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Lipschitz operators represented by vector measures

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Dedication

I have the great pleasure of dedicating this modest work :
To my dear parents:

To the fountain that never tire of giving To the one whose heart woven my happiness with threads To my dear mother "Freiha".
To the one who sought and struggled to enjoy comfort and contentment, who did not skimp on anything in order to push me on the path of success, who taught me to ascend the ladder of life with wisdom and patience,

To my dear father"Rabah".
For their patience, love, support and encouragement

To my dear brothers and sisters
my fiancé "Tarek"
To all my friends and relative

Thank God.
Ouafa.

Acknowledgments

In the name of Allah the Most Merciful

Thank God,who gave me the strength and courage to complete this work.

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"Abdelhamid TALLAB ".
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Contents

List of symbols

- $\mathbb{K}$ The field of real or complex numbers.
- p^* The conjugate of the number p $(1 \leqslant p < \infty)$.
- E^* The topological dual of Banach space E.
- B_E The closed unit ball of Banach space E.
- $\mathcal{L}(E, F)$ The sets of all continuous linear operators.
- $\mathbb{R}_+$ The field of non negative real numbers.
- $Lip_0(X, E)$ The set of all Lipschitz operators that vanish at 0.
- $X^\#$ The Lipschitz dual of the pointed metric space X.
- $\mathcal{M}(X)$ The vector space of all molecule son the metric space X.
- $m_{xx'}$The molecule defined by $m_{xx'} = \mathcal{X}_{\{x\}} - \mathcal{X}_{\{x'\}}$ for $x, x' \in X$, where $\mathcal{X}_A$ is the characteristic function of the set A.
- Æ(X) The Arens-Eells space of X.
- T_L The linearization of the operator T.
- $\mathcal{W}$ The set of all weakly compact linear operators .
- $\mathcal{I}$ The ideal of all linear operator.
- $\mathcal{I}_{Lip}$ The ideal of Lipschitz operators.
- k_X The isometrically embedding from X to Æ(X).
- Π_p^L The class of all Lipschitz p-summing operators $(1 \leqslant p < \infty)$.

- $I_{\infty,p}$ The formal inclusion map defined between $L_\infty(\mu)$ and $L_p(\mu)$ for $(1 \leqslant p < \infty)$.

- $\mathcal{PI}_p$ The class of all linear Pietsch-p-integral operators.

- $\mathcal{PI}_\infty$ The class of all linear Pietsch-∞-integral operators.

- j_p The canonical inclusion map defined between $C(K)$ and $L_p(\mu)$

- j_∞ The canonical inclusion map defined between $C(B_{X^\#})$ to $L_\infty(\mu)$

- ι_X The Lipschitz isometric embedding, $\iota_X : X \longrightarrow C(B_{E^*})$ given by $\iota_X(x) =< x, . >$.

- i_p The canonical inclusion map defined between $L_\infty(\mu)$ and $L_p(\mu)$.

- m Vector measure.

- $\mid m \mid$ Variation of Vector measure m.

- $\|m\|$ Semivariation of vectors measure m .

- $\mathcal{PI}_p^L$ The class of all Lipschitz Pietsch-p-integral operators .

- $\mathcal{PI}_\infty^L$ The class of all Lipschitz Pietsch-∞-integral operators .

- $\mathcal{N}_p$ The set of all p-nuclear linear operators .

Introduction

The class of p-integral linear operators was introduced in 1969 by Persson and Pietsch [16] (also known as strictly p-integral or Pietsch-p-integral operators), establishing many of its fundamental properties using the theory of vector measures. In 1989, Cardassi studied the factorization properties and some results of coincidences for these operators in [5]. The ideal of p-integral polynomials on Banach spaces has been defined and characterized by Cilia and Gutirrez in [6] for $p = 1$ and in [7] for $p \geqslant 1$, as a natural polynomial extension of Pietsch-p-integral operators.

In this memory, we introduce and study the Lipschitz version of this concept. We define the Lipschitz Pietsch-p-integral operator ($1 \leqslant p \leqslant \infty$) as a Lipschitz mapping between a pointed metric space and a Banach space by an integral representation with respect to a vector measure on the Borel σ-algebra of a compact Hausdorff space K. Special attention is paid to the factorization of these mappings, and we compare our class with some well-known Lipschitz operators defined by a factorization schema or by summability of series. Note that the class of Lipschitz Pietsch-1-integral operators is studied in [4]. In this case, the authors use only factorization schemes to define this concept without using vector measure theory.

We now describe the contents of the present memory. After this introduction, in chapter one, we fix notation and basic concepts related to Lipschitz mappings and vector measures of interest for our purposes. In chapter two, we detail the concept of Pietsch-p-integral operators for Lipschitz mappings with $p \geqslant 1$, and we give the prove of factorization theorem for these mappings through the classical Banach spaces $C(K)$ and $L_p(\mu, K)$ and we give the relation with its linearization. The third chapter is devoted to studying the notion of Lipschitz Pietsch-∞-integral operators. Starting from the representation by a vector measure, we present a characterization given by a factorization through a linear compact operator.

Chapter 1

Preliminaries

We will write $\mathbb{K}$ for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For $1 \leq p \leq \infty$, by p^* we denote the conjugate of p, that is $\frac{1}{p} + \frac{1}{p^*} = 1$. Along In this memory, X, Y will be pointed metric spaces with a base point denoted by 0, and the metric that will be denoted by d. Also, E, F and G denote Banach spaces. Given a Banach space E, B_E is its closed unit ball and S_E is unit sphere of E. By $\mathcal{L}(E, F)$ we denote the Banach space of all continuous linear operators between E and F with the usel supre mun norm.

1.1 Lipschitz Spaces

1.1.1 Lipschitz functions

The notion of metric space was formalized by Maurice Fréchet in [13]. He was among the first who used the word space. Recall that a metric or distance on a non empty set X is a function

$$d : X \times X \longrightarrow \mathbb{R}_+,$$

with the following properties:

1. (Positivity) For all x, $y \in X$, $d(x, y) \geq 0$ with equality if and only if $x = y$.
2. (Symmetry) For all x, $y \in X$, $d(x, y) = d(y, x)$.
3. (Triangle inequality) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

The set X equipped with the distance d is called a metric space.

Definition 1.1. *A map $T : (X, d_X) \longrightarrow (Y, d_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that*

$$\forall x, y \in X, d_Y(T(x), T(y)) \leq C d_X(x, y). \tag{1.1}$$

If $C = 1$, the map is called nonexpansive (and contraction if $C < 1$).

For a Lipschitz map T we define its Lipschitz constant by

$$\begin{aligned} Lip\,(T) &= \sup_{x \neq y} \left\{ \tfrac{d_Y(T(x),T(y))}{d_X(x,y)} \right\} \\ &= \inf \left\{ C : C \text{ verifying } 1.1 \right\}. \end{aligned}$$

If $(X, e_X, d_X), (Y, e_Y, d_Y)$ be pointed metric spaces.We say a map $T : (X, e_X, d_X) \longrightarrow (Y, e_Y, d_Y)$ preserves distinguished point if $T(e_X) = e_Y$.

If E is a Banach space, $Lip_0(X, E)$ is a Banach space under the Lipschitz norm given by

$$Lip\,(T) = \left\{ \frac{\|T(x) - T(x')\|}{d(x, y)}, x \neq y \right\}.$$

For $E = \mathbb{K}$, we designate $Lip_0(X, \mathbb{K}) = Lip_0(X) = X^{\#}$. The Banach space $X^{\#}$ of Lipschitz functions is called also Lipschitz dual it has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case.

1.1.2 The Arens-Eells spaces

in this section we present some concepts about the space molecules for more details see [17].

Definition 1.2. *Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies*

$$\sum_{x \in X} m\,(x) = 0.$$

We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For x, $y \in X$ the molecule m_{xy} is defined by $m_{xy} = \mathcal{X}_{\{x\}} - \mathcal{X}_{\{y\}}$, where $\mathcal{X}_A$ is the characteristic function of the set A. For $m \in \mathcal{M}\,(X)$ we can write

$$m = \sum_{j=1}^{n} \lambda_j m_{x_j y_j},$$

for some suitable scalars λ_j, and we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j|\, d(x_j, y_j), m = \sum_{j=1}^{n} \lambda_j m_{x_j y_j} \right\},$$

where the infimum is taken over all representations of the molecule m.

Denote by Æ(X) the completion of the normed space $\left(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)}\right)$.

Proposition 1.3. *Let X be a pointed metric space*

1. *The map $k_X : X \longrightarrow$Æ(X) defined by $k_X(x) = m_{x0}$, is an isometric embedding of X into Æ(X).*

2. *The map $Q_X : X^{\#} \longrightarrow$Æ$(X)^*$ defined by*

$$Q_X(f) = f_L, \text{ where } f_L(m) = \sum_{x \in X} f(x)m(x),$$

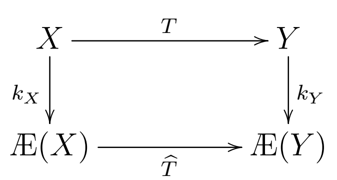
establish an isometric isomorphic between $X^{\#}$ and Æ$(X)^*$. On bounded subsets of $X^{\#}$ its weak * topology agrees with the topology of pointwise convergence (see [17, Theorem 2.2.2]).

The Banach space Æ(X) has some remarkable properties. We mention the followings.

Theorem 1.4. *[17] Let $T \in Lip_0(X, Y)$, then there exists a unique linear operator $\widehat{T} \in \mathcal{L}($Æ$(X)$,Æ$(Y))$ such that*

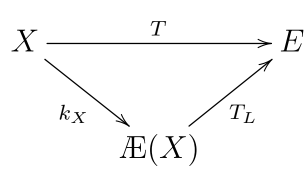
$$\widehat{T} \circ k_X = k_Y \circ T,$$

that is, the diagram



commutes. Furthermore, $\left\|\widehat{T}\right\| = Lip(T)$.

Theorem 1.5. *[17, Theorem 2.2 .4] Let $T \in Lip_0(X, E)$, then there is a unique bounded linear map T_L :Æ(X) $\longrightarrow E$ such that $T = T_L \circ k_X$ that is, the following diagram commutes*



Furthermore $\|T_L\| = Lip(T)$.

The operator T_L *is referred to as the linearization of* T. *The correspondence* $T \longleftrightarrow T_L$ *establishes an isomorphism between the vector spaces* $Lip_0(X, E)$ *and* $\mathcal{L}(Æ(X), E)$.

In particular, the spaces $X^\#$ *and* $Æ(X)^*$ *are isometrically isomorphic via the linearization* $R(f) := f_L$, *where* $f_L(m) = \sum_{x\in X} f(x)m(x)$.

1.2 Ideals of Lipschitz mappings

The notion of Lipschitz operator ideal was introduced by Achour, et all [1]. This can be seen as an extension of the linear operator ideal.

Definition 1.6. *A Lipschitz operator ideal* $\mathcal{I}_{Lip}$ *is a subclass of* $Lip_0(X, E)$ *such that for every pointed metric space* X *and every Banach space* E *the components*

$$\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip},$$

satisfy:

(i) $\mathcal{I}_{Lip}(X, E)$ *is a vector subspace of* $Lip_0(X, E)$.

(ii) $eg \in \mathcal{I}_{Lip}(X, E)$ *for* $e \in E$ *and* $g \in X^\#$.

(iii) The ideal property: if $S \in Lip_0(X, Y)$, $T \in \mathcal{I}_{Lip}(Y, F)$ *and* $u \in \mathcal{L}(F, E)$, *then the composition* $u \circ T \circ S$ *is in* $\mathcal{I}_{Lip}(X, E)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ *is a normed (Banach) Lipschitz operator ideal if there is a function* $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0, +\infty[$ *that satisfies*

(i') For every pointed metric space X *and every Banach space* E, *the pair* $(\mathcal{I}_{Lip}(X, E), \|.\|_{\mathcal{I}_{Lip}})$ *is a normed (Banach) space and* $Lip(T) \le \|T\|_{\mathcal{I}_{Lip}}$ *for all* $T \in \mathcal{I}_{Lip}(X, E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(X, Y)$, $T \in \mathcal{I}_{Lip}(Y, F)$ *and* $u \in \mathcal{L}(F, E)$, *then*

$$\|u \circ T \circ S\|_{\mathcal{I}_{Lip}} \le Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|u\|.$$

Definition 1.7. *(Composition Ideals) Given an operator ideal $\mathcal{I}$, a Lipschitz mapping $T \in Lip_0(X,E)$ belongs to the composition Lipschitz operator ideal $T \in \mathcal{I} \circ Lip_0(X,E)$, if there are a Banach space F, a Lipschitz operator $S \in Lip_0(X,F)$ and an operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal we write*

$$\|T\|_{\mathcal{I}\circ Lip_0} = \inf \|u\|_{\mathcal{I}} \, Lip(S),$$

where the infimum is taken over all u and S as above.

Proposition 1.8. *Let $\mathcal{I}$ be an operator ideal. The following are equivalent for $T \in Lip_0(X,E)$:*

1- $T \in \mathcal{I} \circ Lip_0(X,E)$.

2- $T_L \in \mathcal{I}(Æ(X), E)$.

If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal

$$\|T\|_{\mathcal{I}\circ Lip_0} = \|T_L\|_{\mathcal{I}}\,.$$

1.3 Some class of Lipschitz operators ideal

1.3.1 Lipschitz p-summing operators

The Lipschitz version of p-summing operators ($1 \leqslant p < \infty$) was introduced by Farmer and Johnson in [12].

Definition 1.9. *Let $1 \leqslant p < \infty$. A mapping $T \in Lip_0(X,E)$ is Lipschitz p-summing if there exists a positive constant C such that for all $(x_i)_{i\leqslant n}, (y_i)_{i\leqslant n}$ in X such that*

$$\sum_{i=1}^{n} \|T(x_i) - T(y_i)\|^p \leqslant C^p \sup_{f\in B_{X^\#}} \sum_{i=1}^{n} \mid f(x_i) - f(y_i) \mid^p .$$

The infimum of all such constants $C \geqslant 0$ is denoted by $\pi_p^L(T)$. This class of mappings is denoted by $\Pi_p^L(T)(X,E)$.

It is will known that $\pi_p^L(T)(X,E)$ is a Banach Lipschitz operators ideal.

The next theorem give the characterizing of the Lipschitz p-summing operators

Theorem 1.10. *(Inclusion theorem) If* $1 \leqslant p \leqslant q < \infty$, *then* $\Pi_p^L(X,E) \subset \Pi_q^L(X,E)$. *Moreover,* $\pi_p^L(T) \leqslant \pi_q^L(T)$

For a linear operator $T \in \mathcal{L}(E,F)$ it is clear that $\pi_p^L(T) \leqslant \pi_p(T)$. Farmer and Johnson proved that the reverse inequality is true. This justifies that the notion of Lipschitz p-summing operator is really a generalization of the concept of linear p-summing operator.

Theorem 1.11. *Let* T *be a bounded linear operator from* E *into* F. *If* $1 \leqslant p < \infty$. *then* $\pi_p^L(T) = \pi_p(T)$

Theorem 1.12. *sa [12] The mapping* $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ *is Lipschitz* p*-summing with* $\pi_p^L(i_p) = 1$

1.3.2 Lipschitz compact operators

Following [14], a Lipschitz map $T \in Lip_0(X,E)$ is Lipschitz compact if the set

$$\left\{ \frac{T(x) - T(y)}{d(x,y)} : x, y \in X, x \neq y \right\},$$

is relatively in E. Denote by $Lip_{0\mathcal{K}}(X,E)$ the set of Lipschitz compact operators from X to E. In [14], the relationship between the compactness of a Lipschitz operator $T \in Lip_0(X,E)$ and the compactness of its linearization $T_L \in \mathcal{L}(\text{Æ}(X), E)$ has been established

Proposition 1.13. *Let* $T \in Lip_0(X,E)$. *The following statements are equivalent:*

1. *T is Lipschitz compact.*

2. *T_L is compact from Æ(X) to E.*

Proposition 1.14. *The class* $Lip_\mathcal{K}$ *is closed sub space of* $Lip_0(X,E)$. *Then* $Lip_{0\mathcal{K}}$ *Banach operator ideal, where the ideal norm is the Lipschitz operator norm (Lip).*

1.4 Elementary properties of vector measures

Let us also recall some preliminaries on vector measures. For the general theory of vector measure we refer the reader to the classical monograph (see [10]).

Definition 1.15.

A function $\mathfrak{m}$ from a σ-algebra Σ of subsets of a set Ω to a Banach space E is called a finitely additive vector measure, or simply a vector measure, if whenever A_1 and A_2 are disjoint members of Σ then

$$\mathfrak{m}(A_1 \cup A_2) = \mathfrak{m}(A_1) + \mathfrak{m}(A_2).$$

If, in addition,

$$\mathfrak{m}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathfrak{m}(A_i),$$

in the norm topology of E for all sequences (A_i) of pairwise disjoint members of Σ such that $\cup_{i=1}^{\infty} A_i \in \Sigma$, i.e.,

$$\left\| \mathfrak{m}\left(\cup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{\infty} \mathfrak{m}(A_i) \right\|_E \longrightarrow 0.$$

Then $\mathfrak{m}$ is termed a countably additive vector measure or simply, $\mathfrak{m}$ is countably additive.

Definition 1.16. *A vector measure $\mathfrak{m} : \Sigma \longrightarrow E$ is said to be monatomic if for every set $A \in \Sigma$ with $\mathfrak{m}(A) \neq 0$ there exist $A_1 \neq \emptyset, A_2 \neq \emptyset \in \Sigma$ such that $A_1 \cup A_2 = A$ and $\mathfrak{m}(A_i) \neq 0$ $(i = 1, 2)$.*

Definition 1.17.

Let $\mathfrak{m} : \Sigma \longrightarrow E$ be a vector measure.

- *The variation of $\mathfrak{m}$ is the extended nonnegative function $|\mathfrak{m}|$ whose value on a set $A \in \Sigma$ is given by*

$$|\mathfrak{m}|(A) = \sup \sum_{i=1}^{n} \|\mathfrak{m}(A_i)\|,$$

where the supremum is taken over all partitions $(A_i)_{i=1}^{n}$ of A into a finite number of pairwise disjoint members of Σ. If $|\mathfrak{m}|(\Omega) < \infty$, then will be called a measure of bounded variation.

- *The semivariation of $\mathfrak{m}$ is the extended (i.e., take its value on $[0, +\infty]$) nonnegative function $\|\mathfrak{m}\|$ whose value on a set $A \in \Sigma$ is given by*

$$\|\mathfrak{m}\|(A) = \sup \left\{ |e^*\mathfrak{m}|(A) : e^* \in E^*, \|e^*\| \leq 1 \right\},$$

where $|e^*\mathfrak{m}|$ is the variation of the real-valued measure $e^*\mathfrak{m}$. If $\|\mathfrak{m}\|(\Omega) < \infty$, then $\mathfrak{m}$ will be called a measure of bounded semivariation.

Proposition 1.18. *A vector measure of bounded variation is countably additive if and only if its variation is also countably additive.*

Definition 1.19. *Let Σ be a σ-algebra of subsets of a set Ω and $\mathfrak{m} : \Sigma \longrightarrow E$ be a bounded vector measure. For each $f \in B(\Sigma)$, the integral of f with respect to $\mathfrak{m}$ is defined by*

$$u(f) = \int f d\mathfrak{m},$$

where u is as above. (we say u is representing by $\mathfrak{m}$).

It is easy to see that this integral is linear in f (and also in $\mathfrak{m}$) and satisfies

$$\left\| \int f d\mathfrak{m} \right\| \leq \|f\|_\infty \|\mathfrak{m}\| (\Omega) .$$

Moreover, if $e^* \in E^*$, then $e^* \int f d\mathfrak{m} = \int f d(e^*\mathfrak{m})$ holds; indeed, for simple functions f this equality is trivial and density of simple functions in $B(\Sigma)$ proves the identity for all $f \in B(\Sigma)$.

Chapter 2

Linear Pietsch-p-integral operators

2.1 Linear Pietsch-p-integral operators

For $1 \leqslant p < \infty$, linear Pietsch-p-integral operators were introduced by Persson and Pietsch [16] and deeply studied in [5, 9] among others.

Definition 2.1. *The linear operator $u : E \longrightarrow F$, between Banach spaces E and F, is Pietsch-p-integral* $(1 \leqslant p < \infty)$ *if there are a regular Borel countably additive vector measure* $\mathbf{m}$ *of bounded semivariation on $\mathcal{B}(B_{E^*})$, (where $\mathcal{B}(B_{E^*})$ is the Borel σ algebra of B_{E^*}), and a positive regular Borel measure μ on B_{E^*} such that*

$$u(x) = \int_{B_{E^*}} < x, x^* > d\mathbf{m}(x^*), x \in E, \tag{2.1}$$

and

$$\| \int_{B_{E^*}} f d\mathbf{m} \| \leqslant \left(\int_{B_{E^*}} |f|^p d\mu \right)^{\frac{1}{p}}, \forall f \in C(B_{E^*}).$$

The Banach space of these operators is denoted by $\mathcal{PI}_p(E, F)$ under the norm defined by

$$\|u\|_{\mathcal{PI}_p} = \inf \mathbf{m}(B_{E^*})^{\frac{1}{p}}.$$

where the infimum is taken over all measures μ satisfying the above inequality.

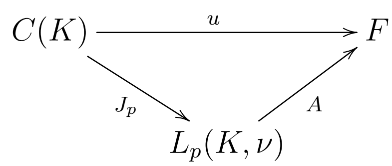
Proposition 2.2. *Let K be a compact Hausdorff space and ν be a positive regular Borel measure on K , then the inclusion J_p of $C(K)$ into $L_p(K, \mu)$ is Pietsch-p-integral and*

$$\|J_p\|_{\mathcal{PI}_p} = \|J_p\| = \mu(K)^{\frac{1}{p}} \ ,$$

Proposition 2.3. *Let (Ω, Σ, μ) be a finite measure space and i_p the inclusion of $L_\infty(\mu)$ into $L_p(\mu)$, then i_p is Pietsch-p-integral and*

$$\|i_p\|_{\mathcal{PI}_p} = \|i_p\| = \mu(\Omega)^{\frac{1}{p}}.$$

Lemma 2.4. *Let K be a compact Hausdorff space. For every $u \in \mathcal{PI}_p(C(K), F)$, we can find a finite nonnegative regular countably additive, Borel measure ν on K and an operator $A \in \mathcal{L}(L_p(K,\nu), F)$ with $\|A\| \leqslant 1$ such that the following diagram commutes*



where J_p is the natural inclusion and

$$\|u\|_{\mathcal{PI}_p} = \|J_p\|_{\mathcal{PI}_p} = \nu(K)^{\frac{1}{p}}$$

The next theorem characterize the Pietsch-p-integral operators in the terms of factorization.

Theorem 2.5. *[5] Let $1 \leqslant p < \infty$. Given $u \in \mathcal{L}(E, F)$, then the following assertions are equivalent:*

1. *u is Pietsch-p-integral.*

2. *There are a compact Hausdorff space K, a linear embedding h from E into $C(K)$, a regular countably additive vector measure $\mathbf{m} : \mathcal{B}(K) \longrightarrow F$ of bounded semivariation, and a regular, Borel measure μ on K such that*

$$u(x) = \int_K h(x) d\mathbf{m}(h), \textit{for all } x \in E \tag{2.2}$$

   and

$$\| \int_K f d\mathbf{m} \| \leqslant \left(\int_K |f|^p d\mu \right)^{\frac{1}{p}}, \textit{ for all } f \in C(K).$$

   In this case

$$\|u\|_{\mathcal{PI}_p} = \inf \|h\| \mu(K)^{\frac{1}{p}}$$

3. *There are a compact Hausdorff space K, a linear embedding h from E into $C(K)$, a regular Borel measure μ on K, and an operator $S \in \mathcal{L}(L_p(K), F)$ such that the following*

diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{\quad u \quad} & F \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle S} \\ C(K) & \xrightarrow[J_p]{} & L_p(\mu) \end{array} \quad , \tag{2.3}$$

where J_p is the natural inclusion,

4. *There are a Borel measure space (Ω, Σ, μ), an operator S from $L_\infty(\mu)$ into F, and an embedding $h^{'} \in \mathcal{L}(E, L_\infty(\Omega, \mu))$ such that the following diagram is commutative*

$$\begin{array}{ccc} E & \xrightarrow{\quad u \quad} & F \\ {\scriptstyle h'}\downarrow & & \uparrow{\scriptstyle S} \\ L_\infty(\Omega, \mu) & \xrightarrow[i_p]{} & L_p(\mu) \end{array} \quad , \tag{2.4}$$

where i_p is the natural inclusion.

If one (and then all) of these assertion holds, we can choose S in (2.3) *and* (2.4) *with* $\|S\| \leqslant .1$ *and then*

$$\|u\|_{\mathcal{PI}} = \inf \|h\| \|\mu(K)^{\frac{1}{p}}$$

where the infimum is taken over $K, h, \mathbf{m}$ and μ as in (2) *Moreover*

$$\|u\|_{\mathcal{PI}} = \inf \|h'\| \|\mu(\Omega)^{\frac{1}{p}},$$

where the infimum is taken over all Ω, h', S and μ as in (2.4)

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3) By [10, Theorem VI.2.1], $\mathbf{m}$ may be seen as the representing measure of an operator $T \in \mathcal{L}(C(K), F)$ given by

$$T(f) = \int_K f d\mathbf{m}, \text{ for all } f \in C(K).$$

If $J_p : C(K) \longrightarrow L_p(K, \mu)$ is the natural inclusion, define the linear map $S : J_p(C(K)) \longrightarrow F$ by

$$S(J_p(f)) := T(f) \text{ for all } f \in C(K).$$

For every $f \in C(K)$, we have

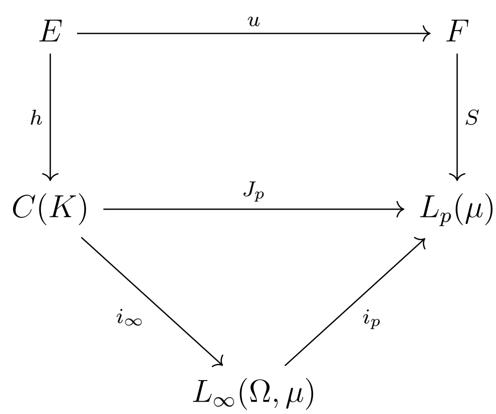
$$\|S(J_p(f))\| := \|T(f)\| = \left\|\int_K f d\mathbf{m}\right\| \leqslant \left(\int_K |f|^p d\mu\right)^{\frac{1}{p}} = \|J_p(f)\|.$$

So S is continuous. Since $J_p(C(K))$ is dense in $L_p(K, \mu)$ [11, Lemma IV.8.19], we can extend S to an operator of norm $\leqslant 1$ on $L_p(K, \mu)$, that we still call S. Then

$$S \circ J_p \circ h(x) = S(J_p \circ h(x)) = \int_K h(x) d\mathbf{m} = u(x)$$

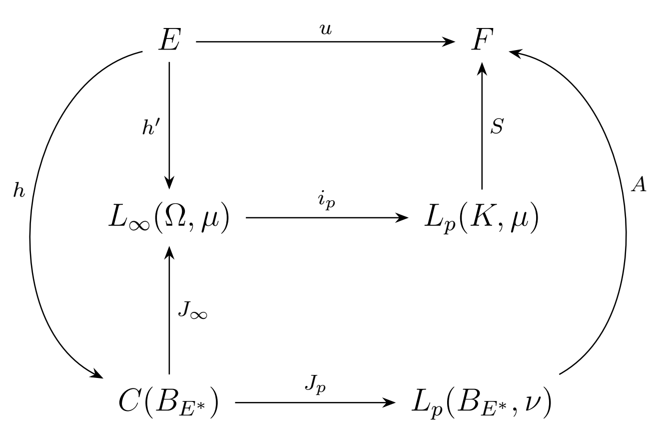
and u factors as in (3)

(3) $\Longrightarrow$ (4) Given the factorization of (2.3), J_p can be factored through $L_\infty(\Omega, \mu)$ in the form $J_p = i_p \circ i_\infty$, where i_∞ and i_p are the natural inclusions as shown in the following diagram



but $h' := i_\infty \circ h$, we obtain the diagram of (4).

(4) $\Longrightarrow$ (1) Given the diagram of (2.4). Let J_∞ is the canonical inclusion map from $C(B_{E^*})$ to $L_\infty(\mu)$. Since $S \circ i_p \circ J_\infty$ is Pietsch-p-integral, by Lemma 2.4, there are a finite nonnegative regular countably additive, Borel measure ν on B_{E^*} and an operator $A \in \mathcal{L}(L_p(B_{E^*}, \nu), F)$ such that the following diagram is commutative, where $\|A\| \leqslant 1$



and

$$\|S \circ i_p \circ J_\infty\| = \|J_p\|_{\mathcal{PI}_p} = \nu(B_{E^*})^{\frac{1}{p}}$$

Let $\mathbf{m}$ be the representing measure of $S \circ i_p \circ J_\infty$. Then

$$u(x) = S \circ i_p \circ J_\infty \circ h(x) = \int_{B_{E^*}} h(x) d\mathbf{m}$$

and u satisfies formula (2.1). Moreover, for all $f \in B_{E^*}$

$$\begin{aligned} \left\| \int_{B_{E^*}} f d\mathbf{m} \right\| &= \|S \circ i_p \circ J_\infty(f)\| = \|A \circ J_p(f)\| \\ &\leqslant \|A\| \|J_p(f)\| \\ &\leqslant \|J_p(f)\| = \left(\int_{B_{E^*}} \mid f \mid^p d\nu \right)^{\frac{1}{p}} \end{aligned}$$

By Definition 2.1, u is Pietsch-p-integral.

We now prove the equalities of the norms. Suppose that $u \in \mathcal{PI}_p(E, F)$ satisfies (2), for suitable K, h, μ, and $\mathbf{m}$. By (2) $\Longrightarrow$ (3), u factors as in (2.3) through an operator S with $\|S\|$ ≤ 1. Following the proofs of (3) $\Longrightarrow$ (4) we obtain a factorization as in diagram (2.4) with K instead of Ω, and

$$\begin{aligned} \|u\|_{\mathcal{PI}_p} &= \nu\left(B_{E^*}\right)^{\frac{1}{p}} = \|S \circ i_p \circ J_\infty\| \leq \|S\| \, \|i_p\|_{\mathcal{PI}_p} \, \|J_\infty\| \\ &\leq \mu\left(K\right)^{\frac{1}{p}} \|h'\| \leq \mu\left(K\right)^{\frac{1}{p}} \|J_\infty \circ h\| \leq \mu\left(K\right)^{\frac{1}{p}} \|h\| \end{aligned}$$

On the other hand, given $\epsilon > 0$, by the definition of the Pietsch-p-integral norm, we can find $\mathbf{m}$ and μ such that formula (2.2) holds and

$$\mu\left(K\right)^{\frac{1}{p}} \leq \|u\|_{\mathcal{PI}_p} + \epsilon$$

If h is the natural embedding of E into $C(B_{E^*})$, we have

$$\|h\| \, \mu(B_{E^*})^{\frac{1}{p}} = \mu(B_{E^*})^{\frac{1}{p}} \leq \|u\|_{\mathcal{PI}_p} + \epsilon$$

Hence

$$\|u\|_{\mathcal{PI}_p} = \inf \|h\| \, \mathbf{m}(K)^{\frac{1}{p}}$$

where the infimum is taken over all m, K, μ, and h as in (2.2).

By similar arguments, it is not difficult to prove the other equalities.

□

Proposition 2.6. *Let $1 \leqslant p \leqslant q < \infty$. Then $\mathcal{PI}_p(E,F) \subset \mathcal{PI}_q(E,F)$ and*

$$\|T\|_{\mathcal{PI}_q} \leqslant \|T\|_{\mathcal{PI}_p} \text{ for all } T \in \mathcal{PI}_p(E,F) \ .$$

Proposition 2.7. *Let $1 \leqslant p < \infty$, the class of p-integral operator constitutes a Banach ideal of linear operator*

Remark 2.8.

1. *It can be show that $\mathcal{N}_p(E,F) \subset \mathcal{PI}_p(E,F)$, with $\|.\|_{\mathcal{PI}_p} \leqslant \|, \|_{\mathcal{N}_p}$, where $\mathcal{N}_p$ is the space of p-nuclear operators.*
2. *It is worthy to note that $\mathcal{PI}_p \subset \mathcal{W}(E,F)$ and $\mathcal{PI}_p \subset \mathcal{C}(E,F)$ where $\mathcal{C}(E,F)$ denotes the set of completely continuous operators from E into F.*

2.2 Linear Pietsch-∞-integral

Definition 2.9. *The linear operator $u : E \longrightarrow F$, is called Pietsch-∞-integral if there is a regular Borel countably additive vector measure $\mathbf{m} : \mathcal{B}(B_{E^*}) \longrightarrow F$ of bounded semivariation such that*

$$u(x) = \int_{B_{E^*}} < x, x^* > d\mathbf{m}(x^*),\ x \in E.$$

In this case,

$$\|T\|_{\mathcal{PI}_\infty} = \inf \|\mathbf{m}\|(B_{E^*}),$$

taking the infimum over all **m** *that satisfying the above equality.*

In the next theorem we give the factorization of Pietsch-∞-integral operators

Theorem 2.10. *Let $u \in \mathcal{L}(E,F)$, then the following assertions are equivalent:*

1. *u is Pietsch -∞-integral.*

2. *There are a compact Hausdorff space K, an embedding $h : E \longrightarrow C(K)$, a regular Borel countably additive vector measure $\mathbf{m} : B(K) \longrightarrow F$ of bounded semivariation and a positive regular Borel measure μ on K such that*

$$u(x) = \int_{B_K} h(x) d\mathbf{m},\ forall,\ x \in E \tag{2.5}$$

3. *There are a compact Hausdorff space K, an embedding h from E into $C(K)$, and a weakly compact operator $S \in \mathcal{L}(C(K), F)$ such that the following diagram is commutative*

$$\begin{array}{ccc} E & \xrightarrow{\quad u \quad} & F \\ & {\scriptstyle h}\searrow \qquad \nearrow{\scriptstyle S} & \\ & C(K) & \end{array} \tag{2.6}$$

4. *There are a finite measure space (Ω, Σ, μ), a weakly compact operator $S \in \mathcal{L}(L_\infty(\Omega, \mu), F)$, and an embedding $h^{'} \in \mathcal{L}(E, \mathcal{L}(L_\infty, \mu)$ such that the following diagram is commutative*

$$\begin{array}{ccc} E & \xrightarrow{\quad u \quad} & F \\ & {\scriptstyle h'}\searrow \qquad \nearrow{\scriptstyle S} & \\ & L_\infty(\Omega, \mu) & \end{array} \tag{2.7}$$

If one (and then all) of these assertions holds, we have

$$\|u\|_{\mathcal{PI}_\infty} = \inf \|h\| \|S\| = \inf \|h'\| \|S\|.$$

where the infimum is taken over all $K, h,$ and S as in (2.6), *and a similar formula holds for the factorization of* (2.7).

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3) It is enough to consider the compact operator $S \in \mathcal{L}(C(K), F)$ associated with the measure $\mathbf{m}$.

(3) $\Longrightarrow$ (4) It is enough to consider the representing measure of the compact operator $S \in \mathcal{L}(C(K), F)$. [10, Theorem VI.2.1, VI.2.5 and Corollary VI.2.14], there is a finite nonnegative countably additive, Borel measure μ on K such that $\mathbf{m}$ vanishes on all sets of μ-measure zeroo. Let $S \in \mathcal{L}(L_\infty(K, \mu), F)$ be given by

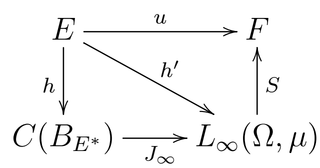
$$S(f) = \int_K f d\mathbf{m} \text{ for all } (f \in L_\infty(K, \mu)).$$

With $\|\mathbf{m}\|(K) = \|S\|$. Since $\mathbf{m}$ is countably additive, S is compact [10, Definition I.1.14 and Theorem VI.1.1].$i_\infty \in \mathcal{L}(C(K), L_\infty(K, \mu))$ be the natural inclusion and let $h' := i_\infty \circ h$. From

$$S \circ h'(x) = S \circ h' = S \circ i_\infty \circ h = \int_K h(x) = u(x), x \in E \ ,$$

we obtain (4).

(4) $\Longrightarrow$ (1) By the injectivity of $L_\infty(\Omega, \mu)$, to an operator J_∞ on $C(B_{E^*})$ with the same norm. Hence, the following diagram is commutative



Since $S \circ J_\infty$ is compact, there is a regular countably additive F-valued, Borel measure $\mathbf{m}$ of bounded semivariation such that

$$S \circ J_\infty(f) = \int_{B_{E^*}} f d\mathbf{m}(f \in C(B_{E^*})) \text{ and } \|\mathbf{m}\|(B_{E^*}) = \|S \circ J_\infty\|,$$

from [10, Theorems VI.2.1 and VI.2.5, and Corollary VI.2.14] it follows that

$$u(x) = S \circ h'(x) = S \circ J_\infty \circ h(x) = \int_{B_{E^*}} h(x) d\mathbf{m}$$

for all $x \in E$, so u is Pietsch- ∞-integral.

We now prove the equalities of the norms. If u satisfies (2.7), following the proof of (4) $\Longrightarrow$ (1), we can find a regular countably additive F-valued, Borel measure $\mathbf{m}$ of bounded semivariation such that

$$u(x) = \int_{B_{E^*}} < x^*, x > d\mathbf{m}, x \in E$$

and

$$\|u\|_{\mathcal{PI}_\infty} \leqslant \|\mathbf{m}\|(B_{E^*}) = \left\| S \circ \widetilde{B} \right\| \leqslant \|S\| \|h\|.$$

Since this is true for every factorization as in (4), we have

$$\|u\|_{\mathcal{PI}_\infty} \leqslant \inf \|h\| \|S\|.$$

On the other hand, if u is Pietsch -∞-integral, for every $\epsilon > 0$ there is a regular countably additive, Borel measure $\mathbf{m}$ on B_{E^*} satisfying Definition 2.9 and $\|\mathbf{m}\|(B_{E^*}) \leqslant \|u\|_{\mathcal{PI}_\infty} + \epsilon$. Let $h \in \mathcal{L}(E, C(B_{E^*}))$ be the natural linaer embedding.

Following the proof of (2) $\Longrightarrow$ (4), we can find a finite nonnegative countably additive, Borel measure μ on B_{E^*} and a compact operator $S \in \mathcal{L}(L_\infty(B_{E^*}, \mu), F)$ whose representing measure is $\mathbf{m}$ such that $u = S \circ h'$ and

$$\|h'\|\|S\| = \|S\| = \|\mathbf{m}\|(B_{E^*}) \leqslant \|u\|_{\mathcal{PI}_\infty} + \epsilon.$$

where $h' \in \mathcal{L}(E, L_\infty(B_{E^*}, \mu))$ is the natural linear embedding. Therefore

$$\|u\|_{\mathcal{PI}_\infty} = \inf \|S\|\|h'\|$$

where the infimum is taken over all h', S and μ as in (2.7). The equality for the factorization of (2.6) is proved in an analogous way. $\square$

Remark 2.11. *For $p = \infty$, we have*

$$\mathcal{W}(C(K), E) = \mathcal{PI}_\infty(C(K), F)$$

and

$$\mathcal{W}(L_\infty(\mu), F) = \mathcal{PI}_\infty(L_\infty(\mu), F)$$

for any banach space F, and $\|.\| = \|.\|_{\mathcal{PI}_\infty}$

Lipschitz Pietsch-p-integral

3.1 Lipschitz Pietsch-p-integral

Definition 3.1. *[15] Let $1 \leq p \leq \infty$ A map $T \in Lip_0(X, E)$ is Lipschitz p-integral if there are a Borel measure space (Ω, Σ, μ), a linear operator $A \in \mathcal{L}(L_p(\mu), E)$ and a Lipschitz operator $B \in Lip_0(X, L_\infty(\mu))$ giving rise to the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle A} \\ L_\infty(\mu) & \xrightarrow[\quad i_p \quad]{} & L_p(\mu) \end{array} \tag{3.1}$$

where $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the canonical mapping. The set of all Lipschitz Pietsch-p-integral mappings from X to E is denoted by $\mathcal{PI}_p^L(X, E)$. With each $T \in \mathcal{PI}_p^L(X, E)$ we associate its Lipschitz Pietsch-p-integral quantity,

$$\|T\|_{\mathcal{PI}_p^L} = \inf \|A\| Lip(B),$$

where the infimum is taken over all μ, A and B as above.

Proposition 3.2. *Let $T \in \mathcal{PI}_p^L(X, E)$ then $T \in Lip_0(X, E)$ and*

$$Lip(T) \leqslant \|T\|_{\mathcal{PI}_p^L}.$$

Proof. Suppose that $T \in \mathcal{PI}_p^L(X, E)$ then T has the factorization (3.1), for all $x, y \in X$ and $\epsilon > 0$ we have

$$\|T(x) - T(y)\| \leqslant \|A\| \|i_p\| \|B(x) - B(y)\|_\infty \leqslant \|A\| Lip(B) d(x, y)$$

this implies that $T \in Lip_0(X, E)$ and

$$\begin{aligned} Lip(T) &\leqslant \|A\| Lip(B) \\ &\leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon \end{aligned}$$

since this holds for all $\epsilon > 0$ we arrive at $Lip(T) \leqslant \|T\|_{\mathcal{PI}_p^L}$. $\square$

Remark 3.3.

1. *Notice that the definition3.1 is the same if we consider a finite regular Borel measure space (Ω, Σ, μ), in this case, for $T \in \mathcal{PI}_p^L(X, E)$ we have*

$$\|T\|_{\mathcal{PI}_p^L} = \inf \|A\| \mu(\Omega)^{\frac{1}{p}} Lip(B)$$

where the infimum is taken over all μ,A and B in (3.1).

2. *We dont know if being Lipschitz Pietsch-p-integrability implies Pietsch-p-integrability whenever the mapping T is linear. The converse is of course true as we see in the following proposition.*

Proposition 3.4. *If X and E are Banach spaces and $T : X \longrightarrow E$ is linear Pietsch-p-integral then T is Lipschitz Pietsch-p-integral and*

$$\|T\|_{\mathcal{PI}_p^L} \leqslant \|T\|_{\mathcal{PI}_p}$$

Proof. For $\epsilon > 0$ choose a regular Borel measure space (Ω, Σ, μ) , two linear operators $A \in \mathcal{L}(L_P(\mu), E)$ and $B \in \mathcal{L}(X, L_\infty(\mu))$ satisfies (2.4) and $\|A\|\|B\| \leqslant \epsilon + \|T\|_{\mathcal{PI}_p}$. It is clear that B is belongs to $Lip_0(X, L_\infty(\mu))$ with $Lip(B) \leq \|B\|$. This means that $T \in \mathcal{PI}_p^L(X, E)$ and

$$\|T\|_{\mathcal{PI}_p^L} \leqslant \|A\| Lip(B) = \|A\|\|B\| \leqslant \epsilon + \|T\|_{\mathcal{PI}_p}$$

Since this holds for all $\epsilon > 0$, we obtain $\|T\|_{\mathcal{PI}_p^L} \leqslant \|T\|_{\mathcal{PI}_p}$

$\square$

We have the following immediate consequence of the Definition 3.1

Proposition 3.5. *(Inclusion Theorem).*

Let $1 \leq p \leqslant q < \infty$. Then $\mathcal{PI}_p^L(X, E) \subset \mathcal{PI}_q^L(X, E)$ and $\|T\|_{\mathcal{PI}_q^L} \leqslant \|T\|_{\mathcal{PI}_p^L}$ for all $T \in \mathcal{PI}_p^L(X, E)$.

Proof. For all $\epsilon > 0$ and $T \in \mathcal{PI}_p^L(X, E)$ we choose the typical factorization $T = A \circ i_p \circ B$ and $\|A\| Lip(B) \leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon$. In the other hand we have

$$i_p = i_{q,p} \circ i_q : L_\infty(\mu) \xrightarrow{i_q} L_q(\mu) \xrightarrow{i_{q,p}} L_p(\nu),$$

where $i_{q,p} : L_q(\mu) \to L_p(\nu)$ is the canonical inclusion map. Then we obtain the factorization

$$T = A \circ i_{q,p} \circ i_q \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_q} L_q(\mu) \xrightarrow{D} E,$$

where $D = A \circ i_{q,p} \in \mathcal{L}(L_q(\mu), E)$, which implies that $T \in \mathcal{PI}_q^L(X, E)$ and

$$\begin{aligned} \|T\|_{\mathcal{PI}_q^L} &\leqslant \|D\| Lip(B) \\ &\leqslant \|A\| \|i_{q,p}\| Lip(B) \\ &\leqslant \|A\| Lip(B) \\ &\leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon. \end{aligned}$$

□

In order to prove the factorization theorem for the class of Lipschitz Pietsch-p-integral operators, $(1 \leqslant p \leqslant \infty)$ we need the following technical lemma.

Lemma 3.6. *Let $J : \mathcal{M}(X) \longrightarrow C(B_{X^\#})$ be the operator defined by*

$$J(m)(f) = \sum_{i=1}^{n} \lambda_i (f(x_i) - f(x_i'))$$

for all $m = \sum_{i=1}^n \lambda_i m_{x_i x_i'} \in \mathcal{M}(X)$ and $f \in B_{X^\#}$. Then this operator is an isometric embedding.

Proof. Since $Æ(X)^*$ and $X^\#$ are isometrically isomorphic via the linearization, for all $f \in X^\#$ there is $m^* \in Æ(X)^*$ such that $f_L = m^*$. For all $m \in \mathcal{M}(X)$ we have

$$\begin{aligned} \|J(m)\|_{C(B_{X^\#})} &= \sup_{B_{X^\#}} \mid J(m)(f) \mid \\ &= \sup_{\|f_L\| \leqslant 1} \left| \sum_{i=1}^{n} \lambda_i (f_L(m_{x_i x_i'}) \right| \\ &= \sup_{\|m^*\| \leqslant 1} |< m, m^* >| \\ &= \|m\|_{Æ(X)} = \|m\|_{\mathcal{M}(X)} \end{aligned}$$

and the proof follows. □

For $x \in X$, we denote by δ_x the functional $\delta_x : X^\# \longrightarrow \mathbb{R}$ defined as $\delta_x(f) = f(x), f \in X^\#$.

Let $\iota_X : X \longrightarrow C(B_{X^\#})$ the natural Lipschitz isometric embedding such that $\iota_X(x)$ is the restriction of δ_x to $B_{X^\#}$, for all $x \in X$.

The following theorem gives a parallel development of the factorization schemes concerning Lipschitz Pietsch-p-integral operators that highlights the role of the space $C(B_{X^\#})$.

Theorem 3.7. *Let $1 \leq p < \infty$ and let $T \in Lip_0(X, E)$. Then T is Lipschitz Pietsch-p-integral if and only if there exist a regular Borel measure ν on $B_{X^\#}$ and an operator $\widetilde{A} \in \mathcal{L}(L_p(\nu), E)$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle \iota_X}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{A}} \\ C(B_{X^\#}) & \xrightarrow[\quad j_p \quad]{} & L_p(\nu) \end{array} \tag{3.2}$$

where j_p is the canonical map. Moreover,

$$\|T\|_{\mathcal{PI}_p^L} = \inf \left\{ \|\widetilde{A}\| : T = \widetilde{A} \circ j_p \circ \iota_X \right\}$$

Proof. We write $\triangle$ the proposed infimum. Suppose that T admits a factorization (3.2). If j_∞ is the canonical inclusion map from $C(B_{X^\#})$ to $L_\infty(\nu)$ we have the factorization
$T = \widetilde{A} \circ i_p \circ j_\infty \circ \iota_X : X \xrightarrow{\iota_X} C(B_{X^\#}) \xrightarrow{j_\infty} L_\infty(\nu) \xrightarrow{i_p} L_p(\nu) \xrightarrow{\widetilde{A}} E$
Denoting by $B = j_\infty \circ \iota_X$, it follows that $B \in Lip_0(X, L_\infty(\nu))$ and $Lip(B) \leqslant 1$, which implies that T is Lipschitz Pietsch-p-integral and

$$\|T\|_{\mathcal{PI}_p^L} \leqslant \|\widetilde{A}\| Lip(B) \leqslant \|\widetilde{A}\|$$

Passing to the infimum we get $\|T\|_{\mathcal{PI}_p^L} \leqslant \triangle$.
Conversely, suppose that $T \in \mathcal{PI}_p^L(X, E)$. Fix $\epsilon > 0$, there are a regular Borel measure space (Ω, Σ, μ), an operator $A \in \mathcal{L}(L_p(\mu), E)$ and a Lipschitz mapping $B \in Lip_0(X, L_\infty(\mu))$ such that

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E$$

and $\|A\| Lip(B) \leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon$. Let $B_L \in \mathcal{L}(Æ(X), L_\infty(\mu))$ be the linearization of the Lipschitz mapping B, that is $B = B_L \circ k_X$ and $\|B_L\| = Lip(B)$. Consider the natural extension of the isometric Lipschitz embedding J, mentioned in Lemma 3.6, to Æ(X) which we denote also by J. The injectivity of $L_\infty(\mu)$ assures the existence of an operator $\widetilde{B}_L \in \mathcal{L}(C(B_{X^\#}, L_\infty(\mu))$ that extends B_L with $\|\widetilde{B}_L\| = \|B_L\|$ that is $B_L = \widetilde{B}_L \circ J$ or the following diagram commutes

$$\begin{array}{ccc} Æ(X) & \xrightarrow{\quad B_L \quad} & L_\infty(\mu) \\ & {\scriptstyle J}\searrow \qquad \nearrow{\scriptstyle \widetilde{B}_L} & \\ & C(B_{X^\#}) & \end{array}$$

The operator $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is p-summing with p-summing norm one then $i_p \circ \widetilde{B}_L$ is to with $\pi_p(i_p \circ \widetilde{B}_L) \leqslant \|\widetilde{B}_L\|$. By [9, Corollary 2.15] there exist a regular Borel measure ν on $B_{X^\#}$ and an operator $S \in \mathcal{L}(L_p(\nu), L_p(\mu))$ such that

$$i_p \circ \widetilde{B}_L = S \circ j_p : C(B_{X^\#}) \xrightarrow{j_p} L_p(\nu) \xrightarrow{S} L_p(\mu)$$

and $\pi_p(i_p \circ \widetilde{B}_L) = \|S\|$. Then

$$T = (A \circ S) \circ j_p \circ (J \circ k_X) : X \overset{J \circ k_X}{\longrightarrow} C(B_{X^\#}) \overset{j_p}{\longrightarrow} L_p(\nu) \overset{A \circ S}{\longrightarrow} E$$

Easy calculations prove that $J \circ k_X = \iota_X$, which implies that T admits a factorization of the form (3.2) with $\widetilde{A} = A \circ S$ and

$$\begin{aligned} \triangle &\leqslant \|\widetilde{A}\| \leqslant \|A\| \pi_p(i_p \circ \widetilde{B}_L) \\ &\leqslant \|A\| \|\widetilde{B}_L\| = \|A\| Lip(B) \\ &\leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon. \end{aligned}$$

Since this holds for all $\epsilon > 0$, $\triangle \leq \|T\|_{\mathcal{PI}_p^L}$ we arrive at $\triangle$. □

The next theorem provides a characterization of the class of Lipschitz Pietsch-p-integral operators, that is an integral representation with respect to a vector measure.

Theorem 3.8. *Let $1 \leq p < \infty$ and let $T \in Lip_0(X, E)$. Then T is Lipschitz Pietsch-p-integral if and only if there are a compact Hausdorff space K, a Lipschitz embedding $\phi : X \longrightarrow C(K)$ with $\phi(0) = 0$ a regular Borel countably additive vector measure $\mathbf{m} : B(K) \longrightarrow E$ of bounded semivariation and a positive regular Borel measure μ on K such that*

$$T(x) = \int_K \phi(x) d\mathbf{m}, x \in X \tag{3.3}$$

and

$$\| \int_K f d\mathbf{m} \| \leqslant \left(\int_K | f |^p \, d\mu \right)^{\frac{1}{p}} \tag{3.4}$$

for all $f \in C(K)$. In this case

$$\|T\|_{\mathcal{PI}_p^L} = \inf Lip(\phi)\mu(K)^{\frac{1}{p}},$$

where the infimum is taken over all $K, \phi, \mathbf{m}$ and μ satisfying (3.3) and (3.4) .

Proof. Suppose that $T \in \mathcal{PI}_p^L(X, E)$, and fix $\epsilon > 0$. There are a regular Borel measure ν on $B_{X^\#}$ and $\widetilde{A} \in \mathcal{L}(L_p(\nu), E)$ such that

$$T = \widetilde{A} \circ j_p \circ \iota_X : X \xrightarrow{\iota_X} C(B_{X^\#}) \xrightarrow{j_p} L_p(\nu) \xrightarrow{\widetilde{A}} E.$$

and $\|\widetilde{A}\| \leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon$. The linear operator $\widetilde{A} \circ j_p : C(B_{X^\#}) \longrightarrow E$ is Pietsch-p-integral with

$$\|\widetilde{A} \circ j_p\|_{\mathcal{PI}_p} \leqslant \|\widetilde{A}\| \|j_p\|_{\mathcal{PI}_p} \leqslant \|\widetilde{A}\| \leqslant T\|_{\mathcal{PI}_p^L} + \epsilon.$$

By Theorem 2.10 (2), there are a compact Hausdorff space K, an Lipschitz embedding $h : C(B_{X^\#}) \longrightarrow C(K)$, a regular Borel countably additive vector measure $\mathbf{m} : \mathcal{B}(K) \longrightarrow E$ of bounded semivariation and a positive regular Borel measure μ on K such that for all $x \in X$,

$$T(x) = \widetilde{A} \circ j_p(\iota_X(x)) = \int_K h(\iota_X(x)) d\mathbf{m},$$

and $\| \int_K f d\mathbf{m} \| \leqslant \left(\int_K | f |^p \, d\mu \right)^{\frac{1}{p}}$ for all $f \in C(K)$ and $\|h\|\mu(K)^{\frac{1}{p}} \leqslant \|\widetilde{A} \circ j_p\|_{\mathcal{PI}_p} + \epsilon$. Which means that (3.3) and (3.4) are true by taking into account that $\phi = h \circ \iota_X$ is a Lipschitz embedding from X to $C(K)$ vanishing at 0 with $Lip(\phi) \leqslant \|h\|$. Moreover

$$Lip(\phi)\mu(K)^{\frac{1}{p}} \leqslant \|\widetilde{A} \circ j_p\|_{\mathcal{PI}_p} + \epsilon \leqslant T\|_{\mathcal{PI}_p^L} + 2\epsilon.$$

Since this holds for every $\epsilon > 0$, it follows that $Lip(\phi)\mu(K)^{\frac{1}{p}} \leqslant T\|_{\mathcal{PI}_p^L}$.

Conversely, suppose that T satisfies the conditions (3.3) and (3.4). By [9, Theorem VI.2.1] there exists $u \in \mathcal{L}(C(K), E)$ such that $u(x) = \int_K f d\mathbf{m}, f \in C(K)$. Consider the canonical mapping

$$j_p = i_p \circ j_\infty : C(K) \xrightarrow{j_\infty} L_\infty(\mu) \xrightarrow{i_p} L_p(\nu),$$

and define $R : j_p(C(K)) \longrightarrow E$ by $R(j_p(f)) := u(f)$. The linear mapping R is well-defined and continuous with norm $\leqslant 1$ since for all $f \in C(K)$,

$$\|R(j_p(f))\| = \|\int_K fd\mathbf{m}\| \leqslant \left(\int_K | f |^p d\mu\right)^{\frac{1}{p}} = \|j_p(f)\|.$$

By [11, Lemma IV.8.19] we have $\overline{j_p(C(K))} = L_p(\mu)$, so R can be extended to a continuous linear operator $\widetilde{R} : L_p(\mu) \longrightarrow E$ with $\|\widetilde{R}\| \leqslant 1$. If we put $B = j_\infty \circ \phi$, we obtain $B \in Lip_0(X, L_\infty(\mu))$ and $Lip(B) \leqslant Lip(\phi)$. On the other hand

$$\widetilde{R} \circ i_p \circ B(x) = \widetilde{R} \circ j_p \circ \phi(x) = u(\phi(x)) = \int_K \phi(x)d\mathbf{m} = T(x)$$

and therefore T factors as in (3.1), that is $T \in \mathcal{PI}_p^L(X, E)$ and

$$\|T\|_{\mathcal{PI}_p^L} \leqslant Lip(B)\|\widetilde{R}\|\mu(K)^{\frac{1}{p}} \leqslant Lip(B)\mu(K)^{\frac{1}{p}}.$$

□

The next theorem give the relationship between the Lipschitz Pietsch-p-integral operator and its linearization.

Theorem 3.9. *Let $T \in Lip_0(X, E)$ and $1 \leqslant p < \infty$, then $T \in \mathcal{PI}_p^L(X, E)$ if and only if $T_L \in \mathcal{PI}_p(Æ(X), E)$. Moreover, we have*

$$\|T\|_{\mathcal{PI}_p^L} = \|T_L\|_{\mathcal{PI}_p}$$

Proof. Suppose that $T_L \in \mathcal{PI}_p$(Æ(X) ,E). According to Theorem 2.5 (4), for every $\epsilon > 0$ we can choose a typical factorization of T_L

$$T_L = A \circ i_p \circ R : Æ(\mathrm{X}) \xrightarrow{R} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

such that $A \in \mathcal{L}(L_p(\mu), E)$ and $R \in \mathcal{L}(Æ(X), L_\infty(\mu))$ with $\|A\|\|R\| \leqslant \|T_L\|_{\mathcal{PI}_p} + \epsilon$. It is clear that the mapping $B := R \circ k_X$ belongs to $Lip_0(X, L_\infty(\mu))$ and $Lip(B) \leqslant \|R\|$. The factorization $T = T_L \circ k_X = A \circ i_p \circ B$ implies that $T \in \mathcal{PI}_p^L(X, E)$ and

$$\|T\|_{\mathcal{PI}_p^L} \leqslant \|A\|Lip(B) \leqslant \|T_L\|_{\mathcal{PI}_p} + \epsilon$$

Conversely, if $T \in \mathcal{PI}_p^L(X, E)$, for $\epsilon > 0$ choose the following factorization of T

$$T = A \circ i_p \circ B : XL_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E$$

such that $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in Lip_0(X, L_\infty(\mu))$ with $\|A\|Lip(B) \leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon$. The uniqueness of the linearization maps gives that

$$T_L = (A \circ i_p \circ B)_L = A \circ i_p \circ B_L$$

Then we have that $T_L \in \mathcal{PI}_p(\text{Æ}(X), E)$ with

$$\|T_L\|_{\mathcal{PI}_p} \leqslant \|A\|\|B_L\| = \|A\|Lip(B) \leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon.$$

□

By Theorem 3.9 and Proposition 1.8, we have the following proposition.

Proposition 3.10. $(\mathcal{PI}_p^L, \|.\|_{\mathcal{PI}_p^L})$ *is the Banach Lipschitz operator ideal generated by the composition method from the Banach operator ideal* $\mathcal{PI}_p$. *In other words*

$$\mathcal{PI}_p^L(X, E) = \mathcal{PI}_p \circ Lip_0(X, E) \text{ isometrically}$$

for every pointed metric space X *and every Banach space* E

We say that a pointed metric space W is 1-injective (or an absolute Lipschitz retract) if for every metric space X, every subset X_0 of $X$and every Lipschitz mapping $T \in Lip_0(X_0, W)$ there is a Lipschitz mapping $\widetilde{T} \in Lip_0(X, W)$ extending T with $Lip(T) = Lip(\widetilde{T})$.

The real Banach space $L_\infty(\mu)$ for a finite measure μ is 1-injective (see [3, Chapter 1]). By the typical Pietsch-p-integral factorization of a Lipschitz mapping T, we can find a Pietsch-p-integral extension $\widetilde{T}$.

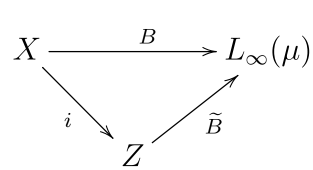
Proposition 3.11. *Let* X *and* Z *be pointed metric spaces with* $X \subset Z$ *and let* E *be a Banach space. Each Lipschitz Pietsch-p-integral operator* $T : X \longrightarrow E$ *admits a Lipschitz Pietsch-p-integral extension* $\widetilde{T} : Z \longrightarrow E$ *with*

$$\|T\|_{\mathcal{PI}_p^L} = \|\widetilde{T}\|_{\mathcal{PI}_p^L}$$

Proof. If $T \in \mathcal{PI}_p^L(X, E)$, then for all $\epsilon > 0$ there are a regular Borel probability measure space (Ω, Σ, μ), $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in Lip_0(X, L_\infty(\mu))$ such that

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E$$

and $Lip(B)\|A\| \leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon$. Since $L_\infty(\mu)$ is 1-injective, B admits an extension $\widetilde{B} \in Lip_0(Z, L_\infty(\mu))$ with $Lip(\widetilde{B}) = Lip(B)$ .i.e., the following diagram commutes



where $i \in Lip_0(X,Z)$ is the natural isometric embedding. This creates a Pietsch-p-integral extension $\widetilde{T} : Z \longrightarrow E$ having the following factorization

$$\widetilde{T} = A \circ i_p \circ \widetilde{B} : Z \xrightarrow{\widetilde{B}} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E.$$

Furthermore,

$$\|\widetilde{T}\|_{\mathcal{PI}_p^L} \leqslant Lip(\widetilde{B})\|A\| = Lip(B)\|A\| \leqslant \|T\|_{\mathcal{PI}_p^L} + \epsilon.$$

Since this holds for all $\epsilon > 0$ we get $\|\widetilde{T}\|_{\mathcal{PI}_p^L} \leqslant \|T\|_{\mathcal{PI}_p^L}$. For the reverse inequality, note that

$$\|T\|_{\mathcal{PI}_p^L} = \|\widetilde{T} \circ i_p\|_{\mathcal{PI}_p^L} \leqslant \|\widetilde{T}\|_{\mathcal{PI}_p^L}$$

□

Proposition 3.12. *Let $1 \leqslant p < \infty$, Every Lipschitz Pietsch-p-integral operator $T : X \longrightarrow E$ is Lipschitz p-summing with*

$$\pi_p^L(T) \leqslant \|T\|_{\mathcal{PI}_p^L}.$$

Proof. If $T \in \mathcal{PI}_p^L(X,E)$, for $\epsilon > 0$ we choose a typical Lipschitz Pietsch-p-integral factorization

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

with $\|A\|Lip(B) \leqslant \epsilon + \|T\|_{\mathcal{PI}_p^L}$. The mapping i_p is linear p-summing with $\pi_p(i_p) = 1$ (see [9, Page 40]). Then it is Lipschitz p-summing with $\pi_p^L(i_p) = 1$ (see [[12, Theorem 2]). By the ideal property we have that $T \in \Pi_p^L(X,E)$ and $\pi_p^L(T) \leqslant \|A\|Lip(B) \leqslant \epsilon + \|T\|_{\mathcal{PI}_p^L}$. □

Proposition 3.13. *Every Lipschitz Piesctch-p-integral is Lipschitz compact*

3.2 Lipschitz-∞-integral

In this section we extend the definition of the class of Pietsch-∞-integral linear operators to the case of Lipschitz operator and we will show a factorization theorem that characterizes these mappings (see[15]).

Definition 3.14. *Let $T \in Lip_0(X, E)$. Then T is Lipschitz Pietsch-∞-integral if there is a regular Borel countably additive vector measure $\mathbf{m} : \mathcal{B}(B_{X^\#}) \longrightarrow E$ of bounded semivariation such that*

$$T(x) = \int_{B_{X\#}} f(x) d\mathbf{m}(f), x \in X. \tag{3.5}$$

We denote by $\mathcal{PI}^L_\infty(X, E)$ the set of all Lipschitz Pietsch-∞-integral mappings and we put

$$\|T\|_{\mathcal{PI}^L_\infty} = \inf \|\mathbf{m}\|(B_{X^\#})$$

taking the infimum over all $\mathbf{m}$ such that (3.5) holds.

Proposition 3.15. *If $T \in \mathcal{PI}^L_\infty(X, E)$, then $Lip(T) \leqslant \|T\|_{\mathcal{PI}^L_\infty}$.*

Proof. Let $T \in \mathcal{PI}^L_\infty(X, E)$, for $\epsilon > 0$ choose $\mathbf{m}$ such that $\|m\|(B_{X^\#}) \leqslant \|T\|_{\mathcal{PI}^L_\infty} + \epsilon$ and for all $x, y \in X$ we have

$$\begin{aligned} \|T(x) - T(y)\| &\leqslant \int_{B_{X\#}} \mid T(x) - T(y) \mid d\mathbf{m}(f) \\ &\leqslant \|\mathbf{m}\|(B_{X^\#}) d(x, y) \\ &\leqslant (\epsilon + \|T\|_{\mathcal{PI}^L_\infty}) d(x, y) \end{aligned}$$

Hence,

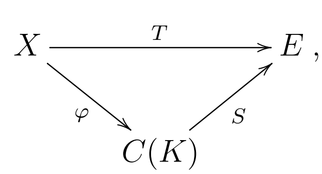
$$Lip(T) \leqslant \|T\|_{\mathcal{PI}^L_\infty}$$

□

The Pietsch-∞-integral Lipschitz operators by means of a factorization scheme through a weakly compact linear operator.

Theorem 3.16. *For a Lipschitz operator $T \in Lip_0(X, E)$ the following statements are equivalent.*

1. *T is Pietsch-∞-integral.*
2. *There are a compact Hausdorff space K a Lipschitz embedding $\varphi \in Lip_0(X, C(K))$ and a weakly compact linear operator $S \in \mathcal{L}(C(K), E)$ such that the following diagram commutes*



(3.6)

3. *There are a regular Borel finite measure space* (Ω, Σ, μ), *a compact operator* $R \in \mathcal{L}(L_\infty(\mu), E)$ *and a Lipschitz embedding* $\phi \in Lip_0(X, L_\infty(\mu))$ *giving rise to the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & E\,, \\ & {\scriptstyle \phi}\searrow \quad \nearrow{\scriptstyle R} & \\ & L_\infty(\mu) & \end{array} \tag{3.7}$$

In addition,

$$\|T\|_{\mathcal{PI}_\infty^L} = \inf \|S\| Lip(\varphi) = \inf \|R\| Lip(\phi).$$

where the first infinmum is taken over all S *and* φ *as in* (3.6) *and the second is taken over all* R *and* ϕ *as in* (3.7).

Proof. (1) $\Longrightarrow$ (2) Take $T \in \mathcal{PI}_\infty^L(X, E)$, For every $\epsilon > 0$ choose m satisfying (3.5) and $\|\mathbf{m}\|(B_{X^\#}) \leqslant \|T\|_{\mathcal{PI}_\infty^L} + \epsilon$. Consider the linear operator $S : C(B_{X^\#}) \longrightarrow E$ defined by

$$S(h) = \int_{B_{X^\#}} h d\mathbf{m}$$

for all $h \in C(B_{X^\#})$ and the natural Lipschitz isometric embedding $\iota_X : X \longrightarrow C(B_{X^\#})$. In this case, for all $x \in X$ we can write

$$S \circ \iota_X(x) = \int_{B_{X^\#}} \iota_X(x)(f) d\mathbf{m}(f) = \int_{B_{X^\#}} f(x) d\mathbf{m}(f) = T(x),$$

[10, Theorem VI.2.5 in] asserts that S is compact with norm $\|S\| = \|\mathbf{m}\|(B_{X^\#})$ and then

$$\|S\| Lip(\iota_X) = \|\mathbf{m}\|(B_{X^\#}) \leqslant \|T\|_{\mathcal{PI}_\infty^L} + \epsilon.$$

(2) $\Longrightarrow$ (3) There is a regular Borel countably additive vector measure $\mathbf{m} : \mathcal{B}(K) \longrightarrow E$ of bounded semivariation such that

$$S(h) = \int_{B_{X^\#}} f d\mathbf{m},$$

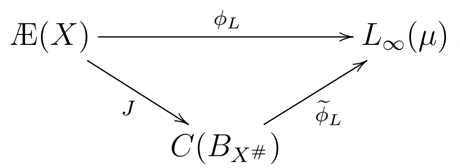
for all $f \in C(K)$ and $\|\mathbf{m}\|(K) = \|S\|$ (see [10, Theorem VI.2.1, Theorem VI.2.5 and Corollary VI.2.14]). It follows that $T(x) = S \circ \varphi(x) = \int_K \varphi(x) d\mathbf{m}$, $x \in X$. On the other hand [10, Corollary I.2.6 and Theorem I.2.1] assures the existence of a regular Borel finite measure μ on $\mathcal{B}(K)$ such that $\mathbf{m}(A) = 0$ for all $A \in \mathcal{B}(K)$ which satisfy that $\mu(A) = 0$. Define the operator $R \in \mathcal{L}(L_\infty(\mu), E)$ by

$$R(h) = \int_K f d\mathbf{m}, f \in L_\infty(\mu),$$

with $\|R\| = \|\mathbf{m}\|(K)$ (see [10, Theorem I.1.13]). This operator is compact (see [10, Definition I.1.14 and Theorem VI.1.1]). Consequently,

$$R \circ (j_\infty \circ \varphi) = \int_K j_\infty \circ \varphi d\mathbf{m} = \int_K = \phi d\mathbf{m} = T$$

(3) $\Longrightarrow$ (1) As in the proof of the second implication of Theorem 3.5 starting from the diagram (3.7), consider the linearization ϕ_L of $\phi \in Lip_0(X, L_\infty(\mu))$ and let $\widetilde{\phi}_L \in \mathcal{L}(C(B_{X^\#}), L_\infty(\mu))$ be the extension of ϕ_L, i.e. the following diagram commutes



The linear operator $R \circ \widetilde{\phi}_L$, that is

$$R \circ \widetilde{\phi}_L = \int_{B_{X\#}} f d\mathbf{m}$$

for all $f \in C(B_{X\#})$ and $\|\mathbf{m}\|(B_{X\#}) = \|R \circ \widetilde{\phi}_L\|$. It follows that

$$\begin{aligned} T(x) &= R \circ \phi(x) = R \circ \widetilde{\phi}_L \circ J \circ k_X(x) \\ &= \int_{B_{X\#}} J \circ_X (x) d\mathbf{m}, \end{aligned}$$

for all $x \in X$ and then $T \in \mathcal{PI}^L_\infty(X, E)$ and

$$\|T\|_{\mathcal{PI}^L_\infty} \leqslant \|\mathbf{m}\|(B_{X\#}) \leqslant \|R\| \|\widetilde{\phi}_L\| = \|R\| Lip(\phi).$$

In order to show the reverse inequality, take $\mathcal{PI}^L_\infty(X, E)$ and $\epsilon > 0$. Then there is $\mathbf{m}$: $\mathcal{B}(B_{X\#}) \longrightarrow E$ as in Definition 3.14 such that (3.5) is true and $\|\mathbf{m}\|(B_{X\#}) \leqslant \epsilon + \|T\|_{\mathcal{PI}^L_\infty}$. Following the proof of (2)$\Longrightarrow$(3), we can find a regular Borel finite measure μ on $B_{X\#}$ and a weakly compact operator $R \in \mathcal{L}(L_\infty(\mu), E)$ represented by $\mathfrak{m}$ such that

$$\|R\| \; Lip(\phi) = \|R\| = \|\mathfrak{m}\| \, (B_{X\#}) \leq \varepsilon + \|T\|_{\mathcal{PI}^L_\infty} \; ,$$

where $\phi \in Lip_0(X, L_\infty(\mu))$, is the Lipschitz embedding defined by $\phi = j_\infty \circ \iota_X$.

The required inequality follows and the second equality follows in a similar way. $\square$

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ملخص: في هذه المذكرة ، نقدم مفهوم المؤثرات الليبشيتزية الممثلة بتكامل من اجل $(1 \leqslant p < \infty)$ ، .يتم تعريف هذه العوامل على أنها مؤثرات ليبشيتزية معرفة
بين مساحة مترية ومساحة بناخ التي يمكن تمثيلها بتكامل فيما يتعلق بمقياس متجه محدد على مساحة هاوسدورف مدمجة مناسبة.نظهر أن هذا النوع من المشغل يتناسب مع نظرية التركيب. ونظرية عوامل غنية لهؤلاء المشغلين ، والتي توفر الكثير من المعلومات عنها. تستند نظرية التحليل هذه إلى مساحات بناخ الكلاسيكية $C(K)$و $L_p(\mu, K)$ و $L_\infty(\mu, K)$ أين K هو مساحة هوزدورف المدمجة.
الكلمات المفتاحية: قياسات المتجهات ، فضاء ايرن ايرس المؤثرات الليبشيتزية ، المثل العليا للمؤثرات الليبشيتزية ،عامل التشغيل،

<u>Resumé :</u>

Dans cette mémoire, le concept des opérateurs Lipschitz Pietsch-p-intégraux, où $(1 \leqslant p < \infty)$ .ces opérateurs sont définis comme des mappages de Lipschitz entre un espace métrique et un espace de Banach. ils peuvent être représentés par une intégrale par rapport à une mesure vectorielle définie sur un espace de Hausdorff compact approprié.
Nous montrons que ce type d'opérateur s'inscrit dans la théorie de la composition des idéaux d'opérateur de Banach Lipschitz. et également une théorie de factorisation riche pour ces opérateurs, qui fournit de nombreuses informations à leur sujet. Cette théorie de factorisation est basée sur les espaces de Banach classiques $C(K), L_p(\mu, K)$ et $L_\infty(\mu, K)$, où K est un espace de Hausdorff compact.
Nous pensons que ce travail fournit une perspective nouvelle et utile sur les opérateurs Lipschitz Pietsch-p-intégraux.
Mots-clés : Mesures vectorielles, espace de Arens-Eells,opérateur lipschitzien,Idéaux de l'opérateur de Lipschitz ,factorisation des opérateurs,

<u>Abstract :</u>

In this memory,the concept of Lipschitz Pietsch-p-integral operators, where $(1 \leqslant p < \infty)$. These operators are defined as Lipschitz mappings between a metric space and a Banach space. They can be represented by an integral with respect to a vector measure defined on a suitable compact Hausdorff space.
We show that this type of operator fits into the theory of composition Banach Lipschitz operator ideals. and a rich factorization theory for these operators, which provides a lot of information about them. This factorization theory is based on the classical Banach spaces $C(K), L_p(\mu, K)$ and $L_\infty(\mu, K)$, where K is a compact Hausdorff space.
We believe that this work provides a new and useful perspective on Lipschitz Pietsch-p-integral operators. We hope that it will be of interest to researchers in functional analysis and operator theory.
Keywords : Vector measures,, Arens-Eells space,Lipschitz operator,Lipschitz operator ideals,Lipschitz mapping, factorization of operators,