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Theme

Haar Wavelets And Its Applications

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I dedicate this thesis
To my parents
To my professor
To my grand mothers
To my sisters and brothers
To my family
To my friends
To all those who support and encourage me.

Imane.

List of symbols and abbreviations

H : The Haar matrix.

P_{2M} : the operational matrix of integration.

x_l : The collocation points.

a_i : The Haar coefficients.

$\Gamma(.)$: The Gamma function.

I^α : The Riemann-Liouville fractional integral.

D^α : The Caputo derivative.

P_{2M}^α : The operational matrix of fractional integration.

A : The Vector of Haar coefficients.

$K(.,.)$: The kernel of the equation.

G_{il} : The discret form of the integral of the Kernel.

F_l : The discret form of the function f.

ODE : Ordinary differential equation.

IVP : Initial value problem.

FDE : Fractional differential equation.

VIE : Volterra integral equation .

VID : Volterra integro-differential.

IDE : Integro-differential equation.

IC : Initial condition.

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Introduction

Wavelet, being a powerful mathematical tool, has been widely used in signal processing and numerical analysis. In the recent years wavelet approach has become more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical solution of initial and boundary value problems. Alfred Haar introduced a group of square waves with magnitude of -1 and 1 in some intervals and zeros elsewhere then we called Haar function.
In fact, Haar wavelets have a number of advantages, including: simplicity, orthogonality and very compact support. The main advantages of the Haar wavelets method are sparse representation, fast transformation and possibility of implementation of fast algorithm in matrix representation. The Haar basis is simplest instance of spline wavelets, resulting when the polynomial degree is set to zero, so computational costs with Haar wavelets is very low. So we use them for solving a variety of problems that is by turning these into a system of linear equations at collocation points and solve its using Matlab .

This thesis consists of five chapters :

In the first chapter, we give notions and preliminary results on Haar functions and its integral. We also give some notions of fractional calculus of Riemann-Liouville and Caputo type, in order to use in the next chapters.

In the second chapter, we study boundary problems associated with ordinary linear and fractional differential equations, using Haar method, the collocation method is used to find the approximate solution.

In the third chapter, we will try to solve the linear integral equations of type Volterra and fredholm by the method described in the preceding chapter, as we study the linear fractional integral equations of type Volterra. Examples will be introduced to test the effectiveness of the method.

In chapter four and five, we present Haar's wavelet method to solve linear integro–differential and fractional integro–differential equations of Volterra and Fredholm type. several examples of different types were tested and compared with the exact solution, to show the validity of the method at certain points of collocation.

CHAPTER 1

BASIC CONCEPTS OF HAAR WAVELETS

In this chapter, we present the definition of the Haar function and its integral and their associated matrices. We also give some basic notions on the fractional calculus, namely, the fractional integral of Riemann-Liouville, the fractional derivative of Caputo, These concepts will be used in the following chapters.

1.1 Haar Wavelets

Let us consider $x \in [a, b]$, where a and b are given constants. We define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[a, b]$ is divised into $2M$ sub–intervals of equal length; the length of each sub-intervals is $\Delta x = \frac{(b-a)}{2M}$. Next, two parameters are introduced: $j = 0, 1, \cdots, J$, and $k = 0, 1, \ldots, m-1$ (the notation $m = 2^j$ is introduced). The wavelets number i is identified as $i = m + k + 1$.
the i-th Haar wavelets is defined as :

$$h_i(x) = \begin{cases} 1 & for \;\; x \in [\xi_1(x) \;\; \xi_2(x)) \,, \\ -1 & for \;\; x \in [\xi_2(x) \;\; \xi_3(x)) \,, \\ 0 & elsewhere. \end{cases} \tag{1.1}$$

where

$$\begin{array}{ll} \xi_1(x) = a + 2k\mu\Delta x, & \xi_2(x) = a + (2k+1)\mu\Delta x, \\ \xi_3(x) = a + 2(k+1)\mu\Delta x, & \mu = \frac{M}{m}. \end{array} \tag{1.2}$$

These equations are valid if $i > 2$. The case $i = 1$ corresponds to the scaling function, where $h_1(x) = 1$ for $x \in [a, b]$.
In the haar wavelets the parameters j and k have a different importance. The width of the i th wavelet is called support and found by

$$\xi_3(x) - \xi_1(x) = a + 2(k+1)\mu\Delta - a + 2k\mu\Delta x = 2\mu\Delta x = \frac{b-a}{m}. \tag{1.3}$$

If we take $a = 0$, $b = 1$, $J = 2$, the first eight haar wavelet functions are shown in Figure(1.1)

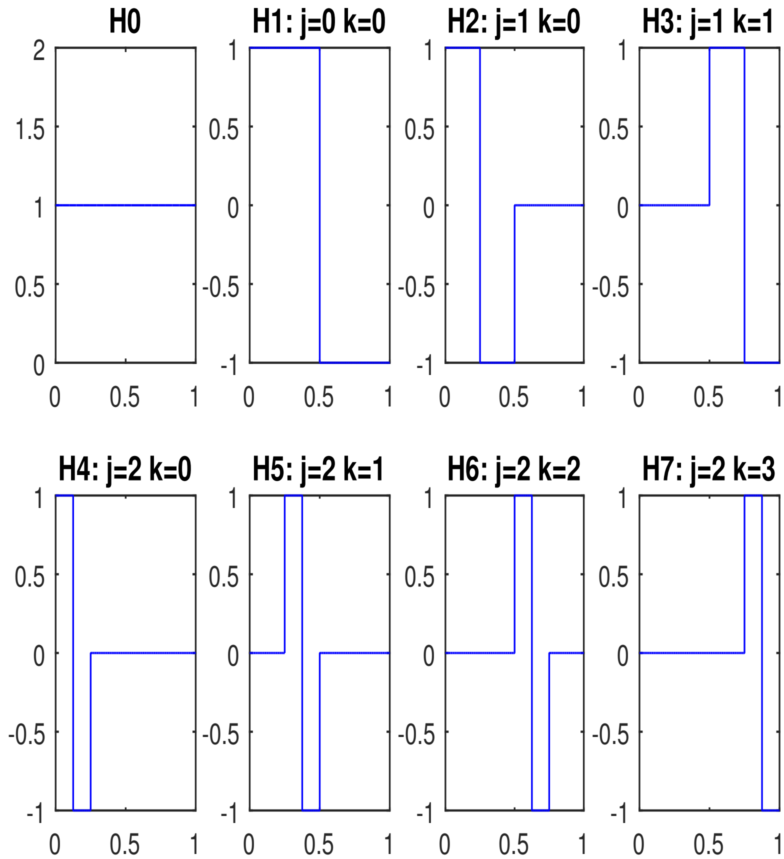

Figure 1.1: Eight first Haar wavelets.

The haar wavelets are orthogonal to each other and we have

$$\int_0^1 h_i(t)h_l(t)dt = \begin{cases} \frac{b-a}{m}, & i = l \\ 0, & i \neq l \end{cases} \tag{1.4}$$

1.2 Haar matrices

If we want to use the Haar wavelets for the numerical solutions, we must put them into a discrete form. There are different ways to do it; in this section the collocation method is applied. Let us denote the grid points by

$$\tilde{x}_l = (a + \Delta x) \ \ for \ \ l = 0, 1, \ldots, 2M. \tag{1.5}$$

For the collocation points we take

$$x_l = \frac{\tilde{x}_{l-1} + \tilde{x}_l}{2} \quad for \;\; l = 0, 1, \ldots, 2M. \tag{1.6}$$

We have that

$$H_{2M}(x) = \begin{bmatrix} h_1(x_l) \\ h_2(x_l) \\ \vdots \\ h_{2M}(x_l) \end{bmatrix} \tag{1.7}$$

Remark 1.1. In the rest of this thesis we take $[a, b] = [0, 1]$

Thus we have

$$H_1 = (1) \;\;, H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Next, by integrating (1.1) α-times we obtain ($\alpha \in \mathbb{N}$)

$$P_{\alpha,i}(x) = \begin{cases} 0 & for \;\; x < \xi_1(i) \\ \frac{1}{\alpha!}[x - \xi_1(x)]^\alpha & for \;\; x \in [\xi_1(i), \xi_2(i)] \\ \frac{1}{\alpha!}\left\{[x - \xi_1(x)]^\alpha - 2[x - \xi_2(x)]^\alpha\right\} & for \;\; x \in [\xi_2(i), \xi_3(i)] \\ \frac{1}{\alpha!}\left\{[x - \xi_1(x)]^\alpha - 2[x - \xi_2(x)]^\alpha - [x - \xi_3(x)]^\alpha\right\} & for \;\; x > \xi_3(i) \end{cases} \tag{1.8}$$

These formulas (1.8) hold for $i > 1$. In the case $i = 1$ we have

$$P_{\alpha,1}(x) = \frac{1}{\alpha!}[x - a]^\alpha. \tag{1.9}$$

The integration of $h(x)$ can be expanded into a Haar series with a Haar coefficient matrix P_{2M} as

$$\int_0^x h(t)dt = P_{2M}h(x) \tag{1.10}$$

The matrix P_{2M} is called the operational matrix of integration.

$$P_{\alpha,2M}(x) = \begin{bmatrix} P_{\alpha,1}(x_l) \\ P_{\alpha,2}(x_l) \\ \vdots \\ P_{\alpha,2M}(x_l) \end{bmatrix} \tag{1.11}$$

so when $\alpha = 1$ we obtain

$$P_1 = (\frac{1}{2}), P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, P_4 = \frac{1}{16}\begin{pmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, P_8 = \frac{1}{64}\begin{pmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 & -4 & 4 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1.3 Fractional calculus

Let's briefly consider some basic formulas on fractional calculus.

Definition 1.1. The Riemann-Liouville fractional integrals of order α are defined by

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt. \ (x > a)([\alpha] > 0) \tag{1.12}$$

and

$$\left(I_{B-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^B f(t)(t-x)^{\alpha-1} dt, \quad (x < B, [\alpha] > 0). \tag{1.13}$$

Definition 1.2. the Caputo fractional derivative is defined by :

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_A^x f^{(n)}(t)(x-t)^{n-1-\alpha} dt \tag{1.14}$$

Here $\Gamma(\alpha)$ is the gamma function and $[\alpha]$ the integer part of α. The integrals (1.13,1.12) are called left-sided and right-sided fractional integrals.

where $f^{(n)}(x) = d^n f/dx^n$ and $n = [\alpha] + 1$. If $\alpha + 1 = n \in N$, then $D^n f(x)$ coincides with the ordinary derivative $f^{(n)}(x)$. The following relations hold

$$I^{\alpha}\left(I^{\beta} f(x)\right) = I^{\beta}\left(I^{\alpha} f(x)\right) = I^{\alpha+\beta} f(x) \tag{1.15}$$

$$D^{\alpha}\left(D^{\beta} f(x)\right) = D^{\beta}\left(D^{\alpha} f(x)\right) = D^{\alpha+\beta} f(x) \tag{1.16}$$

$$D^{\alpha}\left(I^{\beta} f(x)\right) = D^{\alpha-\beta} f(x), \alpha \geq \beta \tag{1.17}$$

$$I^{\alpha}\left(D^{\alpha} f(x)\right) = f(x) + \sum_{k=0}^{n-1} f^{(k)}(0) \frac{x^k}{k!}, x \geq 0 \tag{1.18}$$

1.4 Haar wavelet operational matrix of fractional integration

To obtain the operational matrix of fractional integration for Haar wavelets, which is a generalized form of P_{2M} in (1.10). The fractional integration of order α of $h(x)$ can be expanded into a Haar series with a Haar coefficient matrix P_{2M}^{α} as follows:

$$\frac{1}{\Gamma(\alpha)}\int_0^x h(t)(x-t)^{\alpha-1} = P_{2M}^{\alpha}h(x). \tag{1.19}$$

We call this square matrix P_{2M}^{α} the (generalized) operational matrix of fractional integration. When $\alpha = 0.5$ and $2M = 4$ we obtain

$$P_4^{0.5} = \begin{pmatrix} 0.75238 & -0.2203 & -0.1558 & -0.0820 \\ 0.2203 & 0.3116 & -0.1558 & 0.2296 \\ 0.0410 & 0.1148 & 0.2203 & -0.0350 \\ 0.0779 & -0.0779 & 0 & 0.2203 \end{pmatrix}$$

When $\alpha = 0.75$ and $2M = 4$ we obtain

$$P_4^{0.75} = \begin{pmatrix} 0.6218 & -0.2521 & -0.1499 & -0.1099 \\ 0.2521 & 0.1176 & -0.1499 & 0.1898 \\ 0.0550 & 0.0949 & 0.0699 & -0.0155 \\ 0.0749 & -0.0749 & 0 & 0.0699 \end{pmatrix}$$

1.5 Function Approximations with Haar Wavelet

Any function $y(t)$, $\in L^2[a,b]$, can be expanded into the Haar series as follows

$$y(t) = \sum_0^{\infty} a_i h_i(t) \tag{1.20}$$

The equation (1.20) can be written in matrix form as follows :

$$y(t) = AH \tag{1.21}$$

Where H is the Haar matrix (1.7) and A the vecteur of Haar coefficients.

CHAPTER 2

NUMERICAL SOLUTIONS FOR LINEAR ORDINARY AND FRACTIONAL DIFFERENTIAL EQUATIONS USING HAAR WAVELET METHOD

The idea in this chapter starts from using Haar wavelets to solve linear ordinary and fractional differential equations.

2.1 Numerical solutions for linear ordinary differential equations using Haar wavelet method

Consider the $n-$th order linear differential equation:

$$\sum_{k}^{n} A_k(x) y^{(k)}(x) = f(x) \;\; x \in [a,b] \tag{2.1}$$

With the initial conditions

$$y^{(k)}(x_0) = y_k \;\; k = 0, 1, \ldots, n-1 \tag{2.2}$$

here $A_k(x)$, $f(x)$ are given functions and y_k are a given constants.

2.1.1 Method of solution and numerical examples

We assume that

$$y^{(n)}(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{2.3}$$

Integrate both sides of 2.3 from 0 to x and repeat this n times :

$$\int_0^x y^{(n)}(\tau) d\tau = \int_0^x \sum_{i=1}^{2M} a_i h_i(\tau) d\tau$$

$$y^{(n-1)}(x) = \sum_{i=1}^{2M} a_i P_{1,i}(x) + y^{(n-1)}(x_0)$$

$$\vdots$$

$$y(x) = \sum_{i=1}^{2M} a_i P_{n,i}(x) + \cdots + y(x_0)$$

substituting $y^{(n)}(x), y^{(n-1)}(x), \cdots y(x)$ in (2.1), we shall satisfy all at the collocation points. In this way we get a system of linear equations to seek the wavelet coefficients a_i.

Example 2.1. [6] Consider the second-order homogeneous IVP :

$$y''(x) + \frac{1}{4}y(x) = 0, \quad 0 < x < 1$$
$$y(0) = 1, \quad y'(0) = 0.$$

with the exact solution

$$y(x) = \cos\left(\frac{x}{2}\right)$$

Table (2.1) shows the exact and the approximate solutions at different nodes.

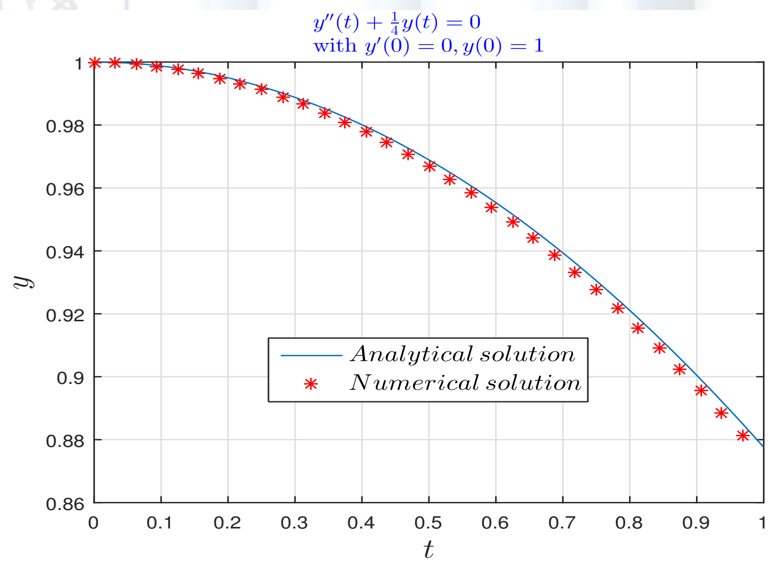

Figure 2.1: Graphical comparison of numerical and exact solutions of the example (2.1) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0000	1.0000	1.0000	1.525856×10^{-5}
0.1094	0.9985	0.9983	2.287328×10^{-4}
0.2344	0.9931	0.9927	4.717916×10^{-4}
0.3594	0.9839	0.9832	7.129686×10^{-4}
0.4844	0.9708	0.9699	9.513225×10^{-4}
0.6094	0.9539	0.9528	1.185923×10^{-3}
0.7344	0.9333	0.9319	1.415855×10^{-3}
0.8594	0.9091	0.9075	1.640220×10^{-3}
0.9844	0.8813	0.8794	1.858145×10^{-3}

Table 2.1: Comparison of exact and approximate solutions with $2M = 64$ of the example (2.1) at some points

Example 2.2. [6] Consider the second-order homogeneous IVP :

$$y''(x) + y(x) = \sin(x) + x\cos(x), 0 < x < 1$$
$$y(0) = 1, y'(0) = 1.$$

with the exact solution

$$y(x) = \cos(x) + \frac{5}{4}\sin(x) + \frac{1}{4}\left(x^2\sin(x) - x\cos(x)\right)$$

Table (2.2) shows the exact and the approximate solutions at different nodes.

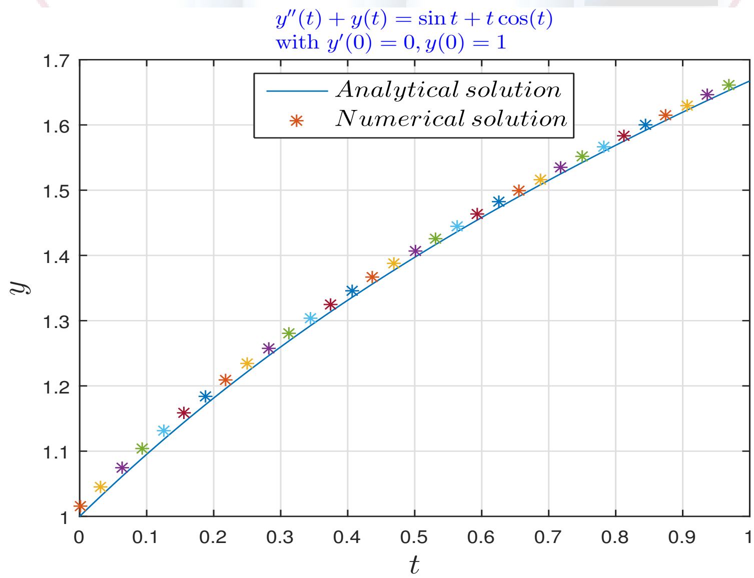

Figure 2.2: Graphical comparison of numerical and exact solutions of the example (2.2) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0000	1.0000	1.0078	7.751945×10^{-3}
0.1094	1.1036	1.1106	6.955127×10^{-3}
0.2344	1.2092	1.2153	6.168356×10^{-3}
0.3594	1.3030	1.3085	5.514662×10^{-3}
0.4844	1.3872	1.3922	4.985999×10^{-3}
0.6094	1.4637	1.4682	4.565548×10^{-3}
0.7344	1.5340	1.5382	4.22857×10^{-3}
0.8594	1.5993	1.6032	3.943556×10^{-3}
0.9844	1.6601	1.6638	3.673555×10^{-3}

Table 2.2: Comparison of exact and approximate solutions with $2M = 64$ of the example (2.2) at some points

Example 2.3. [6] Consider the 4^{th} -order of IVP :

$$y^{(4)}(x) + xy(x) = 16\sin(2x) + x\sin(2x), \quad 0 < x < 1.$$

$$y(0) = 0, y'(0) = 2, y''(0) = 0, y''(0) = -8,$$

with the exact solution

$$y(x) = \sin(2x)$$

Table (2.3) shows the exact and the approximate solutions at different nodes.

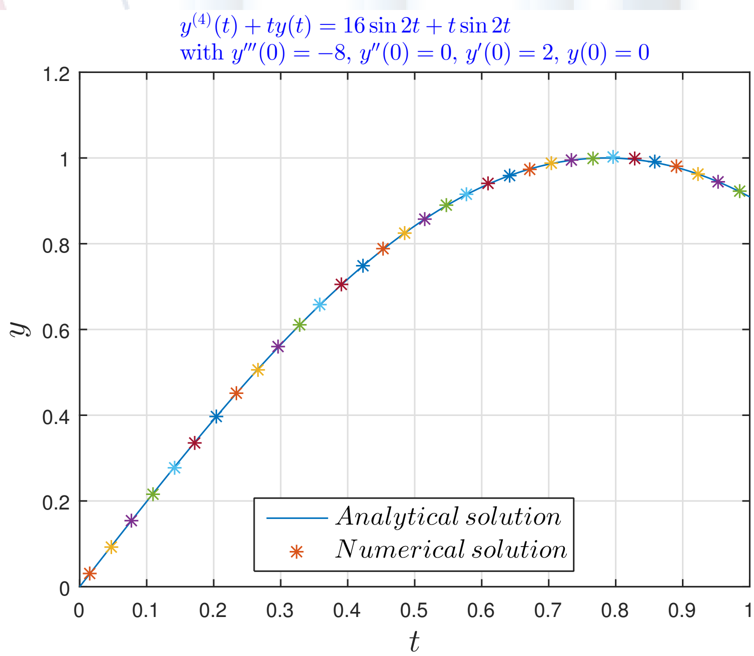

Figure 2.3: Graphical comparison of numerical and exact solutions of the example (2.3) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0000	0.0000	0.0156	1.562437×10^{-2}
0.1094	0.2170	0.2322	1.522633×10^{-2}
0.2344	0.4518	0.4657	1.389076×10^{-2}
0.3594	0.6584	0.6701	1.17026×10^{-2}
0.4844	0.8242	0.8330	8.803749×10^{-3}
0.6094	0.9387	0.9440	5.380338×10^{-3}
0.7344	0.9948	0.9964	1.651303×10^{-3}
0.8594	0.9891	0.9869	2.145249×10^{-3}
0.9844	0.9219	0.9161	5.766817×10^{-3}

Table 2.3: Comparison of exact and approximate solutions with $2M = 64$ of the example (2.3) at some points

Example 2.4. [13] Consider the second-order homogeneous IVP :

$$y''(x) + y(x)' + y(x) = \sin x \tag{2.4}$$

with boundary conditions

$$y(0) = 0, y(0.2) = 1$$

The exact solution is given by

$$y(x) = \frac{\exp(-0.5x)\sin(0.5\sqrt{3}x)(\cos(0.2)+1)}{\exp(-0.5x)(\sqrt{3}\cos(0.2\sqrt{3})+\sin(0.2\sqrt{3}))} + \frac{\exp(-0.5x)\cos(0.5\sqrt{3}x)(\cos(0.2)+1)}{\exp(-0.5x)(\sqrt{3}\cos(0.2\sqrt{3})+\sin(0.2\sqrt{3}))} - \cos(x)$$

Using the method mentioned we get the result in table and the figure (2.4)

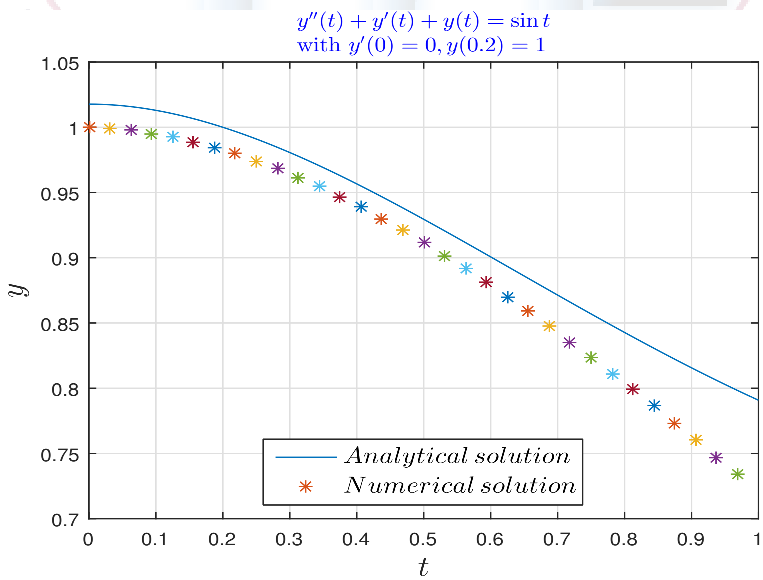

Figure 2.4: Graphical comparison of numerical and exact solutions of the example (2.4) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0000	1.0177	1.0000	1.778472×10^{-2}
0.1094	1.0121	0.9947	1.738966×10^{-2}
0.2344	0.9940	0.9784	1.559971×10^{-2}
0.3594	0.9668	0.9525	1.429762×10^{-2}
0.4844	0.9339	0.9187	1.519449×10^{-2}
0.6094	0.8979	0.8782	1.968740×10^{-2}
0.7344	0.8614	0.8326	2.884983×10^{-2}
0.8594	0.8264	0.7830	4.342950×10^{-2}
0.9844	0.7945	0.7307	6.385343×10^{-2}

Table 2.4: Comparison of exact and approximate solutions with $2M = 64$ of the example (2.4) at some points

2.2 Numerical solutions for linear fractional differential equations using Haar wavelet method

From [9], we consider a non-homogeneous multi-term fractional differential equation

$$c_1 D^{\alpha} y(x) + c_2 D^{\beta} y(x) + c_3 y(x) = g(x), \quad x \in [0,1], \quad 0 < \beta < \alpha \leq 2 \tag{2.5}$$

With $y(0) = y_0, y'(0) = y_1$ where $c_1 \neq 0$ and $c_2, c_3 \in R$.

2.2.1 Method of solution and numerical examples

we shall expand the highest derivative in this equation into the Haar series

$$D^{(\alpha)} y(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

Using the property 1.18, we calculate the lower derivatives and the $y(x)$ function with the initial conditions, so we have

$$D^{\beta} y(t) = \left(I^{\alpha-\beta} D^{\alpha} y\right)(t) = A.P^{\alpha-\beta}.H$$

$$y(t) = A.P^{\alpha}.H + y_1 x + y_0$$

Substituting these results in the differential equation at the collocation points, we obtain an algebraic system of equations to evaluate the Haar wavelet coefficients a_i.

Example 2.5. Consider the homogeneous FDE:

$$D^{0.5}y(x) = x^2, \quad x \in [0,1] \tag{2.6}$$

with the initial conditions

$$y(0) = 0$$

The exact solution is given by

$$y(x) = \frac{16}{15\sqrt{\pi}} x^{\frac{5}{2}}$$

the result obtained are shown in table and figure (2.5)

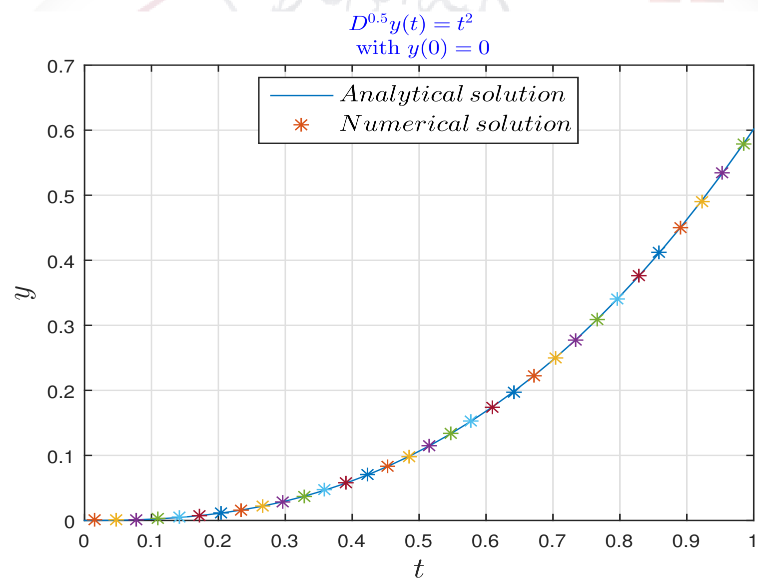

Figure 2.5: Graphical comparison of numerical and exact solutions of the example (2.5) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0000	0.0000	0.0000	5.739233×10^{-6}
0.1094	0.0024	0.0028	4.627560×10^{-4}
0.2344	0.0160	0.0174	1.388627×10^{-3}
0.3594	0.0466	0.0492	2.600331×10^{-3}
0.4844	0.0983	0.1023	4.041484×10^{-3}
0.6094	0.1744	0.1801	5.680051×10^{-3}
0.7344	0.2781	0.2856	7.494594×10^{-3}
0.8594	0.4120	0.4215	9.469452×10^{-3}
0.9844	0.5786	0.5902	1.159253×10^{-2}

Table 2.5: Comparison of exact and approximate solutions with $2M = 64$ of the example (2.5) at some points

Example 2.6. [4] Consider the Bagley-Torvik equation

$$D^2 y(x) + D^{1.5} y(x) + y(x) = 1 + x, x \in [0,1], \tag{2.7}$$

subject to the initial conditions

$$y(0) = 1, y'(0) = 1$$

This equation has the following exact solution :

$$y(x) = x + 1.$$

The maximum absolute errors are given in the table (2.6)

$2M$	max Error
4	1.250000×10^{-1}
8	6.250000×10^{-2}
16	3.125000×10^{-2}
32	1.562500×10^{-2}
64	7.812500×10^{-3}

Table 2.6: Comparison of the maximum absolute Error for $2M = \{4, 8, 16, 32, 64\}$.

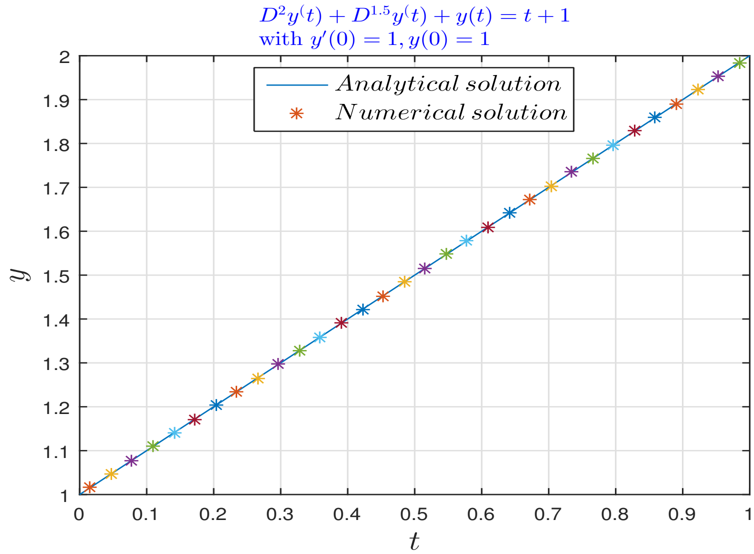

Figure 2.6: Graphical comparison of numerical and exact solutions of the example (2.6) with $2M = 32$

Example 2.7. [15] Consider the linear fractional differential equation :

$$D^{\alpha}y(x) + y(x) = \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha} + x^3, \quad x \in [0,1], \quad \alpha \in \,]1,2] \tag{2.8}$$

with initial conditions: $y(0) = 0, y'(0) = 0.$

The exact solution with $\alpha = 1.9$ is:

$$y(x) = x^2$$

Using the method mentioned above, we get the results in the table and figure(2.7)

x_l	Exact solution	Haar solution	Error
0.0000	1.0177	1.0000	1.778472×10^{-2}
0.1094	1.0121	0.9947	1.738966×10^{-2}
0.2344	0.9940	0.9784	1.559971×10^{-2}
0.3594	0.9668	0.9525	1.429762×10^{-2}
0.4844	0.9339	0.9187	1.519449×10^{-2}
0.6094	0.8979	0.8782	1.968740×10^{-2}
0.7344	0.8614	0.8326	2.884983×10^{-2}
0.8594	0.8264	0.7830	4.342950×10^{-2}
0.9844	0.7945	0.7307	6.385343×10^{-2}

Table 2.7: Comparison of exact and approximate solutions with $2M = 64$ of the example (2.7) at some points

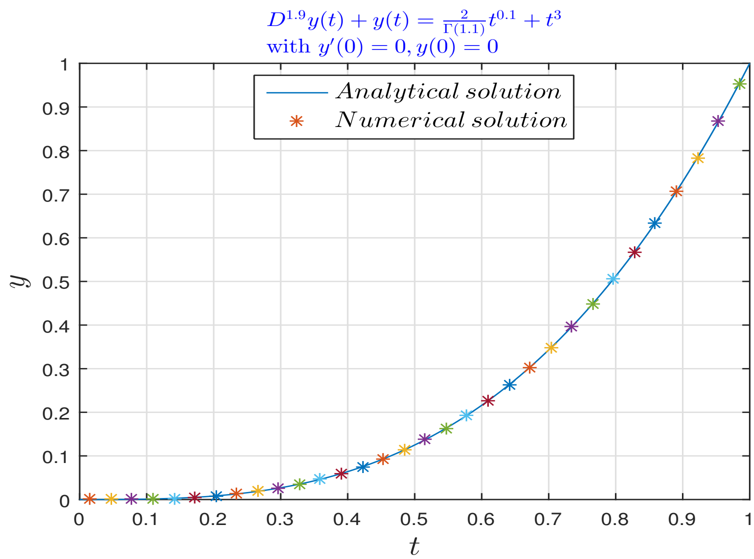

Figure 2.7: Graphical comparison of numerical and exact solutions of the example (2.7) with $2M = 32$

Example 2.8. Let in (2.5) $c_1 = c_2 = 1, c_3 = 0, \alpha = 2, 0 \leq \beta \leq 1, y_0 = y_1 = 0$ and $g(t) = 6x^3\left(\frac{x^{-\alpha}}{\Gamma(4-\alpha)} + \frac{x^{-\beta}}{\Gamma(4-\beta)}\right)$, with that the exact solution in this case is $y(x) = x^3$.
The maximum absolute errors by the proposed method are given in table (2.8).
For example if $\alpha = 2$ and $\beta = 0.5$, we obtain the figure (2.8)

β	$2M$=8	$2M$ =16	$2M$=32	$2M$=64
0	1.264415×10^{-1}	4.675513×10^{-2}	1.475371×10^{-2}	2.288099×10^{-2}
0.1	1.264499×10^{-1}	4.685050×10^{-2}	1.478162×10^{-2}	2.264783×10^{-2}
0.2	1.264840×10^{-1}	4.697195×10^{-2}	1.481724×10^{-2}	2.236719×10^{-2}
0.3	1.265521×10^{-1}	4.712529×10^{-2}	1.486248×10^{-2}	2.203074×10^{-2}
0.4	1.266638×10^{-1}	4.731719×10^{-2}	1.491959×10^{-2}	2.162926×10^{-2}
0.5	1.268303×10^{-1}	4.755521×10^{-2}	1.499126×10^{-2}	2.115269×10^{-2}
0.6	1.270639×10^{-1}	4.789571×10^{-2}	1.508058×10^{-2}	2.059038×10^{-2}
0.7	1.273778×10^{-1}	4.843322×10^{-2}	1.519104×10^{-2}	1.993144×10^{-2}
0.8	1.277855×10^{-1}	4.906978×10^{-2}	1.532647×10^{-2}	1.916522×10^{-2}
0.9	1.283003×10^{-1}	4.981553×10^{-2}	1.551089×10^{-2}	1.828198×10^{-2}
1	1.289347×10^{-1}	5.067919×10^{-2}	1.576009×10^{-2}	1.727358×10^{-2}

Table 2.8: Maximum absolute errors for $2M = \{8, 16, 32, 64\}$ and different values of β

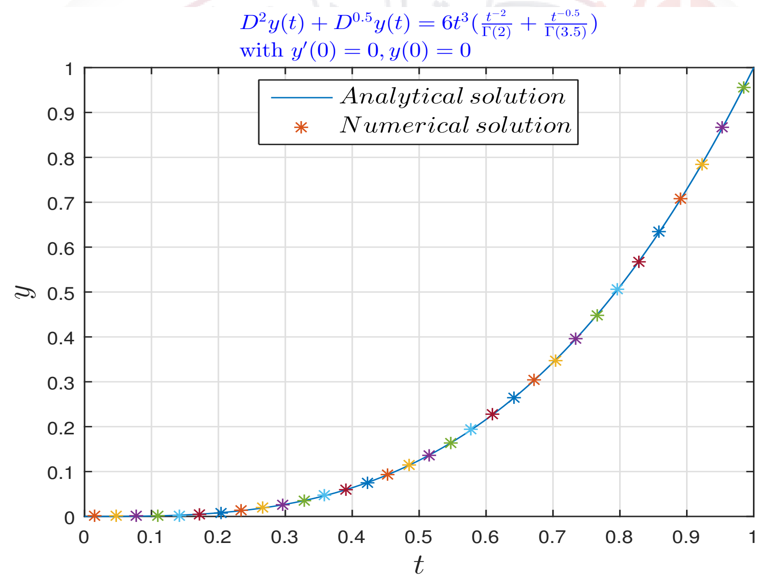

Figure 2.8: Graphical comparison of numerical and exact solutions of the example (2.8) with $2M = 32$

Chapter 3

Numerical solutions for linear integral equation using Haar wavelet method

In this chapter we use the Haar wavelets method to solve linear Volterra and Fredholm integral equation and we use the same method to solve fractional Volterra integral equation.

3.1 Numerical solutions for linear Volterra integral equation using Haar wavelet method

The Volterra integral equation of the second kind has the form

$$y(x)-\int_0^x K(x,t)y(t)dt=f(x), \quad 0\leq x\leq 1 \tag{3.1}$$

3.1.1 Method of solution and numerical examples

We seek the solution in the form

$$y(x)=\sum_{i=1}^{2M} a_i h_i(x) \tag{3.2}$$

The discrete form of (3.1) is

$$y\left(x_l\right)-\int_0^{x_l} K\left(x_l,t\right)y(t)dt=f\left(x_l\right) \tag{3.3}$$

where

$$x_l=\frac{l-0.5}{2M}, \qquad l=1,2,\ldots,2M$$

are the collocation points.

The matrix $G_{il}=G_i\left(x_l\right)$ is now defined as

$$G_{il}=\int_0^{t_l} K\left(x_l,t\right)h_i(t)dt. \tag{3.4}$$

By computing these integrals the following cases should be distinguished:

$$G_{il} = \begin{cases} 0 & x_l < \xi_1, \\ \int_{\xi_1}^{x_l} K(x_l, t)\, dt & \xi_1 \leq x_l \leq \xi_2, \\ \int_{\xi_1}^{\xi_2} K(x_l, t)\, dt - \int_{\xi_2}^{x_l} K(x_l, t)\, dt & \xi_2 \leq x_l \leq \xi_3, \\ \int_{\xi_1}^{\xi_2} K(x_l, t)\, dt - \int_{\xi_2}^{\xi_3} K(x_l, t)\, dt & \xi_3 \leq x_l \leq 1. \end{cases} \tag{3.5}$$

We calculate the wavelet coefficients a_i satisfying (3.2) only at the collocation points, we get a system of linear equations

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) - G_i(x_l)\right] = f(x_l), l = 1, 2, \ldots, 2M \tag{3.6}$$

The matrix form of this system is

$$A(H - G_{il}) = F_l$$

where $G_{il} = G_i(x_l), F_l = f(x_l)$.
so

$$A = F_l (H - G_{il})^{-1}$$

Example 3.1. [16] We consider the following Volterra integral equation

$$y(x) = \frac{3}{4}e^{2x} + x^2 + \frac{1}{2}x - \frac{7}{4} + \int_0^x y(t)(x-t)dt, \tag{3.7}$$

with the exact solution $y(x) = e^{2x} - 2$.

The results of the method are show in table (3.1) and figure (3.1)

x_l	Exact solution	Haar solution	err
0.0313	-0.9355	-0.9346	9.342814×10^{-4}
0.1563	-0.6332	-0.6117	2.149844×10^{-2}
0.2813	-0.2449	-0.1837	6.124779×10^{-2}
0.4063	0.2535	0.3609	1.073809×10^{-1}
0.5313	0.8936	1.0371	1.435369×10^{-1}
0.6563	1.7155	1.8645	1.490683×10^{-1}
0.7813	2.7707	2.8688	9.810095×10^{-2}
0.9063	4.1257	4.0841	4.168197×10^{-2}

Table 3.1: Comparison between the exact and Haar with $2M = 16$ of the example (3.1) at some points

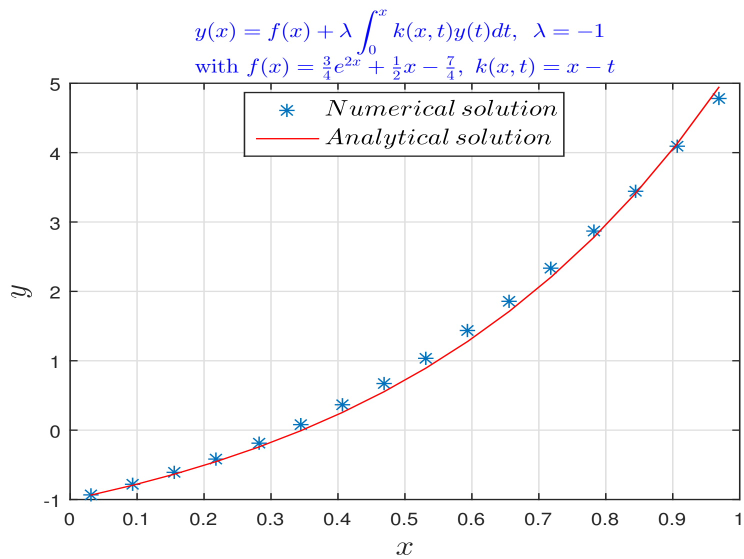

Figure 3.1: Graphical comparison of numerical and exact solutions of the example (3.1) with $2M = 16$

Example 3.2. [16] We consider the following Volterra integral equation

$$y(x) = 2\cosh(x) - \sinh(x) - 2x\sinh(x) + x^2 + \frac{1}{3}x^3 - \frac{3}{2} + \int_0^x y(t)\left(x^2 - t^2\right)dt, \tag{3.8}$$

with the exact solution

$$y(x) = \frac{1}{2} - \sinh(x)$$

The comparison between the exact solution and the Haar solution are shown in table (3.2) and figure (3.2)

x_l	Exact solution	Haar solution	Error
0.0313	0.4687	0.4687	3.974468×10^{-7}
0.1563	0.3431	0.3431	8.049367×10^{-6}
0.2813	0.2150	0.2151	2.601445×10^{-5}
0.4063	0.0825	0.0825	5.445014×10^{-5}
0.5313	-0.0566	-0.0565	9.340539×10^{-5}
0.6563	-0.2044	-0.2042	1.426743×10^{-4}
0.7813	-0.3632	-0.3630	2.016618×10^{-4}
0.9063	-0.5355	-0.5352	2.692711×10^{-4}

Table 3.2: Comparison between the exact and Haar with $2M = 16$ of the example (3.2) at some points

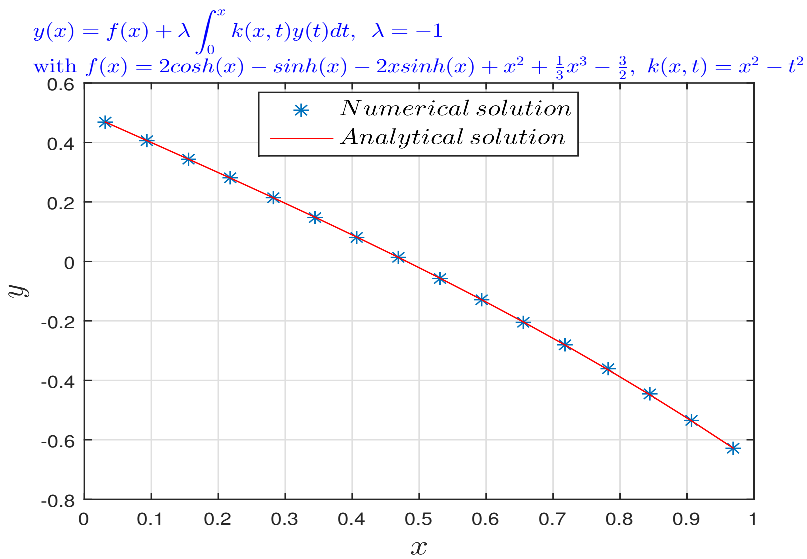

Figure 3.2: Graphical comparison of numerical and exact solutions of the example (3.2) with $2M = 16$

Example 3.3. [7] Consider the VIE of the second kind :

$$y(x) = 1 - x + \frac{x^2}{2} + \int_0^x (t - x)y(t)dt, \quad 0 \leq x \leq 1 \tag{3.9}$$

With exact solution

$$y(x) = 1 - \sin(x)$$

The results by the Haar wavelet method are presented in table and figure (3.3)

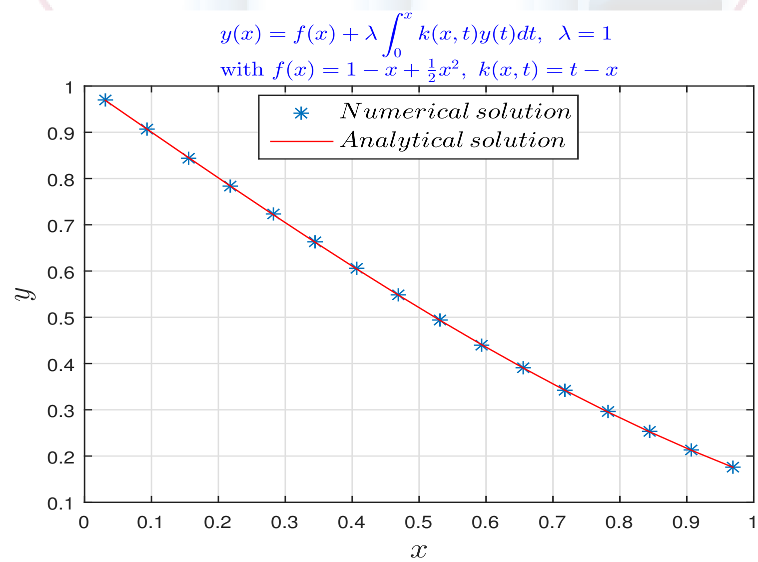

Figure 3.3: Graphical comparison of numerical and exact solutions of the example (3.3) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	0.9688	0.9688	1.016533×10^{-5}
0.1563	0.8444	0.8444	5.052930×10^{-5}
0.2813	0.7224	0.7225	8.971040×10^{-5}
0.4063	0.6048	0.6050	1.267882×10^{-4}
0.5313	0.4934	0.4935	1.608860×10^{-4}
0.6563	0.3898	0.3900	1.911894×10^{-4}
0.7813	0.2958	0.2960	2.169630×10^{-4}
0.9063	0.2128	0.2130	2.375663×10^{-4}

Table 3.3: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.3) at some points

Example 3.4. [7] Consider the VIE of the second kind

$$y(x) = x3^x + \int_0^x -3^{x-t} y(t)dt, \quad 0 \le x \le 1 \tag{3.10}$$

with exact solution

$$y(x) = 3^x \left(1 - e^{-x}\right)$$

The results obtained are shown in table (3.4) and figure (3.4)

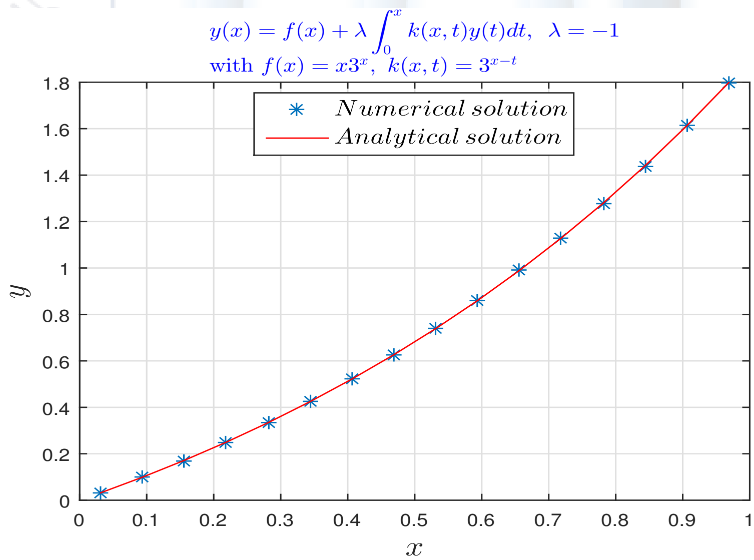

Figure 3.4: Graphical comparison of numerical and exact solutions of the example (3.4) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	0.0318	0.0313	4.964256×10^{-4}
0.1563	0.1717	0.1712	5.318214×10^{-4}
0.2813	0.3339	0.3333	5.717126×10^{-4}
0.4063	0.5217	0.5210	6.167490×10^{-4}
0.5313	0.7388	0.7381	6.676759×10^{-4}
0.6563	0.9896	0.9888	7.253480×10^{-4}
0.7813	1.2790	1.2782	7.907458×10^{-4}
0.9063	1.6129	1.6120	8.649937×10^{-4}

Table 3.4: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.4) at some points

3.2 Numerical solutions for linear Fredholm integral equation using Haar wavelet method

The linear Fredholm integral equation of the second kind has the form

$$y(x) = \int_A^B K(x,t)y(t)dt + f(x), \quad x \in [A,B] \tag{3.11}$$

3.2.1 Method of solution and numerical examples

The solution is sought in the form

$$y(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{3.12}$$

Replacing (3.12) into (3.11) we obtain

$$\sum_{i=1}^{2M} a_i h_i(x) - \sum_{i=1}^{2M} a_i G_i(x) = f(x) \tag{3.13}$$

where

$$G_i(x) = \int_A^B K(x,t)h_i(t)dt. \tag{3.14}$$

We calculate the wavelet coefficients a_i, satisfying (3.13) only at the collocation points, we get a system of linear equations

$$\sum_{i=1}^{2M} a_i \left[h_i\left(x_l\right) - G_i\left(x_l\right)\right] = f\left(x_l\right), l = 1, 2, \ldots, 2M \tag{3.15}$$

The matrix form of this system is

$$A(H - G_{il}) = F_l$$

where

$$G_{il} = G_i\left(x_l\right)$$

$$F_l = f\left(x_l\right)$$

so

$$A = F_l(H - G_{il})^{-1}$$

Example 3.5. [12] Consider Fredholm integral equation of the second kind ,

$$y(x) = x^6 \log(x) + \int_0^1 (x+t)y(t)dt, \quad 0 \leq x \leq 1 \tag{3.16}$$

which has the exact solution

$$y(x) = x^6 \log(x) + 0.3096x + 0.1752$$

.

The results are shown in table (3.5) and figure (3.5)

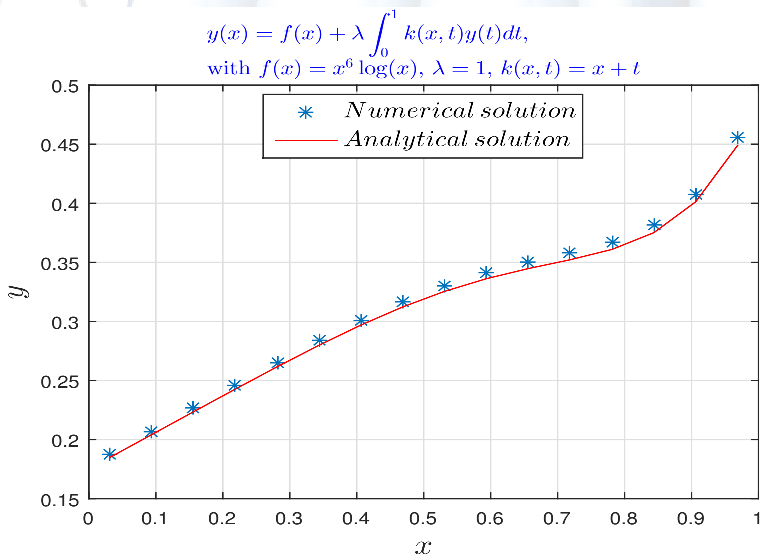

Figure 3.5: Graphical comparison of numerical and exact solutions of the example (3.5) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	0.1849	0.1874	2.546126×10^{-3}
0.1563	0.2235	0.2267	3.105320×10^{-3}
0.2813	0.2616	0.2653	3.664513×10^{-3}
0.4063	0.2969	0.3011	4.223707×10^{-3}
0.5313	0.3255	0.3302	4.782900×10^{-3}
0.6563	0.3447	0.3501	5.342094×10^{-3}
0.7813	0.3609	0.3668	5.901288×10^{-3}
0.9063	0.4012	0.4077	6.460481×10^{-3}

Table 3.5: Comparison between the exact and Haar with $2M = 16$ of the example (3.5) at some points

Example 3.6. [8] Consider Fredholm integral equation

$$y(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 xty(t)dt \tag{3.17}$$

The exact solution is

$$y(x) = x$$

After applying the method, we get the results obtained in figure and table 3.6

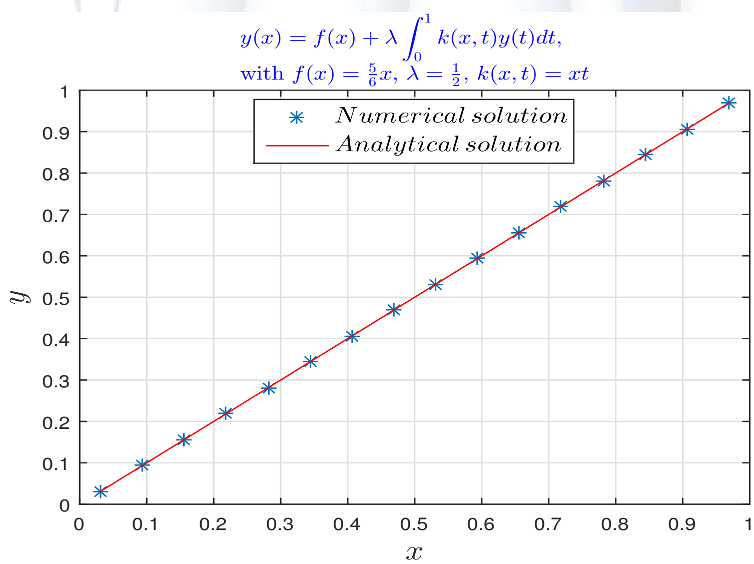

Figure 3.6: Graphical comparison of numerical and exact solutions of the example (3.6) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	0.0313	0.0312	6.102324×10^{-6}
0.1563	0.1563	0.1562	3.051162×10^{-5}
0.2813	0.2813	0.2812	5.492091×10^{-5}
0.4063	0.4063	0.4062	7.933021×10^{-5}
0.5313	0.5313	0.5311	1.037395×10^{-4}
0.6563	0.6563	0.6561	1.281488×10^{-4}
0.7813	0.7813	0.7811	1.525581×10^{-4}
0.9063	0.9063	0.9061	1.769674×10^{-4}

Table 3.6: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.6) at some points

Example 3.7. Consider Fredholm integral equation

$$y(x) = x + \int_0^1 (x^4 - t^4)y(t)dt, \tag{3.18}$$

which has the exact solution

$$y(x) = x$$

Using the method mentioned above, we get the results in table and figure (3.7)

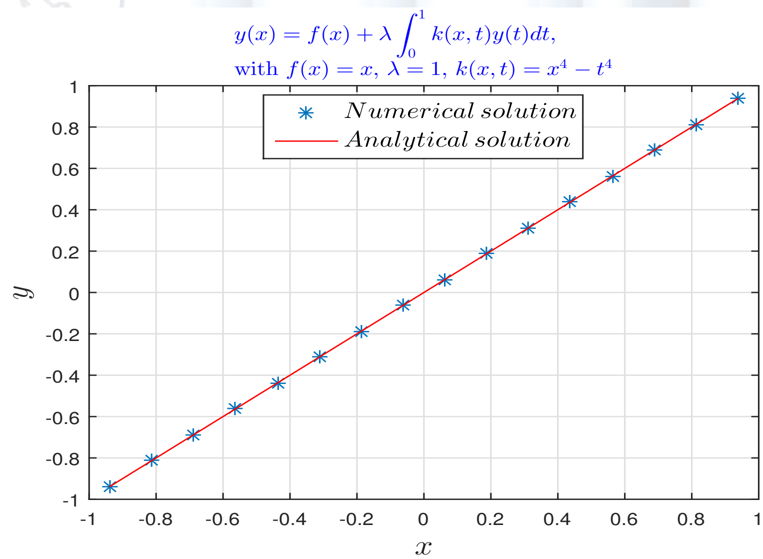

Figure 3.7: Graphical comparison of numerical and exact solutions of the example (3.7) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	0.1951	0.1951	5.551115×10^{-17}
0.1563	0.8315	0.8315	1.110223×10^{-16}
0.2813	0.9808	0.9808	2.220446×10^{-16}
0.4063	0.5556	0.5556	1.110223×10^{-16}
0.5313	-0.1951	-0.1951	1.110223×10^{-16}
0.6563	-0.8315	-0.8315	1.110223×10^{-16}
0.7813	-0.9808	-0.9808	$0.000000\times10^{+00}$
0.9063	-0.5556	-0.5556	2.220446×10^{-16}

Table 3.7: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.7) at some points

Example 3.8. Consider the Fredholm integral equation of the second kind :

$$y(x) = \sin(2\pi x) + \int_0^1 (x^2 - x - t^2 + t)y(t)dt, \tag{3.19}$$

The exact solution is

$$y(x) = \sin(2\pi x)$$

After applying the method, we get the results (see figure and table 3.8)

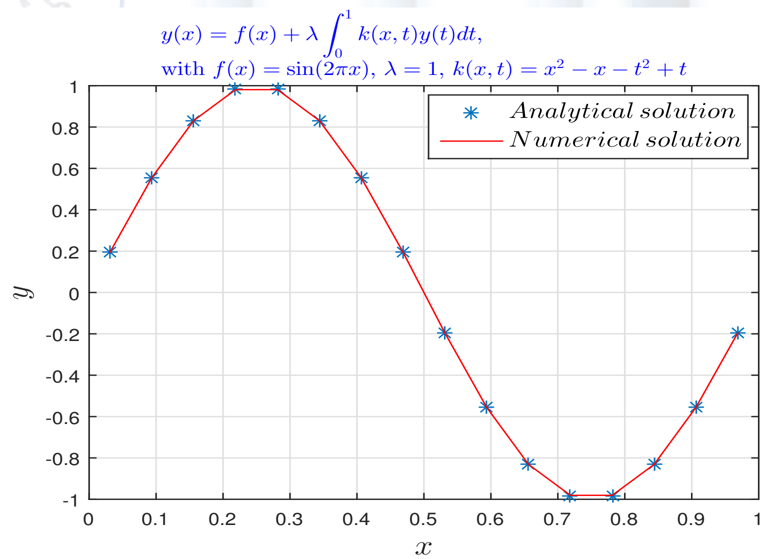

Figure 3.8: Graphical comparison of numerical and exact solutions of the example (3.8) with $2M = 16$

x_l	Exact solution	Haar solution	Error
-0.9375	-0.9375	-0.9375	$0.000000\times10^{+00}$
-0.6875	-0.6875	-0.6875	$0.000000\times10^{+00}$
-0.4375	-0.4375	-0.4375	$0.000000\times10^{+00}$
-0.1875	-0.1875	-0.1875	5.551115×10^{-17}
0.0625	0.0625	0.0625	5.551115×10^{-17}
0.3125	0.3125	0.3125	2.775558×10^{-16}
0.5625	0.5625	0.5625	2.220446×10^{-16}
0.8125	0.8125	0.8125	2.220446×10^{-16}

Table 3.8: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.8) at some points

Example 3.9. [12] Consider the Fredholm integral equation of the second kind :

$$y(x) = e^x + (1-e)x - 1 + \int_0^1 (x+t)y(t)dt, \quad 0 \le x \le 1 \tag{3.20}$$

which has the exact solution

$$y(x) = e^x.$$

The result obtained are shown in table and figure (3.9)

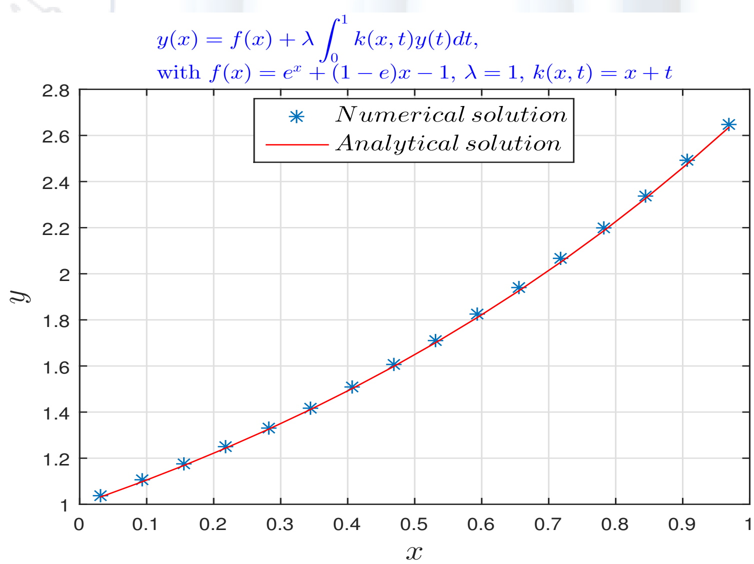

Figure 3.9: Graphical comparison of numerical and exact solutions of the example (3.9) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	1.0317	1.0375	5.794959×10^{-3}
0.1563	1.1691	1.1762	7.092682×10^{-3}
0.2813	1.3248	1.3332	8.390405×10^{-3}
0.4063	1.5012	1.5109	9.688128×10^{-3}
0.5313	1.7011	1.7120	1.098585×10^{-2}
0.6563	1.9276	1.9398	1.228357×10^{-2}
0.7813	2.1842	2.1978	1.358130×10^{-2}
0.9063	2.4750	2.4899	1.487902×10^{-2}

Table 3.9: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.9) at some points

Example 3.10. [12] Consider the Fredholm integral equation of the second kind ,

$$y(x) = e^{2x+\frac{1}{3}} + \int_0^1 \frac{-1}{3} e^{2x-\frac{5}{3}t} y(t)dt, \quad 0 \leq x \leq 1 \tag{3.21}$$

which has the exact solution

$$y(x) = e^{2x}.$$

Using the method mentioned above, we get the results in the table and figure (3.10)

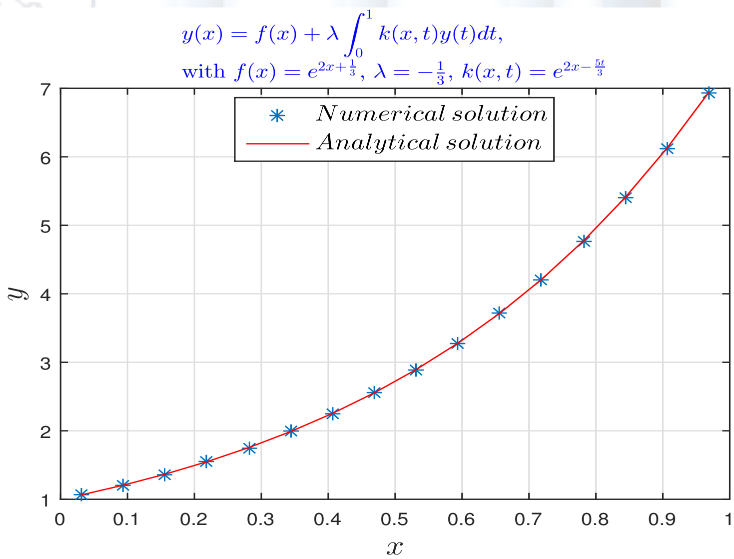

Figure 3.10: Graphical comparison of numerical and exact solutions of the example (3.10) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	1.0645	1.0644	1.309682×10^{-4}
0.1563	1.3668	1.3667	1.681666×10^{-4}
0.2813	1.7551	1.7548	2.159301×10^{-4}
0.4063	2.2535	2.2533	2.772598×10^{-4}
0.5313	2.8936	2.8932	$3.560086e \times 10^{-4}$
0.6563	3.7155	3.7150	$4.571241e \times 10^{-4}$
0.7813	4.7707	4.7701	$5.869589e \times 10^{-4}$
0.9063	6.1257	6.1250	$7.536702e \times 10^{-4}$

Table 3.10: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.10) at some points

Example 3.11. [12] Consider the Fredholm integral equation of the second kind

$$y(x) = e^x - \frac{e^{x+1}-1}{x+1} + \int_0^1 e^{xt} y(t)dt, \quad 0 \le x \le 1. \tag{3.22}$$

which has the exact solution

$$y(x) = e^x.$$

Table and figure (3.11) shows the exact and approximate solutions at different nodes.

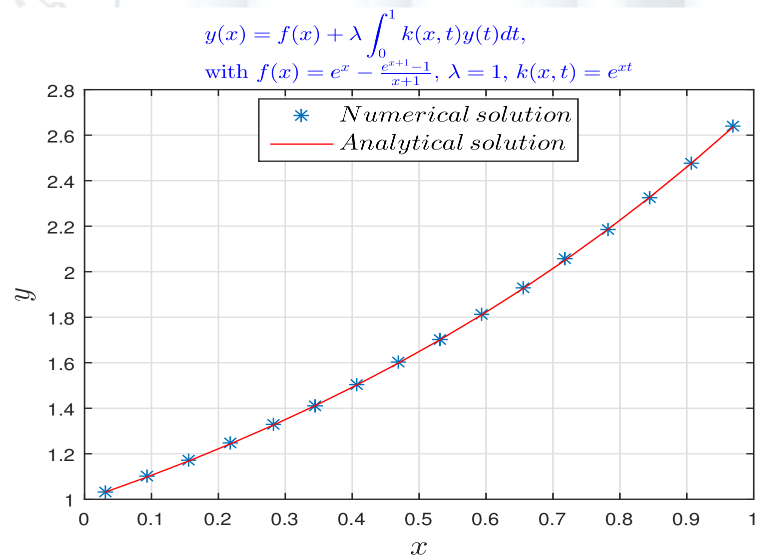

Figure 3.11: Graphical comparison of numerical and exact solutions of the example (3.11) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	1.0317	1.0339	2.180208×10^{-3}
0.1563	1.1691	1.1714	2.249434×10^{-3}
0.2813	1.3248	1.3271	2.319709×10^{-3}
0.4063	1.5012	1.5036	2.390617×10^{-3}
0.5313	1.7011	1.7035	2.461643×10^{-3}
0.6563	1.9276	1.9301	2.532153×10^{-3}
0.7813	2.1842	2.1868	2.601379×10^{-3}
0.9063	2.4750	2.4777	2.668388×10^{-3}

Table 3.11: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.11) at some points

3.3 Numerical solutions for linear fractional Volterra integral equation using Haar wavelet method

The fractional Volterra integral equation has the form

$$y(x) = f(x) + \frac{1}{\Gamma(\alpha)} \int_0^x K(x,t)(x-t)^{\alpha-1} y(t)dt, \quad 0 \leqslant x \leqslant 1. \tag{3.23}$$

The kernel $K(x,t)$ and the right-side function $f(x)$ are given, $\alpha > 0$ is a real number. The value $\alpha = 1$ corresponds to the ordinary (non fractional) Volterra equation(see section 3.1). According to the Haar wavelet method, the solution of (3.23) is sought in the form

$$y(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{3.24}$$

Replacing (3.24) into (3.23) and satisfying this equation in the collocation points we obtain

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) - G_i(x_l)\right] = f(x_l), \quad l = 1, 2, \ldots, 2M. \tag{3.25}$$

Here $G_i(x_l)$ denotes the function defined by

$$G_i(x_l) = \frac{1}{\Gamma(\alpha)} \int_0^{x_l} K(x_l, t)(x_l - t)^{\alpha-1} h_i(t)dt. \tag{3.26}$$

The matrix form of (3.26) is

$$A(H - G_{il}) = F_l,$$

where

$G_{il} = G(i,l) = G_i(x_l)$,

$F_l = f(x_l)$.

finally

$$A = F_l (H - G_{il})^{-1}$$

G_{il} can be written as :

$$G(i,l)=\begin{cases}0 & x_l<\xi_1,\\ \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{x_l}K(x_l,t)(x_l-t)^{\alpha-1}dt & x_l\in[\xi_1,\xi_2],\\ \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{\xi_2}K(x_l,t)(x_l-t)^{\alpha-1}dt-\frac{1}{\Gamma(\alpha)}\int_{\bar{\xi}_2}^{x}K(x_l,t)(x_l-t)^{\alpha-1}dt & x_l\in[\xi_2,\xi_3],\\ \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{\xi_2}K(x_l,t)(x_l-t)^{\alpha-1}dt-\frac{1}{\Gamma(\alpha)}\int_{\xi_2}^{\xi_3}K(x_l,t)(x_l-t)^{\alpha-1}dt & x_l\geqslant\xi_3.\end{cases} \tag{3.27}$$

Example 3.12. [11] Consider the linear Volterra fractional integral equation of the second kind :

$$y(x)=x^{\frac{1}{2}}-\frac{3}{4}\sqrt{\pi}x^2+\frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}}-1+\frac{1}{\sqrt{\pi}}\int_0^x(x-t)^{-\frac{1}{2}}2ty(t)dt \tag{3.28}$$

The exact solution of equation is:

$$y(x)=x^{\frac{1}{2}}-1$$

Using the method mentioned above, we get the results in table and figure (3.12)

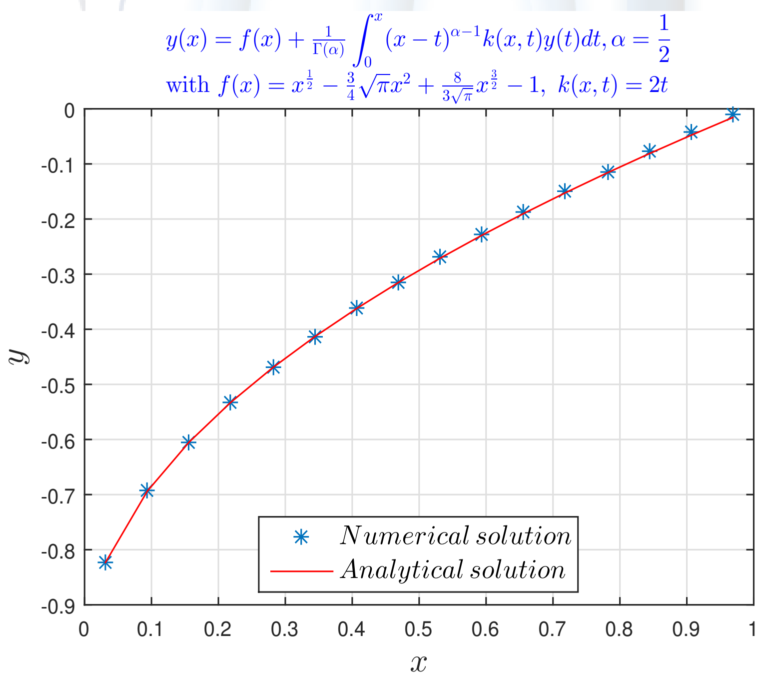

Figure 3.12: Graphical comparison of numerical and exact solutions of the example (3.12) with $2M=16$

x_l	Exact solution	Haar solution	Error
0.0313	-0.8232	-0.8231	1.724934×10^{-4}
0.1563	-0.6047	-0.6044	3.629455×10^{-4}
0.2813	-0.4697	-0.4691	5.731233×10^{-4}
0.4063	-0.3626	-0.3618	8.366148×10^{-4}
0.5313	-0.2711	-0.2699	1.210193×10^{-3}
0.6563	-0.1899	-0.1881	1.791478×10^{-3}
0.7813	-0.1161	-0.1133	2.777171×10^{-3}
0.9063	-0.0480	-0.0434	4.606692×10^{-3}

Table 3.12: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.12) at some points

Example 3.13. [11] Consider the linear Volterra fractional integral equation of the second kind :

$$y(x) = 2x^{\frac{3}{2}} - \frac{x}{2} - \frac{5\sqrt{\pi}}{8}x^{\frac{7}{2}} + \frac{8}{15\sqrt{\pi}}x^3 + \frac{1}{\sqrt{\pi}}\int_0^x (x-t)^{-\frac{1}{2}}x^{\frac{1}{2}}ty(t)dt \tag{3.29}$$

The exact solution is:

$$y(x) = 2x^{\frac{3}{2}} - \frac{x}{2}$$

Table and figure (3.13) shows the exact and the approximate solutions at different collocation points.

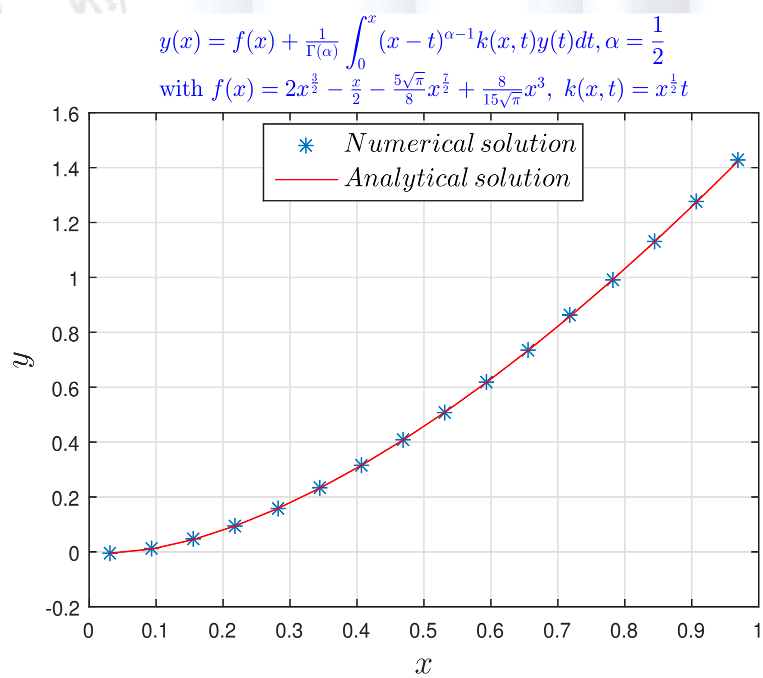

Figure 3.13: Graphical comparison of numerical and exact solutions of the example (3.13) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	-0.0046	-0.0046	1.555736×10^{-7}
0.1563	0.0454	0.0455	5.295896×10^{-5}
0.2813	0.1577	0.1579	2.034341×10^{-4}
0.4063	0.3147	0.3152	4.708469×10^{-4}
0.5313	0.5088	0.5097	8.876856×10^{-4}
0.6563	0.7351	0.7366	1.508177×10^{-3}
0.7813	0.9904	0.9929	2.423121×10^{-3}
0.9063	1.2723	1.2761	3.788175×10^{-3}

Table 3.13: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.13) at some points

Example 3.14. [11] Consider the linear Volterra fractional integral equation of the second kind :

$$y(x) = x - \frac{1}{3} - 0.2615x^{\frac{9}{4}} + 0.0981x^{\frac{5}{4}} + \frac{1}{\Gamma\left(\frac{1}{4}\right)} \int_0^x (x-t)^{-\frac{3}{4}} \frac{t}{3} y(t)dt \tag{3.30}$$

The exact solution is:

$$y(x) = x - \frac{1}{3}$$

Table and figure (3.14) shows the exact and the approximate solutions at different collocation points.

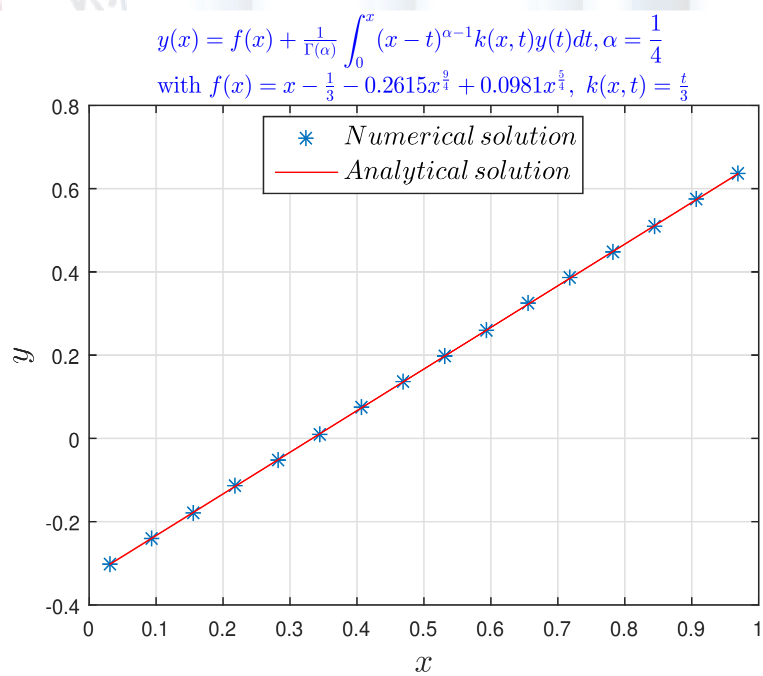

Figure 3.14: Graphical comparison of numerical and exact solutions of the example (3.14) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	-0.3021	-0.3021	1.390475×10^{-5}
0.1563	-0.1771	-0.1770	9.321778×10^{-5}
0.2813	-0.0521	-0.0519	1.776450×10^{-4}
0.4063	0.0729	0.0732	2.689830×10^{-4}
0.5313	0.1979	0.1983	3.685219×10^{-4}
0.6563	0.3229	0.3234	4.776981×10^{-4}
0.7813	0.4479	0.4485	5.981739×10^{-4}
0.9063	0.5729	0.5736	7.319106×10^{-4}

Table 3.14: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.14) at some points

Example 3.15. [11] Consider the Abel integral equation of the second kind :

$$y(x) = \frac{x^2}{3} - x - 0.2399x^{\frac{7}{3}} + 0.8399x^{\frac{4}{3}} + 0.3733 \int_0^x (x-t)^{-\frac{2}{3}} y(t)dt \tag{3.31}$$

The exact solution of this equation is given by:

$$y(x) = \frac{x^2}{3} - x$$

After applying the method, we get the results shown in figure and table3.15

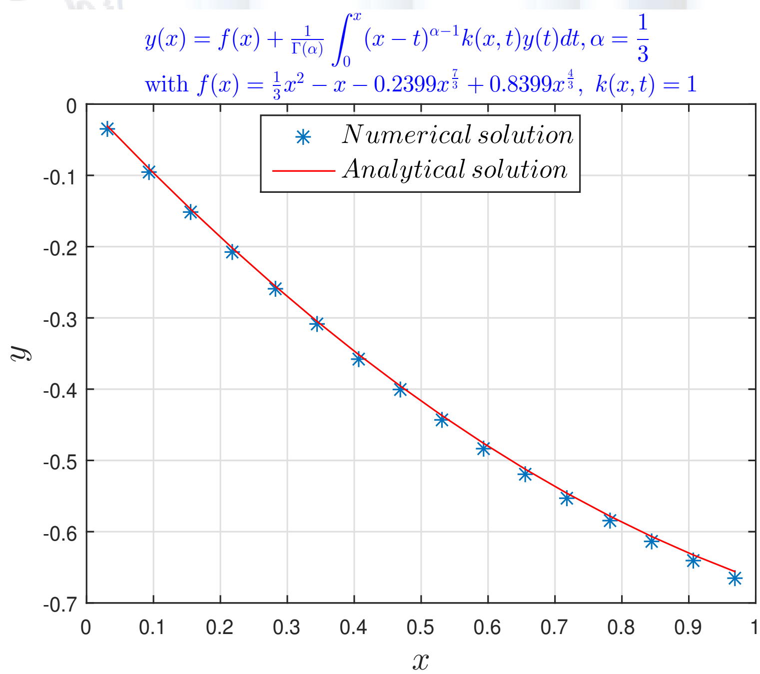

Figure 3.15: Graphical comparison of numerical and exact solutions of the example (3.15) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	-0.0309	-0.0351	4.193800×10^{-3}
0.1563	-0.1481	-0.1525	4.401349×10^{-3}
0.2813	-0.2549	-0.2598	4.953877×10^{-3}
0.4063	-0.3512	-0.3568	5.539212×10^{-3}
0.5313	-0.4372	-0.4433	6.149383×10^{-3}
0.6563	-0.5127	-0.5195	6.790748×10^{-3}
0.7813	-0.5778	-0.5853	7.471334×10^{-3}
0.9063	-0.6325	-0.6407	8.199518×10^{-3}

Table 3.15: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.15) at some points

Example 3.16. [11] Consider the linear Volterra integral equation of the second kind :

$$y(x) = \frac{x^2}{2} - \frac{8}{15}x^{\frac{3}{2}} + \int_0^x (x-t)^{-\frac{1}{2}} y(t)dt \tag{3.32}$$

The exact solution is given by:

$$y(x) = \frac{x^2}{2}$$

After applying the method, we get the results shown in figure and table (3.16)

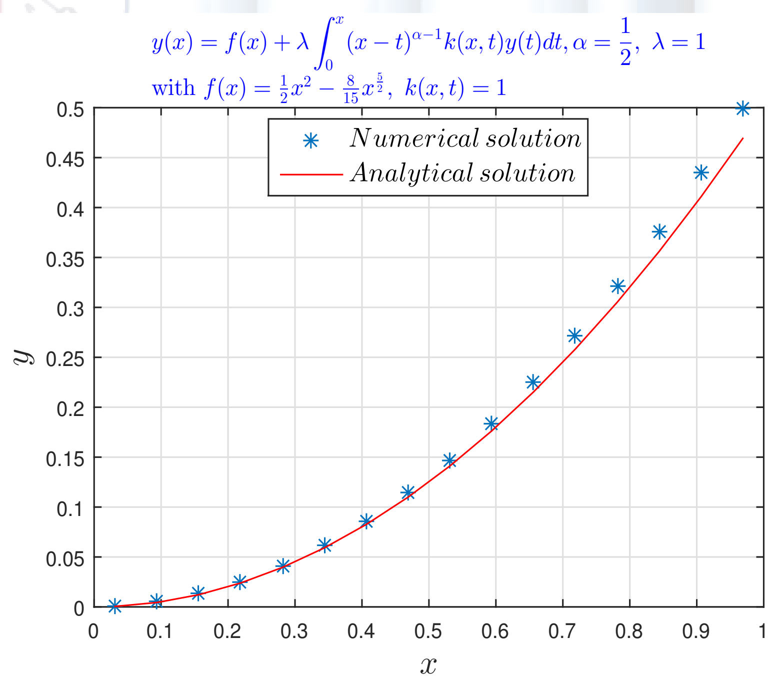

Figure 3.16: Graphical comparison of numerical and exact solutions of the example (3.16) with $2M = 16$

x_l	Exact solution	Haar solution	Error
0.0313	0.0005	0.0006	1.246233×10^{-4}
0.1563	0.0122	0.0130	8.143660×10^{-4}
0.2813	0.0396	0.0415	1.910444×10^{-3}
0.4063	0.0825	0.0861	3.604714×10^{-3}
0.5313	0.1411	0.1473	6.182007×10^{-3}
0.6563	0.2153	0.2254	1.006900×10^{-2}
0.7813	0.3052	0.3211	1.590279×10^{-2}
0.9063	0.4106	0.4353	2.463339×10^{-2}

Table 3.16: Comparison of exact and approximate solutions with $2M = 16$ of the example (3.16) at some points

CHAPTER 4

NUMERICAL SOLUTIONS FOR LINEAR INTEGRO–DIFFERENTIAL EQUATIONS USING HAAR WAVELET METHOD

In this chapter we solve linear Volterra and Fredholm integro–differential equations by using Haar wavelet method

4.1 Numerical solutions for linear Volterra integro differential equation using Haar wavelet method

Consider the form of linear Volterra integro–differential equations :

$$\sum_{k}^{n} A_k(x)y^{(k)}(x) = f(x) + \int_0^x k(x,t)y(t)dt \quad x \in [0,1] \tag{4.1}$$

with the initial condition

$$y^{(k)}(x_0) = y_k, \quad k = 0, 1, \ldots, n-1$$

here $A_k(x)$ and $f(x)$ are given functions and y_k are a given constants.

4.1.1 Method of solution and numerical examples

We assume that the solution is written in the form

$$y^{(n)}(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{4.2}$$

We integrate both sides of 4.2 from 0 to x and repeat this n times we obtain :

$$\int_0^x y^{(n)}(\tau)d\tau = \int_0^x \sum_{i=1}^{2M} a_i h_i(\tau)d\tau$$

$$y^{(n-1)}(x) = \sum_{i=1}^{2M} a_i P_{1,i}(x) + y^{(n-1)}(x_0)$$

$$\vdots$$

$$y(x) = \sum_{i=1}^{2M} a_i P_{n,i}(x) + \cdots + y(x_0)$$

We replace $y^{(k)}(x), \quad k = 0, 1, \cdots, n-1$ in (4.1)

We shall satisfy all at the collocation points x_i. In this way we get a system of linear equations to calculate the wavelet coefficients a_i.

Example 4.1. [14] Consider the following linear VID equation of the second kind

$$y'(x) = 1 - \int_0^t y(t)dt \quad x \in [0, 1] \tag{4.3}$$

with initial condition $\quad y(0) = 0$, the exact solution is given by

$$y(x) = \sin(x)$$

Table and figure (4.1) shows the exact and the approximate solutions at different points of collocation.

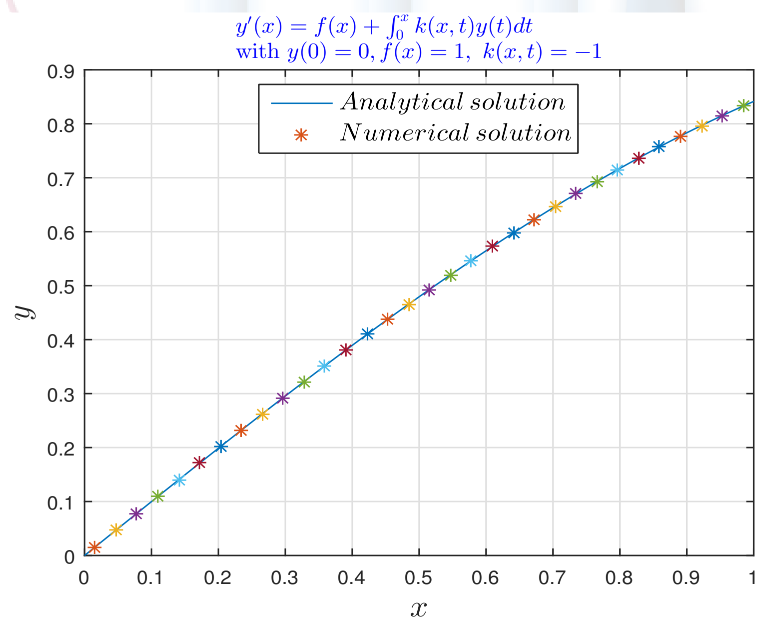

Figure 4.1: Graphical comparison of numerical and exact solutions of the example (4.1) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	0.00000	0.00781	7.812182×10^{-3}
0.1172	0.10916	0.11692	7.758844×10^{-3}
0.2266	0.21701	0.22462	7.612808×10^{-3}
0.3359	0.32227	0.32964	7.375847×10^{-3}
0.4453	0.42368	0.43073	7.050817×10^{-3}
0.5547	0.52002	0.52666	6.641627×10^{-3}
0.6641	0.61015	0.61630	6.153192×10^{-3}
0.7734	0.69299	0.69858	5.591370×10^{-3}
0.8828	0.76754	0.77251	4.962895×10^{-3}
0.9922	0.83293	0.83720	4.275296×10^{-3}

Table 4.1: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.1) at some points

Example 4.2. [14] Consider the equation with initial condition :

$$y'(x) = 1 + \int_0^x y(t)dt, \quad y(0) = 1, \quad x \in [0, 1] \tag{4.4}$$

with the exact solution

$$y(x) = e^x$$

After applying the method, we got the results (4.2)

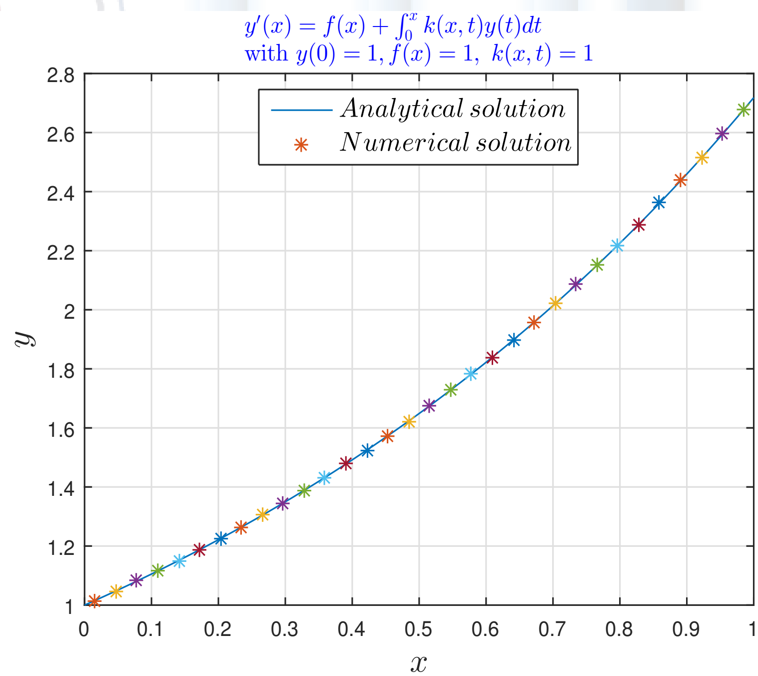

Figure 4.2: Graphical comparison of numerical and exact solutions of the example (4.2) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	1.00000	1.00787	7.873856×10^{-3}
0.1172	1.11558	1.12436	8.784065×10^{-3}
0.2266	1.24452	1.25432	9.799737×10^{-3}
0.3359	1.38836	1.39930	1.093306×10^{-2}
0.4453	1.54883	1.56103	1.219766×10^{-2}
0.5547	1.72785	1.74145	1.360869×10^{-2}
0.6641	1.92755	1.94273	1.518311×10^{-2}
0.7734	2.15034	2.16728	1.693982×10^{-2}
0.8828	2.39888	2.41778	1.889992×10^{-2}
0.9922	2.67614	2.69723	2.108692×10^{-2}

Table 4.2: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.2) at some points

Example 4.3. [14] Consider the equation :

$$y'(x) = \frac{1}{4} + \frac{3}{4}x + \sin x + \int_0^x y(t)dt \quad x \in [0,1] \tag{4.5}$$

with $y(0) = -1$ and the exact solution

$$y(x) = \frac{1}{4}e^x - \frac{3}{4} - \frac{1}{2}\cos(x)$$

Using the method mentioned above, we get the results in table and figure (4.3)

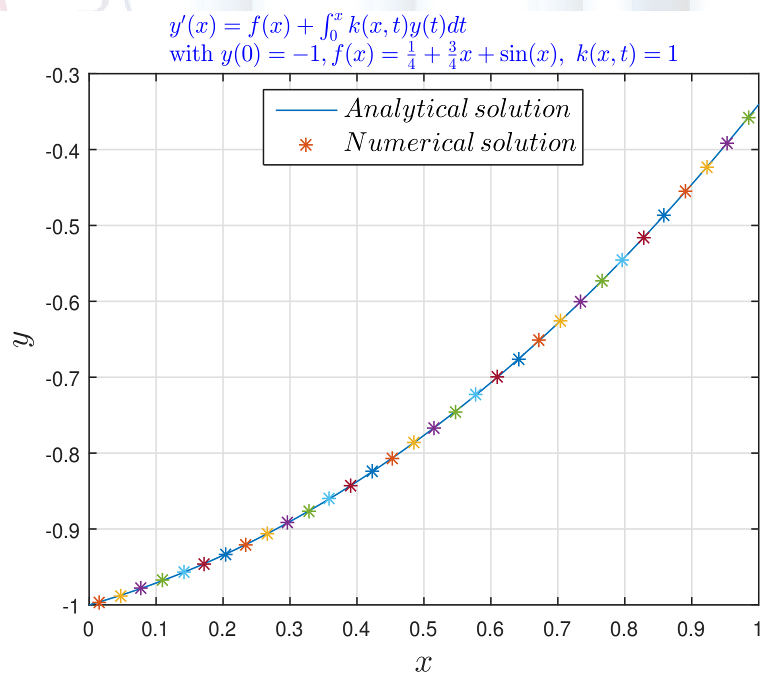

Figure 4.3: Graphical comparison of numerical and exact solutions of the example (4.3) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	-1.00000	-0.99800	1.998982×10^{-3}
0.1172	-0.96812	-0.96546	2.652869×10^{-3}
0.2266	-0.92695	-0.92363	3.327906×10^{-3}
0.3359	-0.87623	-0.87221	4.022114×10^{-3}
0.4453	-0.81570	-0.81097	4.733993×10^{-3}
0.5547	-0.74512	-0.73965	5.462618×10^{-3}
0.6641	-0.66426	-0.65805	6.207747×10^{-3}
0.7734	-0.57289	-0.56592	6.969923×10^{-3}
0.8828	-0.47078	-0.46303	7.750577×10^{-3}
0.9922	-0.35766	-0.34911	8.552146×10^{-3}

Table 4.3: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.3) at some points

Example 4.4. [17] Consider the following linear second order Volterra IDE

$$y''(x) = e^x \left(1 + x - x^2\right) - x + \int_0^x xty(t)dt, \tag{4.6}$$

with the ICs : $y'(0) = y(0) = 1$. The exact solution is :

$$y(x) = e^x$$

Table and figure (4.4) shows the exact and approximate solutions at different points of collocation.

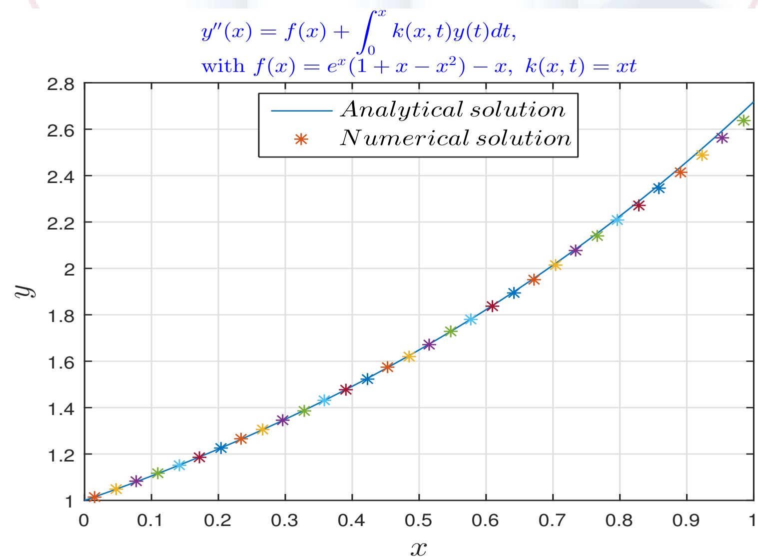

Figure 4.4: Graphical comparison of numerical and exact solutions of the example (4.4) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	1.00000	1.00785	7.853509×10^{-3}
0.1172	1.11558	1.12434	8.762900×10^{-3}
0.2266	1.24452	1.25428	9.761808×10^{-3}
0.3359	1.38836	1.39915	1.078497×10^{-2}
0.4453	1.54883	1.56046	1.163177×10^{-2}
0.5547	1.72785	1.73971	1.186909×10^{-2}
0.6641	1.92755	1.93826	1.070565×10^{-2}
0.7734	2.15034	2.15717	6.830453×10^{-3}
0.8828	2.39888	2.39708	1.793381×10^{-3}
0.9922	2.67614	2.65795	1.819020×10^{-2}

Table 4.4: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.4) at some points

Example 4.5. [17] Consider the second order Volterra IDE

$$y''(x) = 1 + \int_0^x (x-t)y(t)dt \tag{4.7}$$

with the initial conditions $y(0) = 1, \quad y'(0) = 0$. The exact solution is :

$$y(x) = \cos h(x)$$

After applying the method, we get the results shown in figure and table 4.5

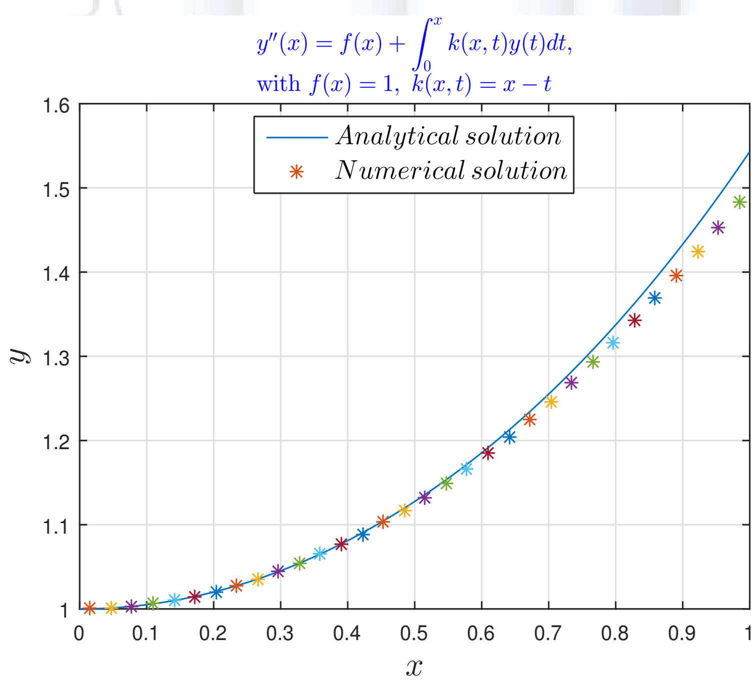

Figure 4.5: Graphical comparison of numerical and exact solutions of the example (4.5) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	1.00000	1.00004	4.069007×10^{-5}
0.1172	1.00599	1.00688	8.889835×10^{-4}
0.2266	1.02402	1.02567	1.651078×10^{-3}
0.3359	1.05432	1.05642	2.105595×10^{-3}
0.4453	1.09724	1.09912	1.882037×10^{-3}
0.5547	1.15330	1.15376	4.575015×10^{-4}
0.6641	1.22317	1.22032	2.847992×10^{-3}
0.7734	1.30769	1.29881	8.877262×10^{-3}
0.8828	1.40787	1.38923	1.864257×10^{-2}
0.9922	1.52491	1.49157	3.333286×10^{-2}

Table 4.5: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.5)

4.2 Numerical solutions for linear Fredholm integro–differential equation using Haar wavelet method

by the same method we can also solve linear Fredholm integro differential equation given by

$$\sum_{k}^{n} A_k(x)y^{(k)}(x) = f(x) + \int_0^1 k(x,t)y(t)dt \quad x \in [0,1] \tag{4.8}$$

with $A_k(x)$ end $f(x)$ are given functions.

Example 4.6. [1] Consider the integro-differential equation

$$y'(x) + 2y(x) = 1 - 5\int_0^1 y(t)dt, y(0) = 0. \tag{4.9}$$

The exact solution is $y(x) = \frac{1}{2}e^{-x}\sin(2x)$. Using the method mentioned above, we get the results in table and figure (4.6)

x_l	Exact solution	Haar solution	Error
0.0000	1.0177	1.0000	1.778472×10^{-2}
0.1094	1.0121	0.9947	1.738966×10^{-2}
0.2344	0.9940	0.9784	1.559971×10^{-2}
0.3594	0.9668	0.9525	1.429762×10^{-2}
0.4844	0.9339	0.9187	1.519449×10^{-2}
0.6094	0.8979	0.8782	1.968740×10^{-2}
0.7344	0.8614	0.8326	2.884983×10^{-2}
0.8594	0.8264	0.7830	4.342950×10^{-2}
0.9844	0.7945	0.7307	6.385343×10^{-2}

Table 4.6: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.6) at some points

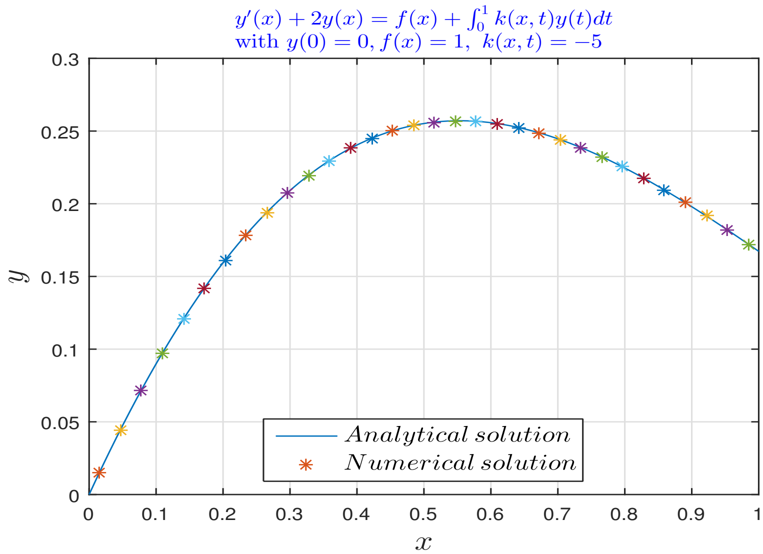

Figure 4.6: Graphical comparison of numerical and exact solutions of the example (4.6) with $2M = 32$

Example 4.7. [2] Consider the Integro-Differential Equation:

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)dt, \quad y(0) = 0 \tag{4.10}$$

with the exact solution

$$y(x) = xe^x.$$

The results obtained are shown in the table and figure (4.7)

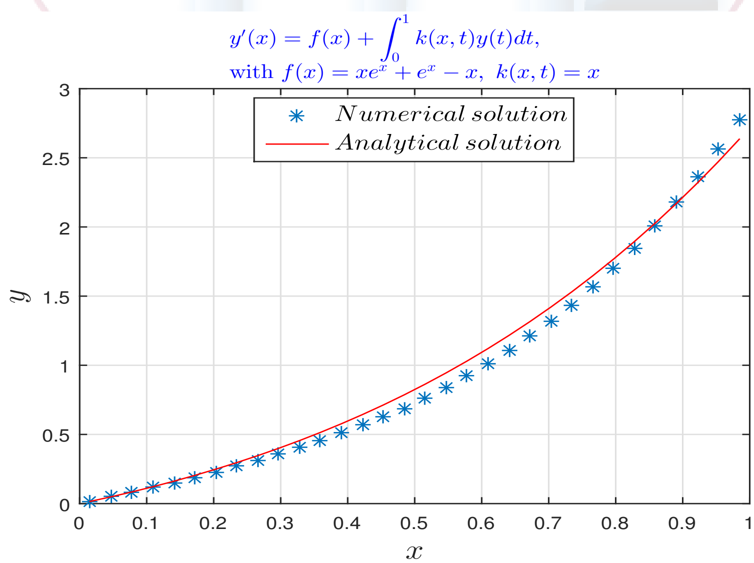

Figure 4.7: Graphical comparison of numerical and exact solutions of the example (4.7) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	0.0079	0.0079	4.824594×10^{-7}
0.1172	0.1318	0.1250	6.777112×10^{-3}
0.2266	0.2842	0.2593	2.486073×10^{-2}
0.3359	0.4701	0.4179	5.212950×10^{-2}
0.4453	0.6951	0.6097	8.546815×10^{-2}
0.5547	0.9659	0.8567	1.091944×10^{-1}
0.6641	1.2901	1.1809	1.091483×10^{-1}
0.7734	1.6762	1.5961	8.014405×10^{-2}
0.8828	2.1344	2.1346	2.744659×10^{-4}
0.9922	2.6761	2.8293	1.532778×10^{-1}

Table 4.7: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.7) at some points

Example 4.8. [5] Consider the Integro-Differential Equation:

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)dt, \quad y(0) = 0, \tag{4.11}$$

with the exact solution

$$y(x) = x$$

After applying the method, we get the results shown in figure and table 4.8

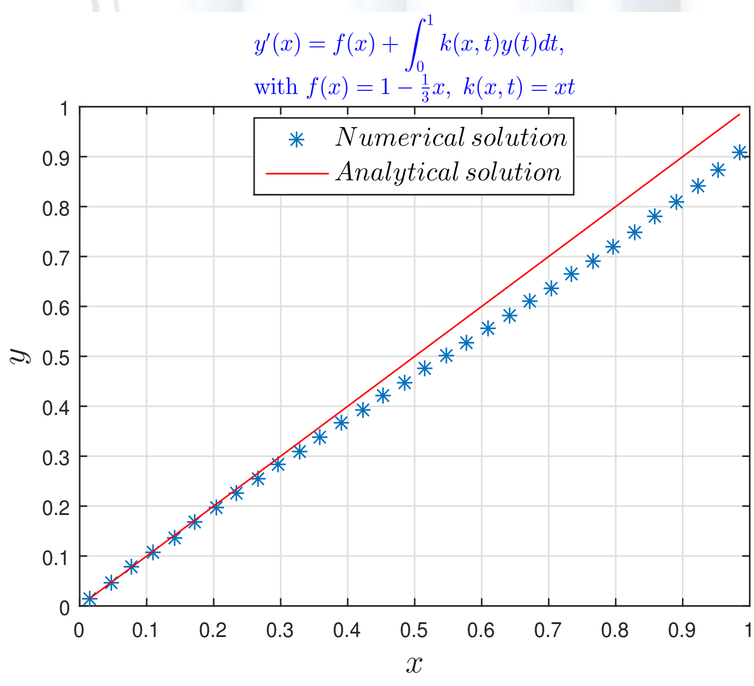

Figure 4.8: Graphical comparison of numerical and exact solutions of the example (4.8) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	0.0078	0.0078	2.034504×10^{-5}
0.1172	0.1172	0.1149	2.296758×10^{-3}
0.2266	0.2266	0.2181	8.507308×10^{-3}
0.3359	0.3359	0.3175	1.841667×10^{-2}
0.4453	0.4453	0.4139	3.142728×10^{-2}
0.5547	0.5547	0.5083	4.641761×10^{-2}
0.6641	0.6641	0.6025	6.157521×10^{-2}
0.7734	0.7734	0.6992	7.422027×10^{-2}
0.8828	0.8828	0.8023	8.048320×10^{-2}
0.9922	0.9922	0.9168	7.536713×10^{-2}

Table 4.8: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.8) at some points

Example 4.9. [17] Consider the following second order linear Fredholm IDE :

$$y''(x) = e^x - \frac{4}{3}x + \int_0^1 xty(t)dt \tag{4.12}$$

with ICs : $y(0) = 1, y'(0) = 2$. The exact solution is

$$y(x) = e^x + x$$

Using the method mentioned above, we get the results(see table and figure) (4.9)

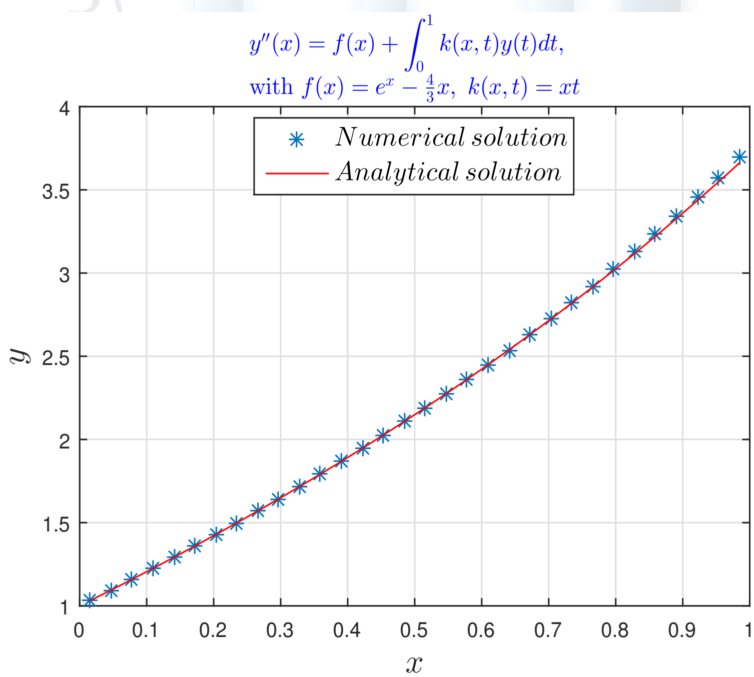

Figure 4.9: Graphical comparison of numerical and exact solutions of the example (4.9) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	1.0157	1.0157	1.030855×10^{-5}
0.1172	1.2415	1.2415	6.070704×10^{-5}
0.2266	1.4808	1.4804	4.099891×10^{-4}
0.3359	1.7352	1.7342	1.021757×10^{-3}
0.4453	2.0063	2.0047	1.582381×10^{-3}
0.5547	2.2961	2.2946	1.448923×10^{-3}
0.6641	2.6067	2.6072	4.426831×10^{-4}
0.7734	2.9406	2.9463	5.632638×10^{-3}
0.8828	3.3005	3.3168	1.630483×10^{-2}
0.9922	3.6893	3.7248	3.551170×10^{-2}

Table 4.9: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.9) at some points

Example 4.10. Consider the integro–Differential Equation.

$$y''(x) - y'(x) + y(x) = x^2 + \frac{7}{4}x + 2 - \int_0^1 xty(t)dt, \quad y(0) = y'(0) = 0. \tag{4.13}$$

With the exact solution

$$y(x) = x^2$$

the result obtained are shown in table and figure (4.10)

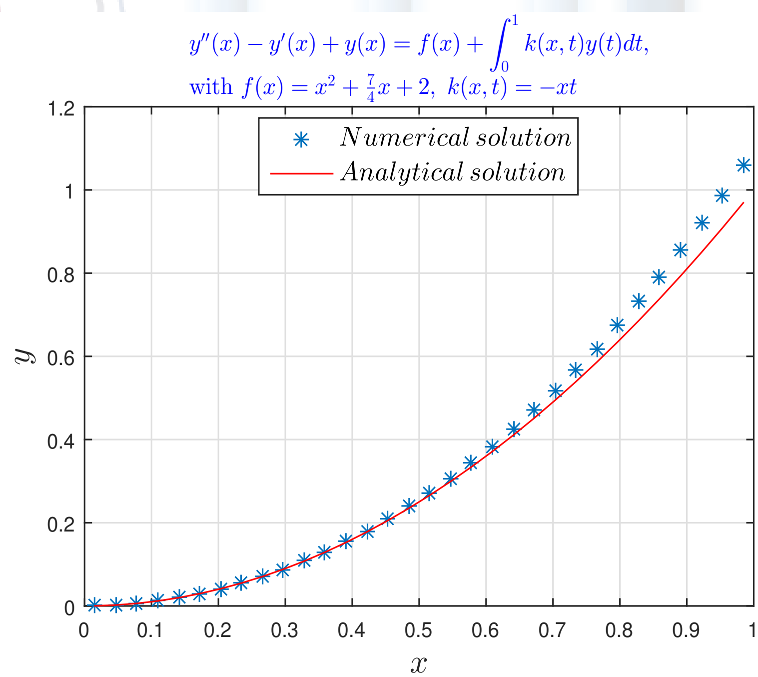

Figure 4.10: Graphical comparison of numerical and exact solutions of the example (4.10) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	0.0001	0.0001	2.027194×10^{-5}
0.1172	0.0137	0.0137	1.427547×10^{-5}
0.2266	0.0513	0.0513	1.796994×10^{-5}
0.3359	0.1129	0.1134	5.491052×10^{-4}
0.4453	0.1983	0.2010	2.700186×10^{-3}
0.5547	0.3077	0.3155	7.842898×10^{-3}
0.6641	0.4410	0.4586	1.766304×10^{-2}
0.7734	0.5982	0.6322	3.400327×10^{-2}
0.8828	0.7794	0.8380	5.867124×10^{-2}
0.9922	0.9844	1.0776	9.319116×10^{-2}

Table 4.10: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.10) at some points

Example 4.11. [3] Consider the Integro-Differential Equation

$$y'''(x) = 1 - e + e^x + \int_0^1 y(t)dt \tag{4.14}$$

Subject to the conditions $y(0) = y'(0) = y''(0) = 1$. The analytical solution is given by:

$$y(x) = e^x$$

After applying the method, we get the results (see tabble and figure) 4.11

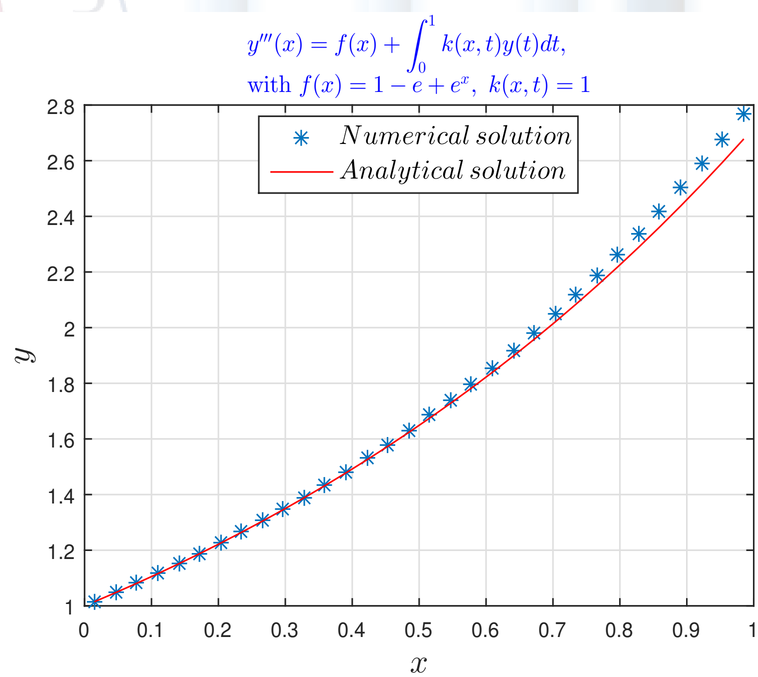

Figure 4.11: Graphical comparison of numerical and exact solutions of the example (4.11) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	1.0078	1.0078	1.265885×10^{-7}
0.1172	1.1243	1.1244	8.556371×10^{-5}
0.2266	1.2543	1.2549	6.654362×10^{-4}
0.3359	1.3993	1.4016	2.355024×10^{-3}
0.4453	1.5610	1.5669	5.947343×10^{-3}
0.5547	1.7414	1.7538	1.243361×10^{-2}
0.6641	1.9427	1.9657	2.303418×10^{-2}
0.7734	2.1672	2.2064	3.923626×10^{-2}
0.8828	2.4177	2.4805	6.283424×10^{-2}
0.9922	2.6971	2.7931	9.598412×10^{-2}

Table 4.11: Comparison of exact and approximate solutions with $2M = 64$ of the example (4.11) at some points

NUMERICAL SOLUTIONS FOR LINEAR FRACTIONAL INTEGRO EQUATION USING HAAR WAVELET METHOD

In this chapter we present the Haar wavelets method to solve linear fractional Volterra and Fredholm integro–differential equations. By using the collocation method, we transforms the problem into a system of linear algebraic equations.

5.0.1 Method of solution and numerical examples

Consider this form of linear fractional Volterra integro-differential equations :

$$D^{\alpha}(x) + Cy(x) = f(x) + \int_0^x k(x,t)y(t)dt \quad x \in [0,1] \tag{5.1}$$

with the initial condition

$$y^{(i)}(0) = 0$$

avec C is a real number.
We assume that the solution is written in the form

$$D^{\alpha}y(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{5.2}$$

Using the relation (1.18) we obtain:

$$y(x) = \sum_{i=1}^{2M} a_i P_{\alpha,i}(x) \tag{5.3}$$

satisfying (5.2) and (5.3) at the collocation points, substitute (5.2) and (5.3) in (5.1) we get:

$$\sum_{i=1}^{2M} a_i h_i(x_l) + C\sum_{i=1}^{2M} a_i h_i(x_l) = f(x_l) + \int_0^{x_l} k(x_l,t)\sum_{i=1}^{2M} a_i P_{\alpha,i}(x_l)dt$$

so

$$\sum_{i=1}^{2M} a_i(h_i(x_l) + CP_{\alpha,i}(x_l)) = f(x_l) + \sum_{i=1}^{2M} a_i \int_0^{x_l} k(x_l,t)P_{\alpha,i}(x_l)dt$$

where

$$G_i(x_l) = \int_0^{x_l} k(x_l,t)P_{\alpha,i}(x_l)dt. \tag{5.4}$$

the discrete form is

$$A(H + C.P^{\alpha} - G_{il}) = F_l$$

where

$$A = \sum_{i=1}^{2M} a_i$$

$$P^{\alpha} = P_{\alpha,i}(x_l)$$

$$G_{il} = G_i\left(x_l\right)$$

$$F_l = f\left(x_l\right)$$

so we have

$$A = F_l(H + C.P^{\alpha} - G_{il})^{-1}$$

with the same procedure we solve the linear Fredholm integro–differential equations :

$$D^{\alpha}(x) = f(x) + \int_0^1 k(x,t)y(t)dt \quad x \in [0,1] \tag{5.5}$$

we get

$$A(H - G_{il}) = F_l$$

where

$$G_i(x_l) = \int_0^1 k(x_l,t)P_{\alpha,i}(x_l)dt. \tag{5.6}$$

so

$$A = F_l(H - G_{il})^{-1}$$

Example 5.1. [5] Consider the following fractional Integro-differential equation :

$$D^{1/2}y(x) = \frac{(\frac{8}{3})x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xty(t)dt \quad x \in [0,1] \tag{5.7}$$

subject to $y(0) = 0$ with the exact solution

$$y(x) = x^2 - x.$$

Using the method mentioned above, we get the results as shown in table and figure (5.1)

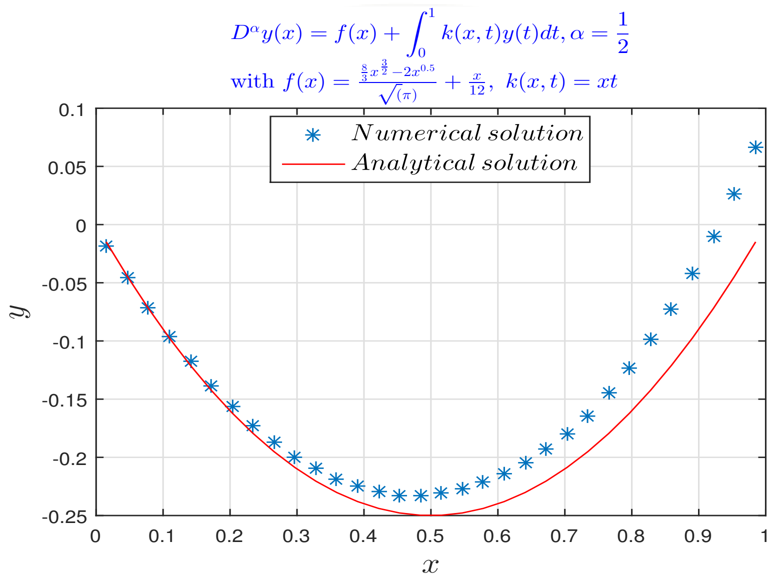

Figure 5.1: Graphical comparison of numerical and exact solutions of the example (5.1) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	-0.0078	-0.0092	1.467921×10^{-3}
0.1172	-0.1035	-0.1011	2.315586×10^{-3}
0.2266	-0.1752	-0.1688	6.405811×10^{-3}
0.3359	-0.2231	-0.2121	1.100639×10^{-2}
0.4453	-0.2470	-0.2318	1.524529×10^{-2}
0.5547	-0.2470	-0.2259	2.111313×10^{-2}
0.6641	-0.2231	-0.1963	2.679926×10^{-2}
0.7734	-0.1752	-0.1392	3.605651×10^{-2}
0.8828	-0.1035	-0.0504	5.307432×10^{-2}
0.9922	-0.0078	0.0764	8.416867×10^{-2}

Table 5.1: Comparison of exact and approximate solutions with $2M = 64$ of the example (5.1) at some points

Example 5.2. [5] Consider the following fractional Integro-differential equation :

$$D^{5/3}y(x) = \frac{3\sqrt{3}\Gamma(2/3)x^{1/3}}{\pi} - \frac{1}{5}x^2 - \frac{1}{4}x + \int_0^1 \left(xt + x^2t^2\right) y(t)dt, \quad x \in [0,1] \tag{5.8}$$

subject to

$$y(0) = y'(0) = 0$$

with the exact solution

$$y(x) = x^2.$$

After applying the method, we get the results (see figure and table) 5.2

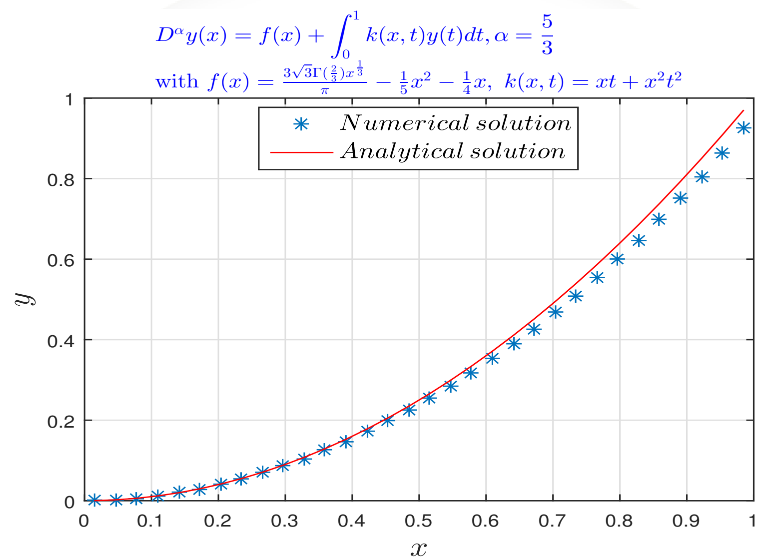

Figure 5.2: Graphical comparison of numerical and exact solutions of the example (5.2) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	0.0001	0.0001	4.665541×10^{-5}
0.1172	0.0137	0.0137	4.908795×10^{-5}
0.2266	0.0513	0.0503	1.065442×10^{-3}
0.3359	0.1129	0.1093	3.574148×10^{-3}
0.4453	0.1983	0.1902	8.104871×10^{-3}
0.5547	0.3077	0.2927	1.500665×10^{-2}
0.6641	0.4410	0.4168	2.416862×10^{-2}
0.7734	0.5982	0.5636	3.464043×10^{-2}
0.8828	0.7794	0.7359	4.341480×10^{-2}
0.9922	0.9844	0.9404	4.405856×10^{-2}

Table 5.2: Comparison of exact and approximate solutions with $2M = 64$ of the example (5.2) at some points

Example 5.3. [10] Consider the following fractional integro-differential equation :

$$D^{\frac{1}{2}}y(x) - y(x) = \frac{8}{3\Gamma(0.5)}x^{1.5} - x^2 - \frac{1}{3}x^3 + \int_0^x y(t)dt \tag{5.9}$$

subject to

$$y(0) = 0$$

with the exact solution

$$y(x) = x^2$$

After applying the method, we get the results (see figure and table) (5.3)

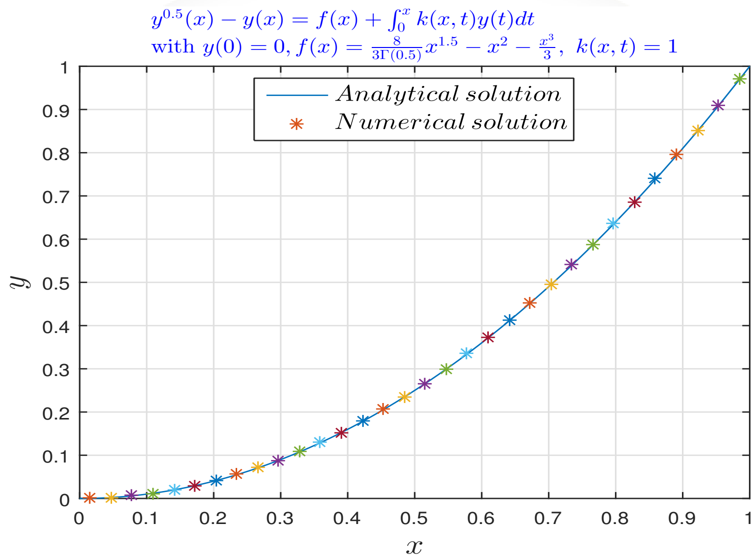

Figure 5.3: Graphical comparison of numerical and exact solutions of the example (5.3) with $2M = 32$

x_l	Exact solution	Haar solution	Error
0.0078	0.00000	0.00010	1.015445×10^{-4}
0.1172	0.01196	0.01380	1.833618×10^{-3}
0.2266	0.04785	0.05141	3.561400×10^{-3}
0.3359	0.10767	0.11296	5.292691×10^{-3}
0.4453	0.19141	0.19844	7.029241×10^{-3}
0.5547	0.29907	0.30785	8.772747×10^{-3}
0.6641	0.43066	0.44119	1.052520×10^{-2}
0.7734	0.58618	0.59847	1.228905×10^{-2}
0.8828	0.76563	0.77969	1.406737×10^{-2}
0.9922	0.96899	0.98486	1.586401×10^{-2}

Table 5.3: Comparison of exact and approximate solutions with $2M = 64$ of the example (5.3) at some points

Conclusion

In this work the Haar wavelet method for solution of linear ordinary and fractional differential equations is proposed and discussed in chapter 2. A method of solution which is applicable for different kind of linear integral and fractional Volterra and Fredholm integral equations, is worked out in chapter 3. Also integro–differential of linear Fredholm and Volterra is considered in chapter 4. A method of solution which is also applicable for linear fractional Volterra and Fredholm integro–differential is presented in chapter 5. Solution, which is based on the collocation technics is proposed. The elaborated method is very simple and – as it follows from the test problems – high precision of results can be obtained with the larger value of $2M$.

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ملخص

مويجات هار وتطبيقاتها

الهدف الرئيسي من هذه الأطروحة هو تقديم وتطبيق طريقة مويجات هار من أجل إيجاد حلول عددية لمسائل متنوعة ، من بين هذه التطبيقات المعادلات التفاضلية الخطية ، المعادلات التفاضلية الكسرية الخطية ، المعادلات التكاملية، المعادلات التكاملية التفاضلية وكذلك المعادلات التكاملية التفاضلية الكسرية. على هذا النحو ، قدمنا العديد من الأمثلة لحل مسائل مختلفة لإظهار صحة الطريقة.

كلمات مفتاحية: مويجات هار ، المعادلات التفاضلية ، المعادلات التفاضلية الكسرية، المعادلات التكاملية ، المعادلة التفاضلية التكاملية ، المعادلات التفاضلية التكاملية الكسرية .

Abstract

Haar wavelets and its applications

The main objective of this thesis is to introduce the application of the Haar wavelet method in order to find numerical solutions to a variety of problems, among these applications we cite: Linear differential equations, Linear fractional differential equations, integral equations, integro-differentials equations and Fractional integro-differentials equations. As such, we have introduced several examples for solving various problems to show the validity of the method.

Key words: Haar wavelet, Differential equations, Fractional differential equations, Integral equations, Integro differentials equations, Fractional integro differentials equations.

Résumé

Les ondelettes de Haar et leurs applications

L'objectif principal de cette thèse est de présenter et d'appliquer la méthode des ondelettes de Haar afin de trouver des solutions numériques à une variété de problèmes, parmi ces applications nous citons : les équations différentielles linéaires, les équations différentielles fractionnaires linéaires, les équations intégrales, les équations integro différentielles et les équations integro différentielles fractionnelles. En tant que tel, nous avons fourni de nombreux exemples de résolution de divers problèmes pour montrer la validité de la méthode.

Mots clé: Wavelet de Haar, équations différentielles linéaires, équations différentielles fractionnaires, équations intégrales, équations integro différentielles, équations integro différentielles fractionnelles.