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Theme

Fuzzy sets: Concepts Results and Uses

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Fuzzy set: Concept Result and Uses

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Introduction

In 1965, a work of Zadeh under the name of fuzzy sets was published in the journal information and control. The aim of ideas and views of the Zadeh's fuzzy set theory was to describe phenomena and concepts that are inaccurate and have more meanings, which are fuzzy. Known mathematical techniques use classical set theory and two-valued logic that do not allow solving problems of this type.

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set, this is described with the aid of a membership function valued in the real unit interval $[0, 1]$. Fuzzy sets generalize classical sets, since the indicator functions (characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets if the latter only take values 0 or 1. In the fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.

The fields of application of the theory of fuzzy sets are very numerous, we have it to control and control blur, in robotics, for perform trajectory planning, in image processing, to attenuate the noise of an image, to do interpolation...etc.

This memory is divided into five chapters. In chapter 1, we recall some notions of fuzzy sets such as truth value, membership function, and the representation of a fuzzy set, along with the basic operations on fuzzy sets. In chapter 2, we give the definitions of triangular norm and triangular conorm, and discuss some particular ones and study the relationships between them. chapter 3 is devoted to fuzzy relations which is a special kind of fuzzy set that is of major importance. In chapter 4, we extend the framework of fuzzy sets from the unit interval to a lattice.

1 Generalities on fuzzy sets

This chapter reviews the concepts and notations of sets, and then introduces the concepts of fuzzy sets. The concept of fuzzy sets is a generalisation of the crisp sets.

1.1. Crisp sets

Before starting the definition of fuzzy subset, we first take care of the classical set and its properties.

The concept of a set is one of the most fundamental in mathematics. Developed at the end of the **19th** century, set theory is now a ubiquitous part of mathematics, and can be used as a foundation from which nearly all of mathematics can be derived.

Etymology: The German word Menge, rendered as "set" in English, was coined by **Bernard Bolzano** in his work The Paradoxes of the Infinite.

Definition 1.1. *A set is a well-defined collection of distinct objects. The objects that make up a set (also known as the set's elements or members) can be anything: numbers, people, letters of the alphabet, other sets, and so on.* ***Georg Cantor****, one of the founders of set theory.*

A set can be written:

In extension: *We give the list of its elements. For example, if* $a_1, a_2, a_3, \ldots, a_n$ *are the elements of set* A*, we write:*

$$A = \{a_1, a_2, a_3, \ldots, a_n\}.$$

In understanding: *We give the property or properties that characterize its elements. For example, if the elements of the set* B *satisfying the conditions* $P_1, P_2, P_3, \ldots, P_n$ *then the set* B *is defined by:*

$$B = \{b/b \text{ satisfied } P_1, P_2, P_3, \ldots, P_n\}.$$

In Characteristic Function: *A classical subset* A *of* X *is defined by a* *characteristic function* χ_A

$$\begin{array}{llll} \chi_A: & X & \longrightarrow & \{0,1\} \\ & x & \longrightarrow & \chi_A(x) \end{array}$$

Notation 1.1.

- $A = \{(x, \chi_A(x)), x \in X\}$ *is crisp set*
- $\mathcal{P}(X) = \{\chi_A / A \subseteq X\}$

Example 1.1. ***(finite case)***

1- The set F of the twenty smallest integers that are four less than perfect squares can be written:

$$F = \{n^2 - 4 : n \textit{ is an integer, and } \ 0 \leq n \leq 19\}$$

2- A is the set whose members are the first four positive integers.

Example 1.2. ***(infinite case)***

Definition 1.2. ***(power set)*** *The power set of a set S is the set of all subsets of S, including S itself and the empty set.*

Remark 1.1. *1. The power set of a set S usually written as $\mathcal{P}(S)$.*

2. The power set of a finite set with n elements has 2^n elements.

3. The power set of an infinite (either countable or uncountable) set is always uncountable.

Example 1.3. *1. The power set of the set $\{1,2,3\}$ is $\{\{1,2,3\},\{1,2\},\{1,3\},$ $\{2,3\},\{1\},\{2\},\{3\},\phi\}$.*

2. The set $\{1,2,3\}$ contains three elements, and the power set shown above contains $2^3 = 8$ elements.

Definition 1.3. ***(cardinality)*** *The cardinality $\mid S \mid$ of a set S is"the number of members of S." For example, if $B = \{blue, white, red\}, \mid B \mid = 3$.*
There is a unique set with no members and zero cardinality, which is called the empty set (or the null set).

The concept of the fuzzy subset was introduced by Zadeh [19] as a generalization of the notion of the classical set.

1.2. Basic concepts of fuzzy sets

1.2.1. Membership functions

Definition 1.4. *[19] A fuzzy set A is characterized by a generalized characteristic function μ_A: $X \longrightarrow [0,1]$, called the membership function of A and defined over a universe of discourse X.*

Remark 1.2.

$$\begin{array}{rccc} \mu_A: & X & \longrightarrow & [0,1] \\ & x & \longrightarrow & \mu_A(x) \end{array}$$

- *μ_A is called the membership function of A*
- *$\mu_A(x)$ is called the membership degree of x in A*

Notation 1.2.

- *$A = \{(x, \mu_A(x)), x \in X\}$ is fuzzy set*
  by convention

  $A = \sum_{x \in X} \frac{\mu_A(x_i)}{x_i}$ *in the discrete case*
  $A = \int \frac{\mu_A(x)}{x}$ *in the continues case*

- *$F(X)$ is the set of all fuzzy subsets of X*

Example 1.4. *$X = \{motorbike, car, train\}$ means of transport,*
A: subset of X, the means of fast transport
$A = \{(motorbike, 0.7), (car, 0.5), (train, 1)\}$

Example 1.5. *[2] Let X the set of all possible ages of people.*

$$Y(x) = \begin{cases} 1 & if \quad x < 25 \\ \frac{40-x}{15} & if \quad 25 \leq x \leq 40 \\ 0 & if \quad 40 < x \end{cases}$$

$Y(x)$ is the degree of belonging of x to the set young people

Example 1.6. *Let's define a fuzzy set $A = \{real\ number\ very\ near\ 0\}$ can*

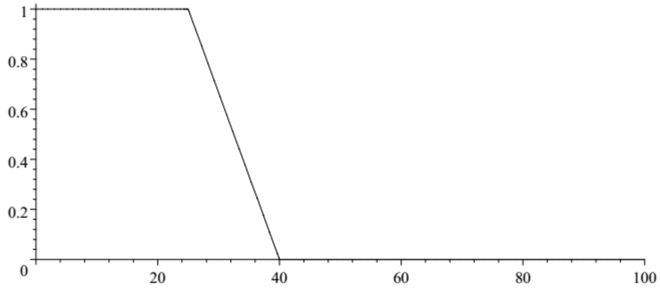


Figure 1.1: A membership function for ”Young”

be defined and its membership function is

$$\mu_A(x) = \Big(\frac{1}{1+x^2}\Big)^2$$

It is easy to calculate $\mu_A(1) = 0.25$, $\mu_A(2) = 0.04$, $\mu_A(3) = 0.01$

Example 1.7. *Consider a universal set* X *which is defined on the age domain.*

$X = \{5, 15, 25, 35, 45, 55, 65, 75, 85\}$, *and* $\mu : X \to [0, 1]$ *the membership function given by*

Age	*Infant*	*Young*	*Adult*	*Senior*
5	*0.00*	*0.00*	*0.00*	*0.00*
15	*0.00*	*0.20*	*0.10*	*0.00*
25	*0.00*	*1.00*	*0.90*	*0.00*
35	*0.00*	*0.80*	*1.00*	*0.00*
45	*0.00*	*0.40*	*1.00*	*0.10*
55	*0.00*	*0.10*	*1.00*	*0.20*
65	*0.00*	*0.00*	*1.00*	*0.60*
75	*0.00*	*0.00*	*1.00*	*1.00*
85	*0.00*	*0.00*	*1.00*	*1.00*

1.3. Fuzzy sets operations

1.3.1. Standard Operations

Let $F(x)$ denote the collection of all fuzzy sets on a given universe of discourse X.
The basic connectives in fuzzy set theory are inclusion, union, intersection, and complementation. When Zadeh introduced these operations, he based union and intersection connectives on the max and min operations.

- **Inclusion**: Let $A, B \in F(X)$. We say that the set A is included in B if
$$A(x) \leq B(x), \forall x \in X.$$
The empty (fuzzy) set $\emptyset$ is defined as $\emptyset(x) = 0, \forall x \in X$, and the total set x is $X(x) = 1, \forall x \in X$.

- **Intersection**: Let $A, B \in F(X)$. The intersection of A and B is the fuzzy set C with
$$C(x) = min\{A(x), B(x)\} = A(x) \wedge B(x), \forall x \in X.$$
We denote $C = A \wedge B$.

- **Union**: Let $A, B \in F(X)$.The union of A and B is the fuzzy set D with
$$D(x) = max\{A(x), B(x)\} = A(x) \vee B(x), \forall x \in X.$$
We denote $D = A \vee B$.

- **Complementation**: Let $A \in F(X)$ be a fuzzy set. The complement of A is the fuzzy set B given by
$$B(x) = 1 - A(x), \forall x \in X.$$
We denote $B = \bar{A}$.

Example 1.8. *If we consider the fuzzy sets*

$$A_1(x) = \begin{cases} 1 & if \quad 40 \leq x < 50 \\ 1 - \frac{x-50}{10} & if \quad 50 \leq x < 60 \\ 0 & if \quad 60 \leq x \leq 100 \end{cases}$$

$$A_2(x) = \begin{cases} 0 & if \quad 40 \leq x < 50 \\ \frac{x-50}{10} & if \quad 50 \leq x < 60 \\ 1 - \frac{x-60}{10} & if \quad 60 \leq x < 70 \\ 0 & if \quad 70 \leq x \leq 100 \end{cases}$$

then their union is

$$(A_1 \vee A_2)(x) = \begin{cases} 1 & if \quad 40 \leq x < 50 \\ 1 - \frac{x-50}{10} & if \quad 50 \leq x < 55 \\ \frac{x-50}{10} & if \quad 55 \leq x \leq 60 \\ 1 - \frac{x-60}{10} & if \quad 60 \leq x \leq 70 \\ 0 & if \quad 70 \leq x \leq 100 \end{cases}$$

The intersection can be expressed as

$$(A_1 \wedge A_2)(x) = \begin{cases} 0 & if \quad 40 \leq x < 50 \\ \frac{x-50}{10} & if \quad 50 \leq x < 55 \\ 1 - \frac{x-50}{10} & if \quad 55 \leq x < 60 \\ 0 & if \quad 60 < x \leq 100 \end{cases}$$

The complement of A_1 can be written

$$\bar{A_1}(x) = \begin{cases} 0 & if \quad 40 \leq x < 50 \\ \frac{x-50}{10} & if \quad 50 \leq x < 60 \\ 1 & if \quad 60 \leq x \leq 100 \end{cases}$$

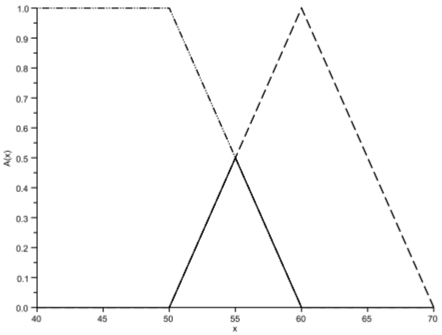


Figure 1.2: Fuzzy Intersection

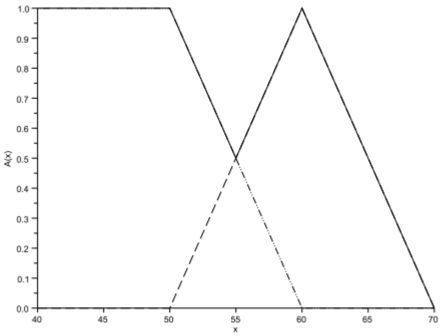


Figure 1.3: Fuzzy Union

1.3.2. Fuzzy complement

Complement set $\bar{A}$ of set A carries the sense of negation. Complement set may be defined by the following function C.

$$C : [0,1] \longrightarrow [0,1]$$

Definition 1.5. *[9] The complement function C is designed to map membership function $\mu_A(x)$of fuzzy set A to* $[0,1]$ *and the mapped value is written as $C\left(\mu_A(x)\right)$. To be a fuzzy complement function, four axioms should be satisfied.*

*(**Axiom C1**)* $C(0)=1$, $C(1)=0$ *(boundary condition)*

*(**Axiom C2**)(monotonic nonicreasing),* $a, b \in [0,1]$

$$\textit{if } a < b, \textit{ then } C(a) \geqslant C(b)$$

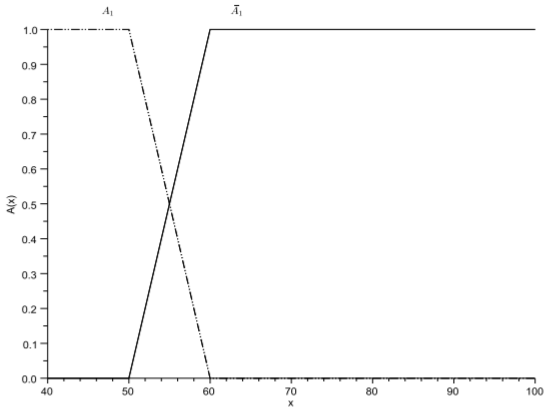


Figure 1.4: The complement of a fuzzy set

*(**Axiom C3**) C is a continuous function.*

*(**Axiom C4**) C is involutive.*

$$C(C(a)) = a \text{ for all } a \in [0,1]$$

Remark 1.3. *$C1$ and $C2$ are fundamental requisites to be a complement function. These two axioms are called "axiomatic skeleton".*

Example of Complement Function

Above four axioms hold in standard complement operator

$$C(\mu_A(x)) = 1 - \mu_A(x) \quad \text{or} \quad \mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

this standard function is shown in (Figure (1.5))

Proposition 1.1. *[9] The function defined by*

$$C_w(a) = (1 - a^w)^{\frac{1}{w}}$$

is a negation, called Yager's function.

Proof.

1. $C_w(0) = 1, C_w(1) = 0.$ (**boundary condition**)

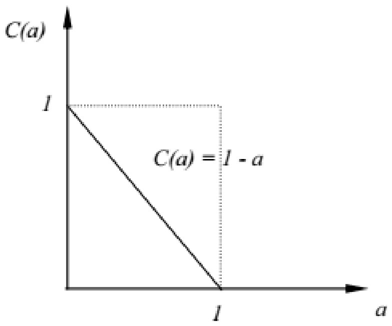


Figure 1.5: Standard complement set function

2. $a, b \in [0,1]$ if $a < b$, then

$$\begin{aligned} a^w < b^w &\Rightarrow 1 - a^w \geq 1 - b^w \\ &\Rightarrow ((1-a^w)^{\frac{1}{w}}) \geq ((1-b^w)^{\frac{1}{w}}) \\ &\Rightarrow C_w(a) \geq C_w(b) \end{aligned}$$

3. C **involutive**

$$\begin{aligned} C_w(C_w(a)) &= C((1-a^w)^{\frac{1}{w}}) \\ &= (1 - [(1-a^w)^{\frac{1}{w}}]^w)^{\frac{1}{w}} \\ &= (1-(1-a^w))^{\frac{1}{w}} \\ &= (a^w)^{\frac{1}{w}} \qquad (\textbf{monotonic nonicreasing}) \end{aligned}$$

4. C is a **continuous function**.

The shape of the function is dependent on the parameter (Figure(1.6)) ☐

Remark 1.4. *(i) When $w = 1$, the Yager's function becomes the standard complement function $c(a) = 1 - a$.*

(ii) The fuzzy complement function C is not unique see Figure(1.6)

Proposition 1.2. ***(Fundamental properties of fuzzy sets operations)*** *Let $A, B, C \in F(X)$, we have the following propriety:*

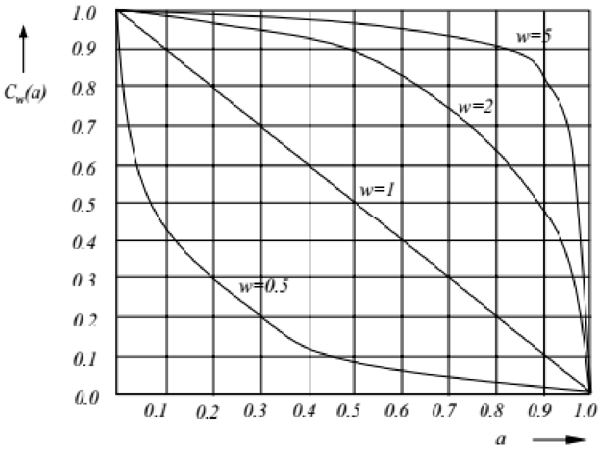


Figure 1.6: Yager complement function

Involution	$\bar{\bar{A}} = A$
Commutativity	$A \cup B = B \cup A, A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$ *and* $(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$\begin{cases} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{cases}$
Absorption	$A \cup (A \cap B) = A, A \cap (A \cup B) = A$
Idempotence	$A \cup A = A, A \cap A = A$
Absorption by X *and* $\emptyset$	$A \times X = X, A \cap \emptyset = \emptyset$
Identity	$A \cup \emptyset = A$
Law of contradiction	$A \cap \bar{A} = \emptyset$
Law of excluded middle	$A \cup \bar{A} = X$
De Morgan's laws	$\overline{A \cap B} = \bar{A} \cup \bar{B}$ *and* $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Remark 1.5. *The two principles of classical logic (the non contradiction and the excluded teirs) no longer remains valid in the theory of fuzzy sets i.e.* $A \cap \bar{A} \neq \emptyset, A \cup \bar{A} \neq X$.

Example 1.9. *let* $X = \{\boldsymbol{smal}, \boldsymbol{medium}, \boldsymbol{large}\}$ *with*
$\mu_A = (x, \mu_A(x)) = \{(\boldsymbol{smal}, 0.3), (\boldsymbol{medium}, 1), (\boldsymbol{large}, 0.6)\}$.
$\mu_{\bar{A}}(x) = 1 - \mu_A(x) = \{(\boldsymbol{smal}, 0.7), (\boldsymbol{medium}, 0), (\boldsymbol{small}, 0.4)\}$.
Hence, $\mu_A \cap \mu_{\bar{A}} = \{(\boldsymbol{smal}, 0.3), (\boldsymbol{medium}, 0), (\boldsymbol{large}, 0.4)\}$.

then, $A \cap \bar{A} \neq \emptyset$*, and* $A \cup \bar{A} \neq X$*. So, min and max is not checked.*

Fuzzy partition

Let A be a crisp set in universal set X and $\bar{A}$ be a complement set of A. The conditions $A \neq \emptyset$ and $A \neq X$ result in couple the $(A, \bar{A})$ which decomposes X into 2 subsets.

Definition 1.6. ***(Fuzzy partition)*** *In the same manner, consider a fuzzy set satisfying* $A \neq \emptyset$ *and* $A \neq X$*. The pair* $(A, \bar{A})$ *is defined as fuzzy partition. Usually, if* m *subsets are defined in* X*,* m*-tuple* $(A_1, A_2, A_3, ..., A_n)$ *holding the following conditions is called a fuzzy partition.*

(i) $\forall i, \quad A_i \neq \emptyset,$

(ii) $A_i \cap A_j = \emptyset \quad for \quad i \neq j,$

(iii) $\forall x \in X, \quad \sum_{i=0}^{m} \mu_{A_i}(x) = 1.$

1.3.3. Characteristics of fuzzy subsets

In this section, we will give definitions for characteristics of fuzzy sets : support, kernel, height and cardinality of a fuzzy subset, and we will give an example and proposition.

Definition 1.7. *[16]* ***(Support of fuzzy subset)*** *Let A be a fuzzy set on a set* X*. The support of A is the crisp subset on* X *given by*

$$Supp(A) = \{x \in X / \mu_A(x) > 0\}$$

.

Definition 1.8. *[16]* ***(Kernel of a fuzzy subset)****Let A be a fuzzy set on a set* X*. The kernel of A is the crisp subset on* X *given by*

$$Ker(A) = \{x \in X / \mu_A(x) = 1\}$$

.

Definition 1.9. *[16]* ***(Height of fuzzy subset)*** *Let A be a fuzzy set on a set* X*. The height of A is the highest value taken by its membership function* *given by*

$$H(A) = sup\{\mu_A(x)/x \in X\}$$

.

Definition 1.10. *A fuzzy subset A is said to be normal whenever $Ht(A) = 1$.*

Definition 1.11. *[19]* ***(Cardinality of a fuzzy subset)*** *The cardinality of a finite fuzzy subset A denoted $\mid A \mid$ is defined by*

$$\mid A \mid = \sum_{x \in X} \mu_A(x)$$

.

Example 1.10. *Let $X = [0,1]$ with $\alpha, \beta \in \mathbb{R}$ and let $a, b \in \mathbb{R}$. We define the fuzzy set A on X by*

$$\mu_A(x) = \begin{cases} 0, \; \textit{if } x < a - \alpha \; \textit{or } b + \beta < x \\ 1, \; \textit{if } a < x < b \\ 1 + \left(\frac{x-a}{\alpha}\right), \quad \textit{if} \quad a - \alpha < x < a \\ 1 - \left(\frac{b-x}{\beta}\right), \quad \textit{if} \quad b < x < b + \beta \end{cases}$$

Then $Ker(A) = [0,1]$, $Supp(A) = [a - \alpha, b + \beta]$ and $H(A) = 1$.

Example 1.11. *Let $X = \{1, 2, \ldots, 6\}$, and A be a fuzzy set of X given by:*

$$A = \{\langle x, \mu_A(x)\rangle\} = \{\langle 1, 0.2\rangle, \langle 2, 0.0\rangle, \langle 3, 0.8\rangle, \langle 4, 1.0\rangle, \langle 5, 0.5\rangle, \langle 6, 1.0\rangle\}.$$

Then $supp(A) = \{1, 3, 4, 5, 6\}, Ker(A) = \{4, 6\}, H(A) = \{1\}, \mid A \mid = 3.5$.

Proposition 1.3. *[21] Let A a fuzzy subset of X. The kernel and support of a fuzzy subset verify the following properties:*

$$supp(A^c) = (ker(A))^c$$
$$ker(A^c) = (supp(A))^c$$

1.3.4. Other fuzzy subset operations

Disjunctive sum

The disjunctive sum is the name of operation corresponding ”exclusive OR” logic. And it is expressed as the following((Figure (1.7))

$$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$$

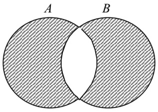


Figure 1.7: Disjunctive sum of two sets

Definition 1.12. *[9]* ***(Simple disjunctive sum)****By means of fuzzy union and fuzzy intersection, the definition of the disjunctive sum in a fuzzy set is allowed just like in the crisp set.*
$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$, *then*
$\mu_{A\oplus B}(x) = Max\{Min[\mu_A(x), 1 - \mu_B(x)], Min[1 - \mu_A(x), \mu_B(x)]\}$
Example 1.12. *Here goes procedures obtaining disjunctive sum of A and B.*
$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$
$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$
consequence,
$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.1)\}$
Definition 1.13. *[9]* ***(Disjoint sum)*** *We can define an operator Δ for the exclusive OR disjoint sum as follows.*

$$\mu_{A\Delta B}(x) =| \mu_A(x) - \mu_B(x) |$$

Difference in Fuzzy Set

The difference in crisp set is defined by

$$A - B = A \cap \bar{B}$$

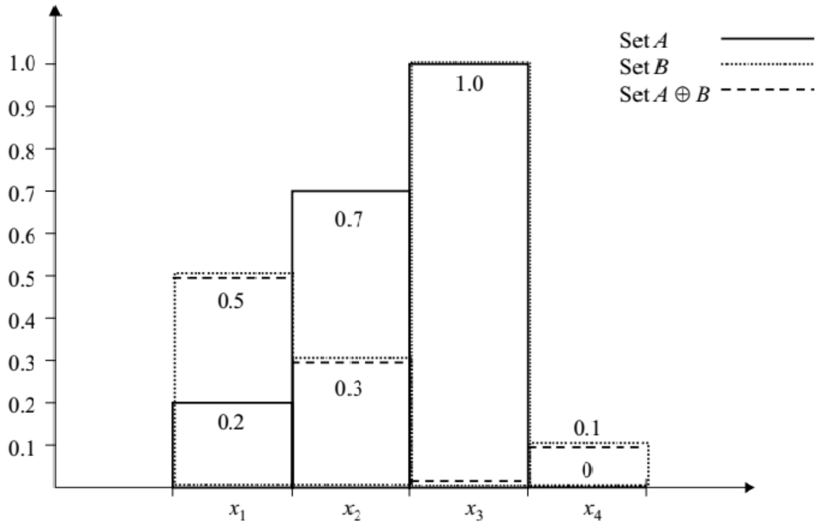


Figure 1.8: Example of simple disjunctive sum

In a fuzzy set, there are two means of obtaining the difference

(1) **Simple difference**

Example 1.13. *By using standard complement and intersection operations, the difference operation would be simple. If we reconsider the previews example,* $A - B$ *would be,(Figure(1.9))*

$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$

$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$

$\bar{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.9)\}$

$A - B = A \cap \bar{B} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0), (x_4, 0)\}$

(2) **Bounded difference**

Definition 1.14. *[9] **(Bounded difference)** For novice-operator* θ*, we define the membership function as,*

$$\mu_{A\theta B}(x) = Max[0, \mu_A(x) - \mu_B(x)]$$

Distance in Fuzzy Set

The concept 'distance' is designated to describe the difference. Measures for distance are defined in the following.

(1)Hamming distance

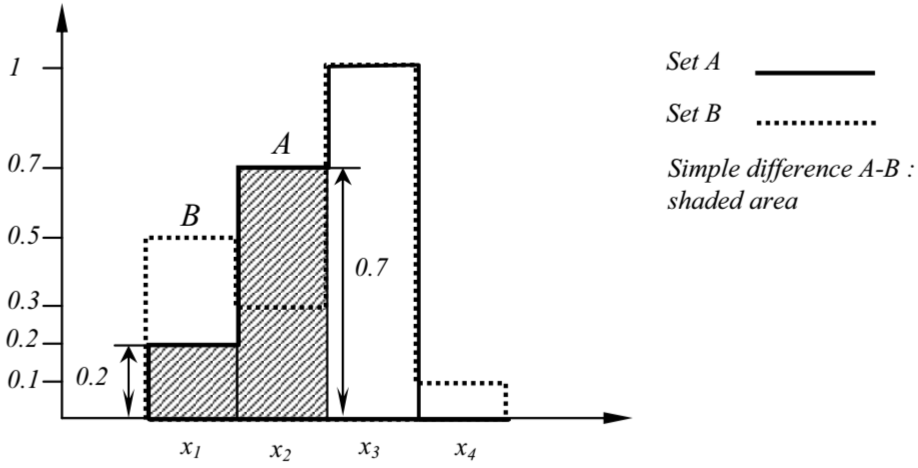


Figure 1.9: Simple difference $A - B$

This concept is marked as,

$$d(A,B) = \sum_{i=0}^{n} \mid \mu_A(x_i) - \mu_B(x_i) \mid$$

Example 1.14. *Following A and B for instance,*
$A = \{(x_1, 0.4), (x_2, 0.8), (x_3, 1), (x_4; 0)\}$
$B = \{(x_1, 0.4), (x_2, 0.3), (x_3, 0), (x_4; 0)\}$
$d(A, B) =\mid 0 \mid + \mid 0.5 \mid + \mid 1 \mid + \mid 0 \mid= 1.5$

Remark 1.6. *Hamming distance contains the usual mathematical senses of 'distance'*

(2)Euclidean distance

$$e(A,B) = \sqrt{\sum_{i=0}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}$$

Example 1.15. *Euclidean distance between sets A and B used for the previous Hamming distance is*

$$e(A,B) = \sqrt{0^2 + 0.5^2 + 1^2 + 0^2}$$

(3)Minkowski distance

$$d_w(A,B) = (\sum_{x\in X} \mid \mu_A(x) - \mu_B(x)) \mid^{w})^{\frac{1}{w}}, \quad w \in [1,\infty]$$

Generalizing Hamming distance and Euclidean distance results in Minkowski distance. It becomes the Hamming distance for $w = 1$ while the Euclidean distance for $w = 2$.

Cartesian product and Projection of fuzzy subsets

Definition 1.15. *[22] **(Cartesian product)** The Cartesian product applied to n fuzzy sets can be defined as follows: Let $\mu_{A_1}(x), \mu_{A_2}(x), \mu_{A_3}(x), ..., \mu_{A_n}(x)$ as membership function of $A_1, A_2, A_3, ..., A_n$. Then, the membership degree of $(x_1, ..., x_n) \in X_1 \times ... \times X_n$ on the fuzzy sets $A_1 \times ... \times A_n$ is,*

$$\mu_{A_1\times A_2\times ...\times A_n} = \min[\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)]$$

Example 1.16. *Lets $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, \}$ and lets A_1, A_2 are two fuzzy subsets respectively defined on X and Y given by :*
$A_1 = \{\langle x_1, 0.1\rangle; \langle x_2, 0.4\rangle; \langle x_3, 0.75\rangle\}$, and $A_2 = \{\langle y_1, 0.2\rangle; \langle y_2, 0.6\rangle\}$.
So, we find:
$\mu_{A_1\times A_2} = \{\langle (x_1, y_1), 0.1\rangle; \langle (x_1, y_2), 0.1\rangle; \langle (x_2, y_1), 0.2\rangle; \langle (x_2, y_2), 0.4\rangle;$
$\langle (x_3, y_1), 0.2\rangle; \langle (x_3, y_2), 0.6\rangle\}$

Definition 1.16. ***(Power of fuzzy sets)** The second power of fuzzy set A is defined by:*

$$\mu_{A^2}(x) = [\mu_A(x)]^2, \quad \forall x \in X.$$

Similarly, m^{th} power of fuzzy set A^m may be computed as,

$$\mu_{A^m}(x) = [\mu_A(x)]^m, \quad \forall x \in X.$$

Let A be a fuzzy subset defined on a universe $X_1 \times X_2$ cartesian product of two reference sets X_1 and X_2.

Definition 1.17. *[11] **(Projection of fuzzy subsets)** The projection on X_1 of the fuzzy set A of $X_1 \times X_2$ is the fuzzy set $Proj_{X_1}(A)$ of X_1, whose the membership function is defined by*

$$\forall x_1 \in X_1, \mu_{Proj_{X_1}(A)}(x_1) = \sup_{x_2\in X_2} \mu_A(x_1, x_2).$$

We define analogously the projection of A on X_2.

1.3.5. Representation of fuzzy subset from classical subsets

Alpha-cuts of a Fuzzy sets

One of the most important concepts of fuzzy sets is the concept of an α-cuts and it's variant.

Definition 1.18. *[9] For a given fuzzy set A on a universe X, The α-cuts of A, written A_α is defined as*

$$A_\alpha = \{x \in X, \mu_A(x) \geq \alpha\}, \quad \textit{for} \;\; \alpha \in [0,1]$$

particular cases:
(1) *if* $\alpha = 0$, *then* $A_0 = X$
(2) *if* $\alpha = 1$, *then* $A_1 = ker(A)$

Remark 1.7. *if A is a crisp set then $supp(A) = ker(A) = A = A_\alpha$*

Example 1.17. *let $X = \{1,2,3,...,10\}$, and A be a fuzzy subset of X given by*
$A = \{< 1; 0.2 >, < 2; 0.5 >, < 3; 0.8 >, < 4; 1 >, < 5; 0.7 >, < 6; 0.3 >, < 7; 0 >, < 8; 0 >, < 9; 0 >, < 10; 0 >\}$
the α-cuts of A:
$A_0 = X$
$A_{0.2} = \{x \in X, A(x) \geq 0.2\} = \{1,2,3,4,5,6\}$
$A_{0.3} = \{x \in X, A(x) \geq 0.3\} = \{2,3,4,5,6\}$
$A_{0.5} = \{x \in X, A(x) \geq 0.5\} = \{2,3,4,5\}$
$A_{0.7} = \{x \in X, A(x) \geq 0.2\} = \{3,4,5\}$
$A_{0.8} = \{x \in X, A(x) \geq 0.2\} = \{3,4\}$
$A_1 = \{x \in X, A(x) \geq 1\} = \{4\}$

Properties 1.1. ***(Basic properties of α-cuts)****Let A, B are two a fuzzy subset on a universe X and $\alpha, \beta \in [0,1]$*
(1) *if* $\alpha \leq \beta$, *then* $A_\beta \subseteq A_\alpha$
(2) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$
(3) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$

Definition 1.19. ***(The strong α-cuts)****For any α of $[0,1]$, we define the* *strong α-cut of the fuzzy subset A as the subset*

$$A^{\alpha} = \{x \in X, \mu_A(x) > \alpha\}$$

.

Remark 1.8. *The strong α-cuts have the same properties as the α-cuts.*

Representation of a fuzzy set by means of its α-cuts

Theorem 1.1. ***(Decomposition theorem)****Any fuzzy subset A of the reference set X is defined from its α-cuts by:*

$$\forall x \in X \;\; \mu_A(x) = \sup_{0<\alpha\leq 1} \; \alpha.\chi_{A_{\alpha}}(x).$$

$\chi_{A^{\alpha}}$ is the characteristic function of A^{α}.

Proof. Let $x \in X$ and put $\mu(x) = \alpha$, $\alpha \in [0,1]$ we have,

$$\begin{cases} \mu_{\alpha}(x) = 1 \quad if \quad \mu_{\alpha}(x) \geq \alpha; \\ \mu_{\alpha}(x) = 0 \quad if \quad \mu_{\alpha}(x) < \alpha; \end{cases}$$

So, $\alpha\mu_{\alpha}(x) = \alpha = \mu(x)$;
From where,

$$\sup_{\alpha\in[0,1]} \; (\alpha\mu_{\alpha}(x)) \geq \mu(x) \;\; ....(*)$$

On the other hand we have:

$$\text{for all} \;\; \alpha \in [0,1], \begin{cases} \mu_{\alpha}(x) = 1 \quad if \quad \mu_{\alpha}(x) \geq \alpha, \\ \mu_{\alpha}(x) = 0 \quad if \quad \mu_{\alpha}(x) < \alpha. \end{cases}$$

we have two cases: $\alpha\mu_{\alpha}(x) \leq \alpha \;\; \forall\alpha \in [0,1]$
Hence,

$$\sup_{\alpha\in[0,1]} \; (\alpha\mu_{\alpha}(x)) \leq \mu(x) \;\; ....(**)$$

According to $(*)$ and $(**)$ then $\forall x \in X \;\; \mu(x) = \sup_{\alpha\in[0,1]} \; (\alpha\mu_{\alpha}(x))$ ☐

Example 1.18. *Let X be the set of some countries*
$X = \{$Germany, Belgium, Spain, France, G-Brittany, Italy$\}$. We can take the fuzzy subset associated with the "southern" property:

$$A = \{< G, 0 >, < B, 0 >, < S, 1 >, < F, 0.8 >, < GB, 0 >, < I, 1 >\},$$

and build it 1*-cut* $A_1 = \{S, I\}$*identical to its core, as well as it* 0.8*-cut* $A_{0.8} = \{S, F, I\}$*, which is identical to all* α*-cuts, for all* $0 < \alpha < 0.8$*. It* 0*-cut* $A_0 = X$ *himself.*
So we get

$$\begin{aligned}
\mu_A(G) &= max(1 \times 0, ..., 0.1 \times 0, 0 \times 1) = 0,\\
\mu_A(B) &= max(1 \times 0, ..., 0, 1 \times 0, 0 \times 1) = 0,\\
\mu_A(S) &= max(1 \times 1, ..., 0 \times 1) = 1.0,\\
\mu_A(F) &= max(1 \times 0, 0.9 \times 0, 0.8 \times 1, ..., 0 \times 1) = 0.8,\\
\mu_A(GB) &= max(1 \times 0, ..., 0.1 \times 0, 0 \times 1) = 0,\\
\mu_A(I) &= max(1 \times 1, ..., 0 \times 1) = 1.
\end{aligned}$$

Which provides the definition of A*.*

2 T-norms and T-conorms

There are two types of operators in fuzzy sets: t-norms and t-conorms. They are often called triangular-norm and triangular-conorm respectively.

2.1. Triangular norms

In the fuzzy sets theory, the conjunction is usually interpreted by a triangular norm.

2.1.1. Definitions and properties

Definition 2.1. *[1] A triangular norm(t-norm) is a binary operation T on the interval $[0,1]$, i.e., $T: \; [0,1] \times [0,1] \longrightarrow [0,1]$ such that for all $x, y, z \in [0,1]$*

(T_1) $T(x,y) = T(y,x)$ *commutativity*

(T_2) $T(x, T(y,z)) = T(T(x,y),z)$ *associativity*

(T_3) $T(x,y) = T(x,z)$ *whenever $y \leq z$ monotonicity*

(T_4) $T(x,1) = x$ *boundary condition*

Example 2.1. *The following are the four basic t-norms T_M, T_P, T_L, and T_D (Figure(2.1)) given by:*

$T_M = (x,y) = min(x,y)$	*(Minimum)*
$T_P(x,y) = x.y$	*(Product)*
$T_L(x,y) = max(x+y-1,0)$	*(Lukasiewiez t-norm)*
$T(x,y) = \begin{cases} 0 & if(x,y) \in [0,1[^2 \\ \min(x,y) & otherwise \end{cases}$	*(Drastic product)*

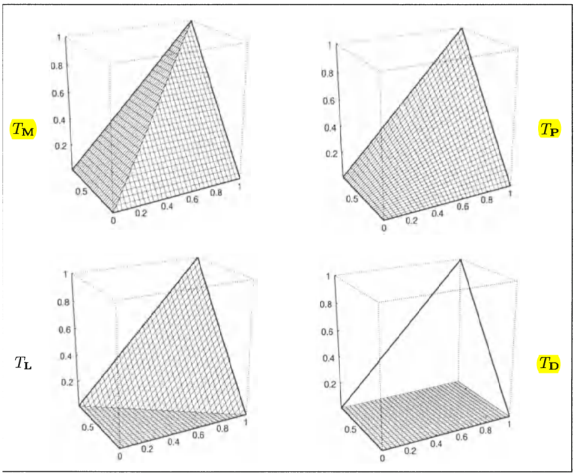


Figure 2.1: $3D$ plots of the four basic t-norm

Example 2.2. ***(Einstein t-norm)***

$$T(x,y)=\frac{xy}{(2-x-y+xy)}$$

Actually,

$$\begin{aligned} T(x,y) &= \tfrac{xy}{(2-x-y+xy)} \\ &= \tfrac{yx}{(2-y-x+yx)} \\ &= T(y,x) \end{aligned}$$

Proposition 2.1. *Any t-norm T satisfies $T(0,x)=T(x,0)=0$, for all $x\in[0,1]$*

En effet: $T(0,x)\in[0,1]\Longrightarrow T(0,x)\geq 0$

$$0\leq T(x,0)\leq T(1,0)=0$$

Hence, $T(x,0)=T(0,x)=0$

Proposition 2.2. *[8] Let A be a set satisfies $]0,1[\ \subseteq A\subseteq[0,1]$, and $F: A^2\longrightarrow A$ is a binary operation on A such that for all $x,y,z\in A$ the properties $(T_1)-(T_3)$and*

$$F(x,y)\leq\min(x,y)$$

are satisfied. Then the function defined by

$$\begin{array}{llll} T: & [0,1]\times[0,1] & \longrightarrow & [0,1] \\ & (x,y)\times(x,y) & \longrightarrow & T(x,y)=\begin{cases} F(x,y) & if \ \ (x,y)\in(A\setminus\{1\})^2 \\ \min(x,y) & otherwise \end{cases} \end{array}$$

is a t-norm.

Definition 2.2. *A function $f:[0,1]^2 \longrightarrow [0,1]$ satisfies, for all $x,y,z \in [0,1]$ the properties $(T_1)-(T_3)$ and $f(x,y)\leq \min(x,y)$ is called t-subnorm.*

Example 2.3.

1- $f(x,y)=0$. *The conditions T_1, T_2, T_3 satisfies and $f(x,y)=0\leq \min(x,y)$.*

2- $f(x,y)=\frac{x.y}{3}$.

3- $f(x,y)=x.y$.

called t-subnorm.

Remark 2.1. *Each t-norm is a t-subnorm, but not vice versa : for example, the zero function, i.e., the function $F:[0,1]^2 \longrightarrow [0,1]$ given by $F(x,y)=0$, is a t-subnorm but not a t-norm.*

Corollary 2.1. *If f is a t-subnorm then the function $T:[0,1]^2 \longrightarrow [0,1]$ defined by*

$$T(x,y)=\begin{cases} f(x,y) & if (x,y)\in[0,1[^2 \\ \\ \min(x,y) & otherwise \end{cases}$$

is a t-norm.

Definition 2.3. *Lets T_1 and T_2 two t-norms.*

(i) T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1, i.e.,

$$T_1(x,y)\leq T_2(x,y), \quad for\ all \ \ x,y\in[0,1]$$

(ii) We shall write $T_1<T_2$ whenever $T_1\leq T_2$ and $T_1\neq T_2$.

Lemma 2.1. *The drastic product T_D is the weakest, and the minimum T_M is the strongest t-norm:*

$$T_D\leq T\leq T_M.$$

proof. Since T is increasing and $1\in[0,1]$ is the neutral element of T, it

follows that $T(x,y) \leq T(x,1) = x$, for any $x,y \in [0,1]$.
In the same way $T(x,y) \leq y$. Thus, $T(x,y) \leq x \bigwedge y = T_M(x,y)$. Hence, $T \leq T_M$. Next, we prove that $T_D \leq T$, for any t-norm T. Let $x,y \in [0,1]$. If $x,y \in [0,1[$, then $T_D(x,y) = 0 \leq T(x,y)$. If $(x,y) \in \{1\} \times [01] \cup [01] \times \{1\}$

$$\begin{aligned} T_D(x,y) &= y = T(x,y) \cdots \text{ case 1} \\ T_D(x,y) &= x = T(x,y) \cdots \text{ case 2} \end{aligned}$$

Hence, $T_D \leq T$. □

Example 2.4. *[1]*

1. $T_0(x,y) = \begin{cases} 0 & if (x,y) \in [0,1[^2; \\ \min(x,y) & otherwise. \end{cases}$
2. $T_1(x,y) = \max(x+y-1,0)$
3. $T_{1.5}(x,y) = \frac{xy}{2-x-y+xy}$
4. $T_2(x,y) = xy$
5. $T_{2.5}(x,y) = \frac{xy}{x+y-xy}$
6. $T_3(x,y) = \min(x,y)$

We have $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Proof.

1. $T_0(x,y) = \begin{cases} 0 & if (x,y) \in [0,1[^2 \\ \min(x,y) & otherwise \end{cases}$
   If $(x,y) \in [0,1[^2$, then $T_0 \leq T_1$.
   If $(x,y) \notin [0,1[$ i.e., $(x,y) \in \{1\} \times [0,1]$ or $[0,1] \times \{1\}$.
   if $(x,y) \in \{1\} \times [0,1]$: $T_0(x,y) = T_0(1,y) = y$ and $T_1(x,y) = T_1(1,y) = y$ then $T_0 \leq T_1$.
   If $(x,y) \in [0,1] \times \{1\}$: $T_0(x,y) = T_0(x,1) = x$ and $T_1(x,y) = T_1(x,1) = x$ then $T_0 \leq T_1$.
   So, $T_0(x,y) \leq T_1(x,y)$
   Hence,
   $$T_0 \leq T_1$$

2. $T_1(x,y) = \max(x+y-1,0)$ there are two cases:
case1 $x+y-1 \leq 0 \Rightarrow T_1(x,y) = \max(x+y-1,0) = 0 \leq T_{1.5}(x,y)$,
case2 $x+y-1 > 0 \Rightarrow T_1(x,y) = \max(x+y-1,0) = x+y-1$.

$$\begin{aligned} T_{1.5}(x,y) - T_1(x,y) &= \tfrac{xy}{2-x-y+xy} - (x+y-1) \\ &= \tfrac{(xy-(x+y-1)(2-(x+y)+xy))}{2-x-y+xy} \end{aligned}$$

Since $(2-x-y+xy) > 0$, it is suffices to determine the sign of the numinator $[xy + (x+y-1)(x+y-xy-2)]$
$(x+y-1)(x+y-xy-2) + xy$
$= (x+y-1)((x+y-1)-(xy+1)) + xy$
$= (x+y-1)^2 - (x+y-1)(xy+1) + xy$
$= (x+y-1)^2 - x^2y - xy^2 + 2xy$
$= (x+y-1)^2 + (xy - x^2y) + (xy - xy^2) \geq 0$
Then, $\quad T_{1.5}(x,y) - T_1(x,y) \geq 0$.
Thus,

$$T_1(x,y) \leq T_{1.5}(x,y)$$

3. $T_2(x,y) = xy$

$$\begin{aligned} T_{1.5}(x,y) - T_2(x,y) &= \frac{xy}{2-x-y+xy} - xy \\ &= \frac{xy - xy(2-(x+y)+xy)}{2-x-y+xy} \end{aligned}$$

Since $2-(x-y)+xy > 0$, then it is enough to determine the sign of the numinator

$$\begin{aligned} xy - xy(2-(x+y)+xy) &= xy + xy(x+y-xy-2) \\ &= xy(x+y-xy-1) \\ &= xy((x-1)+y(1-x)) \\ &= xy(x-1)(1-y) \leq 0 \end{aligned}$$

Then, $\quad T_{1.5}(x,y) - T_2(x,y) \leq 0$.
Thus,

$$T_{1.5}(x,y) \leq T_2(x,y)$$

4. $T_{2.5}(x,y) = \frac{xy}{x+y-xy}$

$$\begin{aligned} T_2(x,y) - T_{2.5}(x,y) &= xy - \frac{xy}{x+y-xy} \\ &= \frac{xy(x+y-xy) - xy}{x+y-xy} \\ &= \frac{xy(x+y-xy-1)}{x+y-xy} \end{aligned}$$

The denominator is positive ($x+y-xy > 0$), the numerator sign should be studied

$$\begin{aligned} xy(x+y-xy-1) &= xy(x(1-y)+(y-1)) \\ &= xy((1-y)(1-x)) \leq 0 \end{aligned}$$

Then, $T_2(x,y) - T_{2.5}(x,y) \leq 0$.
Hence,

$$T_2(x,y) \leq T_{2.5}(x,y).$$

5. Finally,
$T_{2.5}(x,y) - T_3(x,y) = \begin{cases} \frac{xy}{x+y-xy} - x & if x \leq y \\ \frac{xy}{x+y-xy} - y & otherwise \end{cases}$
If $x \leq y$: $T_{2.5}(x,y) \leq \min(x,y)$.
If $x > y$: $T_{2.5}(x,y) \leq \min(x,y)$.
Then, for all $(x,y) \in [0,1[^2$ $T_{2.5}(x,y) \leq T_3(x,y)$

Consequently,

$$T_0(x,y) \leq T_1(x,y) \leq T_{1.5}(x,y) \leq T_2(x,y) \leq T_{2.5}(x,y) \leq T_3(x,y).$$

□

Proposition 2.3. *[8]*

(i) The only t-norm T satisfying $T(x,x) = x$ for all $x \in [0,1]$ is the minimum T_M.

(ii) The only t-norm T satisfying $T(x,x) = 0$ for all $x \in [01[$ is the drastic product T_D.

Proof.

(i) According to (Lemma(2.1)), to show that any idempotent t-norm T is stronger than T_M.
Indeed, $x \wedge y = T(x \wedge y, x \wedge y) \leq T(x, y)$
Hence, $T(x, y) = x \wedge y$.

□

Definition 2.4. *[3] **(Domination of t-norms)** Lets T_1 and T_2 be two t-norms. We say that T_1 dominates T_2 (in symbols $T_1 \gg T_2$) if for all $x, y, u, v \in [0,1]$*

$$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T_1(x, u), T_1(y, v)). \tag{2.1}$$

Lemma 2.2. *[1]*

(i) For each t-norm T, we have $T_M \gg T$ and $T \gg T_D$

(ii) For two t-norms T_1 and T_2. If T_1 dominates T_2 ($T_1 \gg T_2$) then, T_1 is stronger than T_2 ($T_1 \geq T_2$).

(iii) The relation ($\gg$) on the set of all t-norms is reflexive, antisymmetric and non transitive.

Proof. (i) Demonstrate by case separation

(ii) If $T_1 \gg T_2$ then, putting $y = u = 1$ in (Equation(2.1)), we obtain $T_1(x, v) \geq T_2(x, v) \quad \forall x, v \in [0,1]$ donc $T_1 \geq T_2$.

(iii) For each t-norm T and all $x, y, u, v \in [0,1]$.
reflexive: $T \gg T$ i.e., $T(T(x, y), T(u, v)) \geq T(T(x, u), T(y, v))$.
We have

$$\begin{aligned} T(T(x, y), T(u, v)) &= T(T(x, u), T(y, v)) \\ &= T(x, T(y, T(u, v)) \quad \text{(associative)} \\ &= T(x, T(T(y, u), v)) \\ &= T(x, T(T(u, y), v) \quad \text{(commutatif)} \\ &= T(x, T(u, T(y, v)) \quad \text{(associative)} \\ &= T(T(x, u), T(y, v)) \end{aligned}$$

i.e., $T \gg T$, and the assumptions $T_1 \gg T_2$ $(T_1 \geq T_2)$ and $T_2 \gg T_1$ $(T_2 \geq T_1)$ imply, $T_1 = T_2$ antisymmetric. □

Remark 2.2. *If $T_1 \geq T_2$ does not imply $T_1 \gg T_2$.*
consider the t-norm T_P and the t-norm T given by:

$$T(x,y) = \begin{cases} \frac{xy}{2} & if(x,y) \in [0,1[^2 \\ \min(x,y) & otherwise \end{cases}$$

we have $T_P \geq T$ but $T_P \gg T$ is false.

2.2. Triangular conorms

In fuzzy sets theory, the disjunction is usually interpreted by a triangular conorm.

2.2.1. Definitions and properties

Definition 2.5. *A triangular conorm(t-conorm) is a binary operation S on the interval $[0,1]$, i.e., $T : [0,1] \times [0,1] \longrightarrow [0,1]$ such that for all $x, y, z \in [0,1]$*

(S_1) $S(x,y) = S(y,x)$ *commutativity*

(S_2) $S(x, S(y,z)) = S(S(x,y), z)$ *associativity*

(S_3) $S(x,y) = S(x,z)$ *whenever $y \leq z$ monotonicity*

(S_4) $S(x,0) = x$ *boundary condition*

Example 2.5. *The following are the four basic t-conorms S_M, S_P, S_L, and S_D (Figure(2.2)) given by:*

$S_M(x,y) = max(x,y)$	*(Maximum)*
$S_P(x,y) = x + y - x.y$	*(Probabilistic sum)*
$S_L(x,y) = min(x+y, 1)$	*(Lukasiewiez t-conorm)*
$S(x,y) = \begin{cases} 1 & if(x,y) \in]0,1]^2 \\ \max(x,y) & otherwise \end{cases}$	*(Drastic sum)*

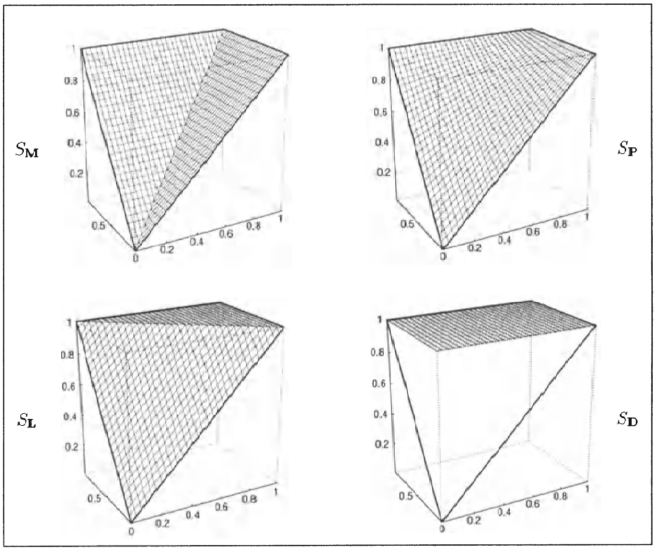


Figure 2.2: $3D$ plots of the four basic t-conorm

Proposition 2.4. *Any t-conorm S satisfies $S(1,x) = S(x,1) = 1$, for all $x \in [0,1]$.*

Proof. We know that $S(x,1) \in [0,1]$, so $S(x,1) \leq 1$, and we use the axiom (S_3), we obtient $S(x,1) \geq S(0,1) = 1$, then $S(x,1) = 1$. Thus, $S(1,x) = S(x,1) = 1$. $\square$

Proposition 2.5. *[7] A function $S : [0,1] \longrightarrow [0,1]$ is a t-conorm if and only if there exists a t-norm T such that for all $(x,y) \in [0,1]^2$ either one of the two equivalent equalities holds:*

$$S(x,y) = 1 - T(1-x, 1-y). \tag{2.2}$$

$$T(x,y) = 1 - S(1-x, 1-y). \tag{2.3}$$

Remark 2.3. *The t-conorm given by (Equation(2.2)) is called the dual t-conorm of T and, analogously, the t-norm given by (Equation(2.3)) is said to be the dual t-norm of S. Obviously, (T_M, S_M), (T_P, S_P), (T_L, S_L) and (T_D, S_D) are pairs of t-norm and t-conorm are mutually dual to each other.*

Definition 2.6. *Let T be a t-norm and S be a t-conorm. Then we say that*

T is distributive over S if for all $x, y, z \in [0,1]$

$$T(x, S(y,z)) = S(T(x,y), T(x,z)).$$

and that S is distributive over T if for all $x, y, z \in [0,1]$

$$S(x, T(y,z)) = T(S(x,y), S(x,z)).$$

Proposition 2.6. *Let T be a t-norm and S a t-conorm. Then we have:*

(i) S is distributive over T if and only if $T = T_M$.

(ii) T is distributive over S if and only if $S = S_M$.

(iii) (T, S) is a distributive pair if and only if $T = T_M$ and $S = S_M$.

Remark 2.4. *If $T_1 \leq T_2$, then $S_1 \geq S_2$. Indeed*

$$\begin{aligned} T_1(1-x, 1-y) &\leq T_2(1-x, 1-y) \\ 1 - T_1(1-x, 1-y) &\geq 1 - T_2(1-x, 1-y) \\ S_1(x,y) &\geq S_2(x,y) \end{aligned}$$

Consequence 2.1. *For each t-conorm S, the maximum S_M is the weakest and the drastic sum S_D is the strongest t-conorm. We have:*

$$S_M \leq S \leq S_D.$$

3 Fuzzy Relation and Composition

In this chapter, we study two types of relations, i.e., **crisp relation** and **fuzzy relation.**

3.1. Crisp Relation

In this section, we recall some basic definitions concerning crisp relations.

3.1.1. Definitions and properties

Definition 3.1. ***(Cartesian Product)****Let X and Y be two non-empty sets, the Cartesian product $X \times Y$ is defined by,*

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

Example 3.1. *When $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2\}$ the Cartesian product given by ((Figure (3.1))*

$$X \times Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2)\}.$$

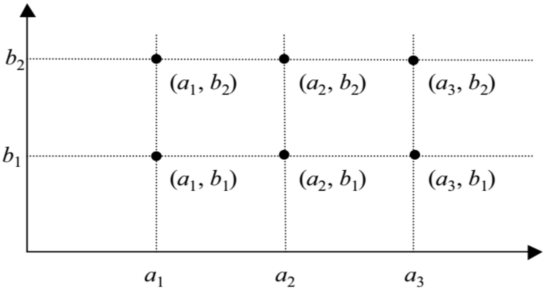


Figure 3.1: Product set $X \times Y$

Definition 3.2. ***(Crisp relation)*** *Let X and Y be two non-empty sets. A subset $\mathcal{R} \subseteq X \times Y$ is a classical relation.*
A classical relation can be characterized by a function $\mathcal{R} : X \times Y \to \{0, 1\}$,

$$\mathcal{R}(x,y) = \begin{cases} 1 & if (x,y) \in \mathcal{R}, \\ 0 & otherwise. \end{cases}$$

This relation can be represented by a matrix.

Example 3.2. *Let $\mathcal{R}$ be "Owning Cars"*
$X = \{Ali, Maher, Kamel\}$, $Y = \{BMW, Chrysler, Ford, Mazda, Fiat\}$.

R	*BMW*	*Chrysler*	*Ford*	*Mazda*	*Fiat*
Ali	*1*	*0*	*0*	*0*	*1*
Maher	*1*	*0*	*1*	*1*	*0*
Kamel	*0*	*1*	*0*	*0*	*0*

Example 3.3. *Let $\mathcal{R}$ be a crisp relation among the two sets $X = \{$dollar, pound, franc, mark$\}$, and $Y = \{$United States, France, Canada, Britain, Germany$\}$, which associates a country with a currency as follows: $\mathcal{R}(X, Y) = \{$(dollar,United States), (franc,France), (dollar,Canada), (pound, Britain),(mark,Germany)$\}$.*

$\mathcal{R}$	*U.S.*	*France*	*Canada*	*Britain*	*Germany*
dollar	*1*	*0*	*1*	*0*	*0*
pound	*0*	*0*	*0*	*1*	*0*
franc	*0*	*1*	*0*	*0*	*0*
mark	*0*	*0*	*0*	*0*	*1*

Definition 3.3. ***(Domain and range)*** *Let $\mathcal{R}$ stands for a relation between A and B. The domain and range of this relation are defined as follows:*

$$dom(\mathcal{R}) = \{x | x \in A, (x, y) \in R \text{ for some } y \in B\}.$$

$$ran(\mathcal{R}) = \{y | y \in B, (x, y) \in R \text{ for some } x \in A\}.$$

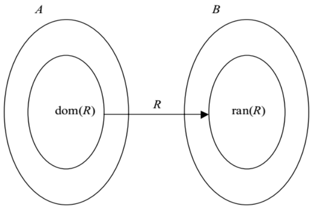


Figure 3.2: Domain and range

3.1.2. Operations on relations

If we assume $\mathcal{R}$ and $\mathcal{S}$ are relations defined on the same space $A \times B$, these relations might have operations of union, intersection, inverse and composition.

1. **Union of relations**
   $\mathcal{T} = \mathcal{R} \cup \mathcal{S}$ is said to be the union of $\mathcal{R}$ and $\mathcal{S}$.
   $$\text{If } (x, y) \in \mathcal{R} \text{ or } (x, y) \in \mathcal{S}, \text{ then } (x, y) \in \mathcal{T}.$$
2. **Intersection of relations**
   $\mathcal{T} = \mathcal{R} \cap \mathcal{S}$ is said to be the intersection of $\mathcal{R}$ and $\mathcal{S}$.
   $$\text{If } (x, y) \in \mathcal{R} \text{ and } (x, y) \in \mathcal{S}, \text{ then } (x, y) \in \mathcal{T}.$$
3. **Complement of relation**
   $A \times B$ represents all possible relations that can occur between two sets. That means it is equivalent to the concept of a universal set.
   $$\bar{\mathcal{R}} = \{(x, y) \in A \times B | (x, y) \notin \mathcal{R}\}.$$
4. **Inverse relation**
   Let $\mathcal{R}$ be a relation from A to B. The inverse $\mathcal{R}^{-1}$ is defined as
   $$\mathcal{R}^{-1} = \{(y, x) \in B \times A | (x, y) \in \mathcal{R}, x \in A, y \in B\}.$$
5. **Composition**
   Let $\mathcal{R} \subseteq A \times B, S \subseteq B \times C$, then the composition $\mathcal{T} = \mathcal{R} \circ \mathcal{S} \subseteq A \times C$
   $$\mathcal{T} = \{(x, z) \text{ such that il exists } y \text{ in } B \text{ satisfying } (x, y) \in \mathcal{R}, (y, z) \in \mathcal{S}\}.$$

Fundamental Properties

Now, we shall see the fundamental properties of a relation defined on the same set A, that is, $\mathcal{R} \subseteq A \times A$. We will review the properties such as reflexive relation, symmetric relation, transitive relation, equivalence relation and order relation in detail.

(i) Reflexive relation : $(x,x) \in \mathcal{R}$ or $\mu_{\mathcal{R}}(x,x) = 1, \quad \forall x \in A$.

(ii) Symmetric relation : $(x,y) \in \mathcal{R} \Rightarrow (y,x) \in \mathcal{R}$ or $\mu_{\mathcal{R}}(x,y) = \mu_{\mathcal{R}}(y,x), \forall x, y \in A$.

(iii) Antisymmetric: $(x,y) \in \mathcal{R}$ and $(y,x) \in \mathcal{R} \Rightarrow x = y, \quad \forall x, y \in A$.

(iv) Transitive relation : $(x,y) \in \mathcal{R}, (y,z) \in \mathcal{R} \Rightarrow (x,z) \in \mathcal{R}, \forall x, y, z \in A$.

Definition 3.4. ***(Equivalence relation)****Relation $\mathcal{R} \subseteq A \times A$ is an equivalence relation if it's reflexive, symmetric, and transitive.*

Definition 3.5. ***(Order relation)*** *A relation $\mathcal{R}$ on a set A is a partial order (or partial ordering) on A if $\mathcal{R}$ is reflexive, antisymmetric and transitive.*

A set A with a partial order is called a partially ordered set, or poset (for short).

Example 3.4. *The natural ordering " $\leq$ " on the set of real numbers $\mathbb{R}$.*

3.2. Fuzzy Relation

In this section, we recall some basic definitions concerning fuzzy relations.

3.2.1. Definitions and properties

Definition 3.6. *[20]* ***(Fuzzy Relation)*** *Let X, Y be two classical sets. A mapping R: $X \times Y \to [0,1]$ is called a* ***fuzzy relation****. The number $R(x,y) \in [0,1]$ can be interpreted as the degree of relationship between x and y.*

$$R = \{| \, (x,y), R(x,y) \mid (x,y) \in X \times Y\}.$$

Particular case:

(i) If $X = Y, R$ is said to be a fuzzy binary relation defined on X.

(ii) If X and Y are finite, a fuzzy relation R defined on both universes X and Y can be written by a matrix M.

Notation 3.1. *A fuzzy relation can be seen as a fuzzy subset of the set $X \times Y$. We denote by $F(X \times Y)$ the family of all fuzzy relations between elements of X and Y.*

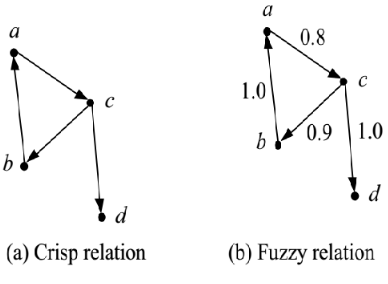
Example 3.5. ***(Finite case)****Let R be a fuzzy relation among the two sets the distance to the target $X = \{far, close, very\ close\}$ and the speed of the car $Y = \{very\ slow, slow, normal, quick, very\ quick\}$, which represents the relational concept "the break must be pressed very strong".*

This relation can also be represented by the following two dimensional membership array:

R	*very slow*	*slow*	*normal*	*quick*	*very quick*
far	*0*	*0*	*0*	*0.1*	*0.2*
close	*0.3*	*0.4*	*0.5*	*0.6*	*0.7*
very close	*0.8*	*0.9*	*1*	*1*	*1*

Example 3.6. *for instance, crisp relation $\mathcal{R}$ in figure (a) reflects a relation in $A \times A$. Expressing this by a membership function, $\mu_{\mathcal{R}}(a, c) = 1.0, \mu_{\mathcal{R}}(c, b) = 1.0, \mu_{\mathcal{R}}(b, a) = 1.0, \mu_{\mathcal{R}}(c, d) = 1.0$.*

If this relation is given as the value between 0 and 1 as in the figure (b), this relation becomes a fuzzy relation. Expressing this fuzzy relation by , $\mu_{\mathcal{R}}(a, c) = 0.8, \mu_{\mathcal{R}}(c, b) = 0.9, \mu_{\mathcal{R}}(b, a) = 1.0, \mu_{\mathcal{R}}(c, d) = 1.0$.



(a) Crisp relation (b) Fuzzy relation

Figure 3.3: Crisp and Fuzzy relation

Example 3.7. ***(Infinite case)*** *Let $\mathbb{N}$ the set of positive integers, and let R be the fuzzy relation defined on $\mathbb{N}$ by*

$$R(x, y) = \begin{cases} 1 & if(x, y) \in x \neq y \\ \frac{|x-y|}{x+y} & otherwise \end{cases}$$

Definition 3.7. *[10] **(Inverse fuzzy relation)** Let R be a fuzzy binary relation between X and Y, the inverse relation for R, denoted by R^{-1}, is defined by:*

$$\mu_{R^{-1}}(x,y) = \mu_R(y,x), \quad \forall x,y \in X \times Y.$$

Definition 3.8. *[1][22] The α-cut, strong α-cut and the support of a fuzzy relations is defined as in fuzzy sets, i.e., the α-cut and strong α-cut of a fuzzy relation $R : X \times X \to [0,1]$ is defined as, for all $x, y \in X$*

$$R_\alpha = \Big\{(x,y) \in X^2 : R(x,y) \geq \alpha\Big\}, \textit{where } \alpha \in]0,1].$$

$$R_\alpha = \Big\{(x,y) \in X^2 : R(x,y) > \alpha\Big\}, \textit{where } \alpha \in]0,1].$$

In the same way, we define the support of a fuzzy relation $S(R)$ as

$$S(R) = \Big\{(x,y) \in X^2 : R(x,y) > 0\Big\}.$$

Definition 3.9. *[11] **(Composition of fuzzy relation)** Two fuzzy relations R and S are defined on the sets A, B, and C. That is, $R \subseteq A \times B, S \subseteq B \times C$. The composition $S \circ R = SR$ of two relations R and S are expressed by the relation from A to C, and this composition is defined by the following. $\forall (x,y) \in A \times B, (y,z) \in B \times C$*

$$\begin{aligned} \mu_{S \circ R}(x,z) &= \mathrm{Max}_y \left[\mathrm{Min}\left(\mu_R(x,y), \mu_S(y,z)\right)\right] \\ &= \vee_y \left[\mu_R(x,y) \wedge \mu_S(y,z)\right] \end{aligned}$$

*This composition is called **Max-Min composition**.*

Remark 3.1. *This definition corresponds to the one traditionally used, but it is possible to replace min with some t-norm and the max by the corresponding t-conorm.*

Example 3.8. *[9] Consider fuzzy relations $R \subseteq A \times B, S \subseteq B \times C$ defined by their tables:*

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

the composition of the relations R and S is the following

S ∘ R	α	β	γ
1	*0.4*	*0.2*	*0.3*
2	*0.3*	*0.3*	*0.3*
3	*0.8*	*0.9*	*0.8*

Definition 3.10. *[1] Let $X \neq \emptyset$, $R : X^2 \to [0,1]$ a fuzzy binary relation on X. R is said to be:*

(i) *Reflexive if* $: \forall x \in X, : \mu_R(x,x) = 1;$

(ii) *Irreflexive if* $: \forall x \in X, : \mu_R(x,x) = 0;$

(iii) *Symmetrical if* $: \forall x, y \in X, \mu_R(x,y) = \mu_R(y,x);$

(iv) *Asymmetrical if*: $\forall x, y \in X, \mu_R(x,y) \wedge \mu_R(y,x) = 0,$ *with* $x \neq y$;

(v) *Antisymmetric if*: $\forall x, y \in X, (\mu_R(x,y) > 0) \wedge (\mu_R(y,x) > 0)$ *then* $x = y$;

(vi) *Transitive if* $: \forall x, y, z \in X, \mu_R(x,z) \geq \sup\{\min_{y \in X}(\mu_R(x,y), \mu_R(y,z))\}.$

Definition 3.11. *Let T be any triangular norm, R a fuzzy relation*

1. *R is said to be T-asymmetric if:*

$$T(R(x,y), R(y,x)) = 0, \quad x \neq y.$$

2. *R is said to be T-transitive if:*

$$T(R(x,y), R(y,z)) \leq R(x,z).$$

Example 3.9. *Let $X = \{a, b, c\}$, and let R be the fuzzy relation defined by the following table*

R	a	b	c
a	1.0	0.4	0.5
b	0.4	1.0	0.8
c	0.1	0.5	1.0

R is T_L antisymmetric, but not T_M antisymmetric.

3.2.2. Particular classes of fuzzy relations

Definition 3.12. *[4]* ***(Fuzzy equivalence relation)****A fuzzy equivalence relations R on X is a fuzzy relations that is:*

1. *Fuzzy reflexive.*

2. *Fuzzy symmetrical.*

3. *Fuzzy transitive.*

Example 3.10. *Let $X = \{a, b, c, d\}$, and let R be the fuzzy relation defined by the following table*

R	a	b	c	d
a	*1.0*	*0.8*	*0.7*	*1.0*
b	*0.8*	*1.0*	*0.7*	*0.8*
c	*0.7*	*0.7*	*1.0*	*0.7*
d	*1.0*	*0.8*	*0.7*	*1.0*

It is easy to verify that, R is a fuzzy equivalence relation

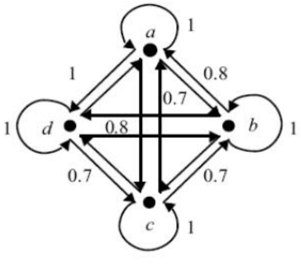


Figure 3.4: Fuzzy equivalence relation

Remark 3.2. *[12] When R is reflexive, then the transitivity can be rewritten by*
R is transitive $\Leftrightarrow R(x,z) = \sup\limits_{y\in X} \min\{R(x,y), R(y,z)\}, \forall x, y, z \in X$
First, we know that $R(x,z) \geq \sup\limits_{y\in X} \min\{R(x,y), R(y,z)\}, \forall x, y, z \in X$. *On the other hand,*

$$\begin{aligned}\sup_{y\in X}\min\{R(x,y),R(y,z)\} &\geq \min\{R(x,x),R(x,z)\}\\ &= \min\{1,R(x,z)\}\\ &= \quad R(x,z)\end{aligned}$$

Therefore, we have $R(x,z)=\sup\limits_{y\in X}\min\{R(x,y),R(y,z)\}$

Definition 3.13. ***(Fuzzy order relation)*** *A fuzzy relation R on X is a partial fuzzy order if the following conditions are satisfies*

1. *Fuzzy reflexive, for all $x\in X, R(x,x)=1$*
2. *Fuzzy antisymmetric, for all $x,y\in X$,* $\begin{cases}\mu_R(x,y)>0\\ \mu_R(y,x)>0\end{cases}\Rightarrow x=y$
3. *Fuzzy transitive, for all $x,y,z\in X, R(x,z)\geq\sup\limits_{y\in X}(\min(R(x,y),R(y,z)))$*

Example 3.11. ***(Finite case)*** *The following membership matrix defines a fuzzy partial ordering R on the set $X=\{a,b,c,d,e\}$:*

R	a	b	c	d	e
a	*1.0*	*0.7*	*0.0*	*1.0*	*0.7*
b	*0.0*	*1.0*	*0.0*	*0.9*	*0.0*
c	*0.5*	*0.7*	*1.0*	*1.0*	*0.0*
d	*0.0*	*0.0*	*0.0*	*1.0*	*0.0*
e	*0.0*	*0.1*	*0.0*	*0.9*	*1.0*

Example 3.12. ***(Infinite case)*** *Let $x,y\in\mathbb{N}$. Then the following fuzzy relation R on $\mathbb{N}$ is a fuzzy order, where*

$$\mu_R(x,y)=\begin{cases}1 & if \quad x=y,\\ 1-\frac{x}{y} & if \quad x<y,\\ 0 & if \quad x>y.\end{cases}$$

Remark 3.3. *The classical order relation $\leq$ is a fuzzy order relation. The fuzzy relation associated with the usual order*

$$\mu_R(x,y)=\begin{cases}1 & if x=y\\ \frac{1}{2} & if x<y\\ 0 & otherwise\end{cases}$$

Proposition 3.1. *[12] Let $\mathcal{R} : X^2 \to [0,1]$ be a fuzzy relation. $\mathcal{R}$ is a fuzzy partial order relation on X if and only if $\mathcal{R}_\alpha$ is partial order relations on X, for all $\alpha \in]0,1]$.*

Proof. $\Rightarrow$) Let $\mathcal{R}$ be a fuzzy partial order relation on X.
Since $\mathcal{R}(x,x) = 1$ for all $x \in X$, then $(x,x) \in \mathcal{R}_\alpha$ for all $\alpha \in]0,1]$. Then, $\mathcal{R}_\alpha$ is reflexive.
Suppose $(x,y) \in \mathcal{R}_\alpha$ and $(y,x) \in \mathcal{R}_\alpha$. Then, $\mathcal{R}(x,y) \geq \alpha > 0$ and $\mathcal{R}(y,x) \geq \alpha > 0$, by the fact that $\mathcal{R}$ is fuzzy antisymmetric, $x = y$. Then, $\mathcal{R}_\alpha$ is antisymmetric.
Suppose $(x,y) \in \mathcal{R}_\alpha$ and $(y,z) \in \mathcal{R}_\alpha$. Then, $\mathcal{R}(x,y) \geq \alpha$ and $\mathcal{R}(y,z) \geq \alpha$, as $\mathcal{R}$ is transitive so $\mathcal{R}(x,z) \geq \sup_{y \in X} \min\{\mathcal{R}(x,y), \mathcal{R}(y,z)\}, \mathcal{R}(x,z) \geq \min\{\mathcal{R}(x,y), \mathcal{R}(y,z)\} \geq \alpha$, that is, $(x,z) \in \mathcal{R}_\alpha$. Hence, $\mathcal{R}_\alpha$ is transitive.
Then, $\mathcal{R}_\alpha$ is a partial order relation on X.
$\Leftarrow$) Suppose that $\mathcal{R}_\alpha$ is a partial order relation, for all $\alpha \in]0,1]$. Hence, $(x,x) \in \mathcal{R}_\alpha$ for all $\alpha \in]0,1]$. Thus $(x,x) \in \mathcal{R}_1$, that is, $\mathcal{R}(x,x) = 1$. Then, $\mathcal{R}$ is reflexive.
Suppose that $\mathcal{R}(x,y) > 0$ and $\mathcal{R}(y,x) > 0$. Then, $\mathcal{R}(x,y) > u > 0$ for some $u \in]0,1]$ and $\mathcal{R}(y,x) > v > 0$ for some $v \in]0,1]$. Take $w = min(u,v)$. Then, $\mathcal{R}(x,y) > w > 0$ and $\mathcal{R}(y,x) > w > 0$. Thus, $\{(x,y),(y,x)\} \in \mathcal{R}_w$. Since $\mathcal{R}_w$ is antisymmetric, $x = y$. Then, $\mathcal{R}$ is antisymmetric ☐

Example 3.13. *Let $X = \{a,b,c\}$, and let $\mathcal{R}$ be the fuzzy order relation defined by the following table*

$\mathcal{R}$	a	b	c
a	1.0	0.0	0.0
b	0.3	1.0	0.0
c	0.2	0.6	1.0

If $0.6 < \alpha \leq 1.0$ then $\mathcal{R}_\alpha = " = "$ is a fuzzy order relation defined by the following table

$\mathcal{R}_\alpha$	a	b	c
a	*1.0*	*0.0*	*0.0*
b	*0.0*	*1.0*	*0.0*
c	*0.0*	*0.0*	*1.0*

3.2.3. Operations of Fuzzy Relation

We know with certainty that this fuzzy relationship is a fuzzy set. So we can apply operations on sets to relationships.

Definition 3.14. *[15] Let $\mathcal{R}$ and $\mathcal{S}$ be two fuzzy relations between X and Y, for all $(x,y) \in X \times Y$ we can define:*

1. $R = S \Leftrightarrow \mu_{\mathcal{R}}(x,y) = \mu_{\mathcal{S}}(x,y)$.
2. $\mathcal{R} \leq \mathcal{S} \Leftrightarrow \mu_{\mathcal{R}}(x,y) \leq \mu_{\mathcal{S}}(x,y)$.
3. $\mathcal{R} \cup \mathcal{S} = \{\langle (x,y), \max(\mathcal{R}(x,y), \mathcal{S}(x,y))\rangle\}$.
4. $\mathcal{R} \cap \mathcal{S} = \{\langle (x,y), \min(\mathcal{R}(x,y), \mathcal{S}(x,y))\rangle\}$.
5. $\mathcal{R}^c = \{\langle (x,y), 1 - \mu_{\mathcal{R}}(x,y)\rangle\}$.

Example 3.14. *Let $\mathcal{R}$ and $\mathcal{S}$ be two fuzzy relations on $X \times X$ such that $X = \{x,y,z\}$, represented by the following tables*

$\mathcal{R}$	x	y	z
x	1.0	0.8	0.7
y	0.8	1.0	0.7
z	0.7	0.7	1.0

$\mathcal{S}$	x	y	z
x	0.6	0.2	0.7
y	0.9	0.0	1.0
z	0.1	0.7	0.6

The union and intersection relations defined by

$\mathcal{R} \cup \mathcal{S}$	x	y	z
x	1.0	0.8	0.7
y	0.9	1.0	1.0
z	0.7	0.7	0.6

$\mathcal{R} \cap \mathcal{S}$	x	y	z
x	0.6	0.2	0.7
y	0.8	0.0	0.7
z	0.1	0.7	0.6

The complementary relation is given by the following table

$\mathcal{R}$	x	y	z
x	0.0	0.2	0.3
y	0.2	0.0	0.3
z	0.3	0.3	0.0

Proposition 3.2. *[15] Let $\mathcal{R}, \mathcal{S}$ and $\mathcal{Q}$ be three fuzzy relations of $X \times Y$ then:*

(i) $\mathcal{R} \subseteq \mathcal{S} \Rightarrow \mathcal{R}^{-1} \subseteq \mathcal{S}^{-1}$;

(ii) $(\mathcal{R} \cup \mathcal{S})^{-1} = \mathcal{R}^{-1} \cup \mathcal{S}^{-1}$;

(iii) $(\mathcal{R} \cap \mathcal{S})^{-1} = \mathcal{R}^{-1} \cap \mathcal{S}^{-1}$;

(iv) $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$;

(v) *if* $\mathcal{R} \geq \mathcal{S}$ *and* $\mathcal{R} \geq \mathcal{Q}$ *then* $\mathcal{R} \geq \mathcal{S} \vee \mathcal{Q}$;

(vi) *if* $\mathcal{R} \leq \mathcal{S}$ *and* $\mathcal{R} \leq \mathcal{Q}$ *then* $\mathcal{R} \leq \mathcal{S} \wedge \mathcal{Q}$.

4 Generality on fuzzy lattices

4.1. Classical Lattice

Definition 4.1. *[6]* ***(Lattice)*** *A lattice is a partially ordered set in which any two elements have a least upper bound (supremum) and a greatest lower bound (infimum).*

Notation 4.1.

1. *We denote the least upper bound of* a *and* b *by* $a \vee b$ *and the greatest lower bound by* $a \wedge b$.

2. *The lattice is briefly denoted as* $(L, \leq)$.

Example 4.1. *For any set* E*, the ordered set* $(P(E), \subseteq)$ *is a lattice with*

$$A \vee B = A \cup B \text{ and } A \wedge B = A \cap B, \quad \forall A, B \in P(E)$$

Definition 4.2. *A lattice* $\boldsymbol{L} = (L, \leq)$ *which has top and bottom elements is called a* ***bounded lattice****.*

The Duality Principle for lattice, note:

$$\text{If } (L, \leq) \text{ is a lattice, so is it's dual } (L, \geq).$$

In this section, we view a lattice also as an algebraic structure $(L, \vee, \wedge)$. We stress the connection between $\vee, \wedge$ and $\leq$.

Lemma 4.1. *Let* L *be a lattice and let* $x, y \in L$*. Then the following are equivalent:*

- *(i)* $x \leq y$;

- *(ii)* $x \vee y = y$;

- *(iii)* $x \wedge y = x$.

Proposition 4.1. *[6] An algebraic structure* $L = (L, \wedge, \vee)$ *where* L *is a nonempty set and* $\wedge, \vee$ *are binary operations is a lattice if, for each* $x, y, z \in L$*, the following properties are verified:*

L1. $x \wedge x = x, x \vee x = x$ *(idempotency)*

L2. $x \wedge y = y \wedge x, x \vee y = y \vee x$ *(Commutativity)*

L3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z, x \vee (y \vee z) = (x \vee y) \vee z$ *(Associativity)*

L4. $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ *(Absorption)*

Definition 4.3. *Let L be a lattice. We say L has a top element if there exists $1 \in L$ such that $a = a \wedge 1$ for all $a \in L$. Dually, we say L has a bottom element if there exists $0 \in L$ such that $a = a \vee 0$ for all $a \in L$.*
Remark 4.1. *Let $L = (L, \wedge, \vee)$ be a lattice.*
$(L, \wedge, \vee, 1, 0)$ is a bounded lattice when $0, 1 \in L$ and for all $x \in L, x \wedge 1 = x$ and $x \vee 0 = x$, meaning that 0 and 1 are the bottom and top elements, respectively.
Definition 4.4. ***(Closed lattice)****A lattice L is said to be closed if it has a smaller element denoted (0) and a larger element denoted (1).*
Example 4.2. *The lattice $(D(60), |)$ is closed because $(0) = 1, (1) = 60$ according to Figure (4.1)*

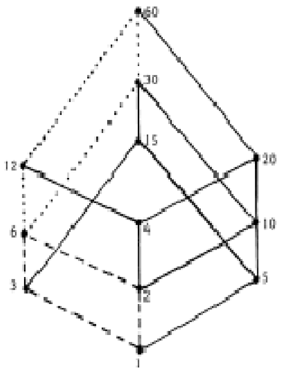


Figure 4.1: Lattice $(D(60), |)$

We defined the ideal and filter of a lattice L as follows.
Definition 4.5. *Let L be a nonempty set and $L = (L, \wedge, \vee)$ a lattice. A nonempty subset I of L is called an* ***ideal*** *of L if for all $x, y \in L$*

(i) If $y \in I$ and $x \leq y$, then $x \in I$;

(ii) $x, y \in I$ implies $x \vee y \in I$.

Using the Duality Principle we define a filter of a lattice as:
Definition 4.6. *Let L be a nonempty set and $L = (L, \wedge, \vee)$ a lattice. A nonempty subset F of L is called a* ***filter*** *of L if for all $x, y \in L$*

(i) If $y \in F$ and $y \leq x$, then $x \in F$;

(ii) $x, y \in F$ *imply* $x \wedge y \in F$.

Remark 4.2. *(i) An ideal (resp. filter) is called* ***proper*** *if it does not coincide with* $L (I \neq L)$.

(ii) The intersection of any filter (resp. ideal) family is a filter (resp. ideal).

(iii) The set of all ideals (resp. filters) of L *is denoted by* $I(L)$ *(resp.* $F(L)$*), and carries the usual inclusion order.*

Example 4.3. *In a lattice* $(\mathbb{N}^*, |)$, $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *, the subset* $F = \{2, 6, 10, 30\}$ *is a filter and* $I = \{1, 3, 5, 15\}$ *is an ideal.*

Definition 4.7. *Let* L *be a lattice. A proper ideal* I *of* L *is* ***prime*** *if and only if* $x, y \in L$, *and* $x \wedge y \in I$ *imply that* $x \in I$ *or* $y \in I$.
Dually, A proper filter F *of* L *is* ***prime*** *if and only if* $x, y \in L$ *and* $x \vee y \in F$ *imply that* $x \in F$ *or* $y \in F$.

Definition 4.8. *For each* $x \in L$, *the set* $\uparrow x = \{y \in L, x \leq y\}$ *is a filter, it is known as the* ***principal filter*** *generated by* x. *Dually,* $\downarrow x = \{y \in L, y \leq x\}$ *is a* ***principal ideal****.*

Example 4.4. *In the lattice* $D(60)$ *all filters and ideals are principals, on the Figure (4.1) we have represented the principal filter* $F_6 = \{6, 12, 30, 60\}$, *and the principal ideal* $I_6 = \{1, 2, 3, 6\}$.

Definition 4.9. ***(Ultra-filter)****A proper filter* F *of* L *is said to be a* ***maximal*** *if for each filter* X *of* $L, F \subseteq X \subseteq L \Rightarrow X = F$ *or* $X = L$.

Proposition 4.2. *Let* F *a proper filter then the following two assertions are equivalent:*

1. F *is an ultrafilter,*

2. $\forall x \notin F, \exists y \in F : x \wedge y = 0$.

Remark 4.3. *Dually we define a* ***maximal ideal****.*

4.1.1. Lattice Homomorphisms

In this section, we define the morphism of lattice, examples and their properties

Definition 4.10. ***(Morphisme of lattice)****Let* L *and* L' *be the two lattices* f *is an application of* L *in* L'. *We say that* f *is a lattice morphism if* $\forall x, y \in L$*:*

1. $f(x \wedge y) = f(x) \wedge f(y)$,

2. $f(x \vee y) = f(x) \vee f(y)$.

Remark 4.4. *[17]*

(i) A lattice isomorphism is a bijective morphism between two lattices.

(ii) A lattice morphism is a fortiori an increasing application.

(iii) if $x \leq y \Rightarrow x \vee y = y$

$$f(x \vee y) = f(x) \vee f(y) = f(y) \Rightarrow f(x) \leq f(y)$$

***Conclusion**: a lattice morphism is an order morphism but the reverse is false.*

Example 4.5. $E_1 = (\mathbb{N}^*, |), E_2 = (\mathbb{N}^*, \leq)$

$$\begin{array}{rccc} f: & E_1 & \longrightarrow & E_2 \\ & x \mid y & \longrightarrow & x \leq y \end{array}$$

$$\begin{aligned} f(x \wedge y) \neq f(x) \wedge f(y) \Rightarrow \quad & pgcd(x, y) \neq x \wedge y \\ & pgcd(2, 3) \neq 2 \wedge 3 \\ & \qquad 1 \neq 2 \end{aligned}$$

Definition 4.11. *[5] Let L be a lattice. We now formulate the following two distributive laws:*

(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in L$

(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \forall x, y, z \in L$

Example 4.6. *1. Let* $\mathbb{N}$ *be a set of natural numbers. If* $a \leq b$ *for natural numbers* a *and* b *means* $a|b(a$ *is a divisor of* $b)$*, then* $(\mathbb{N}, |)$ *is a distributive lattice.*

2. The lattice N_5 *(pentagon) and* M_3 *(Diamond) is not distributive.*

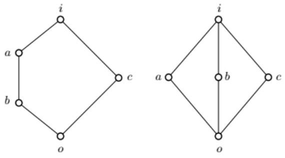


Figure 4.2: The lattice N_5 and M_3

Theorem 4.1. *A lattice L is non-distributive if and only if it contains a lattice of the form N_5 or M_3.*
Corollary 4.1. *Any chain is a distributive lattice.*

4.1.2. Boolean algebra

Definition 4.12. ***(Complemented lattice)****Let L be a lattice with has identity element 1 and zero element 0, $x, y \in L$. If $x \wedge y = 0, x \vee y = 1$, then y is called a complement of x. If every element of L has complements, then L is called a complemented lattice.*
Example 4.7.

1. *In the lattice $L = \{1, 2, 4, 5, 20\}$ ordered by divisibility: 5 has complement 2 and 4.*
2. *In a chain, an element other than 0 and 1 never has a complement.*
3. *Generally, 0 and 1 are always complementary to each other.*

Definition 4.13. *A* ***boolean algebra*** *(or boolean lattice) is a lattice with an identity element 1 and zero element 0 which is distributive and complemented.*
Example 4.8.

1. *Any lattice $(P(E), \subseteq)$ is a boolean algebra.*
2. *$(D(6), |)$ is a boolean algebra.*

Definition 4.14. *Lets B_1, B_2 be two boolean algebra. $f : B_1 \longrightarrow B_2$ is a* ***boolean morphism*** *if for all $x, y \in B_1$*

(i) $f(x \wedge y) = f(x) \wedge f(y)$.

(ii) $f(x \vee y) = f(x) \vee f(y)$.

(iii) $f(\bar{x}) = \bar{f}(x)$.

Properties 4.1.

1. $\bar{0} = 1, \bar{1} = 0$.
2. $\bar{\bar{x}} = x$ *(involution).*
3. $\overline{x \wedge y} = \bar{x} \vee \bar{y}$ *and* $\overline{x \vee y} = \bar{x} \wedge \bar{y}$ *(De morgan's laws).*

Corollary 4.2. *In a Boolean algebra B, we have for all $x, y \in B$,*

$$x \leq y \Longleftrightarrow x' \geq y'.$$

Proof. $x \leq y \iff x \vee y = y \iff x' \wedge y' = (x \vee y)' = y' \iff x' \geq y'$. □

4.2. Fuzzy Lattice

Definition 4.15. *[4] A fuzzy poset* $(X, \mathcal{R})$ *is called a **fuzzy lattice** if* $x \vee y$ *and* $x \wedge y$ *exist, for all* $x, y \in X$*. We simply note a fuzzy lattice* $(X, \mathcal{R})$ *by* $\mathcal{L}$.

Example 4.9. *[18]*

1. *Let* $E = \{x_0, y_0, z_0\}$ *and* $\mathcal{R}_\mathcal{L}$ *a fuzzy partial order relationship on* E *such that :*

$\mathcal{R}_\mathcal{L}(x,y)$	x_0	y_0	z_0
x_0	1	0.2	0.1
y_0	0	1	0.1
z_0	0	0	1

$x \wedge_\mathcal{L} y$	x_0	y_0	z_0
x_0	x_0	x_0	x_0
y_0	x_0	y_0	y_0
z_0	x_0	y_0	z_0

$x \vee_\mathcal{L} y$	x_0	y_0	z_0
x_0	x_0	y_0	z_0
y_0	y_0	y_0	z_0
z_0	z_0	z_0	z_0

The ordered set $\mathcal{L} = (E, \mathcal{R}_\mathcal{L})$ *is a fuzzy lattice (see **the figure** 4.3).*

2. *Let* $F = \{x_1, y_1, z_1, w_1\}$ *and* $\mathcal{R}_\mathcal{M}$ *a fuzzy partial order relationship on* F *such that :*

$\mathcal{R}_\mathcal{M}(x,y)$	x_1	y_1	z_1	w_1
x_1	1	0.3	0.1	0.4
y_1	0	1	0	0.1
z_1	0	0	1	0.3
w_1	0	0	0	1

$x \wedge_{\mathcal{M}} y$	x_1	y_1	z_1	w_1
x_1	x_1	x_1	x_1	x_1
y_1	x_1	y_1	x_1	y_1
z_1	x_1	x_1	z_1	z_1
w_1	x_1	y_1	z_1	w_1

$x \vee_{\mathcal{M}} y$	x_1	y_1	z_1	w_1
x_1	x_1	y_1	z_1	w_1
y_1	y_1	y_1	w_1	w_1
z_1	z_1	w_1	z_1	w_1
w_1	w_1	w_1	w_1	w_1

The ordered set $\mathcal{M} = (F, \mathcal{R}_{\mathcal{M}})$ is a fuzzy lattice (see ***the figure*** *4.3).*

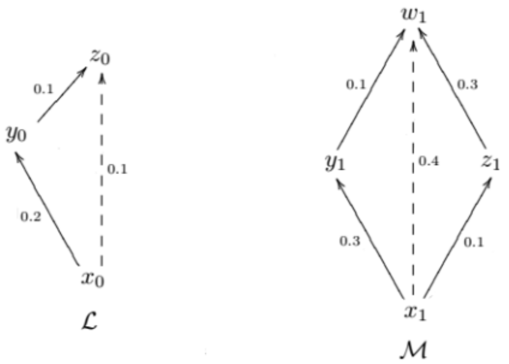


Figure 4.3: Fuzzy Lattice

Definition 4.16. *Let $\mathcal{L} = (X, \mathcal{R})$ and $\mathcal{M} = (Y, \mathcal{S})$ two fuzzy lattices. A mapping $h : X \to Y$ is a fuzzy homomorphism from $\mathcal{L}$ into $\mathcal{M}$ if, for all $x, y \in X$, it satisfies the following conditions:*

1. $h(x \wedge_{\mathcal{L}} y) = h(x) \wedge_{\mathcal{M}} h(y)$;
2. $h(x \vee_{\mathcal{L}} y) = h(x) \vee_{\mathcal{M}} h(y)$;
3. $h(0_{\mathcal{L}}) = 0_{\mathcal{M}}$;
4. $h(1_{\mathcal{L}}) = 1_{\mathcal{M}}$.

If h is a bijection, then h is said to be an isomorphism.

Example 4.10. *Let $\mathcal{L} = (X, \mathcal{R})$ be a fuzzy lattice where $X = \{x, y, z, w\}$ and $\mathcal{R} : X \times X \to [0, 1]$ is a fuzzy relation given by*

$\mathcal{R}$	x	y	z	w
x	1	0.1	0.4	0.8
y	0	1	0.2	0.5
z	0	0	1	0.3
w	0	0	0	1

Let $\mathcal{M} = (Y, \mathcal{S})$ *be a fuzzy lattice where* $Y = \{x', y', z', v', w'\}$ *and* $\mathcal{S} : Y \times Y \to [0,1]$ *is a fuzzy relation given by*

$\mathcal{S}$	x'	y'	z'	v'	w'
x'	1.0	0.1	0.4	0.7	0.9
y'	0.0	1.0	0.0	0.5	0.7
z'	0.0	0.0	1.0	0.5	0.6
v'	0.0	0.0	0.0	1.0	0.3
w'	0.0	0.0	0.0	0.0	1.0

Then, a fuzzy homomorphism h *from* $\mathcal{L}$ *into* $\mathcal{M}$ *can be defined by* $h(x) = x', h(y) = y', h(z) = z'$ *and* $h(w) = w'$.

Definition 4.17. *[13] Let* $(X, \mathcal{R})$ *be a fuzzy lattice, and* F *be a non-empty subset of* X. F *is a filter of* $(X, \mathcal{R})$ *if it satisfies the following conditions:*

(i) If $x \in X, y \in F$ *and* $\mathcal{R}(y, x) > 0$, *then* $x \in F$.

(ii) If $x, y \in F$, *then* $x \wedge y \in F$.

Definition 4.18. *[13] Let* $(X, \mathcal{R})$ *be a fuzzy lattice, and* I *be a non-empty subset of* X. I *is an ideal of* $(X, \mathcal{R})$ *if it satisfies the following conditions:*

(i) If $x \in X, y \in I$ *and* $\mathcal{R}(y, x) > 0$, *then* $x \in I$.

(ii) If $x, y \in I$, *then* $x \vee y \in I$.

Proposition 4.3. *[14] Let* $\mathcal{R} : X^2 \to [0,1]$ *be a fuzzy relation. If* $\mathcal{R}$ *is a fuzzy partial order relation on* X, *then* $S(\mathcal{R})$ *is a partial order relation on* X.

Proof. Let $\mathcal{R}$ be a fuzzy partial order relation on X. For all $x \in X$, $\mathcal{R}(x, x) = 1 > 0$ so $(x, x) \in S(\mathcal{R})$. Then, $S(\mathcal{R})$ is reflexive. If $(x, y) \in S(\mathcal{R})$ and $(y, x) \in S(\mathcal{R})$ i.e., $\mathcal{R}(x, y) > 0, \mathcal{R}(y, x) > 0$, as $\mathcal{R}$ is antisymmetric so $x = y$. Then $S(\mathcal{R})$ is antisymmetric. If $(x, y) \in S(\mathcal{R})$ and $(y, z) \in S(\mathcal{R})$ i.e., $\mathcal{R}(x, y) > 0$ et $\mathcal{R}(y, z) > 0$, then $\mathcal{R}(x, z) > 0$, since $(x, z) \in S(\mathcal{R})$. Then, $S(\mathcal{R})$ is transitive.
Hence, $S(\mathcal{R})$ is a partial order relation on X. □

Proposition 4.4. *[13] Let* $(X, \mathcal{R})$ *be a fuzzy lattice and let* $x, y, z \in X$. *Then:*

1. $\mathcal{R}(x, x \vee y) > 0$, $\mathcal{R}(y, x \vee y) > 0$, $\mathcal{R}(x \wedge y, x) > 0$, $\mathcal{R}(x \wedge y, y) > 0$.
2. $\mathcal{R}(x, z) > 0$ *and* $\mathcal{R}(y, z) > 0 \Longrightarrow \mathcal{R}(x \vee y, z) > 0$.
3. $\mathcal{R}(z, x) > 0$ *and* $\mathcal{R}(z, y) > 0 \Longrightarrow \mathcal{R}(z, x \wedge y) > 0$.
4. $\mathcal{R}(x, y) > 0 \Longleftrightarrow x \vee y = y$.
5. $\mathcal{R}(x, y) > 0 \Longleftrightarrow x \wedge y = x$.
6. *if* $\mathcal{R}(y, z) > 0 \Longrightarrow \mathcal{R}(x \wedge y, x \wedge z) > 0$ *and* $\mathcal{R}(x \vee y, x \vee z) > 0$.

Proof. Let $(X, \mathcal{R})$ be a fuzzy lattice and let $x, y, z \in X$.

1. $x \vee y$ is the least upper bound of $\{x, y\}$, then $x \vee y$ is an upper bound of x and y. So $\mathcal{R}(x, x \vee y) > 0, R(y, x \vee y) > 0$. Similarly, we prove that $\mathcal{R}(x \wedge y, x) > 0, R(x \wedge y, y) > 0$.
2. $\mathcal{R}(x, z) > 0$ and $\mathcal{R}(y, z) > 0$, mean that z is an upper bound of $\{x, y\}$, since $x \vee y$ is the least upper bound. So, $\mathcal{R}(x \vee y, z) > 0$.
3. $\mathcal{R}(z, x) > 0$ and $\mathcal{R}(z, y) > 0$, prove that z is a lower bound of $\{x, y\}$, since $x \wedge y$ is the greatest lower bound. So $\mathcal{R}(z, x \wedge y) > 0$.
4. Direct implication, suppose $\mathcal{R}(x, y) > 0$, and like $\mathcal{R}(y, y) = 1 > 0$, therefore of (2) $\mathcal{R}(x \vee y, y) > 0$. On the other hand by (1) $\mathcal{R}(y, x \vee y) > 0$. So, by the antisymmetric of $\mathcal{R}$, then $x \vee y = y$.
   Conversely suppose $x \vee y = y$. Then $\mathcal{R}(x, y) = \mathcal{R}(x, x \vee y) > 0$ of (1) .
5. The proof is similar to that of (4).
6. Suppose $\mathcal{R}(y, z) > 0$. Then:
   $\mathcal{R}(x \wedge y, z) \geqslant sup\{min_{p \in E}(\mathcal{R}(x \wedge y, p), \mathcal{R}(p, z))\} \geqslant min(\mathcal{R}(x \wedge y, y), \mathcal{R}(y, z)) > 0$.
   As of (1) $\mathcal{R}(x \wedge y, x) > 0$ thus from (3) one obtains $\mathcal{R}(x \wedge y, x \wedge z) > 0$.
   And : $\mathcal{R}(y, x \vee z) \geqslant sup\{min_{p \in E}(\mathcal{R}(y, p), \mathcal{R}(p, x \vee z))\} \geqslant min(\mathcal{R}(y, z), \mathcal{R}(z, x \vee z)) > 0$.
   As of (1) $\mathcal{R}(x, x \vee z) > 0$ thus from (2) one obtains $\mathcal{R}(x \vee y, x \vee z) > 0$.

□

Corollary 4.3. *Let $\mathcal{R}$ a fuzzy partial order relation on X. If $(X, \mathcal{R})$ is a fuzzy lattice, then $(X, S(\mathcal{R}))$ is a lattice.*

Remark 4.5. *The converse is not necessarily true*

Example 4.11. *Let* $X = \{a, b, c\}$*, and let* $\mathcal{R}$ *be the fuzzy relation defined by the following table:*

$\mathcal{R}$	a	b	c
a	1.0	0.3	0.1
b	0.0	1.0	0.8
c	0.0	0.0	1.0

Then $S(\mathcal{R})$ *defined by the following table:*

$S(\mathcal{R})$	a	b	c
a	1.0	1.0	1.0
b	0.0	1.0	1.0
c	0.0	0.0	1.0

We can verify that $S(\mathcal{R})$ *is a partial order on* X*, but* $\mathcal{R}$ *is not a fuzzy partial order on* X *because it is not transitive.*

General conclusions and future research

In this work, we saw the concept of the fuzzy set as a generalization of the concept of classical set, we studied the basic notions of a fuzzy subset and its fundamental properties. Also, we introduced the notion of a fuzzy lattice and fuzzy complete lattice and some characterizations have been expressed in terms of the supremum and the infimum of its subsets.
This work gives a great possibility to generalize some classical results in particular the different classes of the fuzzy lattice and we introduced the notions of a fuzzy ideal and filter on a fuzzy lattice a generalization of the notions of a fuzzy ideal and filter given by **I. Mezzomo, B.C. Bedregal and R.H.N. Santiago.**

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ملخص

في هذه المذكرة ، درسنا أولاً بعض مفاهيم المجموعات الكلاسيكية ، ثم المجموعات الضبابية والعمليات علي المجموعات الضبابية ، درسنا النظم المثلثية، وكذا النظم المثلثية الثنوية مع تقديم اهم خصائص ونتائج هذه المفاهيم. ثالثا ، تطرقنا إلى نوعين من العلاقات العلاقات الكلاسيكية و الضبابية مع ذكر أهم الخصائص و النتائج. أخيرا، قدمنا مفهوم الشبكة الضبابية ، حيث درسنا الشبكات الكلاسيكية ومثل ومرشحات هذه الشبكات ودراسة مزاياها وخصائصها المختلفة كما قمنا بتوسيع هذه المفاهيم على الشبكة الضبابية ، حيث قدمنا خصائصها.

كلمات مفتاحية

مجموعة ضبابية ، تكملة ضبابية ، قسم ضبابي ، الجداء الديكارتي ، علاقة ضبابية ، شبكة ضبابية ، مرشح ، مثالية ، تماثل شبكات.

Abstract

In this memory, we first studied some concepts of classical sets, then fuzzy sets and operations on them. Second, we studied triangular norm, triangular conorm with presenting important properties and results for these concepts. Third, we touched on two types of relationships: classical and fuzzy relationships with mentioning the most important characteristics and results. Finally, we introduced the concept of fuzzy lattice, where we studied the classical lattice, ideals and filters, as well as a study of its various advantages and characteristics, then we expanded it on the fuzzy lattice, where we presented important characteristics of these concepts.

Key words

Fuzzy set, Fuzzy complement, Fuzzy partition, Cartesian product, Fuzzy Relation, Fuzzy Lattice, filter, ideal, lattices homomorphism.

Résumé

Dans ce mémoire, nous avons d'abord étudié quelques notions d'ensembles classiques, puis des ensembles flous et des opérations sur eux. Deuxièmement, nous avons étudié la norme triangulaire, conorm triangulaire avec présentant des propriétés importantes et résultats pour ces concepts. Troisièmement, nous avons abordé deux types de relations: relations classiques et floues avec mention des caractéristiques et résultats les plus importants. Enfin, nous avons introduit le concept de treilli flou, où nous avons étudié le treilli classique, les idéaux et les filtres, ainsi qu'une étude de ses différents avantages et caractéristiques, puis nous avons développé sur le treilli flou, où nous avons présenté des caractéristiques importantes de ces notions

Mot-clés

Ensemble flou, complément flou, partition floue, produit cartésien, relation floue, treilli flou, filtre, idéal, homomorphisme des treillis.