	α	α	α	α	α	α	α
$\{x\}$	0	1	0	0	α	α	0	α
$\{y\}$	0	0	1	0	α	0	α	α
$\{z\}$	0	0	0	1	0	α	α	α
$\{x, y\}$	0	0	0	0	1	0	0	α
$\{x, z\}$	0	0	0	0	0	1	0	α
$\{y, z\}$	0	0	0	0	0	0	1	α
X	0	0	0	0	0	0	0	1

.

The set $T(L(X))$ is equal to $\{f_1, f_2, f_3\}$.

$L(X)$	$f_1(X_i)$	$f_1(X_i)$	$f_1(X_i)$
ϕ	0	0	0
$\{x\}$	1	0	0
$\{y\}$	0	1	0
$\{z\}$	0	0	1
$\{x,y\}$	1	1	0
$\{x,z\}$	1	0	1
$\{y,z\}$	0	1	1
X	1	1	1

.

r_2 is given by

r_2	f_1	f_2	f_3
f_1	1	0	0
f_2	0	1	0
f_3	0	0	1

.

The isomorphism $G_X : X \to T(L(X))$ is defined as follows:

X	$G_X(X_i) : X_i \in X$
x	f_1
y	f_2
z	f_3

.

Example 3.12. *Let (X, τ, r) be a Priestley space, where $X = \{x, y, z, t, s\}$ and r is given by*

r	x	y	z	t
x	1	0	0.5	0.6
y	0	1	0.4	0.5
z	0	0	1	0.4
t	0	0	0	1

.

and $L(X) = \{\phi, \{t\}, \{z,t\}, \{x,z,t\}, \{y,z,t,\}, X\}$, *where* r_1 *is given by*

r_1	ϕ	$\{t\}$	$\{z,t\}$	$\{x,z,t\}$	$\{y,z,t\}$	X
ϕ	1	0.4	0.4	0.4	0.4	0.4
$\{t\}$	0	1	0.4	0.4	0.4	0.4
$\{z,t\}$	0	0	1	0.4	0.4	0.4
$\{x,z,t\}$	0	0	0	1	0	0.6
$\{y,z,t\}$	0	0	0	0	1	0.5
X	0	0	0	0	0	1

.

and $T(L(X)) = \{f_1, f_2, f_3, f_4\}$ *such that*

r_1	f_1	f_2	f_3	f_4
ϕ	0	0	0	0
$\{t\}$	0	0	0	1
$\{z,t\}$	0	0	1	1
$\{x,z,t\}$	0	1	1	1
$\{y,z,t\}$	1	0	1	1
X	1	1	1	1

.

The isomorphism $G_X : X \to T(L(X))$ *is defined as follows*

X	$G_X(X_i), X_i \in X$
x	f_1
y	f_2
z	f_3
t	f_4

.

Chapter 4

Priestley duality for intuitionistic fuzzy perfect distributive lattices: finite case

The results obtained in this chapter have been published in the Journal of Fuzzy Set Valued Analysis [5].

In the following we recall some definitions of the intuitionistic fuzzy sets, intuitionistic fuzzy relations [7], [11].

Definition 4.1. *[13] Let $L^* = \{(a_1, a_2) \in [0,1]^2 : a_1 + a_2 \leq 1\}$ and $\leq_{L^*}$ be the order in L^* defined by $\forall (a_1, a_2)$, $(b_1, b_2) \in L^* : (a_1, a_2) \leq_{L^*} (b_1, b_2) \Leftrightarrow (a_1 \leq b_1$ and $a_2 \geq b_2)$. $(L, \leq_{L^*})$ is a complete lattice. $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ are the units of L^*.*

Let X be a given non-empty set. An intuitionistic fuzzy set in X is an expression A given by $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \diagup x \in X\}$ where $\mu_A : X \longrightarrow [0,1]$, $\nu_A : X \longrightarrow [0,1]$ with the condition $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$ (We will denote by $A = (\mu_A, \nu_A)$). The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the element x in the set A. We will denote by $IFS_S(X)$ the set of all intuitionistic fuzzy sets on X. In particular, $\tilde{0}$ and $\tilde{1}$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in X defined by $\tilde{0}(x) = (0,1)$ and $\tilde{1}(x) = (1,0)$ for each $x \in X$, respectively.

Obviously, when $\nu_A(x) = 1 - \mu_A(x)$ for every x in X, the set A is fuzzy set.

We will denote the set of all $IFSs$ in X by $IFS(X)$.

Definition 4.2. *The following expressions are defined in [7] for all intuitionistic fuzzy sets A, B in X;*

1. $A \subseteq B$ *if and only if* $\mu_A \leq \mu_B$ *and* $\nu_A \geq \nu_B$,
2. $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$,
3. $A^c = (\nu_A, \mu_A)$,
4. $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$,

Definition 4.3. $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

In the following we give an Example of applications of intuitionistic fuzzy sets to sociometry.

Example 4.4. *[32] Sociometric questionnaire*

Every pupil obtains questionnaire. He have to write to every name the sign + (accept) or − (nonaccept) or nothing. So, for every pupil x two numbers are obtained;

$A(x)$ =*number of accepts,*

$N(x)$ =*number of non-accepts.*

An IF-set is a pair of mappings $\mu : X \to [0, 1]$, $\vartheta : X \to [0, 1]$ *such that*

$\mu(x) + \vartheta(x) \leq 1$ *for any* $x \in X$:

In our case X is the set of all pupils in the considered class. If $A(x)$ is the number of acceptation of the pupil x (hence $A(x) \in \{0, 1, .., n\}$ where n is the number of pupils in the class), then we put $\mu(x) = \frac{A(x)}{n}$. Similarly $\vartheta(x) = \frac{N(x)}{n}$ where $N(x)$ is the numbers of non-acceptation of the pupil x. Since $A(x) + N(x) \leq n$, we obtain

$\mu(x) + \vartheta(x) = \frac{A(x)}{n} + \frac{N(x)}{n} \leq 1$, *hence the pair (μ, ϑ) is an example of an IF-set.*

An intuitionistic fuzzy relation (for short, IFR) R is an intuitionistic fuzzy subset of $X \times Y$ given by the expression

$R = \{\langle (x, y), \mu_R(x, y), \vartheta_R(x, y)\rangle / x \in X,\ y \in Y\}$, $(R = (\mu_R, \vartheta_R)$ for short$)$ where $\mu_R : X \times Y \longrightarrow [0, 1]$ and $\vartheta_R : X \times Y \longrightarrow [0, 1]$ satisfy the condition $\mu_R(x, y) + \vartheta_R(x, y) \leq 1$ for every $(x, y) \in X \times Y$.In particular, if R is an intuitionistic fuzzy relation from X to itself, then R is called a binary intuitionistic fuzzy relation on X, and we will denote the set of all intuitionistic fuzzy relations on X by $IFR(X)$.

Definition 4.5. *[11] A fuzzy relation* $R = (\mu_R, \vartheta_R)$ *on a nonempty set* X *is called:*

1. *reflexive if and only if for all* $x \in X$, $\mu_R(x,x) = 1$ *and* $\vartheta_R(x,y) = 0$,

2. *perfect antisymmetrical, if for every* $(x,y) \in X \times X$ *with* $x \neq y$ *and*

$$\begin{cases} \mu_R(x,y) > 0, \\ or \\ \mu_R(x,y) = 0 \ and \ \vartheta_R(x,y) < 1, \end{cases} \quad then \begin{cases} \mu_R(y,x) = 0, \\ \quad and \\ \vartheta_R(y,x) = 1, \end{cases}$$

3. *transitive if and only if for all* $x, y, z \in X$,

$$\begin{cases} \mu_R(x,y) \wedge \mu_R(y,z) \leq \mu_R(x,z), \\ \qquad and \\ \vartheta_R(x,y) \vee \vartheta_R(y,z) \geq \vartheta_R(x,z). \end{cases}$$

A reflexive, perfect antisymmetric and transitive intuitionistic fuzzy relation is called an intuitionistic fuzzy perfect partial ordering relation. An intuitionistic fuzzy perfect partial order relation R is an intuitionistic fuzzy perfect total order relation if and only if $(\mu_R(x,y) > 0$ or $(\mu_R(x,y) = 0$ and $\vartheta_R(x,y) < 1))$ or $(\mu_R(y,x) > 0$ or $(\mu_R(y,x) = 0$ and $\vartheta_R(y,x) < 1))$ for all $x, y \in X$. A set equipped with an intuitionistic fuzzy perfect partial order relation is called an intuitionistic fuzzy perfect poset. The height of R, denoted by $h(R)$, is defined by $h(R) = \vee_{\leq_{L^*}\{(x,y) \in X^2 : x \neq y\}} R(x,y)$.

4.1 Intuitionistic fuzzy perfect lattices

In this section, we first extend the concept of fuzzy lattices studied in [12], to intuitionistic fuzzy perfect case. Hence, we extend some results in this direction.

The fo1lowing definition introduce the intuitionistic fuzzy perfect lattice as a relational structure.

Definition 4.6. *Let* (X, R) *be an intuitionistic fuzzy perfect poset and let* A *be a subset of* X*. An element* $u \in X$ *is said to be an upper bound for* A *if and only if* $\mu_R(a,u) > 0$ *or* $(\mu_R(a,u) = 0$ *and* $\vartheta_R(a,u) < 1)$*, for all* $a \in A$*. An upper bound* u_0 *for* A *is the least upper bound of* A *if and only if* $\mu_R(u_0,u) > 0$ *or* $(\mu_R(u_0,u) = 0$ *and* $\vartheta_R(u_0,u) < 1)$*, for every upper bound* u *of* A*. An element* $l \in X$ *is said to be a lower bound for* A *if and only if* $\mu_R(l,a) > 0$

or $(\mu_R(l,a) = 0$ *and* $\vartheta_R(l,a) < 1)$*, for all* $a \in A$*. A lower bound* l_0 *of* A *is the greatest lower bound of* A *if and only if* $\mu_R(l,l_0) > 0$ *or* $(\mu_R(l,l_0) = 0$ *and* $\vartheta_R(l,l_0) < 1)$*, for every lower bound* l *of* A*.*

The least upper bound and the greatest lower bound of a set $\{x, y\}$ are denoted by $x \vee y$ and $x \wedge y$ respectively.

Definition 4.7. *Let* (X, R) *be an intuitionistic fuzzy perfect poset.* (X, R) *is an intuitionistic fuzzy perfect lattice if and only if* $x \vee y$ *and* $x \wedge y$ *exist for all* $x, y \in X$*.*

Remark 4.8. *Since* R *is perfect antisymmetric, then the least upper (greatest lower) bound, if it exists, is unique.*

Proposition 4.9. *Let* (X, R) *be an intuitionistic fuzzy perfect lattice and* $x, y, z \in X$*. Then*

1. $(\mu_R(x, x \vee y) > 0$ *or* $(\mu_R(x, x \vee y) = 0$ *and* $\vartheta_R(x, x \vee y) < 1))$ *and* $(\mu_R(x \wedge y, x) > 0$ *or* $(\mu_R(x \wedge y, x) = 0$ *and* $\vartheta_R(x \wedge y, x) < 1)$*;*

2. $(\mu_R(x, z) > 0$ *or* $(\mu_R(x, z) = 0$ *and* $\vartheta_R(x, z) < 1))$ *and* $(\mu_R(y, z) > 0$ *or* $\mu_R(y, z) = 0$ *and* $\vartheta_R(y, z) < 1)$ *implies* $[\mu_R(x \vee y, z) > 0$ *or* $\mu_R(x \vee y, z) = 0$ *and* $\vartheta_R(x \vee y, z) < 1]$*;*

3. $(\mu_R(z, x) > 0$ *or* $(\mu_R(z, x) = 0$ *and* $\vartheta_R(z, x) < 1)$ *and* $(\mu_R(z, y) > 0$ *or* $(\mu_R(z, y) = 0$ *and* $\vartheta_R(z, y) < 1)$ *implies* $(\mu_R(z, x \wedge y) > 0$ *or* $(\mu_R(z, x \wedge y) = 0$ *and* $\vartheta_R(z, x \wedge y) < 1))$*;*

4. $(\mu_R(x, y) > 0$ *or* $(\mu_R(x, y) = 0$ *and* $\vartheta_R(x, y) < 1))$ *if and only if* $x \vee y = y$*;*

5. $(\mu_R(x, y) > 0$ *or* $(\mu_R(x, y) = 0$ *and* $\vartheta_R(x, y) < 1))$ *if and only if* $x \wedge y = x$*;*

6. *If* $\mu_R(y, z) > 0$ *or* $(\mu_R(y, z) = 0$ *and* $\vartheta_R(y, z) < 1)$*, then* $(\mu_R(x \wedge y, x \wedge z) > 0$ *or* $(\mu_R(x \wedge y, x \wedge z) = 0$ *and* $\vartheta_R(x \wedge y, x \wedge z) < 1))$ *and* $(\mu_R(x \vee y, x \vee z) > 0$ *or* $(\mu_R(x \vee y, x \vee z) > 0$ *and* $\vartheta_R(x \vee y, x \vee z) < 1))$*.*

Proof. (1), (2), and (3) are straightforward.

(4) Suppose $(\mu_R(x, y) > 0$ or $(\mu_R(x, y)$ and $\vartheta_R(x, y) < 1))$. Since $\mu_R(y, y) > 0$, then $(\mu_R(x \vee y, y) > 0$ or $(\mu_R(x \vee y, y) = 0$ and $\vartheta_R(x \vee y, y) < 1))$ by (2). Since $(\mu_R(y, x \vee y) > 0$ or $(\mu_R(y, x \vee y) = 0$ and $\vartheta_R(y, x \vee y) < 1))$ by (1), by the perfect antisymmetrical of R it follows $x \vee y = y$.

Conversely, suppose $x \vee y = y$. Then, $\mu_R(x,y) = \mu_R(x, x \vee y) > 0$ or $(\mu_R(x,y) = \mu_R(x, x \vee y) = 0$ and $\vartheta_R(x,y) = \vartheta_R(x, x \vee y) < 1)$ by (1).

(5)The proof is similar to that of (4).

(6) Suppose $\mu_R(y,z) > 0$ or $(\mu_R(y,z)) = 0$ and $\vartheta_R(y,z) < 1)$. Then,

$\mu_R(x \wedge y, z) \geq \mu_R(x \wedge y, y) \wedge \mu_R(y,z) > 0$ or $(\mu_R(x \wedge y, z) = 0$ and

$\vartheta_R(x \wedge y, z) < \vartheta_R(x \wedge y, y) \vee \vartheta_R(y,z) < 1)$.

Since $\mu_R(x \wedge y, x) > 0$ or $(\mu_R(x \wedge y, x) = 0$ and $\vartheta_R(x \wedge y, x) < 1)$ by (1), $x \wedge y$ is a lower bound of $\{x, z\}$.

Since $x \wedge z$ is the greatest lower bound of $\{x, z\}$, $\mu_R(x \wedge y, x \wedge z) > 0$ or $(\mu_R(x \wedge y, x \wedge z) = 0$ and $\vartheta_R(x \wedge y, x \wedge z) < 1)$.

$\mu_R(y, x \vee z) \geq \mu_R(y,z) \wedge \mu_R(z, x \vee z) > 0$ or $(\mu_R(y, x \vee z) = 0$ and

$\vartheta_R(y, x \vee z) < \vartheta_R(y,z) \vee \vartheta_R(z, x \vee z) < 1)$.

Since $\mu_R(x, x \vee z) > 0$ or $(\mu_R(x, x \vee z) = 0$ and $\vartheta_R(x, x \vee z) < 1)$ by (1), $\mu_R(x \vee y, x \vee z) > 0$ or $(\mu_R(x \vee y, x \vee z) = 0$ and $\vartheta_R(x \vee y, x \vee z) > 0)$ by (2). □

Proposition 4.10. *Let (X, R) be an intuitionistic fuzzy perfect lattice and $x, y, z \in X$. Then*

1) $x \vee x = x$, $x \wedge x = x$;

2) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$;

3) $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;

4) $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$.

Proof. (1) and (2) are straightforward.

(3) Since $\mu_R(x, x \vee (y \vee z)) > 0$ or $(\mu_R(x, x \vee (y \vee z)) = 0$ and $\vartheta_R(x, x \vee (y \vee z)) < 1)$

$\mu_R(y, x \vee (y \vee z)) \geq \mu_R(y, y \vee z) \wedge \mu_R(y \vee z, x \vee (y \vee z)) > 0$ or $(\mu_R(y, x \vee (y \vee z)) = 0$ and $\vartheta_R(y, x \vee (y \vee z)) \leq \vartheta_R(y, y \vee z) \vee \vartheta_R(y \vee z, x \vee (y \vee z)) < 1)$.

$\mu_R(x \vee y, x \vee (y \vee z)) > 0$ or $(\mu_R(x \vee y, x \vee (y \vee z)) = 0$ and $\vartheta_R(x \vee y, x \vee (y \vee z)) < 1)$ by (2) of Proposition 4.9.

Since $\mu_R(z, x \vee (y \vee z)) \geq \mu_R(z, y \vee z) \wedge \mu_R(y \vee z, x \vee (y \vee z)) > 0$ or $(\mu_R(z, x \vee (y \vee z)) = 0$ and $\vartheta_R(z, x \vee (y \vee z)) \leq \vartheta_R(z, y \vee z) \vee \vartheta_R(y \vee z, x \vee (y \vee z)) < 1)$.

$\mu_R((x \vee y) \vee z, x \vee (y \vee z)) > 0$ or $(\mu_R((x \vee y) \vee z, x \vee (y \vee z)) = 0$ and $\vartheta_R((x \vee y) \vee z, x \vee (y \vee z)) < 1)$ by (2) of Proposition 4.9.

Similarly we may show $\mu_R(x \vee (y \vee z), (x \vee y) \vee z) > 0$ or $(\mu_R(x \vee (y \vee z), (x \vee y) \vee z) = 0$ and $\vartheta_R(x \vee (y \vee z), (x \vee y) \vee z) < 1)$. By the perfect antisymmetrical of R,

$(x \vee y) \vee z = x \vee (y \vee z)$. In the same way, we show that $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

(4) Let $B = \{x \vee y, x\}$.Since $(\mu_R(x, x \vee y) > 0$ or $(\mu_R(x, x \vee y) = 0$ and $\vartheta_R(x, x \vee y) < 1)$ and $(\mu_R(x, x) > 0$ and $\vartheta_R(x, x) < 1)$, x is a lower bound of B. If z is a lower bound of B, then $\mu_R(z, x) > 0$ or $(\mu_R(z, x) = 0$ and $\vartheta_R(z, x) < 1)$. Thus x is the greatest lower bound of B. Hence $(x \vee y) \wedge x = x$. Similarly we may show $(x \wedge y) \vee x = x$. □

In the following, we give some characterizations of intuitionistic fuzzy perfect distributive lattices.

Definition 4.11. *Let (X, R) be an intuitionistic fuzzy perfect lattice. (X, R) is distributive if and only if*

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and* $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$ *for all* $x, y, z \in X$.

Theorem 4.12. *Let (X, R) be an intuitionistic fuzzy perfect totally ordered set. Then (X, A) is an intuitionistic fuzzy perfect distributive lattice.*

Proof. Straightforward. □

Definition 4.13. *Let $(X, R, \wedge, \vee)$ be an intuitionistic fuzzy perfect lattice, and F be a nonempty crisp subset of X. F is a filter of $(X, R, \wedge, \vee)$ if for all $x, y \in X$, it holds that*

(i) *If $y \in F$ and $\mu_R(y, x) > 0$ or $(\mu_R(y, x) = 0$ and $\vartheta_R(y, x) < 1)$, then $x \in F$.*

(ii) *If $x, y \in F$, then $x \wedge y \in F$.*

Definition 4.14. *Let $(X, R, \wedge, \vee)$ be an intuitionistic fuzzy perfect lattice and F be a filter of $(X, R, \wedge, \vee)$. Then F is called prime filter if F is proper $(F \neq X)$ and for all $x, y \in X$, $x \vee y \in F$ imply $x \in F$ or $y \in F$.*

Definition 4.15. *Let (X, R) be an intuitionistic fuzzy lattice, and I be a nonempty crisp subset of X. I is a ideal of (X, R) if for all $x, y \in X$, it holds that*

(i) *If $y \in I$ and $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$and $\vartheta_R(x, y) < 1)$, then $x \in I$.*

(ii) *If $x, y \in I$, then $x \vee y \in I$.*

Definition 4.16. *Let (X, R) be an intuitionistic fuzzy perfect lattice and I be an ideal of (X, R). Then I is called prime ideal if I is proper $(I \neq X)$ and for all $x, y \in X$, $x \wedge y \in I$ imply $x \in I$ or $y \in I$.*

For every crisp subset δ of X, the smallest filter of X (with respect to the inclusion) which contains δ is said to be the filter generated by δ and will be denoted by $\langle\delta\rangle$.

Proposition 4.17. *If δ is a non-empty subset of an intuitionistic fuzzy perfect lattice $(X, R, \wedge, \vee)$, then*

$\langle\delta\rangle =$

$\{x \in X/\mu_R(\wedge_{i=1}^n a_i, x) > 0$ *or* $(\mu_R(\wedge_{i=1}^n a_i, x) = 0$ *and* $\vartheta_R(\wedge_{i=1}^n a_i, x) < 1),\ a_1, .., a_n \in \delta\}$

Proof. Let $\langle\delta\rangle = \{x/\mu_R(\wedge_{i=1}^n a_i, x) > 0$ or $\vartheta_R(\wedge_{i=1}^n a_i, x) < 1$, for some $a_1, a_2, .., a_n \in \delta\}$.

First, we prove that $\langle\delta\rangle$ is non-empty. Let $a \in \delta$, since $\mu_R(a, a) > 0$, then $a \in \langle\delta\rangle$, hence $\langle\delta\rangle \neq \emptyset$. To proof that $\langle\delta\rangle$ is a filter. Let $x \in \langle\delta\rangle$, $y \in X$ such that $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\vartheta_R(x, y) < 1)$, there exists $a_1, a_2, .., a_n$ such that $\mu_R(\wedge_{i=1}^n a_i, x) > 0$ or $(\mu_R(\wedge_{i=1}^n a_i, x) = 0$ and $\vartheta_R(\wedge_{i=1}^n a_i, x) < 1)$. Then, $\mu_R(\wedge_{i=1}^n a_i, y) > 0$ or $(\mu_R(\wedge_{i=1}^n a_i, y) = 0$ and $\vartheta_R(\wedge_{i=1}^n a_i, y) < 1)$, then $y \in F$.

On the other hand, let $x, y \in \langle\delta\rangle$, there are $a_1, a_2, .., a_n, b_1, b_2, .., b_m$ such that $(\mu_R(\wedge_{i=1}^n a_i, x) > 0$ or $(\mu_R(\wedge_{i=1}^n a_i, x) = 0$ and $\vartheta_R(\wedge_{i=1}^n a_i, x) < 1))$ and $(\mu_R\left(\wedge_{j=1}^m b_j, y\right) > 0$ or $(\mu_R\left(\wedge_{j=1}^m b_j, y\right) = 0$ and $\vartheta_R(\wedge_{i=1}^n a_i, y) < 1))$. Then, $\mu_R\left((\wedge_{i=1}^n a_i) \wedge (\wedge_{j=1}^m b_j), x \wedge y\right) > 0$ or $(\mu_R\left((\wedge_{i=1}^n a_i) \wedge (\wedge_{j=1}^m b_j), x \wedge y\right) = 0$ and $\vartheta_R\left((\wedge_{i=1}^n a_i) \wedge (\wedge_{j=1}^m b_j), x \wedge y\right) < 1)$. Therefore $x \wedge y \in \langle\delta\rangle$.

Next, let $a \in \delta$, since $\mu_R(a, a) > 0$, we have $a \in \langle\delta\rangle$. Then $\delta \subseteq \langle\delta\rangle$.

Finally, suppose that F is a filter with $\delta \subseteq F$. Then for any $x \in \langle\delta\rangle$, then there exists $a_1, a_2, .., a_n$ such that

$\mu_R(\wedge_{i=1}^n a_i, x) > 0$, then $x \in F$. Therefore $\langle\delta\rangle \subseteq F$. □

If x is an element of the intuitionistic fuzzy perfect lattice X, then $\uparrow x = \{y \in X : \mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\vartheta_R(x, y) < 1)\}$ $(\downarrow x = \{y \in X : \mu_R(y, x) > 0$ or $(\mu_R(y, x) = 0$ or $\vartheta_R(y, x) > 0)\})$ is the principal filter (ideal) generated by x. We denote the set of all principal filters generated by elements of X by $PF(X)$.

Proposition 4.18. *$(PF(X), \subseteq, \wedge, \vee)$ is a lattice with $(\uparrow x) \wedge (\uparrow y) = (\uparrow x) \cap (\uparrow y) = \uparrow (x \vee y)$ and $(\uparrow x) \vee (\uparrow y) = \langle(\uparrow x) \cup (\uparrow y)\rangle$ for any $\uparrow x, \uparrow y \in PF(X)$. Furthermore, $\uparrow (x \wedge y) = \langle(\uparrow x) \cup (\uparrow y)\rangle$.*

Proof. Let $a \in X$,

$$a \in \uparrow (x \vee y) \Leftrightarrow \mu_R(x \vee y, a) > 0 \text{ or } (\mu_R(x \vee y, a) = 0 \text{ and } \vartheta_R(x \vee y, a) < 1)$$
$$\Leftrightarrow \begin{cases} \mu_R(x,a) > 0 \text{ or } (\mu_R(x,a) = 0 \text{ and } \vartheta_R(x,a) < 1) \\ \text{and} \\ \mu_R(y,a) > 0 \text{ or } (\mu_R(y,a) = 0 \text{ and } \vartheta_R(y,a) < 1) \end{cases}$$
$$\Leftrightarrow a \in (\uparrow x) \cap (\uparrow y)$$
$$\Leftrightarrow a \in (\uparrow x) \wedge (\uparrow y).$$

and

$$\begin{cases} R(x \wedge y, x) >_{L^*} (0,1) \\ R(x \wedge y, y) >_{L^*} (0,1) \end{cases} \Rightarrow \begin{cases} \uparrow x \subseteq \uparrow (x \wedge y) \\ \uparrow y \subseteq \uparrow (x \wedge y) \end{cases}$$
$$\Rightarrow (\uparrow x) \cup (\uparrow y) \subseteq \uparrow (x \wedge y)$$
$$\Rightarrow \langle (\uparrow x) \cup (\uparrow y) \rangle \subseteq \uparrow (x \wedge y).$$

Let $a \in \uparrow (x \wedge y)$, hence $R(x \wedge y, a) >_{L^*} (0,1)$, it follows that $R(x,a) >_{L^*} (0,1)$ and $R(y,a) >_{L^*} (0,1)$, therefore $a \in \uparrow x$ and $a \in \uparrow y$, it follows that $a \in (\uparrow x) \cup (\uparrow y)$, then $a \in \langle (\uparrow x) \cup (\uparrow y) \rangle$. Then, $\uparrow (x \wedge y) \subseteq \langle (\uparrow x) \cup (\uparrow y) \rangle$.

□

Theorem 4.19. *Let X be an intuitionistic fuzzy perfect distributive lattice, F a filter and I an ideal of X. If $F \cap I = \phi$ then there is a prime filter P such that $F \subseteq P$ and $P \cap I = \phi$.*

Proof. Let G be the family of those filters F' which satisfy $F \subseteq F'$ and $F' \cap I = \phi$. It follows from the Zorn's lemma that G has a maximal element P. Since $P \in G$ it remains to prove that the filter P is prime. P is proper because $P \cap I = \phi$. Suppose P is not prime. Then there exist $a, b \in X$ such that $a \vee b \in P$, $a \notin P$ and $b \notin P$. Let $\delta = P \cup \{a\}$. Then $\langle \delta \rangle \cap I \neq \phi$, otherwise

$P \subseteq \langle \delta \rangle \in G$ contradicting the maximality of P. Take $x \in \langle \delta \rangle \cap I$. Then Proposition 4.17 implies easily the existence of $p \in P$ such that $\mu_R(p \wedge a, x) > 0$ or $(\mu_R(p \wedge a, x) = 0$ and $\vartheta_R(p \wedge a, x) < 1)$ and since $x \in I$ it follows that $p \wedge a \in I$. Similarly there is $q \in P$ such that $q \wedge b \in I$. Then $(p \wedge a) \vee (q \wedge b) \in I$ and on the other hand $(p \wedge a) \vee (q \wedge b) = (p \vee q) \wedge (p \vee b) \wedge (a \vee q) \wedge (a \vee b) \in P$, therefore $I \cap P \neq \phi$, a contradiction. □

Corollary 4.20. *Let X be an intuitionistic fuzzy perfect distributive lattice. If I is an ideal and $a \in X - I$ there is a prime filter P such that $a \in P$ and $P \cap I = \phi$.*

Proof. Let I an ideal and $a \in X - I$. Take $F = \uparrow a$ it follows $F \cap I = \phi$. By Theorem 4.19 there is a prime filter P such that $F \subseteq P$ and $P \cap I = \phi$. □

4.2 Intuitionistic fuzzy perfect lattices isomorphisms

In the following, we extend the concept of fuzzy lattices isomorphism studied in [36], to intuitionistic fuzzy perfect case.

Definition 4.21. *Let $(L, r, \wedge, \vee)$, and $(M, R, \wedge, \vee)$ be two intuitionistic fuzzy perfect lattices.*

1. *A function $f : L \to M$ is called increasing if for all $x, y \in L$, $r(x, y) \leq_{L^*} R(f(x), f(y))$, i.e. $\mu_r(x, y) \leq \mu_R(f(x), f(y))$ and $\vartheta_r(x, y) \geq \vartheta_R(f(x), f(y))$.*

2. *Let $f : L \to M$ be a increasing function between intuitionistic fuzzy perfect lattices. Then f is called a lattice homomorphism if for any $x, y \in L$, $f(x \wedge y) = f(x) \wedge f(y)$, and $f(x \vee y) = f(x) \vee f(y)$. If f is a bijection, then f is said to be intuitionistic fuzzy perfect lattices isomorphism.*

Proposition 4.22. *Let $(X, R, \wedge, \vee)$ be an intuitionistic fuzzy perfect lattice and F be a subset of $(X, R, \wedge, \vee)$. The following conditions are equivalents,*

1. *F is a prime filter;*

2. *There is a surjective lattice homomorphism $f : X \to \{0, 1\}$ such that $F = f^{-1}(\{1\})$.*

Proof. Similar to Proposition 1.47. □

Corollary 4.23. *Let $(X, R, \wedge, \vee)$ be an intuitionistic fuzzy perfect distributive lattice. If $a, b \in X$ are such that $R(a, b) = (0, 1)$ (i.e., $\mu_R(a, b) = 0$ and $\vartheta_R(a, b) = 1$) there is a prime filter F such that $a \in F$ and $b \notin F$.*

Proof. Similar to Corollary 1.48. □

4.3 Intuitionistic fuzzy perfect Priestley spaces

In this section we extend some results obtained by A. Amroune and B. Davvaz in [2] in the intuitionistic fuzzy perfect case.

Definition 4.24. *Let (X, R) be an intuitionistic fuzzy perfect ordered set. A subset E of X is called increasing if x belongs to E and $\mu_R(x, y) > 0$ or $(\mu_R(x, y) = 0$ and $\vartheta_R(x, y) < 1)$ (y is an upper bound of x), then y belongs to E (a decreasing set is defined in a similar way).*

Definition 4.25. *An intuitionistic fuzzy perfect ordered space is a triplet* (X, τ, R), *where* X *is a non empty set,* τ *is a topology on* X *and* R *is an intuitionistic fuzzy perfect order on* X.

Definition 4.26. *An intuitionistic fuzzy perfect ordered space* (X, τ, R) *is called perfect totally order disconnected if for* $x, y \in X$, $\mu_R(x, y) = 0$ *and* $\vartheta_R(x, y) = 1$, *there exist an increasing* $\tau-$*clopen* U *and a decreasing* $\tau-$*clopen* V *such that* $U \cap V = \emptyset$ *with* $x \in U$ *and* $y \in V$.

We recall that a $\tau-$clopen set in a topological space is a set which is both open and closed.

Definition 4.27. *An intuitionistic fuzzy perfect ordered space* (X, τ, R) *is called an intuitionistic fuzzy perfect Priestley space if it is compact and perfect totally order disconnected.*

4.4 Intuitionistic fuzzy perfect Priestley spaces isomorphism

Definition 4.28. *Let* (X, τ, r), *and* (X', τ', r') *be two intuitionistic fuzzy perfect Priestley spaces.*

1. *A function* $f : X \to X'$ *is called increasing if for all* $x, y \in X$, $r(x, y) \leq_{L^*} r'(f(x), f(y))$, *i.e.* $\mu_r(x, y) \leq \mu_{r'}(f(x), f(y))$ *and* $\vartheta_r(x, y) \geq \vartheta_{r'}(f(x), f(y))$.
2. *Let* $f : X \to X'$ *be a increasing function between intuitionistic fuzzy perfect Priestley spaces. Then* f *is called a Priestley spaces homomorphism if it is continuous. If* f *is a bijection, then* f *is said to be intuitionistic fuzzy perfect Priestley spaces isomorphism.*

4.5 Priestley duality for intuitionistic fuzzy perfect distributive lattices

In this section, we extend the concept of Priestley duality for distributive lattices studied in [2, 29, 30], to the intuitionistic fuzzy perfect case. In this way, some results are obtained.

Throughout this section, all fuzzy intuitionistic lattices are finite perfect distributive lattices and homomorphisms preserve first (0) and last (1) elements. If $(A, \vee, \wedge, R)$ is a finite intuitionistic fuzzy perfect distributive lattice, then we show that its dual is a finite intuitionistic fuzzy perfect priestley space defined by $T(A) = (X, \tau, R_1)$, where X is the set of $0-1$ homomorphisms from A onto $\{0,1\}$, and τ be the topology induced by the product topology of $\{0,1\}^A$ and R_1 is the intuitionistic fuzzy perfect order adequately chosen on X. Indeed, R_1 is constructed from R see Lemma 3.1.

If $\delta = (X, \tau, r)$ is an intuitionistic fuzzy perfect Priestley space, then we show that its dual is an intuitionistic fuzzy perfect distributive lattice defined by $(L(\delta), \vee, \wedge, r_1)$, where $L(\delta) = \{Y \subseteq X | Y \text{ is increasing and } \tau\text{−clopen}\}$ and r_1 is an intuitionistic fuzzy perfect order adequately chosen.

4.5.1 Dual and bidual of an intuitionistic fuzzy perfect distributive lattice

Lemma 4.29. *If $(A, \vee, \wedge, R)$ is an intuitionistic fuzzy finite perfect distributive lattice, then there exist two intuitionistic fuzzy perfect orders R_1, R_2 such that:*

1. *$T(A) = (X, \tau, R_1)$ is an intuitionistic fuzzy perfect Priestley space,*
2. *$(L(T(A)), \vee, \wedge, R_2)$ is an intuitionistic fuzzy perfect distributive lattice.*

Proof. (1) Let $R_1 = (\mu_{R_1}, \vartheta_{R_1})$ be the relation defined by

$$\mu_{R_1}(f,g) = \begin{cases} \mu_R\left(\wedge g^{-1}(1), \wedge f^{-1}(1)\right) & \text{if } f^{-1}(1) \subseteq g^{-1}(1), \\ 0 & \text{otherwise.} \end{cases}$$

$$\vartheta_{R_1}(f,g) = \begin{cases} \vartheta_R\left(\wedge g^{-1}(1), \wedge f^{-1}(1)\right) & \text{if } f^{-1}(1) \subseteq g^{-1}(1), \\ 1 & \text{otherwise.} \end{cases}$$

where the symbol $\wedge$ stands for an infimum with respect to the intuitionistic fuzzy perfect relation R. We show that R_1 is an intuitionistic fuzzy perfect order. We have $\mu_{R_1}(f,f) = \mu_R\left(\wedge f^{-1}(1), \wedge f^{-1}(1)\right) = 1$ and $\vartheta_{R_1}(f,f) = \vartheta_R\left(\wedge f^{-1}(1), \wedge f^{-1}(1)\right) = 0$ for all $f \in X$, then R_1 is reflexive.

On the other hand, for all $f, g \in X$ such that $f \neq g$, if $\mu_{R_1}(f,g) > 0$, then $f^{-1}(1) \subseteq g^{-1}(1)$, which imply $g^{-1}(1) \nsubseteq f^{-1}(1)$, it follows that $\mu_{R_1}(g,f) = 0$ and $\vartheta_{R_1}(g,f) = 1$, the case $\mu_R(f,g)=0$ and $\vartheta_R(f,g)<1$, is impossible case. Hence, R_1 is perfect antisymmetric relation.

In order to verify the transitivity of R_1, let $f,g,h\in X$, we show that $\mu_{R_1}(f,g)\wedge\mu_{R_1}(g,h)\leq\mu_{R_1}(f,h)$ and $\vartheta_{R_1}(f,g)\vee\vartheta_{R_1}(g,h)\geq\vartheta_{R_1}(f,h)$. We use the following truth table, where the proposition (P) is $\mu_{R_1}(f,g)\wedge\mu_{R_1}(g,h)\leq\mu_{R_1}(f,h)$ and the proposition (P') is $\vartheta_{R_1}(f,g)\vee\vartheta_{R_1}(g,h)\geq\vartheta_{R_1}(f,h)$

$f^{-1}(1)\subseteq g^{-1}(1)$	$g^{-1}(1)\subseteq h^{-1}(1)$	$f^{-1}(1)\subseteq h^{-1}(1)$	(P)
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

and

$f^{-1}(1)\subset g^{-1}(1)$	$g^{-1}(1)\subset h^{-1}(1)$	$f^{-1}(1)\subset h^{-1}(1)$	(P')
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

.

The only case for investigating is when $f^{-1}(1)\subset g^{-1}(1)$ and $g^{-1}(1)\subset h^{-1}(1)$. By the transitivity of R, for every a,b,c in A, we have $\mu_R(a,b)\wedge\mu_R(b,c)\leq\mu_R(a,c)$ and $\vartheta_R(a,b)\vee$

$\vartheta_R(b,c) \geq \vartheta_R(a,c)$. This yields

$$\mu_R\left(\wedge g^{-1}(1), \wedge f^{-1}(1)\right) \wedge \mu_R\left(\wedge h^{-1}(1), \wedge g^{-1}(1)\right) \leq \mu_R\left(\wedge h^{-1}(1), \wedge f^{-1}(1)\right)$$

and

$$\vartheta_R\left(\wedge g^{-1}(1), \wedge f^{-1}(1)\right) \vee \vartheta_R\left(\wedge h^{-1}(1), \wedge g^{-1}(1)\right) \geq \vartheta_R\left(\wedge h^{-1}(1), \wedge f^{-1}(1)\right).$$

Then for all $f,g,h \in X$, $\mu_{R_1}(f,g) \wedge \mu_{R_1}(g,h) \leq \mu_{R_1}(f,h)$ and $\vartheta_{R_1}(f,g) \vee \vartheta_{R_1}(g,h) \geq \vartheta_{R_1}(f,h)$ are hold, i.e. R_1 is transitive. Hence, R_1 is an intuitionistic fuzzy perfect order and by [29, 30] $T(A) = (X, \tau, R_1)$ is an intuitionistic fuzzy perfect Priestley space.

(2) Let $M = (M_0, M_1)$ where

$M = \wedge_{\leq_{L^*}}\{R(x,y)/x,y \in X,\ x \neq y \text{ and } R(x,y) \neq (0,1)\}$.

We define $R_2 = (\mu_{R_2}, \vartheta_{R_2})$ by

$$\mu_{R_2}(H,D) = \begin{cases} 1 & \text{if } H = D, \\ \mu_R\left(\wedge \cap_{f\in H} f^{-1}(1), \wedge \cap_{g\in D} g^{-1}(1)\right) & \text{if } H \subset D \text{ and } H \neq \phi, \\ M_0 & \text{if } H = \phi, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vartheta_{R_2}(H,D) = \begin{cases} 0 & \text{if } H = D, \\ \vartheta_R\left(\wedge \cap_{f\in H} f^{-1}(1), \wedge \cap_{g\in D} g^{-1}(1)\right) & \text{if } H \subset D \text{ and } H \neq \phi, \\ M_1 & \text{if } H = \phi, \\ 1 & \text{otherwise.} \end{cases}$$

for all $H, D \in L(T(A))$,

where the symbol $\wedge$ stands for an infimum with respect to the perfect fuzzy relation R.

First, we show that R_2 is an intuitionistic fuzzy perfect order. Since, $\mu_{R_2}(H,D) + \vartheta_{R_2}(H,D) \leq 1$ for all $H, D \in L(T(A))$, R_2 is an intuitionistic fuzzy perfect relation. To show the reflexivity we have $\mu_{R_2}(H,H) = 1$ and $\vartheta_{R_2}(H,H) = 0$, then $R_2(H,H) = (1,0)$. To proof the perfect antisymmetrical of R_2 let $H, D \in L(T(A))$ such that $H \neq D$, if $\mu_{R_2}(H,D) > 0$ then $\mu_{R_2}(D,H) = 0$ and $\vartheta_{R_2}(D,H) = 1$, if $[\mu_{R_2}(H,D) = 0 \text{ and } \vartheta_{R_2}(H,D) < 1]$, then $\mu_{R_2}(D,H) = 0$ and $\vartheta_{R_2}(D,H) = 1$.

In order to verify the transitivity, we use the following truth table, where the proposition (P) is $\mu_{R_2}(H,D) \wedge \mu_{R_2}(D,E) \leq \mu_{R_2}(H,E)$ and the proposition (P') is $\vartheta_{R_2}(H,D) \vee \vartheta_{R_2}(D,E) \geq \vartheta_{R_2}(H,E)$

$H \subseteq D$	$D \subseteq E$	$H \subseteq E$	(P)
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

and

$H \subseteq D$	$D \subseteq E$	$H \subseteq E$	(P')
1	1	1	1
1	1	0	Impossible case
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

First, if one of the three elements H, D, E is empty, then the transitivity is a trivial fact.

If $H \neq \phi$ and $D \neq \phi$ and $E \neq \phi$, the only case that need investigation is when $H \subset D$ and $D \subset E$. Setting $\alpha = \wedge \cap_{f \in H} f^{-1}(1)$, $\beta = \wedge \cap_{g \in D} g^{-1}(1)$ and $\gamma = \wedge \cap_{h \in E} h^{-1}(1)$, since

$\mu_R(\alpha,\beta) \wedge \mu_R(\beta,\gamma) \leq \mu_R(\alpha,\gamma)$ and $\vartheta_R(\alpha,\beta) \vee \vartheta_R(\beta,\gamma) \geq \vartheta_R(\alpha,\gamma)$, we have $\mu_{R_2}(H,D) \wedge \mu_{R_2}(D,E) \leq \mu_{R_2}(H,E)$ and $\vartheta_{R_2}(H,D) \vee \vartheta_{R_2}(D,E) \geq \vartheta_{R_2}(H,E)$.

Hence, R_2 is transitive.

Finally, the least upper and greatest lower bounds of H and D (with respect of the intuitionistic fuzzy perfect ordering relation R_2) are denoted by $H \vee_{R_2} D$ and $H \wedge_{R_2} D$, respectively, to show that $H \vee_{R_2} D = H \cup D$ and $H \wedge_{R_2} D = H \cap D$. We have that $H \cup D$ is an upper bound of $\{H, D\}$ because $(\mu_{R_2}(H, H \cup D) > 0$ or $(\mu_{R_2}(H, H \cup D) = 0$ and $\vartheta_{R_2}(H, H \cup D) < 1))$, and $(\mu_{R_2}(D, H \cup D) > 0$ or $(\mu_{R_2}(D, H \cup D) = 0$ and $\vartheta_{R_2}(D, H \cup D) < 1))$, if C is the least upper bound of $\{H, D\}$ we have four cases:

1) if $H = \phi$ and $D = \phi$, it follows that $\mu_{R_2}(C, H \cup D) = \mu(C, \phi)$ and $\vartheta_{R_2}(C, H \cup D) = \vartheta_R(C, \phi)$, which imply that $(\mu_{R_2}(C, H \cup D), \vartheta_{R_2}(C, H \cup D))$ different to $(0, 1)$ if and only if $C = \phi$, hence $C = H \cup D$.

2) if $H = \phi$, $D \neq \phi$ and $D \subseteq C$ (because $\mu_R(D, C) > 0$ or $(\mu_R(D, C) = 0$ and $\vartheta_R(D, C) < 1))$, it follows that $\mu_{R_2}(C, H \cup D) = \mu_R(C, D)$ and $\vartheta_{R_2}(C, H \cup D) = \vartheta_R(C, D)$, which imply that $(\mu_{R_2}(C, H \cup D), \vartheta_{R_2}(C, H \cup D))$ different to $(0, 1)$ if and only if $C \subseteq D$, hence $C = D = H \cup D$.

3) if $H \neq \phi$, $D = \phi$ and $H \subseteq C$ (because $\mu_R(H, C) > 0$ or $(\mu_R(H, C) = 0$ and $\vartheta_R(H, C) < 1))$, it follows that $\mu_{R_2}(C, H \cup D) = \mu_R(C, H)$ and $\vartheta_{R_2}(C, H \cup D) = \vartheta_R(C, H)$, which imply that $(\mu_{R_2}(C, H \cup D), \vartheta_{R_2}(C, H \cup D))$ different to $(0, 1)$ if and only if $C \subseteq H$, hence $C = H = H \cup D$.

4) if $H \neq \phi$, $D \neq \phi$ and $H \subseteq C$ and $D \subseteq C$, which imply $H \cup D \subseteq C$, it follows that $(\mu_{R_2}(C, H \cup D), \vartheta_{R_2}(C, H \cup D))$ different to $(0, 1)$ if and only if $C \subseteq H \cup D$, hence $C = H \cup D$.

Similarly, we prove that $H \wedge_{R_2} D = H \cap D$, its known that $H \cup D$ and $H \cap D$ are increasing and $\tau-$clopens. This shows that $(L(T(A)), \vee, \wedge, R_2)$ is a fuzzy distributive lattice. □

4.5.2 Dual and bidual of an intuitionistic fuzzy perfect Priestley space

Lemma 4.30. *If $\delta = (X, \tau, r)$ is an intuitionistic fuzzy finite perfect Priestley space, then there exist two fuzzy perfect orders r_1 and r_2 such that:*

1. *$(L(\delta), \vee, \wedge, r_1)$ is an intuitionistic fuzzy perfect distributive lattice,*
2. *$(T(L(\delta)), \tau, r_2)$ is an intuitionistic fuzzy perfect Priestley space.*

Proof. (1) (i) If $h(r) = (0, 1)$, then X is an antichain and we can write r_1 as follows:

$r_1 = (\mu_{r_1}, \vartheta_{r_1})$ such that

$$\mu_{r_1}(A,B) = \begin{cases} 1 & \text{if } A = B, \\ 1 - \frac{cardA}{cardB} & \text{if } A \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vartheta_{r_1}(A,B) = \begin{cases} 0 & \text{if } A = B, \\ \delta \ \frac{cardA}{cardB} & \text{if } \ A \subset B, \text{ where } \delta \in \left]0,1\right[, \\ 1 & \text{otherwise.} \end{cases}$$

r_1 is an intuitionistic perfect fuzzy relation. It is easy to show that r_1 is an intuitionistic fuzzy perfect order, and $A \vee_{r_1} B = A \cup B$, $A \wedge_{r_1} B = A \cap B$ exist for every A and B from $L(\delta)$, and $A \cup B$, $A \cap B$ are increasing and τ-clopens sets of $L(\delta)$, where $(L(\delta), \vee, \wedge, r_1)$ is an intuitionistic fuzzy perfect distributive lattice.

If $h(r) \neq 0$, then X is not an antichain, setting $M = (M_0, M_1)$ where $M = \wedge_{\leq_{L^*}} \{R(x,y) / x, y \in X, x \neq y$ and $R(x,y) \neq (0,1)\}$. Then, $M \neq (0,1)$ and we can take $r_1 = (\mu_{r_1}, \vartheta_{r_1})$ such that for every A and B from $L(\delta)$

$$\mu_{r_1}(A,B) = \begin{cases} 1 & \text{if } A = B, \\ Max\left(M_0, \vee_{a \in A, b \in B_{a \neq b}} \mu_r(a,b)\right) & \text{if } A \subset B \text{ and } A \neq \phi, \\ M_0 & \text{if } A = \phi, \\ 0 & \text{othewise.} \end{cases}$$

and

$$\vartheta_{r_1}(A,B) = \begin{cases} 0 & \text{if } A = B, \\ Min\left(M_1, \wedge_{a \in A, b \in B_{a \neq b}} \vartheta_r(a,b)\right) & \text{if } A \subset B \text{ and } A \neq \phi, \\ M_1 & \text{if } A = \phi, \\ 1 & \text{othewise.} \end{cases}$$

Similar to the previous lemma, r_1 is an intuitionistic fuzzy perfect order and we can assume that $A \vee_{r_1} B = A \cup B$ and $A \wedge_{r_1} B = A \cap B$, for every A and B from $L(\delta)$, where $(L(\delta), \vee, \wedge, r_1)$ is an intuitionistic fuzzy perfect distributive lattice.

(2) To proof the second assertion, let $r_2 = (\mu_{r_2}, \vartheta_{r_2})$, such that

$$\mu_{r_2}(f,g) = \begin{cases} 1 & \text{if } f = g, \\ \mu_r\left(\wedge \wedge_{A\in f^{-1}(1)} A, \wedge \wedge_{B\in g^{-1}(1)} B\right) & \text{if } f^{-1}(1) \subset g^{-1}(1), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vartheta_{r_2}(f,g) = \begin{cases} 0 & \text{if } f = g, \\ \vartheta_r\left(\wedge \wedge_{A\in f^{-1}(1)} A, \wedge \wedge_{B\in g^{-1}(1)} B\right) & \text{if } f^{-1}(1) \subset g^{-1}(1), \\ 1 & \text{otherwise.} \end{cases}$$

where the first infimum $\wedge$ is in the sense of the intuitionistic fuzzy perfect ordering relation r and the second infimum $\wedge$ is in the sense of the intuitionistic fuzzy perfect ordering relation r_1. Note that r_2 is well defined: $A_1 = \wedge_{A\in f^{-1}(1)} A$, where the symbol $\wedge$ stands for an infimum with respect to the intuitionistic fuzzy perfect ordering relation r_1, it exists because $L(\delta)$ is a lattice and $a = \wedge A_1$, where the symbol $\wedge$ stands for an infimum with respect to the intuitionistic perfect fuzzy relation r. Then, a exists because A_1 is a finite increasing τ-clopen, if A_1 has two minimal elements x, y, then $r(x,y) = (0,1)$, there exist an increasing $\tau-$clopen U and a decreasing $\tau-$clopen V such that $U \cap V = \emptyset$ with $x \in U$ and $y \in V$. It is easy to see that $A_1 \subset U$, then $U \cap V \neq \emptyset$ contradiction. By definition r_2 is an intuitionistic perfect fuzzy relation. It is easy to show that r_2 is an intuitionistic fuzzy perfect ordering relation. Furthermore, By [29, 30] $(T(L(\delta)), \tau, r_2)$ is an intuitionistic fuzzy perfect Priestley space. □

In the following we shows that the category of intuitionistic fuzzy perfect Priestley spaces is equivalent to the dual of the category of intuitionistic fuzzy perfect distributive lattices.

4.5.3 Bidual of an intuitionistic fuzzy perfect distributive lattice and isomorphisms

Lemma 4.31. *Let A be an intuitionistic fuzzy perfect distributive lattice. The map F_A : $A \mapsto L(T(A))$ defined by*

$$F_A(a) = \{f \in X / f(a) = 1\}$$

is an intuitionistic fuzzy perfect lattice isomorphism.

Proof. It is easy to see that for all $a, b \in A$ we have $F_A(a \wedge b) = F_A(a) \wedge_{R_2} F_A(b)$ and $F_A(a \vee b) = F_A(a) \vee_{R_2} F_A(b)$.

Suppose that $a \neq b$, it follows $R(a,b) = (0,1)$ or $R(b,a) = (0,1)$.

If $R(a,b) = (0,1)$ then, (by Corollary 4.23) there exist a prime filter F such that $a \in F$ and $b \notin F$, (by Proposition 4.22) it follows that there exists a surjection $f : A \to \{0,1\}$ such that $a \in f^{-1}(\{1\})$ and $b \notin f^{-1}(\{1\})$, hence $f(a) = 1$ and $f(b) = 0$ i.e., $R_2(F_A(a), F_A(b)) = (0,1)$.

Similarly if $R(b,a) = (0,1)$ we have $R_2(F_A(b), F_A(a)) = (0,1)$. Hence, $a \neq b$ imply $F_A(a) \neq F_A(b)$ i.e., F_A is injective.

To prove that F_A is surjective let $U \in L(T(A))$, for all $f \in U$ and $g \in L(T(A)) - U$ we have $g < f$, it follows that $\exists a_{fg} \in A$ such that $f(a_{fg}) = 1$ and $g(a_{fg}) = 0$. Then, $f \in F_A(a_{fg})$ and $g \in L(T(A)) - F_A(a_{fg})$. For fixed $f \in U$ we have $g \in L(T(A)) - U \subseteq \bigcup_{i=1}^{n} (L(T(A)) - F_A(a_{fgi})) = L(T(A)) - F_A(\bigwedge_{i=1}^{n} a_{fgi})$. Setting $\bigwedge_{i=1}^{n} a_{fgi} = a_f$, it follows $F_A(a_f) = F_A(\bigwedge_{i=1}^{n} a_{fgi}) \subset U$, on the other hand $f(a_f) = 1$ then $f \in F_A(a_f)$. Setting $U = \cup_{f \in U} F_A(a_f)$, it follows $U = \bigcup_{j=1}^{n} F_A(a_{fj}) = F_A(\bigvee_{j=i}^{n} a_{fj}) \in L(T(A))$, hence $\exists a = \bigvee_{j=i}^{n} a_{fj}$ such that $U = F_A(a)$ i.e., F_A is surjective.

Let us show that the map $F_A(a) = \{f \in X / f(a) = 1\}$ is increasing.

We show that: $R(x,y) \leq_{L^*} R_2(F_A(x), F_A(y))$, i.e. $\mu_R(x,y) \leq \mu_{R_2}(F_A(x), F_A(y))$ and $\vartheta_R(x,y) \geq \vartheta_{R_2}(F_A(x), F_A(y))$ for all $x, y \in A$, where

$$\mu_{R_2}(F_A(x), F_A(y)) = \begin{cases} 1 & \text{if } F_A(x) = F_A(y), \\ \mu_R\left(\wedge \cap_{f \in F_A(x)} f^{-1}(1), \wedge \cap_{g \in F_A(y)} g^{-1}(1)\right) & \text{if } F_A(x) \subset F_A(y), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vartheta_{R_2}(F_A(x), F_A(y)) = \begin{cases} 0 & \text{if } F_A(x) = F_A(y), \\ \vartheta_R\left(\wedge \cap_{f \in F_A(x)} f^{-1}(1), \wedge \cap_{g \in F_A(y)} g^{-1}(1)\right) & \text{if } F_A(x) \subset F_A(y), \\ 1 & \text{otherwise.} \end{cases}$$

Where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation R.

Note that $F_A(x) \neq \phi$ for all $x \in A$.

If $x = y$, it follows that $R(x,y) = R_2(F_A(x), F_A(y))$.

If $x \neq y$, we consider two cases

1. If $R(x,y)=(0,1)$, then we have $R(x,y)\leq_{L^*} R_2(F_A(x),F_A(y))$.

2. If $\mu_R(x,y)>0$ or ($\mu_R(x,y)=0$ and $\vartheta_R(x,y)<1$), it follows that $F_A(x)\subset F_A(y)$, which implies that $R(x,y)=R_2(F_A(x),F_A(y))$ (Since, $\wedge\cap_{f\in F_A(x)} f^{-1}(1)=x$ and $\wedge\cap_{g\in F_A(y)} g^{-1}(1)=y$, because if $\wedge\cap_{f\in F_A(x)} f^{-1}(1)\neq x$, it follows that $\cap_{f\in F_A(x)}f^{-1}(1)$ has minimal element $z\neq x$, then $r(x,z)=(0,1)$, there exists an increasing $\tau-$clopen U and a decreasing $\tau-$clopen V such that $U\cap V=\emptyset$ with $x\in U$ and $z\in V$. It is easy to see that $\cap_{f\in F_A(x)}f^{-1}(1)\subset U$, then $U\cap V\neq\emptyset$, contradiction). It follows that the map F_A is an intuitionistic fuzzy perfect lattice isomorphism. □

Lemma 4.32. *If $f:A_1\mapsto A_2$ is an intuitionistic fuzzy perfect lattice homomorphism, then the map $T(f):T(A_1)\mapsto T(A_2)$ defined by $T(f)(g)=g\circ f$ is an homomorphism of intuitionistic fuzzy perfect Priestley space, i.e., a continuous and increasing map .*

Proof. For all $g_1,g_2\in T(A_1)$ $g_1\leq g_2\Rightarrow g_1\circ f\leq g_2\circ f$, hence $T(f)$ is increasing.

The continuity of $T(f)$ follows from the fact that for every $a\in A_1$,

$$\begin{aligned}
T(f)^{-1}(F_{A_1}(a)) &= \{g\in T(A_2)/T(f)(g)\in F_{A_1}(a)\}\\
&= \{g\in T(A_2)/g\circ f(a)=1\}\\
&= \{g\in T(A_2)/g(f(a))=1\}\\
&= F_{A_2}(f(a)).
\end{aligned}$$

Hence, $T(f)$ is continuous. □

4.5.4 Bidual of an intuitionistic fuzzy perfect Priestley space and isomorphisms

Lemma 4.33. *If $\delta=(X,\tau,r)$ is a finite intuitionistic fuzzy perfect Priestley space, then the map $G_\delta:\delta\mapsto T(L(\delta))$ defined by*

$$G_\delta(x)(Y)=\begin{cases}1 & \text{if } x\in Y,\\ 0 & \text{if } x\notin Y.\end{cases}$$

for all $Y\in L(\delta)$ is an isomorphism of intuitionistic fuzzy perfect Priestley space, i.e., a bijection, continuous and increasing map.

Proof. Let $G_\delta:\delta\mapsto T(L(\delta))$ defined by

$$G_\delta(x)(Y)=\begin{cases}1 & \text{if } x\in Y,\\ 0 & \text{if } x\notin Y.\end{cases}$$

To prove the surjection of G_δ, let $f \in T(L(\delta))$ and setting $U = \{Y \in L(\delta) : f(Y) = 1\}$, $V = \{Z \in L(\delta) : f(Z) = 0\}$, $A = \cap_{Y \in U} Y$ and $B = \cup_{Z \in V} Z$. To show that $A - B \neq \phi$, suppose that $A - B = \phi$, it follows that $(\cap_{Y \in U} Y) \cap (\cup_{Z \in V} Z)^c = \phi$, then $(\cap_{Y \in U} Y) \cap (\cap_{Z \in V} Z^C) = \phi$, since X is compact we have $(\overset{n}{\underset{i=1}{\cap}} Y_i) \cap (\overset{m}{\underset{j=1}{\cap}} Z_j^C) = \phi$, it follows that $\overset{n}{\underset{i=1}{\cap}} Y_i \subseteq \overset{m}{\underset{j=1}{\cup}} Z_j$, then $f(\overset{m}{\underset{j=1}{\cup}} Z_j) = 1$,

conduction since $f(\overset{m}{\underset{j=1}{\cup}} Z_j) = \overset{m}{\underset{j=i}{\vee}} f(Z_j) = 0$, hence $A - B \neq \phi$. Then, there exist $x \in A - B$ such that $G_\delta(x) = f$ and therefore

$G_\delta(x)(Y) = 1 \Leftrightarrow x \in Y \Leftrightarrow Y \in U \Leftrightarrow f(Y) = 1$

hence G_δ is surjective.

To prove the injectivity, let $x_1, x_2 \in \delta$,

$x_1 \neq x_2 \Rightarrow r(x_1, x_2) = (0, 1)$ or $r(x_2, x_1) = (0, 1)$.

If $r(x_1, x_2) = (0, 1)$, then since $L(\delta)$ is totally disconnected, there exist $Y_0 \in L(\delta)$ such that $x_1 \in Y_0$ and $x_2 \notin Y_0$, hence $G_\delta(x_1)(Y_0) \neq G_\delta(x_2)(Y_0)$.

If $r(x_2, x_1) = (0, 1)$ then, there exist $Y_1 \in L(\delta)$ such that $x_2 \in Y_1$ and $x_1 \notin Y_1$, hence $G_\delta(x_2)(Y_1) \neq G_\delta(x_1)(Y_1)$. It follows that $x_1 \neq x_2 \Rightarrow G_\delta(x_1)(Y) \neq G_\delta(x_2)(Y)$, hence G_δ is injective.

To prove that G_δ is continuous, let Z a $\tau-$clopen of $T(L(\delta))$. Then, there exist $y \in L(\delta)$ such that $Y = F_{L(\delta)}(y)$.

$$\begin{aligned} G_\delta^{-1}(Y) &= G_\delta^{-1}(F_{L(\delta)}(y)) \\ &= \left\{x \in X / G_\delta(x) \in F_{L(\delta)}(y)\right\} \\ &= \{x \in X : G_\delta(x)(y) = 1\} \\ &= \{x \in X : x \in y\} \\ &= X \cap y \\ &= y \ (\tau - \text{clopen}) \end{aligned}.$$

Hence, G_δ is continuous.

To prove that G_δ is increasing it suffices to show that $r(x, y) \leq_{L^*} r_2(G_\delta(x), G_\delta(y))$. then

$$\mu_{r_2}(G_\delta(x), G_\delta(y)) = \begin{cases} 1 & \text{if } G_\delta^{-1}(x) = G_\delta^{-1}(y), \\ \mu_r(\wedge\wedge_{A \in G_\delta^{-1}(x)(1)} A, \wedge\wedge_{B \in G_\delta(y)^{-1}(1)} B) & \text{if } G_\delta^{-1}(x)(1) \subset G_\delta^{-1}(y)(1), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vartheta_{r_2}\left(G_\delta(x), G_\delta(y)\right) = \begin{cases} 0 & \text{if } G_\delta^{-1}(x) = G_\delta^{-1}(y), \\ \vartheta_r(\wedge\wedge_{A\in G_\delta^{-1}(x)(1)} A, \wedge\wedge_{B\in G_\delta(y)^{-1}(1)} B) & \text{if } G_\delta(x)^{-1}(1) \subset G_\delta^{-1}(y)(1), \\ 1 & \text{otherwise.} \end{cases}$$

If $x = y$, then so $r(x, y)) = r_2(G_\delta(x), G_\delta(y))$. If $x \neq y$, then there are two cases as follows:

Case 1: if $r(x, y) = (0, 1)$, then we have $r(x, y) \leq_{L^*} r_2(G_\delta(x), G_\delta(y))$.

Case 2: if $\mu_r(x, y) > 0$ or ($\mu_r(x, y) = 0$ and $\mu_r(x, y) < 1$), then y belongs to each τ-clopen which contains x, so,

$G_\delta^{-1}(x)\,(1) \subset G_\delta^{-1}(y)\,(1)$, it follows that $\wedge\wedge_{A\in G_\delta^{-1}(x)(1)} A = x$ and $\wedge\wedge_{B\in G_\delta^{-1}(y)(1)} B = y$. Then, $\mu_{r_2}\left(G_\delta(x), G_\delta(y)\right) = \mu_r\left(\wedge\wedge_{A\in G_\delta^{-1}(x)(1)} A, \wedge\wedge_{B\in G_\delta^{-1}(y)(1)} B\right) = \mu_r\left(x, y\right)$ and $\vartheta_{r_2}\left(G_\delta(x), G_\delta(y)\right) = \vartheta_r\left(\wedge\wedge_{A\in G_\delta^{-1}(x)(1)} A, \wedge\wedge_{B\in G_\delta^{-1}(y)(1)} B\right) = \vartheta_r\left(x, y\right)$, hence $r_2(G_\delta(x), G_\delta(y)) = r(x, y)$. □

Lemma 4.34. *If $h : \delta_1 \mapsto \delta_2$ is a homomorphism of intuitionistic fuzzy perfect Priestley space, then the map $L(h) : L(\delta_2) \mapsto L(\delta_1)$ defined by $L(h)(y) = h^{-1}(y)$ for every $y \in L(\delta_2)$ is a fuzzy perfect lattices homomorphism.*

Proof. For all $y \in L(\delta_2)$ we have $L(h)(y) \in L(\delta_1)$.

For all $y, z \in L(\delta_2)$ since h^{-1} commutes with set-theoretical operations we have,

$$\begin{aligned} L(h)(y \cup z) &= h^{-1}(y \cup z) \\ &= h^{-1}(y) \cup h^{-1}(z) \\ &= L(h)(y) \cup L(h)(z). \end{aligned}$$

and

$$\begin{aligned} L(h)(y \cap z) &= h^{-1}(y \cap z) \\ &= h^{-1}(y) \cap h^{-1}(z) \\ &= L(h)(y) \cap L(h)(z). \end{aligned}$$

and for all $y, z \in L\left(\delta_2\right)$

$$\begin{aligned} y \subseteq z &\Rightarrow h^{-1}\left(y\right) \subseteq h^{-1}\left(z\right) \\ &\Rightarrow L\left(h\right)\left(y\right) \subseteq L\left(h\right)\left(z\right). \end{aligned}$$

Hence, $L(h)$ is fuzzy perfect lattices homomorphism. □

Theorem 4.35. *If $f : A_1 \mapsto A_2$ is an intuitionistic fuzzy perfect lattice homomorphism, then*

$$L\left(T(f)\right) \circ F_{A_1} = F_{A_2} \circ f.$$

$$\begin{array}{ccccc} & & f & & \\ & A_1 & --- \quad ---- & \rightarrow & A_2 \\ & | & & & | \\ F_{A_1} & | & & & | \; F_{A_2} \\ & \downarrow & & & \downarrow \\ & L(T(A_1)) & ---- & \rightarrow & L(T(A_2)) \\ & & L(T(f)) & & \end{array}$$

Proof. For all $a \in A_1$,

$$\begin{aligned} (L(T(f)) \circ F_{A_1})(a) &= L(T(f))(F_{A_1}(a)) \\ &= T^{-1}(f)(F_{A_1}(a)) \\ &= \{g \in T(A_2) : T(f)(g) \in F_{A_1}(a)\} \\ &= \{g \in T(A_2) : g \circ f \in F_{A_1}(a)\} \\ &= \{g \in T(A_2) : g(f(a)) = 1\} \\ &= F_{A_2}(f(a)) \\ &= F_{A_2} \circ f(a). \end{aligned}$$

□

Theorem 4.36. *If $h : \delta_1 \mapsto \delta_2$ is an homomorphism of intuitionistic fuzzy perfect Priestley space, then*

$$T(L(h)) \circ G_{\delta_1} = G_{\delta_2} \circ h.$$

$$\begin{array}{ccccccc} & & & h & & & \\ & \delta_1 & & ---- & \rightarrow & \delta_2 & \\ & | & & & & | & \\ G_{\delta_1} & | & & & & | & G_{\delta_2} \quad . \\ & \downarrow & & & & \downarrow & \\ & T(L(\delta_1)) & & ---- & \rightarrow & T(L(\delta_2)) & \\ & & & T(L(h)) & & & \end{array}$$

Proof. For all $f \in \delta_1$,

$$\begin{aligned}(T(L(h))\circ G_{\delta_1})(f) &= T(L(h))(G_{\delta_1}(f))\\ &= G_{\delta_1}(f)\circ L(h)\,(\text{since } T(f)(g)=g\circ f)\end{aligned}$$

hence for all $y \in L(\delta_2)$,

$$\begin{aligned}(T(L(h))\circ G_{\delta_1})(f)(y) &= (G_{\delta_1}(f)\circ L(h))(y)\\ &= G_{\delta_1}(f)(h^{-1}(y))\\ &= \begin{cases} 1 \text{ if } f\in h^{-1}(y)\\ 0 \text{ if } f\notin h^{-1}(y)\end{cases}\\ &= \begin{cases} 1 \text{ if } h(f)\in y\\ 0 \text{ if } h(f)\notin y\end{cases}\\ &= G_{\delta_2}(h(f))(y)\\ &= (G_{\delta_2}\circ h)(f)(y).\end{aligned}$$

□

Theorem 4.37. *The dual of the category of intuitionistic fuzzy perfect distributive lattices is equivalent to the category of intuitionistic fuzzy perfect Priestley spaces.*

Proof. Lemmas 4.31, 4.33, Theorems 4.35, 4.36 establish the functorial isomorphisms. □

Example 4.38. *Let $(A, \vee, \wedge, R)$ be an intuitionistic fuzzy perfect distributive lattice, where $A = \{a, b, c, d, e, f\}$ and R is an intuitionistic fuzzy perfect relation defined by*

μ_R	a	b	c	d	e	f
a	1	0.1	0.2	0.2	0.3	0.4
b	0	1	0	0.3	0.4	0.5
c	0	0	1	0.3	0.4	0.5
d	0	0	0	1	0.4	0.4
e	0	0	0	0	1	0.3
f	0	0	0	0	0	1

and

ϑ_R	a	b	c	d	e	f
a	0	0.5	0.4	0.4	0.3	0.2
b	1	0	1	0.3	0.2	0.1
c	1	1	0	0.4	0.3	0.2
d	1	1	1	0	0.2	0.2
e	1	1	1	1	0	0.1
f	1	1	1	1	1	0

.

Then its dual is: $T(A) =$ The set of $0\!-\!1$ *homomorphisms from* A *onto* $\{0,1\} = \{f_1, f_2, f_3, f_4\}$

A	f_1	f_2	f_3	f_4
a	0	0	0	0
b	0	0	1	0
c	0	0	0	1
d	0	0	1	1
e	0	1	1	1
f	1	1	1	1

and its bidual is: $L(T(A)) = \{\phi, \{f_3\}, \{f_4\}, \{f_3, f_4\}, \{f_2, f_3, f_4\}, X\}$, *where* R_2 *is given by*

μ_{R_2}	ϕ	$\{f_3\}$	$\{f_4\}$	$\{f_3, f_4\}$	$\{f_2, f_3, f_4\}$	X
ϕ	1	0.1	0.1	0.1	0.1	0.1
$\{f_3\}$	0	1	0	0.3	0.4	0.5
$\{f_4\}$	0	0	1	0.3	0.4	0.5
$\{f_3, f_4\}$	0	0	0	1	0.4	0.4
$\{f_2, f_3, f_4\}$	0	0	0	0	1	0.3
X	0	0	0	0	0	1

ϑ_{R_2}	ϕ	$\{f_3\}$	$\{f_4\}$	$\{f_3, f_4\}$	$\{f_2, f_3, f_4\}$	X
ϕ	0	0.5	0.5	0.5	0.5	0.5
$\{f_3\}$	1	0	1	0.3	0.2	0.1
$\{f_4\}$	1	1	0	0.4	0.3	0.2
$\{f_3, f_4\}$	1	1	1	0	0.2	0.2
$\{f_2, f_3, f_4\}$	1	1	1	1	0	0.1
X	1	1	1	1	1	0

Finally, $F_A : A \mapsto L(T(A))$ *is given by*

A	$F_A(a_1)/i = 1\ to\ 6$
a	ϕ
b	$\{f_3\}$
c	$\{f_4\}$
d	$\{f_3, f_4\}$
e	$\{f_2, f_3, f_4\}$
f	X

.

Example 4.39. *Let (X, τ, r) be an intuitionistic perfect Priestley space, where $X = \{x, y, z\}$ and r is given by*

μ_r	x	y	z
x	1	0	0
y	0	1	0
z	0	0	1

and

ϑ_r	x	y	z
x	0	1	1
y	1	0	1
z	1	1	0

.

Then $L(X) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}$ and r_1 is given by $r_1 = (\mu_{r_1}, \vartheta_{r_1})$ such that

$$\mu_{r_1}(A, B) = \begin{cases} 1 & \text{if } A = B, \\ 1 - \frac{cardA}{cardB} & \text{if } A \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vartheta_{r_1}(A, B) = \begin{cases} 0 & \text{if} A = B, \\ \delta\ \frac{cardA}{cardB} & \text{if } A \subset B \text{ where } \delta \in \left]0, 1\right[, \\ 1 & \text{otherwise.} \end{cases}$$

Then r_1 will be given by

μ_{r_1}	ϕ	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$	X
ϕ	1	1	1	1	1	1	1	1
$\{x\}$	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$
$\{y\}$	0	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$
$\{z\}$	0	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$
$\{x,y\}$	0	0	0	0	1	0	0	$\frac{2}{3}$
$\{x,z\}$	0	0	0	0	0	1	0	$\frac{2}{3}$
$\{y,z\}$	0	0	0	0	0	0	1	$\frac{2}{3}$
X	0	0	0	0	0	0	0	1

and

ϑ_{r_1}	ϕ	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$	X
ϕ	0	0	0	0	0	0	0	0
$\{x\}$	1	0	1	1	$\frac{1}{4}$	$\frac{1}{4}$	1	$\frac{1}{6}$
$\{y\}$	1	1	0	1	$\frac{1}{4}$	1	$\frac{1}{4}$	$\frac{1}{6}$
$\{z\}$	1	1	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
$\{x,y\}$	1	1	1	1	0	1	1	$\frac{1}{3}$
$\{x,z\}$	1	1	1	1	1	0	1	$\frac{1}{3}$
$\{y,z\}$	1	1	1	1	1	1	0	$\frac{1}{3}$
X	1	1	1	1	1	1	1	0

.

$\left(Chosen\ \delta = \frac{1}{2}\right)$

and the set of $0-1$ *homomorphisms from* $L(X)$ *onto* $\{0,1\}$, *i.e.,* $T(L(X))$ *is equal to* $\{f_1, f_2, f_3\}$

$L(X)$	$f_1(X_i)$	$f_2(X_i)$	$f_3(X_i)$
ϕ	0	0	0
$\{x\}$	1	0	0
$\{y\}$	0	1	0
$\{z\}$	0	0	1
$\{x, y\}$	1	1	0
$\{x, z\}$	1	0	1
$\{y, z\}$	0	1	1
X	1	1	1

.

And r_2 will be given by

μ_{r_2}	f_1	f_2	f_3
f_1	1	0	0
f_2	0	1	0
f_3	0	0	1

and

ϑ_{r_2}	f_1	f_2	f_3
f_1	0	1	1
f_2	1	0	1
f_3	1	1	0

and the isomorphism G_X is defined by$G_X : X \to T(L(X))$, where

X	$G_X(X_i)/X_i \in X$
x	f_1
y	f_2
z	f_3

.

Example 4.40. *Let (X, τ, r) be a Priestley space, where $X = \{x, y, z, t\}$ and r is given by*

μ_r	x	y	z	t
x	1	0.2	0.3	0
y	0	1	0.4	0
z	0	0	1	0
t	0	0	0	1

and

ϑ_r	x	y	z	t
x	0	0.5	0.4	1
y	1	0	0.5	1
z	1	1	0	1
t	1	1	1	0

and $L(X) = \{\phi, \{t\}, \{z\}\{y, z\}\{z, t\}, \{x, y, z\}, \{y, z, t\}, X\}$, where r_1 is given by

μ_{r_1}	ϕ	$\{t\}$	$\{z\}$	$\{t,z\}$	$\{y,z\}$	$\{y,z,t\}$,	$\{x,y,z\}$	X
ϕ	1	0.2	0.2	0.2	0.2	0.2	0.2	0.2
$\{t\}$	0	1	0	0.2	0	0.2	0	0.2
$\{z\}$	0	0	1	0.2	0.2	0.2	02	0.2
$\{t,z\}$	0	0	0	1	0	0.2	0	0.2
$\{y,z\}$	0	0	0	0	1	0.4	0.4	0.4
$\{y,z,t\}$	0	0	0	0	0	1	0	0.4
$\{x,y,z\}$	0	0	0	0	0	0	1	0.4
X	0	0	0	0	0	0	0	1

and

ϑ_{r_1}	ϕ	$\{t\}$	$\{z\}$	$\{t,z\}$	$\{y,z\}$	$\{y,z,t\}$	$\{x,y,z\}$	X
ϕ	0	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$\{t\}$	1	0	1	0.5	1	0.5	1	0.5
$\{z\}$	1	1	0	0.5	0.5	0.5	0.5	0.5
$\{t,z\}$	1	1	1	0	1	0.5	1	0.5
$\{y,z\}$	1	1	1	1	0	0.5	0.5	0.5
$\{y,z,t\}$	1	1	1	1	1	0	1	0.5
$\{x,y,z\}$	1	1	1	1	1	1	0	0.5
X	1	1	1	1	1	1	1	0

and $T(L(X)) = \{f_1, f_2, f_3, f_4\}$ *such that*

$L(X)$	$f_1(X_i)$	$f_2(X_i)$	$f_3(X_i)$	$f_4(X_i)$
ϕ	0	0	0	0
$\{t\}$	0	0	0	1
$\{z\}$	0	0	1	0
$\{t,z\}$	0	0	1	1
$\{y,z\}$	0	1	1	0
$\{y,z,t\}$	0	1	1	1
$\{x,y,z\}$	1	1	1	0
X	1	1	1	1

.

The isomorphism G_X is defined as follows: $G_X : X \mapsto T(L(X))$

X	$G_X(X_i)\, X_i \in X$
x	f_1
y	f_2
z	f_3
t	f_4

.

Conclusion

In this thesis, we have proposed a way to represent fuzzy distributive lattices and intuitionistic fuzzy perfect distributive lattices. This, by constructing adequate fuzzy orders and intuitionistic fuzzy perfect orders.

In this context a study of representation of fuzzy distributive lattices and intuitionistic finite fuzzy perfect distributive lattices is presented. We have shown that the category of fuzzy Priestley spaces is equivalent to the dual of the category of fuzzy bounded distributive lattices. We have also shown that the category of intuitionistic finite fuzzy perfect Priestley spaces is equivalent to the dual of the category of finite intuitionistic fuzzy perfect distributive lattices.

Future work is anticipated in multiple directions. Firstly, is it possible to obtain such representation for an infinite intuitionistic fuzzy perfect bounded distributive lattices?

Secondly, is it possible to obtain such representation if we change the definition of fuzzy set? In other words, what happens if we replace the unite interval $[0, 1]$ by any complete lattice L?

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الخلاصة

في هذه الاطروحة قمنا بتعميم بعض النتائج المحصل عليها من طرف الاستاذين عبد العزيز عمرون (A. Amroune) و بيجان دافاز (B. Davvaz) وذلك من خلال البحث المقدم من طرفهما في المجلة المسماة" Iranian Journal of Fuzzy Systems" بعنوان "Fuzzy ordered sets and duality for finite fuzzy distributive Lattices ".

في هذا الاتجاه قمنا بتقديم نظرية التمثيل للشبكات التوزيعية المغلقة الضبابية في الحالة الغير منتهية، اي سنبرهن ان الفئة الثنوية لفضاءات بريستلي تكافئ فئة الشبكات التوزيعية المغلقة الضبابية الغير منتهية.

من جهة ثانية قمنا ايضا بالتعميم الى نظرية تمثيل الشبكات التوزيعية المثالية الضبابية الحدسية المنتهية. في هذا السياق قمنا باستعمال الشبكات التوزيعية المثالية الضبابية الحدسية المنتهية وفضاءات بريستلي المثالية الضبابية الحدسية المنتهية من أجل اثبات التكافؤ بين الفئة الثنوية لفضاءات بريستلي المثالية الضبابية الحدسية المنتهية وفئة الشبكات التوزيعية المثالية الضبابية الحدسية المنتهية.

Abstract

In this thesis, we extend some results obtained by A. Amroune and B. Davvaz in their paper entitled "Fuzzy ordered sets and duality for finite fuzzy distributive lattices", published in "Iranian Journal of Fuzzy Systems". In this way a representation theorem for the infinite fuzzy distributive lattices is given. More precisely, we show that the category of infinite fuzzy Priestley spaces is equivalent to the dual of the category of infinite fuzzy distributive lattices.

We have also developed a representation theory of intuitionistic fuzzy perfect distributive lattices in the finite case. To that end, we have introduced the notion of intuitionistic fuzzy perfect distributive lattices and the one of fuzzy perfect Priestley spaces to show the equivalence between the category of finite intuitionistic fuzzy perfect Priestley spaces and the dual of the category of finite intuitionistic fuzzy perfect distributive lattices

Résumé

Dans cette thèse, nous avons développé quelques résultats obtenus par Pr: A. Amroune et Pr: B. Davvaz dans leur article intitulé "Fuzzy ordered sets and duality for finite fuzzy distributive lattices", publié dans "Iranian Journal of Fuzzy Systems". Ainsi nous avons donné un théorème de représentation pour les treillis distributive flous infini. Plus précisément, nous avons montré que la catégorie des espaces de Priestley flous est équivalente au dual de la catégorie des treillis distributifs flous fermés dans le cas infini. D'autre part nous avons développé une théorie de représentation des treillis distributifs flous intuitionnistes parfaits dans le cas fini. Pour cela, nous avons introduit la notion de treillis distributif flous intuitionnistes parfaits et les espaces de Priestley flous intuitionnistes parfaits pour montrer l'équivalence entre la catégorie des espaces de Priestley flous intuitionnistes parfait fini et le dual de la catégorie des treillis distributifs flous intuitionnistes parfaits finis.

Keywords: Fuzzy lattice, fuzzy Priestley space, homomorphism of fuzzy lattices, homomorphism of fuzzy Priestley spaces, intuitionistic fuzzy perfect ordered set, intuitionistic fuzzy perfect lattice, homomorphism of an intuitionistic fuzzy perfect lattices, Homomorphism of an intuitionistic fuzzy perfect Priestley spaces.