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A study of a nonrelativistic energy spectrum produced of an isotropic potential in the framework of extended quantum mechanics symmetries: the case of Varshni potential

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بسم الله الرحمن الرحيم

Dedication and Acknowledgment

Dedication

To the ones that I am indebted to for eternity: my dear father and mother who have encouraged me all the way and whose encouragement, to this day, has made sure that I give it all it takes to finish that which I have started.

Khalaf Salah Eddin

Acknowledgme

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General introduction

General introduction

Although a century has passed since the appearance of Schrödinger's non-relativistic equation, it is still of wide interest to researchers and scholars in institutes and universities all over the world. The special importance of this equation is evident through its success in finding solutions to some physical problems accurately, similar to the two cases of the harmonic oscillator [1], as well as the study of some central potentials, such as the Coulomb potential [2]. In addition, the Schrödinger equation succeeded in finding approximate solutions to many physical problems described by different potentials, such as exponential potentials as an example the Hulthen potential, generalized exponential Coulomb potential, generalized inverse quadratic Yukawa potential [3,4,5], and others such as Killingbeck potential, Cornell potential, [6,7,8,9] .

The physical importance of Schrödinger's non-relativistic equation lies through its research in the bound state eigenvalues for CO, NO, CH, and N2 diatomic molecules, and the mass spectra of heavy quarkonium systems so on the determination of the dynamics of non-relativistic particles in quantum mechanics such as the thermodynamic properties of the system, mass spectra of mesons, among others [10,11,12,13].

The Varshni potential is considered among potentials that have received great attention in recent years, and this importance is due to its wide applications in various fields such as nuclear physics, particle physics, and molecular physics (For more details, see the introduction to Chapter Two).

This importance is evident through the study of this potential within the framework of the basic equations (Shrodinger, Klein-Gordon, and Dirac equations) known in the literature within the limits of non-relativistic and relativistic quantum mechanics.

As for our contribution through this Master's thesis in theoretical physics for the academic year

2022 and 2023, we continue the research path that was launched to see the effect of the deformation of the topological properties of the phase space on the energy bands resulting from the Varshni potential within the framework of the symmetries of expanded mechanics in the field of the deformed Schrödinger equation.

The structure of this master memory is as follows. The non-commutative quantum theory is described in chapter one. The Schrödinger equation is updated in chapter two using the Varshni potential. We examine the impact of non-commutativity characteristics on the Varshni potential in chapter three.

Chapter 1

The non-commutative phase-space formalism

Introduction

The postulates and hypotheses that identify the quantum and physical structures of the non-commutative phase-space and its physical structures will be discussed in this chapter. The following are the foundational principles that we shall discuss:

-General principles of quantum mechanics

-The non-commutative phase-space postulates

-The Moyal-Weyl star product

-Bopp's Shift method

General Principles of quantum mechanics

Through the methods of the results of the photoelectric effect experiments, where he considered that light consists of particles called light quanta, every quanta known with photo with energy $E_\gamma = h\nu$, here $h \approx 6{,}6262.10^{-34}$js. Currently, ordinary quantum mechanics is formulated on the commutative space of the coordinates of variables $\{x_i\}$ and the canonical moment of hermetic operators $\{p_i\}$, as follows [14,15,16,17]:

$$\begin{cases} [x_i, p_j] = \imath\hbar\delta_{ij} \\ [x_i, x_j] = 0 \\ [p_i, p_j] = 0 \end{cases} \tag{1.1}$$

Here, $\hbar$ is the reduced Plank constant and equal to $\frac{\hbar}{2\pi}$, and δ_{ij} is the usual Kronecker symbol which is equal to 0 if $i \neq j$ and one if $i = j$. The above algebra can be generalized to the Dirac picture as follows:

$$\begin{cases} [x_i(t), p_j(t)] = \imath\hbar\delta_{ij} \\ [x_i(t), x_j(t)] = 0 \\ [p_i(t), p_j(t)] = 0 \end{cases} \tag{1.2}$$

where the usual canonical coordinates (x_i, p_i) and the corresponding time-dependent $x_i(t)$ and $p_i(t)$ are determined from the projection relations:

$$\begin{cases} x_i(t) = exp\left(\frac{\iota}{\hbar}H(t-t_0)\right)x_i\, exp\left(-\frac{\iota}{\hbar}H(t-t_0)\right) \\ p_i(t) = exp\left(\frac{\iota}{\hbar}H(t-t_0)\right)p_i\, exp\left(-\frac{\iota}{\hbar}H(t-t_0)\right) \end{cases} \tag{1.3}$$

Here, $\{x_i(t)\}$,$\{p_i(t)\}$ and H are Hermitian operators on a Hilbert space of physical states, which each satisfy the Heisenberg equation of motions. We get the following:

$$\begin{cases} \frac{dx_i}{dt} = \frac{\iota}{\hbar}[H, x_i(t)] \\ \frac{dp_i}{dt} = \frac{\iota}{\hbar}[H, p_i(t)] \end{cases} \tag{1.4}$$

Both related concepts relating to energy E and impulsion p_i are satisfied by the quantization procedure:

$$\begin{cases} E \rightarrow i\hbar\frac{\partial}{\partial t} \\ p_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x_i} \end{cases} \tag{1.5}$$

It is well known that the classical energy E of a particle of mass μ subjected to the external forces produced by a potential $V(\vec{r}, t)$, in a classical mechanic is given by:

$$E = \frac{\overleftrightarrow{p}^2}{2\mu} + V(\vec{r}, t) \tag{1.6}$$

The quantization procedure in Eq. (1.5) permitted to obtain the Shrödinger equation known in the framework of quantum mechanics known in the literature:

$$\left(-\frac{\hbar^2}{2m}\Delta + V(\vec{r}, t)\right)\psi\left(\overleftrightarrow{r}, t\right) = i\hbar\frac{\partial\psi(\overleftrightarrow{r}, t)}{\partial t} \tag{1.7}$$

Here Δ is the Laplacian operator in spherical coordinates $\overleftrightarrow{r}\,(r, \theta, \phi)$:

$$\Delta = \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2}{\partial\phi^2} \tag{1.8}$$

Which can be expressed in Cartesian coordinates $\overleftrightarrow{r}\,(x, y, z)$ as follows:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.9}$$

while $\psi(\overleftrightarrow{r}, t)$ Denote to the complex wave function. The probability of finding the particle at a time t in elementary volumes (d^3r, d^3p), rounding the point r as follows:

$$dp = \begin{cases} \left|\psi(\overleftrightarrow{r}, t)\right|^2 d^3r \\ \text{In the configuration space} \\ \left|\psi(\overleftrightarrow{p}, t)\right|^2 d^3p \\ \text{In the momentum space} \end{cases} \tag{1.10}$$

where (d^3r, d^3p) are equal to $(r^2 \sin\theta\, d\theta d\phi dr, p^2 \sin\theta\, d\theta d\phi dp)$, respectively.

Non-commutative phase-space

Quantum mechanics on non-commutative space was first proposed by Heisenberg in 1930 [18] and then developed by Snyder in late 1947 [19]. The proposal of extended quantum mechanics came as a possible solution to many physical problems that non-relativistic and relativistic quantum mechanics were unable to find solutions to the divergence problem in the standard model, string theory, and quantum gravity [20,21,22,23,24]. The idea of non-commutativity developed and took a positive turn through the research work of Connes et al. from the eighties to decades (see, for example, [25,26,27,29]). The simplest commutation relation that described the non-commutativity idea satisfies the following algebra (see refs. [30,31,32,33,34,35,36]):

$$\begin{cases} [x_i, x_j] = [x_i(t), x_j(t)] = \imath\hbar_{eff}\theta_{ij} \\ [p_i, p_j] = [p_i(t), p_j(t)] = \imath\bar{\theta}_{ij} \end{cases} \tag{1.11}$$

where $(\theta_{ij}, \bar{\theta}_{ij}) = -(\theta_{ji}, \bar{\theta}_{ji}) = \varepsilon_{ij}(\theta, \bar{\theta})$ are constants anti-symmetric tensors of dimensions $[x]^2$and $[p]^2$, $((\theta, \bar{\theta})$ are the NC parameters and ε_{ij} is just an anti-symmetric number ($\varepsilon_{ij} = -\varepsilon_{ji} = 1$ with$i \neq j$and$\varepsilon_{jj} = 0$) and $\hbar_{eff} = \hbar\left(1 + \frac{\theta\bar{\theta}}{4\hbar^2}\right)$ is the effective constant of Planck. The non-commutative coordinates (x_i, p_i) take the form:

$$\begin{cases} x_i \to x_i = f(x_i, p_i) \\ p_i \to p_i = f(x_i, p_i) \end{cases} \tag{1.12}$$

In this work, we are interred by the phase-phase has three dimensions $N = 3$, therefore the indices take the values $(i, j = 1{,}3)$. In this particular case, the rules of canonical commutations become:

$$\begin{cases} [x_1, p_2] = 0 \\ [x_1, p_3] = 0 \\ [x_2, p_3] = 0 \\ [x_1, x_2] = i\theta_{12} \\ [x_1, x_3] = i\theta_{13} \\ [x_2, x_3] = i\theta_{23} \end{cases} \tag{1.13}$$

and

$$\begin{cases} [x_1, p_2] = i\hbar_{eff} \\ [x_1, p_3] = i\hbar_{eff} \\ [x_2, p_3] = i\hbar_{eff} \\ [p_1, p_2] = i\bar{\theta}_{12} \\ [p_1, p_3] = i\bar{\theta}_{13} \\ [p_2, p_3] = i\bar{\theta}_{23} \end{cases} \tag{1.14}$$

Weyl's quantization:

The fundamentals of quantum physics inspired many of the broad principles behind non-commutative geometry. Weyl proposed an elegant formulation for mapping quantum operators to classical functions of phase-space variables within the framework of canonical quantification. This method establishes a systematic approach to modeling non-commutative spaces in general and examining ancient ideas based on them [37,38,39,40,41,42,43,44]. Weyl quantization is a technique for describing quantum physics using classical mechanics' phase space. It is a rule that allows a quantum operator to be associated with a classical function that is dependent on phase space variables. The Weyl quantification also applies to commutative relations in a general form. Consider a $f(x,p)$ and $g(x,p)$ a general two functions, their product in the notion of non-commutative phase-space can be expressed as a new product called the star product or the Weyl-Moyal star product defined on phase space,

$$f(x,p) * g(x,p) =$$

$$f(x,p)g(x,p) + \frac{i}{2}\sum_{m} \theta^{mn} \frac{\partial}{\partial x^m} f(x,p) \frac{\partial}{\partial x^n} +$$

$$+\frac{i}{2}\sum_{m} \bar{\theta}^{mn} \frac{\partial}{\partial p^m} f(x,p) \frac{\partial g(x,p)}{\partial p^n} + O(\theta^2, \bar{\theta}^2) \tag{1.15}$$

The formalism of the star product initiated by Weyl and Wigner to allow a description of quantum mechanics in terms of phase-space, is articulated not around non-commuting operators, as in the operational approach, but around the deformation of the product between the phase space variables. We will see how this formalism can be used in the context of non-commutative quantum mechanics (NCQM) symmetries.

Properties of the Star product

The formalism of the star product was initiated by Weyl and Wigner to allow a description of quantum mechanics in terms of phase space; the properties of the star product are presented as follows [37,38,39,40,41,42,43,44,45,46,47]:

-When $\left(\theta, \overline{\theta}\right) = (0,0)$

$$f(x) * g(x) = f(x)g(x) \tag{1.16}$$

Thus we find the commutative case.

-The star product between exponential:

$$e^{\iota kx} * e^{\iota qx} = e^{\iota(k+q)x} e^{-\frac{\iota}{2}(k?q)} \quad \text{with } k \cap q = k^{\iota} q^{j} \theta_{ij} \tag{1.17}$$

-Not commutative:

$$f(x,p) * g(x,p) \neq g(x,p) * f(x,p) \tag{1.18}$$

-Associative:

$$\big(f(x,p) * g(x,p)\big) * h(x,p) = f(x,p)(g(x,p) * h(x,p)) \tag{1.19}$$

-The relation of the complex conjugate:

$$\big(f(x,p) * g(x,p)\big)^{*} = g(x,p)^{*} * f(x,p) \tag{1.20}$$

-The integral relation:

$$\begin{cases} \int d^{D}x(f * g) = \int d^{D}x(g * f)(x,p) \\ \quad = \int d^{D}x f(x,p) g(x,p) \end{cases} \tag{1.21}$$

-Cyclic permutation:

$$\int d^{D}x(f * g * h) = \int d^{D}(g * f * h) = \int d^{D}x(g * f * h) \tag{1.22}$$

-Satisfies Leibniz's rule:

$$\partial_{\mu}(f * g) = \partial_{\mu} f * g + f * \partial_{\mu} g \tag{1.23}$$

Boop's shift method

It is well known that the physicist Fritz Bopp was the first to examine pseudo-differential operators derived from a symbol using quantization methods [48,49,50,51,52]:

$$\begin{cases} x \to x + \frac{1}{2}\iota\hbar\partial_{p} \\ p \to p - \frac{1}{2}\iota\hbar\partial_{x} \end{cases} \tag{1.24}$$

Instead of the usual correspondence$\left(x \to x,\ p \to -\frac{1}{2}\iota\hbar\partial_{x}\right)$, the operators ($x \to x + \frac{1}{2}\iota\hbar\partial_{p}$ and $p \to p - \frac{1}{2}\iota\hbar\partial_{x}$) are known as Bopp's shifts, and this quantization procedure is known as the Bopp quantization procedure. This quantization leads us to obtain the following:

$$\begin{cases} x^i = x^i - \sum_j \frac{\theta^{ij}}{2} p_j \\ p^i = p^i + \sum_j \frac{\overline{\theta}^{ij}}{2} x_j \end{cases} \tag{1.25}$$

To write the Schrödinger equation in the non-commutative phase-space, we apply these steps:

1- The ordinary three-dimensional Hamiltonian operators$H(p_i, x_i)$will be replaced with a new Hamiltonian operator $H(p_i, x_i)$.

2- The ordinary complex wave function $\psi\left(\vec{r}\right)$ becomes a new complex wave function $\hat{\psi}\left(\vec{r}\right)$.

3- The ordinary energy E will be replaced with new values E_{nc}.

4- We replace the ordinary product with the star product.

Hence, we get the following Schrödinger equation in the non-commutative space :

$$\begin{cases} H(p_i, x_i)\hat{\psi}\left(\vec{r}\right) = E_{nc}\hat{\psi}\left(\vec{r}\right) \\ \Rightarrow \\ H\left(x^i, p^i\right) * \Psi(\vec{r}, t) = E_{nl}\Psi(\vec{r}, t) \end{cases} \tag{1.26}$$

Bopp's shifts method allows us to reduce the above deformed Schrödinger equation to the new translated form applicable to the principles of quantum mechanics known in the literature:

$$\begin{cases} H\left(x^i, p^i\right) * \Psi(\vec{r}, t) = E_{nl}\Psi(\vec{r}, t) \\ \Rightarrow \\ H\left(x^i \to x_i, p^i \to p_i\right)\Psi(\vec{r}, t) = E_{nl}\Psi(\vec{r}, t) \end{cases} \tag{1.27}$$

Thus, the Hamiltonian operator takes the three varieties forms as follows [52,53,54]:

$$\begin{cases} H(p_i, x_i) = H\left(p_i = p_i + \sum_j \frac{\bar{\theta}^{ij}}{2} x_j, \quad \mathrm{x}_i = x_i - \sum_j \frac{\theta^{ij}}{2} p_i\right) \\ \text{For Non-commutative phase-space} \\ H(p_i, x_i) = H\left(p_i = p_i, \quad \mathrm{x}_i = x_i - \sum_j \frac{\theta^{ij}}{2} p_j\right) \\ \text{For Non-commutative space-space} \\ H(p_i, x_i) = H\left(p_i = p_i + \sum_j \frac{\bar{\theta}^{ij}}{2} x_j, \quad \mathrm{x}_i = x_i\right) \\ \text{For Non-commutative phase-phase} \end{cases} \tag{1.28}$$

The first variety corresponds to non-commutative phase-space NCSP symmetries, which correspond to the new Hamiltonian operator $H\left(p_i = p_i + \frac{\bar{\theta}^{ij}}{2} x_j, x_i = x_i - \frac{\theta^{ij}}{2} p_i\right)$ in Eq. (1.28):

$$\begin{cases} p_i \to p_i = p_i - \frac{\bar{\theta}^{ij}}{2} x_j \\ x_i \to x_i = x_i - \frac{\bar{\theta}^{ij}}{2} p_j \end{cases} \tag{1.29.1}$$

The second variety corresponds to non-commutative space-space symmetries, which correspond to the new Hamiltonian operator $H\left(p_i = p_i, x_i = x_i - \frac{\theta^{ij}}{2} p_j\right)$ In Eq. (1.28):

$$\begin{cases} p_i \to p_i = p_i \\ x_i \to x_i = x_i - \sum_j \frac{\theta^{ij}}{2} p_j \end{cases} \tag{1.29.2}$$

The third variety corresponds to non-commutative phase-phase (NCPP) symmetries which correspond to the new Hamiltonian operator $H\left(p_i = p_i + \frac{\bar{\theta}^{ij}}{2} x_j, x_i = x_i\right)$ in Eq. (1.28):

$$\begin{cases} p_i \to p_i = p_i - \sum_j \frac{\bar{\theta}^{ij}}{2} p_j \\ x_j \to x_i = x_i \end{cases} \tag{1.30}$$

In our current master memoir, we are interested in applying the following general procedure to NCSP symmetries which correspond to the first variety of Eq. (1.28). The three-generalized coordinates $(x = x_1, \hat{y} = x_2, \hat{z} = x_3)$ in the non-commutative phase-space were depended on corresponding three-usual generalized positions (x, y, z) and three momentum coordinates

(p_x, p_y, p_z) :

$$\begin{cases} i_1 = 1 & \mathrm{x}_1 = x_{p_1} = p_x \\ i_2 = 2 & \mathrm{x}_2 = x_{p_2} = p_y \\ i_3 = 3 & \mathrm{x}_3 = x_{p_3} = p_z \end{cases} \tag{1.31}$$

It is important to notice that the new operators $(x_i$ and $p_i)$ in (3D-NCQM) were dependent on ordinary operators x_i and p_i from the projection relations :

$$\begin{cases} x_1 = x_1 - \frac{\theta^{12}}{2} p_2 - \frac{\theta^{13}}{2} p_3 \\ x_2 = x_2 - \frac{\theta^{21}}{2} p_1 - \frac{\theta^{23}}{2} p_3 \\ x_3 = x_3 - \frac{\theta^{31}}{2} p_1 - \frac{\theta^{32}}{2} p_2 \end{cases} \tag{1.32}$$

and

$$\begin{cases} p_1 = p_1 + \frac{\bar{\theta}^{12}}{2} x_2 + \frac{\bar{\theta}^{13}}{2} x_3 \\ p_2 = p_2 + \frac{\bar{\theta}^{12}}{2} x_1 + \frac{\bar{\theta}^{23}}{2} x_3 \\ p_3 = p_3 + \frac{\bar{\theta}^{31}}{2} x_1 + \frac{\bar{\theta}^{32}}{2} x_2 \end{cases} \tag{1.33}$$

The non-vanish 9-commutators in (3D-NCQM) can be determined as follows :

$$\begin{cases} [x, p_x] = [\hat{y}, p_y] = [\hat{z}, p_z] = i \\ [x, \hat{y}] = i\theta_{12}, \\ [x, \hat{z}] = i\theta_{13}, \\ [\hat{y}, \hat{z}] = i\theta_{23}, \\ [p_x, p_y] = i\bar{\theta}_{12}, \\ [p_y, p_z] = i\bar{\theta}_{23}, \\ [p_x, p_z] = i\bar{\theta}_{13}. \end{cases} \tag{1.34}$$

The square of the operators $(\vec{r}, \vec{p})$ can be found by applying the following expressions as follows [52,53,54]:

$$\begin{cases} \vec{\hat{r}}^{\,2} = \vec{\hat{r}}_x^{\,2} + \vec{\hat{r}}_y^{\,2} + \vec{\hat{r}}_z^{\,2} \\ \vec{p}^{\,2} = \vec{p}_x^{\,2} + \vec{p}_y^{\,2} + \vec{p}_z^{\,2} \end{cases} \tag{1.35}$$

To get the solution to the non-commutative Schrödinger equation, we added the star product

introduced by Bopp's shift method. That is a consequence of the star product between the potential operator $\hat{V}(x)$ and the complex wave function $\hat{\Psi}(\hat{r})$ [48,49,50,51,52]:

$$\begin{cases} \left[\frac{\vec{p}^2}{2m} + \hat{V}(\hat{r})\right] * \hat{\Psi}(\hat{r}) = E_{nc}\hat{\psi}(\vec{r},t) \\ \Rightarrow \\ \left[\frac{\overleftrightarrow{p}_{nc}^2}{2m} + V(\hat{r})\right]\Psi(\overleftrightarrow{r}) = E\Psi(\overleftrightarrow{r}) \end{cases} \tag{1.36}$$

The two operators x and $p,$ when on a non-commutative three-dimensional phase-space, can be written as follows [52,53,54]:

$$\begin{cases} \hat{r}^2 = r^2 - \vec{L}\vec{\Theta} \\ \frac{\overleftrightarrow{p}_{nc}^2}{2\mu} = \frac{\overleftrightarrow{P}^2}{2\mu} + \frac{\vec{L}\vec{\overline{\theta}}}{2\mu} \end{cases} \tag{1.37}$$

Where the two couplings $(\vec{L}\vec{\Theta}$ and $\vec{L}\vec{\overline{\theta}})$ are given by the following relations respectively:

$$\begin{cases} \vec{L}\vec{\Theta} = L_x\theta_{12} + L_y\theta_{23} + L_z\theta_{13} \\ \vec{L}\vec{\overline{\theta}} = L_x\overline{\theta}_{12} + L_y\overline{\theta}_{23} + L_z\overline{\theta}_{13} \end{cases} \tag{1.38}$$

Chapter 2

Reviewed Schrödinger equation with Varshni potential in 3D-NRQM

2.1 Introduction

We revised the Varshni potential within the framework of ordinary quantum mechanics in this chapter. Within the context of the three-dimensional Schrödinger equation, we also attempt to revise the associated wave function and energy eigenvalue.

2.2 Schrödinger equation with the Varshni potential

The Varshni potential is known by Varshni itself in 1957 [55]. This potential play a vital role in many fields, such as nuclear physics, particle physics, and molecular physics. Furthermore, this potential is used to describe bound states of the interaction of systems and has been applied in both classical and molecular physics. The Varshni potential was studied by Lim using the 2-body Kaxiras-Pandey parameters. He observed that Kaxiras and Pandey used this potential to describe the two-body energy portion of multi-body condensed matter [56]. Etido P. Inyang et al. (Ephraim P. Inyanga, Eddy S. Williama, and Etebong E. Ibekwe) (2021), studied the Schrödinger equation solutions for the Varshni potential using the Nikiforov-Uvarov method and obtained the energy eigenvalues and the corresponding eigenfunction in terms of Laguerre polynomials in the non-relativistic regime. Also, they applied their results to calculate heavy-meson masses of charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ [57]. Etido P. Inyang et al. (with other researchers: J. E. Ntibi, E. P. Inyang, F. Ayedun, E. A. Ibanga, E. E. Ibekwe, and E. S. William) studied the radial Schrödinger equation analytically with Varshni potential model using the Nikiforov-Uvarov method and obtained the energy equation and corresponding wave function, in addition, they predicted the analytical energy expression of the mass spectra of heavy quarkonia such as charmonium $c\overline{c}$ and bottomonium $b\overline{b}$, also obtained thermodynamic properties such as free energy, mean energy, entropy, and specific heat [58]. Etido P. Inyang and his research team consisting of Joseph E. Ntibib, Effiong O. Obisung, Eddy S. William, Etebong E. Ibekwe, Ita O. Akpanb, and Ephraim P. Inyang in (2022) solved the Klein-Gordon equation with Varshni potential through the Nikiforov-Uvarov method. By employing the Greene and Aldrich approximation schemes to overcome the centrifugal barrier, they obtained energy eigenvalues in relativistic and non-relativistic regimes, as well as the corresponding normalized wave function [59]. Eddy S. Williama, with collaborations (Etido P. Inyanga, Joseph E. Ntibib, Joseph A. Obub, and Ephraim P. Inyang) (2022) obtained the approximate solutions of the Schrödinger equation for the Varshni-Hellmann potential within the framework of the Nikiforov- Uvarov method by employing the Greene-Aldrich approximation scheme to deal with the centrifugal term, also they obtained the numerical results of the rovibrational energies and normalized wave function in terms of Jacobi polynomials for various quantum states of the diatomic molecules of LiH, T iH, CrH and ScN [60]. The Varshni potential takes the form:

$$V(r) = a - b\,\frac{exp(-\alpha r)}{r} \tag{2.1}$$

where a and b are potential strengths for Varshni potential, α is the screening parameter.

2.3 Reviewing the eigenfunctions and the energy eigenvalues for Varshni 's potential

Schrödinger equation is a fundamental equation of quantum mechanics that describes the evolution of the wave function of a physical system over time. It is a first-order partial differential equation concerning time and a second-order partial differential equation concerning the coordinates of ordinary space. It takes the following form:

$$H\psi\left(\vec{r},t\right) = E\psi\left(\vec{r},t\right) \tag{2.2}$$

here $\psi\left(\vec{r}\right)$ is the complex wave function that satisfies the stationary Schrödinger equation, and E is a non-relativistic eigenvalue of the Hamiltonian $H,$ which can be written in the form :

$$H = \frac{p^2}{2\mu} + a - b\frac{exp(-\alpha r)}{r} \tag{2.3}$$

where μ is the reduced mass for the quarkonium particle such as charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ that can be equal to $\frac{m_c m_{\overline{c}}}{m_c + m_{\overline{c}}}$ and $\frac{m_b m_{\overline{b}}}{m_b + m_{\overline{b}}}$, P represents the impulse $\overleftrightarrow{P} = -i\hbar\overleftrightarrow{\nabla}$, and $\overleftrightarrow{\nabla}$ represents the operator of partial derivatives (Nabla). Etido P. Inyanga et al. carried out Taylor series expansion of the exponential term in Eq. (2.1) up to order three in order to make the potential interact in the quark-antiquark system and this yields [57,58]:

$$\frac{exp(-\alpha r)}{r} = \frac{1}{r} - \alpha + \frac{\alpha^2}{2}r - \frac{\alpha^3}{6}r^2 + \ldots \tag{2.4}$$

Substitute Eq. (2.4) into Eq. (2.1) and obtain:

$$V(r) = -\frac{B}{r} - Cr + Dr^2 + A \tag{2.5}$$

where the new parameters B , C, D and A are given by:

$$\begin{cases} B = ab, \\ C = \frac{ab\alpha^2}{2}, \\ D = \frac{ab\alpha^3}{6}, \\ A = a + ab\alpha. \end{cases} \tag{2.6}$$

In Cartesian coordinates, it is defined by:

$$\overleftrightarrow{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{2.7}$$

Hence, Schrödinger's equation becomes:

$$i\hbar\frac{\partial\psi(\overleftrightarrow{r},t)}{\partial t}=\left(-\frac{\hbar^2}{2\mu}\Delta-\frac{B}{r}-Cr+Dr^2+A\right)\psi(\overleftrightarrow{r},t) \quad (2.8)$$

Since the Varshni potential does not depend on time, solutions can be written separately as a part that is only position-dependent and an only time-dependent part:

$$\Psi\left(\vec{r},t\right)=exp(-iE_{nl}/\hbar)\Psi\left(\vec{r}\right) \quad (2.9)$$

And by substituting into the Schrödinger equation, we find:

$$\left(\frac{-\hbar^2}{2\mu}\Delta-\frac{B}{r}-Cr+Dr^2+A\right)\Psi\left(\vec{r}\right)=E_{nl}\Psi\left(\vec{r}\right) \quad (2.10)$$

Using the spherical coordinate system $\vec{r}(r,\theta,\phi)$, the complex wave function $\Psi\left(\vec{r}\right)$ can be written as:

$$\Psi(r,\theta,\phi)=R_{nl}(r)Y_{lm}(\theta,\phi) \quad (2.11)$$

where $R_{nl}(r)$ is the radial part of the wave function that depends only on the radius r, $Y_{l,m}(\theta,\phi)$ represented by the angular part that depends on the angles (θ,ϕ) , and n is the principal quantum number, l the orbital quantum number, and m the magnetic quantum number that is determined from the interval $(-l\leq m\leq +l)$. The Schrödinger equation in the spherical coordinate can be expressed as:

$$\left(\frac{-\hbar}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\hat{L}^2}{r^2}+\frac{B}{r}+Cr-Dr^2-A\right)\right)R_{nl}(r)=E_{nl}R_{nl}(r) \quad (2.12)$$

In quantum mechanics, the classical momentum obtains the forms $\vec{L}$ is the orbital angular momentum. The total moment $\vec{J}$ is given by:

$$\begin{cases}\vec{J}=\vec{L}+\vec{S}\\ \vec{L}=\vec{r}\wedge\vec{p}\end{cases} \quad (2.13)$$

here $\vec{S}$ is the spin. The components $(L_x,\ L_y$ and $L_z)$ of $\vec{L}$ which are expressed in Cartesian coordinates (x,y,z) as:

$$\begin{cases} L_x = \frac{\hbar}{\iota}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y = \frac{\hbar}{\iota}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z = \frac{\hbar}{\iota}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{cases} \tag{2.14}$$

In the spherical coordinate system $\vec{r}(r,\theta,\phi)$, the components $(L_x,\ L_y$ and $L_z)$ of $\vec{L}$ are expressed as:

$$\begin{cases} L_x = \frac{\hbar}{\iota}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ L_y = \frac{\hbar}{\iota}\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ L_z = \frac{\hbar}{\iota}\frac{\partial}{\partial\phi} \end{cases} \tag{2.15}$$

Note that the operators $(H,\ L^2$ and $L_z)$ commute with each other, and they have formed a common set of eigenfunctions $\psi(r,\theta,\phi)$; however, the three components of the angular momentum $\left(L_{x,}L_y, L_z\right)$ do not commute with each other:

$$\begin{cases} [H, L^2] = [H, L_z] = 0 \\ \left[L_i, L_j\right] = \iota\hbar\xi_{ijk}L_k \end{cases} \tag{2.16}$$

here L^2 is the square of the angular momentum :

$$L^2 = -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right) + \hbar^2\frac{\partial^2}{\partial\phi^2} \tag{2.17}$$

the eigenvalues of $(L^2$ and $L_z)$ are determined from :

$$\begin{cases} \hat{L}^2\psi(r,\theta,\phi) = \hbar^2 l(l+1)\psi(r,\theta,\phi) \\ \hat{L}_z\psi(r,\theta,\phi) = m\hbar\psi(r,\theta,\phi) \end{cases} \tag{2.18}$$

We introducing the wave function: we have:

$$R_{n,l}(r) = \frac{u_{n,l}(r)}{r} \tag{2.19}$$

Thus, the new radial part $u_{n,l}(r)$will be satisfying the following equation:

$$\frac{d^2u_{n,l}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{nl}+\frac{B}{r}+Cr+Dr^2-A-\frac{l(l+1)}{r^2}\right)u_{n,l}(r)=0 \quad (2.20)$$

The energy eigenvalues E_{nl} and corresponding eigenfunctions in closed forms were obtained using the parametric Nikiforov-Uvarov approach by the authors of Refs. [57,58]. They further show that these results are consistent with those obtained previously in other studies using different approaches. They also discovered that when the Varshni potential's screening parameter is set to zero, the energy levels of the familiar pure Coulomb potential energy levels are:

$$E_{nl}=a(1-b\alpha)-\frac{3ab\alpha^2}{2\delta}+\frac{ab\alpha^3}{\delta^3}-\frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu ab}{\hbar^2}-\frac{3\mu ab\alpha^2}{\hbar^2\delta^2}+\frac{16\mu ab\alpha^3}{6\hbar^2\delta^2}}{n+\frac{1}{2}+\sqrt{(l+\frac{1}{2})^2-\frac{\mu ab\alpha^3}{\hbar^2\delta^3}+\frac{16\mu ab\alpha^3}{6\hbar^2\delta^4}}}\right]^2 \quad (2.21)$$

and the wave function of the Varshni potential[57,58]:

$$\Psi(y)=N_{n,l}\,y^{\frac{-a}{2\sqrt{\varepsilon}}}e^{\frac{-\varepsilon}{y\sqrt{\varepsilon}}}\;L_n^{\frac{a}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{y\sqrt{\varepsilon}}\right) \quad (2.22)$$

where

$$\begin{cases} y=\frac{1}{r} \\ -\varepsilon=\left(\frac{2\mu E_{n,l}}{\hbar^2}-\frac{2\mu A}{\hbar^2}+\frac{6\mu C}{\hbar^2\delta}-\frac{12\mu D}{\hbar^2\delta^2}\right) \\ \alpha=\left(\frac{2\mu B}{\hbar^2}-\frac{2\mu C}{\hbar^2\delta^2}+\frac{16\mu D}{\hbar^2\delta^3}\right) \end{cases} \quad (2.23)$$

and $N_{n,l}$ is a normalization constant which can be obtained from

$$\int_0^{+\infty}|\Psi(y)|^2dr=1 \quad (2.24)$$

The complex wave function $\Psi\left(\vec{r}\right)$ can be written as:

$$\Psi\left(\vec{r}\right)=\frac{N_{n,l}}{r}r^{\frac{a}{2\sqrt{\varepsilon}}}exp\left(-\frac{\varepsilon}{\sqrt{\varepsilon}}r\right)\ L_n^{\frac{a}{\sqrt{\varepsilon}}}(\frac{2\varepsilon}{\sqrt{\varepsilon}}r)Y_{lm}(\theta,\phi) \tag{2.25}$$

The Rodrigues' formula of the associated Laguerre polynomials $L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}(\frac{2\varepsilon}{\sqrt{\varepsilon}}r)$ is:

$$L_n^{\frac{a}{\sqrt{\varepsilon}}}(\frac{2\varepsilon}{y\sqrt{\varepsilon}})=\frac{1}{n!}exp\left(\frac{2\varepsilon}{y\sqrt{\varepsilon}}\right)y^{\frac{\alpha}{\sqrt{\varepsilon}}}\frac{d^n}{dy^n}\left(exp\left(\frac{2\varepsilon}{y\sqrt{\varepsilon}}\right)y^{2n-\frac{\alpha}{\sqrt{\varepsilon}}}\right) \tag{2.26}$$

In mathematics, the Laguerre polynomials, named after Edmond Laguerre, are solutions of Laguerre's differential equation [61,62,63]

$$\begin{cases} xy^{''}+(1-x)y^{'}+ny=0, \\ y=y(x). \end{cases} \tag{2.27}$$

The first few generalized Laguerre polynomials are:

$$\begin{cases} L_0^{(\alpha)}=1 \\ L_1^{(\alpha)}=-x+\alpha+1 \\ L_2^{(\alpha)}=\frac{x^2}{2}+(\alpha+2)x+\frac{(\alpha+1)(\alpha+2)}{2} \end{cases} \tag{2.28}$$

The simple Laguerre polynomials are the special case $\alpha=0$ of the generalized Laguerre polynomials.

$$L_n^{(0)}(x)=L_n(x) \tag{2.29}$$

where

$$\begin{cases} L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \\ L_n(0) = 1, L_n^{'}(0) = n, L_n^{''}(0) = \frac{n(n-1)}{2} \end{cases} \tag{2.30}$$

Chapter 3

The Effect of Non-commutativity on the energy spectrum produced by the Varshni potential in 3D-NCQM

3.1 Introduction

The purpose of this chapter is to study the modified Schrödinger equation of Varshni potential in non-commutative three-dimensional phase space. Accordingly, we use Bopp's shift method instead of solving the modified Schrödinger equation directly, thus, using the star product and the perturbation theorem to find the corresponding energy correction.

The Schrödinger equation on a Non-commutative space-time

3.2 New solutions of deformed Schrödinger equation with modified Varshni potential in 3D-NCQM symmetries

In this section, we aim to search for new solutions to the deformed Schrödinger equation in the framework of three-dimensional non-commutative quantum mechanics (3D-NCQM) symmetries under the influence of the modified Varshni potential. Pr. Maireche had several types of research related to the topic being dealt with, and we will follow the same method for our treatment of the deformed Schrödinger equation under the modified Varshni potential. We simply replace the wave function products (or fields) with the star product or the Moyal product. The Schrödinger equation for a non-commutative space-time has the form:

$$\begin{cases} H(p_i, x_i)\widehat{\Psi}\left(\overleftrightarrow{r}\right) = E_{nc}\widehat{\Psi}\left(\overleftrightarrow{r}\right) \\ \Rightarrow \\ H(p_i, x_i) * \Psi\left(\overleftrightarrow{r}\right) = E_{nc}\Psi\left(\overleftrightarrow{r}\right) \end{cases} \tag{3.1}$$

According to the method of Bopp's Shift, which we have seen in the first chapter, the above equation can be simplified into the following form:

$$\begin{cases} H(p_i, x_i) * \Psi\left(\overleftrightarrow{r}\right) = E_{nc}\Psi\left(\overleftrightarrow{r}\right) \\ \Rightarrow \\ H_{nc}^{vp}(p_i, x_i)\Psi(\overleftrightarrow{\hat{r}}) = E_{nl}\Psi(\overleftrightarrow{\hat{r}}) \end{cases} \tag{3.2}$$

where

$$H_{nc}^{vp}(p_i, x_i) = H_{nc-vp}\left(p_i = p_i - \sum_{i=1}^{3} \frac{\bar{\theta}^{ij}}{2} x_j, \quad \mathrm{x}_i = x_i + \sum_{i=1}^{3} \frac{\theta^{ij}}{2} p_i\right)$$

$$= \frac{\vec{p}^2}{2\mu} + V(\hat{r}) \tag{3.3}$$

and

$$\begin{cases} V(\hat{r}) = -\frac{B}{\hat{r}} - C\hat{r} + D\hat{r}^2 + A \\ \frac{\vec{P}^2}{2\mu} = \frac{P^2}{2\mu} + \frac{\vec{L}\vec{\theta}}{2\mu} \end{cases} \tag{3.4}$$

Using Eq.(1.37)**,** we can obtain ($-\frac{B}{\hat{r}}$, $-C\hat{r}$ and $D\hat{r}^2$) as the sum of corresponding values ($-\frac{B}{r}$, $-Cr$ and Dr^2) in the symmetries of non-relativistic quantum mechanics plus the induced terms ($\frac{B}{2r^3}\vec{L}\vec{\Theta}$, $\frac{C}{2r}$ and$-D\vec{L}\vec{\Theta}$) with the effect of deformed proprieties of space-space, as follows (More information We refer the reader and researcher to see the references [51,64]):

$$\begin{cases} -\frac{B}{r} \rightarrow \frac{B}{\hat{r}} = -\frac{B}{r} + \frac{B}{2r^3}\vec{L}\vec{\Theta} + O(\Theta^2) \\ -Cr \rightarrow -C\hat{r} = -Cr + \frac{C}{2r}\vec{L}\vec{\Theta} + O(\Theta^2) \\ Dr^2 \rightarrow D\hat{r}^2 = Dr^2 - D\vec{L}\vec{\Theta} + O(\Theta^2) \end{cases} \tag{3.5}$$

Allow us to get the Varshni potential in the non-commutative phase-space as follows:

$$V(r) \rightarrow V(\hat{r}) = -\frac{B}{r} - Cr + Dr^2 + A + \left(\frac{B}{2r^3} + \frac{C}{2r} - D\right)\vec{L}\vec{\Theta} + O(\Theta^2) \tag{3.6}$$

The global Hamiltonian operator $H_{nc}^{vp}(p_i, x_i)$in non-commutative three-dimensional phase-space can be written in the following form:

$$H(p_i, x_i) \rightarrow H_{nc}^{vp}(p_i, x_i) = H(p_i, x_i) + \left(\frac{B}{2r^3} + \frac{C}{2r} - D\right)\vec{L}\vec{\Theta} + \frac{\vec{L}\vec{\theta}}{2\mu} \tag{3.7}$$

1-The first two terms in the Hamiltonian operator $H_{nc}^{vp}(p_i, x_i)$, which corresponds to the Varshni potential in Eq. (2.3) and the Kinetic term or dynamic $\frac{\vec{p}^2}{2m}$ in ordinary commutative space, which formed the usual Hamiltonian operator:

$$H(p_i = p_i, x_i = x_i) = \frac{\vec{p}^2}{2m} + -\frac{B}{r} - Cr + Dr^2 + A \tag{3.8}$$

The second and third terms are formed by the new Hamiltonian operator or the additive-created term H_{nc-vp}^{ind} which represents the contributions of the non-commutative phase-space:

$$H_{nc-vp}^{ind} = \left(\frac{B}{2r^3} + \frac{C}{2r} - D\right)\vec{L}\vec{\Theta} + \frac{\vec{L}\vec{\overline{\theta}}}{2\mu} \tag{3.9}$$

where $\vec{L}\vec{\Theta}$ and $\vec{L}\vec{\overline{\theta}}$ are $L_x\theta_{12} + L_y\theta_{23} + L_z\theta_{13}$ and $L_x\overline{\theta}_{12} + L_y\overline{\theta}_{23} + L_z\overline{\theta}_{13}$, respectively, determined from Eq. (1.38) in the first chapter. According to the mathematical forms of the two couplings ($\vec{L}\vec{\Theta}$ and $\vec{L}\vec{\overline{\theta}}$) observed in Eq.(3.8), it is physically possible to replace ($\vec{L}\vec{\Theta}$ and $\vec{L}\vec{\overline{\theta}}$) with ($\mu\Theta\vec{L}\vec{S}$ and$\mu\bar{\theta}\vec{L}\vec{S}$), respectively:

$$\begin{cases} \vec{L}\vec{\Theta} \to \mu\Theta\vec{L}\vec{S} \\ \vec{L}\vec{\overline{\theta}} \to \mu\bar{\theta}\vec{L}\vec{S} \end{cases} \tag{3.10}$$

With $\overleftrightarrow{S}$ denote to the spin of the particles such as charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ which interacted with Varshni's potential, and ε is a new constant of proportionality. This enables rewriting Eq. (3.11) as follows:

$$H_{nc-vp}^{ind} = \varepsilon\left[\frac{\bar{\theta}}{2\mu} + \left(\frac{B}{2r^3} + \frac{C}{2r} - D\right)\Theta\right]\vec{L}\vec{S} \tag{3.11}$$

The parameters (Θ and $\overline{\theta}$) are given by:

$$\begin{cases} \Theta = (\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2)^{\frac{1}{2}} \\ \overline{\theta} = (\bar{\theta}_{12}^2 + \theta_{23}^2 + \bar{\theta}_{13}^2)^{\frac{1}{2}} \end{cases} \tag{3.12}$$

In ordinary quantum mechanics, we have sets of operators $(\hat{A}, \hat{B}, \hat{C}$ ......) which form a complete set of complete observable commutes (ECOC). We apply this rule to the sets of operators. $\left(\overleftrightarrow{J}^2, \overleftrightarrow{S}^2, \overleftrightarrow{L}^2 \text{ and } J_z\right)$, i.e.:

$$\begin{cases} \left[\overleftrightarrow{J}^2, \overleftrightarrow{L}^2\right] = 0 \\ \left[\overleftrightarrow{J}^2, \overleftrightarrow{S}^2\right] = 0 \\ \left[\overleftrightarrow{J}^2, J_z\right] = 0 \end{cases} \tag{3.13}$$

And the corresponding eigenvalues are$j(j+1)$,$l(l+1)$, $s(s+1)$ and $m(-l \leq m \leq +l)$ in the system $(c = \hbar = 1)$, so:

$$\begin{cases} \overleftrightarrow{J}^2 \Psi_{n,l,m_l}(r,\theta,\phi) = j(j+1)\Psi_{n,l,m_l}(r,\theta,\phi) \\ \overleftrightarrow{L}^2 \Psi_{n,l,m_l}(r,\theta,\phi) = l(l+1)\Psi_{n,l,m_l}(r,\theta,\phi) \\ \overleftrightarrow{S}^2 \Psi_{n,l,m_l}(r,\theta,\phi) = s(s+1)\Psi_{n,l,m_l}(r,\theta,\phi) \\ J_z\Psi_{n,l,m_l}(r,\theta,\phi) = m\Psi_{n,l,m_l}(r,\theta,\phi) \end{cases} \tag{3.14}$$

With $\vec{J}$ being the geometric sum of the moments $\vec{L}$ and $\vec{S}$. This allows us to find the spin-orbit coupling $\vec{L}\vec{S}$ as follows:

$$\vec{L}\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{S}^2 - \vec{L}^2\right) \tag{3.15}$$

An immediate result is:

$$\vec{L}\vec{S}\Psi = \frac{1}{2}[j(j+1) - l(l+1) - s(s+1)]\Psi \tag{3.16}$$

With $|l-s| \leq j \leq |l+s|$, and

$$j = |l-s|, |l-s|+1, \ldots\ldots, |l+s| \tag{3.17}$$

For the two extreme values of the total angular momentum, we can write for spin-1 :

$$\vec{L}\vec{S}\Psi = \begin{cases} \frac{1}{2}\{(l+1)(l+2) - l(l+1) - 2\}\Psi \\ \qquad \equiv k_1\Psi \quad \text{if} \quad j = |l+1| \\ \frac{1}{2}\{l(l+1) - l(l+1) - 2\}\Psi \\ \qquad \equiv k_2\Psi \quad \text{if} \quad j = |l| \\ \frac{1}{2}\{(l-1)l - l(l+1) - 2\}\Psi \\ \qquad \equiv k_3\Psi \quad \text{if} \quad j = |l-1| \end{cases} \tag{3.18}$$

The Varshni Hamiltonian is extended by including new additive potential H_{nc-vp}^{ind} expressed to the radial terms $(\frac{B\Theta}{2r^3}, \frac{C\Theta}{2r})$ and constant numbers $((-D\Theta), \left(\frac{\bar{\theta}}{2\mu}\right)$) to become the modified Varshni potential in non-commutative three-dimensional phase-space symmetries. This generated new potential H_{nc-vp}^{ind} is also proportional to the infinitesimal parameters $(\Theta$ and $\bar{\theta})$. This allows us to consider the new additive part of the potential H_{nc-vp}^{ind} as perturbation potential compared with the main potential $V(r)$. That is all physical justifications for applying the time-independent perturbation theory become satisfied to calculate the expectation values of previous radial terms. This allows us to give a complete prescription for determining the energy level of the generalized $(n,l,m)^{th}$ excited state. The exact spectrum produced by the spin-orbit effect for the Varshni potential in the three-dimensional non-commutative space-phase $E_{nc\_nl}^{vp}$ is the sum of the energy corresponding to ordinary space E_{nl} and the corrections ΔE_{per}^{vp}

$$E_{nc\_nl}^{vp} = E_{nl} + \Delta E_{per}^{vp} \tag{3.19}$$

The perturbation theorem allows us to obtain the first-order corrections as follows:

$$\Delta E_{nc-vp}(\Theta,\bar{\theta}) = \left\langle \Psi\left(\vec{r}\right) \middle| H_{nc-vp}^{ind} \middle| \Psi\left(\vec{r}\right) \right\rangle \tag{3.20}$$

We can write the equation (3.20) in the form:

$$\Delta E_{per}^{vp}(\Theta,\bar{\theta}) = \int \quad \Psi\left(\overleftrightarrow{r}\right) H_{nc-vp}^{ind}(r,\Theta,\theta) \Psi\left(\overleftrightarrow{r}\right) d\tau \tag{3.21}$$

where $d\tau$ represent the volume element in spherical coordinates r, which is given by:

$$d\tau = r^2 dr d\Omega \tag{3.22}$$

With the solid angle

$$d\Omega = \sin\theta \, d\theta d\phi$$

And the nonperturbative complex wave function is defined by :

$$\Psi(\overleftrightarrow{r}) = R_{n,l}(r) Y_l^m(\theta,\varphi) \tag{3.23}$$

So, we can write the equation (3.21) in the form:

$$E_{nc-vp}(\Theta,\bar{\theta}) = \left\langle \vec{L}\vec{S} \right\rangle$$

$$\int_0^\infty R_{n,l}^*(r) H_{nc-vp}^{ind} R_{n,l}(r) r^2 dr \int_0^\pi \int_0^{2\pi} Y_l^{*m_l}(\theta,\varphi) Y_l^m(\theta,\varphi) d\Omega \tag{3.24}$$

The normalized wave function $\Psi(\overleftrightarrow{r})$ Allows us to write :

$$\int_0^\pi \int_0^{2\pi} Y_l^{*m_l}(\theta,\varphi) Y_l^m(\theta,\varphi) d\Omega = 1 \tag{3.25}$$

This reduces the corrections found by (3.29) to the form:

$$\Delta E_{nc-vp}(\Theta,\bar{\theta}) = \left\langle \vec{L}\vec{S} \right\rangle \int_0^\infty R_{n,l}^*(r) H_{nc-vp}^{ind} R_{n,l}(r) r^2 dr \tag{3.26}$$

We substituted the spin-orbit coupling term $V_{nc}^{py}(r)$, and we find:

$$\begin{gathered} \Delta E_{nc-vp}(\Theta,\bar{\theta}) = \varepsilon \left\langle \vec{L}\vec{S} \right\rangle \Big(\frac{\mu\bar{\theta}}{2\mu} \int_0^\infty R_{n,l}^*(r) R_{n,l}(r) r^2 dr \\ + \frac{B}{2} \int_0^\infty R_{n,l}^*(r) \frac{1}{r^3} R_{n,l}(r) r^2 dr + \frac{C}{2} \int_0^\infty R_{n,l}^*(r) \frac{1}{r} R_{n,l}(r) r^2 dr \\ - D \int_0^\infty R_{n,l}^*(r) R_{n,l}(r) r^2 dr\Big) \end{gathered} \tag{3.27}$$

If we apply the following known formula:

$$\int_0^\infty R_{n,l}^*(r) R_{n,l}(r) r^2 dr = 1, \tag{3.28}$$

The corrections in Eq. (3.32) will be simplified to the form :

$$\Delta E_{nc-vp}(\Theta,\bar{\theta}) = \varepsilon\left\langle \vec{L}\vec{S}\right\rangle (\frac{\mu\bar{\theta}}{2\mu} - D + + \frac{B}{2}\int_0^\infty R_{n,l}^*(r)\frac{1}{r^3}R_{n,l}(r)r^2dr + \frac{C}{2}\int_0^\infty R_{n,l}^*(r)\frac{1}{r}R_{n,l}(r)r^2dr) \tag{3.29}$$

If we replace the radial part $R_{n,l}(r)$ which is obtained from Eq. (2.25) as:

$$R_{nl}(s) = \frac{N_{n,l}}{r}r^{\frac{a}{2\sqrt{\varepsilon}}}exp\left(-\frac{\varepsilon}{\sqrt{\varepsilon}}r\right)\ L_n^{\frac{a}{\sqrt{\varepsilon}}}(\frac{2\varepsilon}{\sqrt{\varepsilon}}r) \tag{3.30}$$

The corrections in Eq. (3.30) will be simplified to the form :

$$\Delta E_{nc-vp}(\Theta,\bar{\theta}) = \varepsilon\left\langle \vec{L}\vec{S}\right\rangle (\frac{\mu\bar{\theta}}{2\mu} - D + + \frac{BN_{n,l}^2}{2}\int_0^\infty r^{\frac{a}{\sqrt{\varepsilon}}-3}exp\left(-\frac{2\varepsilon}{\sqrt{\varepsilon}}r\right)\left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(\frac{2\varepsilon}{\sqrt{\varepsilon}}r)\right]^2dr + \frac{CN_{n,l}^2}{2}\int_0^\infty r^{\frac{a}{\sqrt{\varepsilon}}-1}exp\left(-\frac{2\varepsilon}{\sqrt{\varepsilon}}r\right)\left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(\frac{2\varepsilon}{\sqrt{\varepsilon}}r)\right]^2dr) \tag{3.31}$$

We introduce a new variable $z = \frac{2\varepsilon}{\sqrt{\varepsilon}}r$, A direct calculation gives:

$$\int_0^\infty r^{\frac{a}{\sqrt{\varepsilon}}-3}exp\left(-\frac{2\varepsilon}{\sqrt{\varepsilon}}r\right)\left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(\frac{2\varepsilon}{\sqrt{\varepsilon}}r)\right]^2dr = \left(\frac{\sqrt{\varepsilon}}{2\varepsilon}\right)^{\frac{a}{\sqrt{\varepsilon}}-2}\int_0^\infty z^{\frac{a}{\sqrt{\varepsilon}}-3}exp(-z)\left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(z)\right]^2dz \tag{3.32}$$

and

$$\int_0^\infty r^{\frac{a}{\sqrt{\varepsilon}}-1}exp(-z)\left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(z)\right]^2dr = \left(\frac{\sqrt{\varepsilon}}{2\varepsilon}\right)^{\frac{a}{\sqrt{\varepsilon}}}\int_0^\infty z^{\frac{a}{\sqrt{\varepsilon}}-1}exp(-z)\left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(z)\right]^2dz \tag{3.33}$$

The corrections in Eq. (3.31) will be simplified to the form :

$$\Delta E_{nc-vp}(\Theta,\bar{\theta}) = \varepsilon \left\langle \vec{L}\vec{S} \right\rangle \Big(\frac{\mu\bar{\theta}}{2\mu} - D + \\ + \frac{BN_{n,l}^2}{2}\left(\frac{\sqrt{\varepsilon}}{2\varepsilon}\right)^{\frac{a}{\sqrt{\varepsilon}}-2} \int_0^{\infty} z^{\frac{a}{\sqrt{\varepsilon}}-3} exp(-z) \left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(z)\right]^2 dz \\ + \frac{CN_{n,l}^2}{2}\left(\frac{\sqrt{\varepsilon}}{2\varepsilon}\right)^{\frac{a}{\sqrt{\varepsilon}}} \int_0^{\infty} z^{\frac{a}{\sqrt{\varepsilon}}-1} exp(-z) \left[L_n^{\frac{a}{\sqrt{\varepsilon}}}(z)\right]^2 dz\Big) \tag{3.34}$$

Comparing Eq.(3.35) with the integral of the form [63]:

$$\int_0^{+\infty} z^{\beta+K} exp(-z)\,[L_n^K(z)]^2 dz = \sum_{i=0}^{n} \binom{\beta}{n-1}^2 \frac{\Gamma(\beta+K+1+i)}{i!} \tag{3.35}$$

where $\binom{\beta}{n-1}$ is a generalized binomial coefficient which is computed by the multiplicative formula:

$$\binom{\beta}{n-1} = \frac{\beta!}{(n-1)!(\beta-n+1)!} \tag{3.36}$$

We obtain the energy correction for the generalized $(n,l,m)^{th}$ excited state:

$$\Delta E_{nc-vp}(\Theta,\bar{\theta}) = \varepsilon \left\langle \vec{L}\vec{S} \right\rangle \Big(\frac{\mu\bar{\theta}}{2\mu} - D + \\ + \frac{BN_{n,l}^2}{2}\left(\frac{\sqrt{\varepsilon}}{2\varepsilon}\right)^{\frac{a}{\sqrt{\varepsilon}}-2} \sum_{i=0}^{n} \binom{3}{n-1} \frac{\Gamma\left(\frac{a}{\sqrt{\varepsilon}}-2+i\right)}{i!} \\ + \frac{CN_{n,l}^2}{2}\left(\frac{\sqrt{\varepsilon}}{2\varepsilon}\right)^{\frac{a}{\sqrt{\varepsilon}}} \sum_{i=0}^{n} \binom{1}{n-1} \frac{\Gamma\left(\frac{a}{\sqrt{\varepsilon}}+i\right)}{i!}\Big) \tag{3.37}$$

The global energy E_{nc-nl}^{vp} for the generalized $(n,l,m)^{th}$ excited states is the energy spectrums:

$$E_{nc-nl}^{vp} = E_{nl} + \Delta E_{nc-vp}(\Theta,\bar{\theta}) \tag{3.38}$$

Where E_{nl} is determined from Eq.(2.21), which we have seen in the second chapter:

$$E_{nl} = a(1-b\alpha) - \frac{3ab\alpha^2}{2\delta} + \frac{ab\alpha^3}{\delta^3} - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu ab}{\hbar^2} - \frac{3\mu ab\alpha^2}{\hbar^2\delta^2} + \frac{16\mu ab\alpha^3}{6\hbar^2\delta^2}}{n + \frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 - \frac{\mu ab\alpha^3}{\hbar^2\delta^3} + \frac{16\mu ab\alpha^3}{6\hbar^2\delta^4}}}\right]^2$$

For heavy-meson masses of charmonium $c\overline{c}$ and bottomonium $b\overline{b}$, $\left\langle \vec{L}\vec{S} \right\rangle$ Is determined from:

$$\left\langle \vec{L}\vec{S} \right\rangle = \begin{cases} \frac{1}{2}\{(l+1)(l+2) - l(l+1) - 2\}\Psi \\ \equiv k_1 \quad \text{if} \quad j = |l+1| \\ \frac{1}{2}\{l(l+1) - l(l+1) - 2\} \\ \equiv k_2 \quad \text{if} \quad j = |l| \\ \frac{1}{2}\{(l-1)l - l(l+1) - 2\} \\ \equiv k_3 \quad \text{if} \quad j = |l-1| \end{cases} \tag{3.39}$$

It is clear that the following physical limit procedure:

$$\begin{cases} \underset{(\Theta,\bar{\theta})\to(0,0)}{Lim} E_{nc-nl}^{vp} = E_{nl} \\ \underset{(\Theta,\bar{\theta})\to(0,0)}{Lim} \Delta E_{nc-vp}(\Theta,\bar{\theta}) = 0 \end{cases} \tag{3.40}$$

Gives us all the results of physical treatments which we have seen in the standard references [57,58].

Conclusion

Conclusion

Through this Master's memory in physics, theoretical specialty: Promotion 2022-2023. The non-relativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of Varshni potential as a model.

This memory aims to study physical systems within the framework of the modified Schrödinger equation with the modified Varshni potential in three-dimensional non-commutative quantum mechanics.

In the first chapter, we have represented the mathematical and physical formalisms of the non-commutative three-dimensional phase-space and apply these principles to the atoms of modified Varshni potential.

In the second chapter, we reviewed the Schrödinger equation under the Varshni potential based on many works.

In the third chapter, we studied the effect of the non-commutativity of the three-dimensional phase-space, by applying the generalized Bopp shift method and standard perturbation theory at the first order of parameters $(\Theta, \bar{\theta})$, the modifications on the energy corresponding to the ground state are obtained. We can conclude that the application of the non-commutativity in this work on the modified Varshni potential includes the spin-orbit coupling effect automatically. This is in contrast to what we observe in the framework of quantum mechanics known in the literature, where the spin-orbit interaction appears by external addition and not through spontaneous birth as a result of space deformation.

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Abstract

In our work on this master memory, in theoretical physics (2022/2023): A study of a non-relativistic energy spectrum produced with an isotropic potential in the framework of 3-dimensional non-relativistic quantum mechanics symmetries in the context of deformed Schrödinger equation: the case of Varshni potential. We have studied the deformed Schrödinger equation with the Varshni potential in non-commutative three-dimensional spaces and phases by applying Boop's Shift method to the first order of the parameters $(\Theta, \bar{\theta})$, in addition to the standard perturbation theory, to obtain the modified spectrum of energy of this system, which is changing radically, and replaced by new degenerate new states depending on the discrete atomic quantum numbers (j, n, l, s), the potential strengths (a,b), and the screening parameter α.

Keywords: Schrödinger equation, Varshni potential, non-commutative quantum mechanics, star product, Boop's shift method.

Contents