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Theme

The Leray-schauder Principle and its Applications in Integral Equation Theory

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By accepting to chair and examine this work :

GAGUI Bachir : President , DIAIDJA Noui : Examiner

Finally, and at this point, they must already feel forgotten , I think of my mother ,may God have mercy on her , my father , my husband and my little chick Ritadje my sister, my brothers and all my family . Their love, affection and support are above all the thanks.

ملخص

الهدف من هذه المذكرة دراسة مبدأ ليراي شودر و تطبيقاته في برهان وجود حل للمعادلات التكاملية الغير خطية ذات الشكل العام

$$u(x) = f(x) + \int_{\Omega} K\big(x, y, u(y)\big)\, dy$$

حيث $K(x,y)$ تسمى نواة المعادلة التكاملية , $f(x)$, $K(x,y)$ دوال معطاة .

و نخص بالذكر معادلات فولتيرا التكاملية, معادلات فريدهولم التكاملية و معادلات تكاملية مع تأخير

الكلمات المفتاحية: مبدا ليراي شودر, معادلة تكاملية, معادلة فولتيرا , معادلة فريدهولم , معادلة تكاملية مع تأخير وشرط الحدود

Abstract

The aim of this memoir is to study the Leray- schauder principle and its applications in proving the existence of a solution to nonlinear integral equations, which have the general forme

$$u(x) = f(x) + \int_{\Omega} K(x, y, u(y))\, dy \qquad x \in \Omega$$

where K is called the Kernel of the integral equation, and both the Kernel $K(x, y)$ and the function $f(x)$ in the integral equations functions.

In particular, we mention the integral equations of Volterra , the integral equations of Fredholm and integral equations with Delay.

Keywords : the Leray- schauder principle, integral equation , Volterra equations, Fredholm equations, integral equations with Delay and the boundary condition.

Resumé

L'objectif de cette memoire est d'étudier le principe de Leray-Schauder et ses applications pour prouver l'existence d'une solution des équations intégrales non linéaires, qui ont la forme générale suivante

$$u(x) = f(x) + \int_{\Omega} K(x, y, u(y))\, dy \qquad x \in \Omega$$

ou K est appelée le noyau de l'equation integrale, et le noyan $K(x, y)$ et la fonction $f(x)$ dans l'equation intégrale sont des fonctions données.

En particulier, nous mentionnons les équations intégrales de Volterra , les équations intégrales de Fredholm et les équations intégrales avec retard.

Mots clés : principe de Leray-Schauder, équation intégrale, équations de Volterra, équations de Fredholm, équations intégrales avec retard et la condition aux limites.

Notation

$\mathbb{R}_+$	set of all nonnegative real numbers
$\mathbb{R}^n$	set of all n − tuples $x = x_1, x_2, \ldots, x_n$
$\mathbb{R}^n_+$	set of all $x \in \mathbb{R}^n$ with $x_i \geq 0$ for all i
$\|x\|$	Euclidian norm of $x \in \mathbb{R}^n, \|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$
μ	Lebesgue measure in $\mathbb{R}^n$
$\|\cdot\|$	norm in X, also denoted by $\|\cdot\|_X$
$\overline{U}$	closure of U
∂U	boundary of $U : \partial U = \overline{U}/intU$
$conv\ A$	convex hull of A
$\overline{conv}A$	closed convex hull of A
$C^k(\Omega; \mathbb{R}^n)$	set of k − times continuously differentiable functions $u : \Omega \to \mathbb{R}^n$ ($\Omega \subset \mathbb{R}^n$ open)
$C^k(\overline{\Omega}; \mathbb{R}^n)$	space of all functions $u \in C^k(\Omega; \mathbb{R}^n), u = (u_1, u_2, \ldots, u_n)$ such that $D^\alpha u_i$ admits a continuous extension to $\overline{\Omega}$ for all i
$\|u\|_\infty$	$\max_{x\in\overline{\Omega}} \|u(x)\|$ ($\Omega \subset \mathbb{R}^n$ bounded open , $u \in C(\overline{\Omega}; \mathbb{R}^n)$)
$L^p(\Omega; \mathbb{R}^n)$	space of all measurable functions $u : \Omega \to \mathbb{R}^n$ with $\int_\Omega \|u(x)\|^p dx \prec \infty$ ($\Omega \subset \mathbb{R}^n$ open, $1 \leq p \prec \infty$)
$\|\cdot\|_\infty$	norm in $L^\infty(\Omega; \mathbb{R}^n)$, $\|u\|_\infty = \inf\{c : \|u(x)\| \leq c \quad a.e. \quad \text{on } \Omega\}$

Table of Contents

Introduction

The fixed point theorems are the basic mathematical tools in demonstrating the existence of solution in various kinds of equations. The fixed point theory is at the heart of nonlinear analysis applied to integral equations and it provides the necessary tools to have existence theorems in many different nonlinear problems.

The aim of this memory is to study the Leray-Schauder principle theorem, also called the continuity theorem, is a powerful existence as well as the uniqueness of solution of the nonlinear integral equation tool, when studing operator equations.

Integral equation are very useful mathematical tools in both pure and applied mathematics, appear in various fields of scince and numerous application such that elasticity, plasticity, heat and mass transfer, oscillation theory, fluids dynamics, filtration theory, electrostatics, electrodtnamics, biomachnics, game theory control, queuing theory , electrical engineering economics, medicine, etc.

An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign.

A general form of an integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x, y, u(y))\, dy$$

where $K(x, y)$ is called the kernel of the integral equation ,$\alpha(x)$ and $\beta(x)$ are the limits of integration .It is to be noted here that both the kernel $K(x, y)$ and the function $f(x)$ in the integral eguation are given functions; and λ is a constant parameter.

If the lower limit of integration is constant and uper one is variable we are in the case of Volterra[Vito Volterra (1860-1940) Italian mathematician] integral equation and If the lower limit of integration is constant and uper one is constant we are in the case of Fridholm[Ivan Fredholm (1866-1927)Swedish mathematician] .

In **chapter** 1, we define the notation of a completely continuous operator from a Banach space and we present the Ascoli-Arzèla theorem .

Next we present some fixed point theorems (Banach Fixed Point Theorem ,Schauder's Fixed point theorem, Shaefer's Fixed point theorem and The Leray-Schauder Principle)

Finally we prove The Leray-Schauder PrincipleTheorem .

chapter 2 is an introduction to the terminology and classification of integral equations, such as presenting a classification for linear and nonlinear integra equation, as well as giving examples of these equations, finally we study the existence and the uniqueness of the solution of the linear integral equation of Volterra and Fredholm by some methods (Resolvent and successive approximation .

Chapter 3 is devoted to the Leray-Schauder principle and its applications to integral equations. In section 3.1 we present au elementary proof (based on Urysohn's lemma and Schauder's fixed point theorem) of the following version of the **Leary-Schauder principle.**

Using Theorem 1.3.6 in section 3.2 we establish an existence principle for the Fredholm integral equation which, in particular, yields existence results for Hammerstein integral equation and two point boundary value problems for second order differential equations. Similar results for Volterra integral equations and, in particular, for Volterra-Hammerstein equation and the cauchy problem for a first order differentail system are established in Section 3.3. In Section 3.4 and 3.5 we focus on the delay integral equation

$$u(t) = \int_{t-\tau}^{t} f(s, u(s))\, ds$$

which comes from biomathematics Wediscuss the existence of solutions to the intial value problem and of periodic solution.

Chapter 1

Some Basic notions

In this chapter we define the notation of a completely continuous operator from a Banach space and we present the Ascoli-Arzèla theorem .

Next we represent contains basic theory of fixed point we will recall some important and different theorem such that Banach fixed point theorem, Schauder's Fixed point theorem, Shaefer's Fixed point theorem and the Leray-Schauder Principle, these theorem are very useful in the next chapters.

1.1 Completely Continuous Operator

Definition 1.1.1 *Let X be a banach space, An operator $T : X \longrightarrow X$ is called totally bounded if for every bounded subset S of X , $T(S)$ is compact.Moreveover , T is said to be completely continuous over X if it is continuous and totally bounded over X.*

1.2 The Ascoli-Arzèla Theorem

Let (K, d) be a compact metric space and $C(K; \mathbb{R}^n)$ be the Banach space of all continuous functions from K to $\mathbb{R}^n$; under the sup-norm $|\cdot|_\infty$

Theorem 1.2.1 [11] *A subset Y of $C(K; \mathbb{R}^n)$ is relatively compact if and only if the following conditions are stisfied :*

1) *Y is bounded i.e ; there exists a constant $c \succ 0$ such that*

$$|u(x)| \leq c \text{ for all } x \in K \quad \text{and } u \in Y$$

2) *Y is equicontinuous , i.e , for every $\varepsilon \succ 0$ there exists a $\delta \succ 0$ such that for all $u \in Y$*

$$\left|u(x) - u\left(x^{'}\right)\right| \prec \varepsilon \quad \text{whenever } x, x' \in K \text{ and } d(x, x') \prec \delta$$

proof. see [11] ■

Corollary 1.2.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$. Every bounded sub-set of space $\left(C^1\left(\overline{\Omega}; \mathbb{R}^n\right), |\cdot|_\infty\right)$ is relatively compact in $\left(C\left(\overline{\Omega}; \mathbb{R}^n\right), |\cdot|_\infty\right)$.*

1.3 Fixed Point Theorems

The theory of fixed point is concerned with the conditions which guarantee that a map $T : X \to X$ of a topological space X into itself admits one or more fixed points, that is, points φ of X for which $\varphi = T\varphi$.

Definition 1.3.1 *let T be an operator of a Banach space E in itself ; the operator T is said a contraction operator if it can be exist a positive constant k , such that $0 \prec k \prec 1$ and $\|T\varphi_1 - T\varphi_2\| \leq k \|\varphi_1 - \varphi_2\| \quad \forall \varphi_1, \varphi_2 \in E$*

1.3.1 Banach Fixed Point Theorem

Theorem 1.3.1 *let T be a contraction operator in a Banach space E the equation $T\varphi = \varphi$ admits a unique solution φ in E this solution is said to be a Fixed point of the operator T.*

Theorem 1.3.2 (*Banach Fixed Point Theorem*)

Let X be a non-empty closed subset of Banach space E , $X \subset E, X \neq \varnothing$.

And let $T : X \longrightarrow X$ be a contraction operator with constant $k \prec 1$.

Then the sequence of successive approximations $\{f_n / f_n = T f_{n-1}, f_0 \in X\}$ converges to the unique fixed point $f \in X$, $f = Tf$ for any starting point $f_0 \in X$

$$f_n \longrightarrow f$$

and the following estimate is valid

$$\|f_n - f\| \leq \frac{1}{k}\|f_n - f_{n+1}\| \leq \frac{k^n}{1-k}\|f_0 - f_1\|$$

1.3.2 Schauder's Fixed point theorem

Theorem 1.3.3 (*Schauder*)*: let X be a Banach Space , $K \subset X$ a nonempty convex compact set and let $T : K \longrightarrow K$ be a continuous operator, then T has at least one fixed point.*

proof. see [11] ■

The following variant of Schauder's theorem is most useful in application .

Theorem 1.3.4 (*Schauder*) [11] : *let X be a Banach Space , $D \subset X$ a nonempty convex bounded closed set and let $T : D \longrightarrow D$ be a completely continuous operator.Then T has at least one fixed point.*

We can derive Theorem 1.3.4 from theorem 1.3.3 via the following result

Lemma 1.3.1 (*Mazur*) *The convex hull of any relatively compact subset of a Banach space is relatively compact .*

1.3.3 Schaefer's Fixed point theorem

Theorem 1.3.5 [7] *let X be a Banach space and let $T : X \longrightarrow X$ be a completeley continuous operator.then either:*

1)the operator equation $u = \lambda T u$ has a solution for $\lambda = 1$.

or

2)the set $\varepsilon = \{u \in X \;\; ; \quad u = \lambda T u \;\; \lambda \in (0,1)\}$ is unbounded.

1.3.4 The Leray-Schauder Principle

Theorem 1.3.6 (*Leray – Schauder*) [11] *let X be a Banach space, $K \subset X$ a closed convex subset, $U \subset K$ a bounded set, open in K and $u_0 \in U$ a fixed element. Assume that the operator $T : \overline{U} \to K$ is completely continuous and satisfies the boundary condition*

$$u \neq (1-\lambda)\, u_0 + \lambda T(u) \quad \textit{for all} \;\; u \in \partial U, \lambda \in (0,1) \tag{1.1}$$

Then T has at least one fixed point in $\overline{U}$.

proof. For the proof we use Granas'fixed point approach (see Dugundji- Granace[3] , Granas[5] , O'Regan-Precup [10] and Zeidler[13]). Notice the property of U of being open as well as the boundary ∂U are understood with respect to the topology of K.We may assume that (1.1) holds on ∂U for all $\lambda \in [0,1]$. Indeed, this is obvious for $\lambda = 0$ since $u_0 \in U$, whilst if (1.1) does not hold for $\lambda = 1$, then the theorem is proved.

Let

$$S = \left\{ u \in \overline{U} : u = (1-\lambda) u_0 + \lambda T(u) \text{ for some } \lambda \in [0,1] \right\}.$$

Obviously S is nonempty (since $u_0 \in S$) , closed, and $S \cap \partial U = \emptyset$.By Urysohn's lemma (see Dugundji-Granas [3]) , there existe a function $\varphi \in C\left(\overline{U}; [0,1]\right)$ such that

$$\varphi(u) = \begin{cases} 0 \text{ for } & u \in \partial U, \\ 1 \text{ for } & u \in S. \end{cases}$$

we now define the operator $\widetilde{T} : K \longrightarrow K$ by

$$\widetilde{T}(u) = \begin{cases} (1-\varphi(u)) u_0 + \varphi(u) T(u) & \text{for } u \in U \\ u_0 & \text{for } u \in K \setminus U. \end{cases}$$

It is clear that $\widetilde{T}$ is continuous and

$$\widetilde{T}(K) \subset conv\left(\{u_0\} \cup T\left(\overline{U}\right)\right).$$

Since T is completely continuous, $T\left(\overline{U}\right)$ is relatively compact. Hence by Mazur's lemma the following subset of K,

$$D = \overline{conv}\left(\{u_0\} \cup T\left(\overline{U}\right)\right),$$

is convex and compact . In addition

$$\widetilde{T}(D) \subset D.$$

Hence Theorems 1.3.3 applies and guarantees the existeence of a $u \in D$ with $\widetilde{T}(u) = u$. By the definition of $\widetilde{T}, u$ muste lie in U. Then

$$u = (1-\varphi(u)) u_0 + \varphi(u) T(u).$$

This shows that $u \in S$ and so $\varphi(u) = 1$.As a result, $u = T(u)$. ■

Notice that the essential idea of the Leray-Schauder principle consists in joining the operator T to the constant operator u_0 by means of the homotopy $H : \overline{U} \times [0,1] \to K$

$$H(u,\lambda) = (1-\lambda)\,u_0 + \lambda T(u)$$

in a such way that the unique fixed point of $H(.,0)$, namely u_0, can be 'continued' in a fixed point of $H(.,\lambda)$ for each $\lambda \in [0,1]$, and, in particular, in a fixed point of $H(.,1) = T$. This continuation process is possible if all operators $H(.,\lambda)$ for $\lambda \in [0,1]$ are fixed point free on boundary of U .

In applications the Leray-Schauder principle is usually used together with the so called 'a priori bounds technique' :

Suppose we wish to solve the operator equation

$$u = T(u), \qquad u \in K, \tag{1.2}$$

where K is closed ,convex subset of a Banach space $(X, |.|_X)$, and $T : K \to K$ is completely continuous. Then we look at the set of all solution to the one-paramater family of equation

$$u = (1-\lambda)\,u_0 + \lambda T(u), \qquad u \in K, \tag{1.3}$$

when $\lambda \in (0,1)$. Here $u_0 \in K$ is fixed (in most cases $u_0 = 0$). If this set is bounded, i.e.,there exists $R \succ 0$ such that

$$|u - u_0|_X \prec R$$

(strictly !) whenever u solves (1.3)for some $\lambda \in (0,1)$, then we let U be the intersection of K with the open ball $B_R(u_0, X)$. Thus Theorem 1.3.6 applies and guarantees the existence of a solution to (1.2).

For other result of Leray-Schauder type we refer the reader to the monograph O'Regan-Precup [10] .

Theorem 1.3.7 (*Leray-schauder principle*) [6] .*Let* $(X, |.|)$ *be a Banach space and suppose that* $T \in C(X,X)$ *and compact . Suppose that any solution* u *of* $u = \lambda T u$, $0 \leq \lambda \leq 1$ *satisfies the a priori bound* $|u| \leq R$ *for some constant* $R \succ 0$, *then* T *has a fixed point .*

From above we conclude that the difference between applying Schauder's theorem and applying banach's theorem, namely to apply Banach's theorem we have to show that a mapping is sufficiently small, while to apply Schauder's theorem we have to prove that a mapping is compact. This means that, in $C(\Omega)$ or $L^p(\Omega)$ case, we have to show that the image set for the mapping consists of more" regular " functions.

Chapter 2

Classifications and theory of integral equations

In this chapter we present integral equations and we illustrate different criterious of classification of these equations , also we discuss the existence of solution of some kind of integral equation and some solving methods (mothod of successive approximation and the method of successive substitutionsthe(resolvent method)).

2.1 Definition of Integral Equations

Integral equations are equation in which the unknown function is under the integral sign . the typical integral equation for unknown function $u(x)$,$x \in E \subset \mathbb{R}^n$

include integral term in the form of integral operator with the kernel $k(x,t,u(t))$

$$\int_E K(x,t,u(t))\,dt = \lambda u(x) + f(x) \tag{2.1}$$

Where E is a measured space, $f(x)$ a given measurable function on E , λ a given scalar which can be real or complex

With all these data, our problem is to find the function u that satisfies the equation (2.1) .

Remark 2.1.1 1.*For the study of following integral equation :*

$$\int_E K(x,t,u(t))\,dt = \lambda u(x) + f(x) \qquad ,x \in E \qquad (*)$$

On restrict ourselves to spaces $L_p(E)$ with $(1 \leq p \leq \infty)$.

Implicitly, for the function $f \in L_p(E)$, we look for the function u in $L_p(E)$ the verification of this equation, this means that in this restriction , we only use $K(x,t)$ kernels for which $T(u)$ is in $L_p(E)$ when u is .

2.If we take

$$K(x,t,u(t)) = K(x,t)\,u(t) \ ,$$

the equation$(*)$ becomes linear, i.e

$$f(x) = \int_E K(x,t)\,u(t) - \lambda u(x)$$

is otherwise becomes a non-linear integral equation .

3. Note that the equation can be written as an operator

$$Tu = \lambda u + f$$

where the operator T is written as

$$Tu(x) = \int_E K(x,t,u(t))\,dt.$$

4.The most general type of integral equation is :

$$h(x)\,u(x) = f(x) + \lambda \int_E K(x,t)\,u(t)\,dt.$$

the function $h(x)$ determines the type of the equation .

2.2 Classification of Integral Equations

The classification of integral equation is centered on three basic characteristics that describle their global structure , it is useful to quote them before going into details .

a.The type (Kind) of an equation refers to the location of the unknown function .

For the equation of the first kind , the unknown function appears only inside the integral sign. However , for the second kind equations , the unknown function the integral sign .

b.The historical description of Fredholm and Volterra concerns the integration terminals.

In a Fredholm equation , the integration bounds are fixed , in the Volterra equation the integration bounds are undefined .

c.The adjective singular is sometimes used on the one hand , when the integration is improper the one hand if one of the integration bounds are infinite or if the integrant is unbounded equation can be singular in both directions .

2.2.1 Linear integral equations

a- Fredholm Inegral Equation

We call a fredholm lineair integral equation such that the two limits of integration are constant an equation , with one unknown $u(x)$ of the form

$$h(x)\,u(x) = f(x) + \lambda \int_a^b K(x,t)\,u(t)\,dt, \tag{2.2}$$

Here $K(x,t)$ is a given function of two variable (The Kernel) $f\ (x)$ is a given function and λ is a non-null real or complex parameter .

the function $h(x)$ determines the type of the integral equation .

1. if $h(x) = 0$, the equation(2.2) is written

$$f(x) + \lambda \int_a^b K(x,t)\,u(t)\,dt = 0\ . \tag{2.3}$$

and is called the Fredholm integral equation of the first kind .

2.if $h(x) = c$ =constant$\neq 0$, the equation (2.2) is written

$$cu(x) = f(x) + \lambda \int_a^b K(x,t)\,u(t)\,dt\ . \tag{2.4}$$

and is called the Fredholm integral equation of the second kind .

3.if $h(x) \neq 0$, so the formula (2.2) is called the third kind of Fredholm integral equation

.

Remark 2.2.1 *1. if* $f(x) = 0$ *, the equation* (2.2) *is said to be homogeneous*

2. if $f(x) \neq 0$ *, the equation* (2.2) *is said to be non- homogeneous*

b-Voltera Integral Equation

We call aVolterra lineair integral equation such that one of the two limits of integration is variable, an equation of the form

$$h(x)\,u(x) = f(x) + \lambda \int_a^x K(x,t)\,u(t)\,dt \tag{2.5}$$

1. we call Volterra integtal equation of the first kind , if $h(x) = 0$, So the equation (2.5) is written

$$f(x) + \lambda \int_a^x K(x,t)\,u(t)\,dt = 0 \tag{2.6}$$

2. we call Volterra integtal equation of the second kind , if $h(x) = c$ =constant$\neq 0$, So the equation (2.5) is written

$$cu(x) = f(x) + \lambda \int_a^x K(x,t)\,u(t)\,dt \tag{2.7}$$

3.if $h(x) \neq 0$, So the formula (2.5) is called Volterra integtal equation of the third kind

Remark 2.2.2 1) *if* $f(x) = 0$ *, the equation* (2.5) *is said to be homogeneous*

2)*if* $f(x) \neq 0$ *, the equation* (2.5) *is said to be non- homogeneous*

Remark 2.2.3 *Note that Volterra 's equation can be viewed as a spesial case of Fredholm's equation with* $K(x,t) = 0$ *for* $0 \prec x \prec t \prec a$ *(it is called a Volterra karnel)* .

2.2.2 Non-linear integral equation

a- fredholm integral equation

Consider the nonlinear Fredholm integral equation of the 1^{st} kind :

$$f(x) + \lambda \int_a^b K(x,t,u(t))\,dt = 0 \tag{2.8}$$

An integral equation

$$cu(x) = f(x) + \lambda \int_a^b K(x, t, u(t))\, dt \tag{2.9}$$

is called the nonlinear Fredholm integral equation of the 2^{nd} kind, such that c =constant$\neq 0$ and third kind , of form

$$h(x)\, u(x) = f(x) + \lambda \int_a^b K(x, t, u(t))\, dt \tag{2.10}$$

b- Volterra Integral Equation

Consider the nonlinear Volterra integral equation of the 1^{st} kind :

$$f(x) + \lambda \int_a^x K(x, t, u(t))\, dt = 0 \tag{2.11}$$

An integral equation

$$cu(x) = f(x) + \lambda \int_a^x K(x, t, u(t))\, dt \tag{2.12}$$

is called the nonlinear Volterra integral equation of the 2^{nd} kind, such that c =constant$\neq 0$ and third kind , of the form

$$h(x)\, u(x) = f(x) + \lambda \int_a^x K(x, t, u(t))\, dt \tag{2.13}$$

2.2.3 Mixed integral equation

1) **Fredholm-Volterra integral equation**

The Fredholm-Volterra equation is an equation of the form

$$h(x)\, u(x,t) + \lambda \int_a^b K(x,y)\, u(y,t)\, dy + \lambda \int_0^t F(t,s)\, u(x,s)\, ds = f(x,t) \quad , t \in [0,T], T \prec \infty \tag{2.14}$$

2)**Volterra-Fredholm integral equation**

The Volterra-Fredholm equation is an equation of the form

$$h(x)\, u(x,t) + \int_0^t \int_a^b K(x,t)\, F(t,s)\, u(y,s)\, dy ds = f(x,t) \ , \quad t \in [0,T], T \prec \infty \tag{2.15}$$

2.2.4 Singular integral equations

An integral equation is said to be singular if one or both limits of integration are infinite, or the kernel becomes infinite in the neighborhood of the limits of integration

Definition 2.2.1 *consider the following integral equation*

$$u(x) = f(x) + \int_T M(x,t)\, K(x,t)\, u(t)\, dt \tag{2.16}$$

We say that $(2,16)$ is singular if $M(x,t)$ admits a singularity or the domain T is not bounded .

Singularity Of Type Volterra and Fredholm

Consider the second kind integral equation of the forme

$$u(x) = f(x) + \int_a^x M(x,t)\, K(x,t)\, u(t)\, dt, a \leq x \prec \infty \tag{2.17}$$

where $K(x,t)$ is weakly singular, in general $K(x,t)$ is given by

$$K(x,t) = \begin{cases} |x-t|^{-\alpha}, 0 \prec \alpha \prec 1 \\ \log|x-t| \end{cases}$$

So

1. Equation $(2,17)$ is Volterra .
2. If $x = b$ Equation $(2,17)$ is of Fredholm.
3. The case where $|x-t|^{-\alpha}, 0 \prec \alpha \prec 1$ is called algebraic singularity .
4. The case where $\log|x-t|$ is called logarithmic singularity .

2.2.5 Integral equation with Delay

We define the integral equation with Delay as follows

$$u(t) = \int_{t-\tau}^{t} f(s, u(s))\, ds$$

In this equation $u(t)$ is the proportion of infective in a population at time t, τ is the length of time an individual remains infections, and $f(t, u(t))$ is the proportion of new infectives per unit time.

2.3 Solution Methods for Linear Integral Equation

In this part we will present some solution methods for the volterra and fredholm integral equation .

2.3.1 The Method of Successive Approximation

1.**for Fredholm 's integral equation**

For the integral equation

$$u = \lambda K u + f$$

the following iterations of the method of Successive Approximation are set by :

$$u_0(x) = f(x)$$

$$u_n(x) = \lambda K u_{n-1} + f \qquad n = 1, 2, ...$$

Lemma 2.3.1 $u_n(x) = \sum_{k=0}^{n} \lambda^k K^k f \quad$ *where* $K^k = \underbrace{K(K(...K))}_{k\ times}$

proof. by mathematical induction (assume that the formula for n is true)

$n = 0$
$$u_o(x) \underbrace{=}_{confirmed} \lambda^0 K^0 f = f(x)$$

$n = n + 1$
$$u_{n+1}(x) \underbrace{=}_{by\ definition} \lambda K u_n + f$$
$$\underbrace{=}_{by\ assumption} \lambda K \left(\sum_{k=0}^{n} \lambda^k K^k f \right) + f$$
$$\underbrace{=}_{linearity} f + \sum_{k=0}^{n} \lambda^{k+1} K^{k+1} f$$
$$\underbrace{=}_{\text{change of index}\ p=k+1} f + \sum_{p=1}^{n+1} \lambda^p K^p f$$
$$= \lambda^0 K^0 f + \sum_{p=1}^{n+1} \lambda^p K^p f$$
$$= \sum_{p=0}^{n+1} \lambda^p K^p f$$

$$= \sum_{k=0}^{n+1} \lambda^k K^k f$$ change of index $p = k$ ■

Neumann Series $\sum_{k=0}^{\infty} \lambda^k K^k f$ is called to be the Neumann Series

Theorem 2.3.1 *Fredholm 's integral equation* [1]

$$u = \lambda K u + f$$

with $|\lambda| \prec \frac{1}{M(b-a)}$ and continuous Kernel has a unique solution $u(x) \in C[a,b]$ for any $f(x) \in C[a,b]$.

This solution is given by a convergent Neumann series

$$\sum_{k=0}^{\infty} \lambda^k K^k f$$

and satisfies

$$\|u(x)\|_C \leq \frac{\|f\|_C}{1 - |\lambda| M(b-a)}$$

if $|\lambda| \prec \frac{1}{M(b-a)}$, then there exists an inverse operator $(1 - \lambda K)^{-1}$

b. For the Volterra's Integral equation

Consider the Volterra Integral equation of the 2^{nd} kind

$$u(x) = f(x) + \lambda \int_0^x K(x,t) u(t) dt$$

where $K(x,t)$ is a continuous Kernel , $K(x,t) \in C([a,b] \times [a,b])$

The method of Successive Approximation is defined by the following iterations :

$u_0(x) = f(x)$

$u_n(x) = \sum_{k=0}^{n} \lambda^k K^k f = \lambda K u_{n-1} + f$

Theorem 2.3.2 [1] *the Volterra Integral equation of the* 2^{nd} *kind*

$$u(x) = f(x) + \lambda \int_0^x K(x,t) u(t) dt$$

with continuous Kernel $K(x,t)$ *and with any* $\lambda \in \mathbb{R}$ *has a unique solution* $u(x) \in C([0,a])$ *for any* $f(x) \in C([0,a])$.

This solution is given by a uniformly convergent Neumann series

$$u(x) = \sum_{k=0}^{\infty} \lambda^n (K^n f)(x)$$

and its norm satisfies

$$\|u(x)\|_C \le \|f\|_C \, e^{|\lambda| Ma}.$$

2.3.2 the method of successive substitutions(the resolvent method)

a.**for fredholm's integral equation**

Iterated kernel : let integral operator K has a continuous kernel $K(x,t)$, then define :

repeated operator $$K^n = K\left(K^{n-1}\right) = \left(K^{n-1}\right)K \qquad n = 2,3,...$$

it has a kernel $$K_n(x,t) = \int_G K(x,y)\, K_{n-1}(y,t)\, dy$$

indeed $$\left(K^1 f\right)(x) = \int_G \underbrace{K(x,t)}_{K_1(x,t)} f(t)\, dt$$

$$\begin{aligned}
\left(K^2 f\right)(x) &= [K(Kf)](x) \\
&= \int_G K(x,y) \left[\int_G K(y,t)\, f(t)\, dt\right] dy \\
&= \int_G \underbrace{\left[\int_G K(x,y)\, K(y,t)\, dy\right]}_{K_2(x,t)} f(t)\, dt
\end{aligned}$$

$$\cdots$$

Kernel
$$K_n(x,t) = \int_G K(x,y)\, K_{n-1}(y,t)\, dy = \int_G K_{n-1}(x,y)\, K(y,t)\, dy$$

is called an iterated kernel .Kernels $K(x,t)$ are continuous, and if domain $G=(a,b)$, then

$$|K_n(x,t)| \leq M^n (b-a)^{n-1}$$

Function defined by the infinite series

$$R(x,t,\lambda) = \sum_{k=0}^{\infty} \lambda^k K_{k+1}(x,t)$$

is called a resolvent .

Theorem 2.3.3 *Solution of integral equation $u = \lambda K u + f$ with continuous kernel $K(x,t)$ is unique in $C[a,b]$*

for $|\lambda| \prec \frac{1}{M(b-a)}$, and for any $f \in C[a,b]$ is given by

$$u(x) = f(x) + \lambda \int_a^b R(x,t,\lambda)\, f(y)\, dt$$

i.e. there exists inverse operator

$$(I - \lambda K)^{-1} = I + \lambda R\ , \quad |\lambda| \prec \frac{1}{M(b-a)}$$

b. For the Volterra Integral equation

We consider the Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt \tag{2.18}$$

where $K(x,t)$ is a continuous function for $0 \leq x \leq a$, $0 \leq t \leq x$, and $f(x)$ is continuous when $0 \leq x \leq a$.

Theorem 2.3.4 *Let $K(x,t)$ is a continuous function for $0 \leq x \leq a$, $0 \leq t \leq x$,and $f(x)$ is continuous when $0 \leq x \leq a$.*

The equation$(2, 18)$ has a unique and continuous solution given by the formula

$$u(x) = f(x) + \lambda \int_a^x R(x,t,\lambda) f(t)\, dt$$

Theorem 2.3.5 [1] *Let be the Volterra integral equation of the first kind*

$$f(x) = \int_a^x K(x,t) u(t)\, dt \tag{2.19}$$

such that f, K continuous functions, derivable on $[a,b]$

$$K(x,x) \neq 0 \quad \text{and} \quad \int_a^b \int_a^b |K(x,t)|^2 \, dxdt \prec \infty$$

Then, there is a unique and continuous solution of the equation $(2, 19)$.

proof. First, we notice that we

$$f(a) = \int_a^a K(x,t) u(t)\, dt = 0$$

Equation (2.19) can be transformed into a volterra equation of the second kind by using the leibniz rule

$$\frac{\partial}{\partial x} \int_a^x K(x,t) u(t)\, dt = K(x,x) u(x) + \int_a^x \frac{\partial}{\partial x} K(x,t) u(t)\, dt = f'(x)$$

As $K(x,x) \neq 0$,then

$$u(x) = \frac{f'(x)}{K(x,x)} - \int_a^x \frac{K'_x(x,t)}{K(x,x)} u(t)\, dt.$$

which is a Volterra equation of the second kind, and by the theorem we obtain the existence and the uniqueness of the solution u. ■

Chapter 3

Application of the Leray-Schauder Principle in integral equations

.In this chapter we shall apply the Leray-Schauder principle in order to obtain existence results for continuous solution of integral equation.In particular, we give results on the existence of continuous solutions of initial value and two-point boundary value problems for nonlinear ordinary differential equations in $\mathbb{R}^n$.

3.1 Existence Results for Fredholm Integral Equations

Theorem 3.1.1 [11] *Let* $h : \overline{\Omega}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping. Then the Fredholm operator associated to* $h, T : C\left(\overline{\Omega}; \mathbb{R}^n\right) \to C\left(\overline{\Omega}; \mathbb{R}^n\right)$ *given by*

$$T(u)(x) = \int_{\Omega} h(x, y, u(y))\, dy, x \in \overline{\Omega} \tag{3.1}$$

is completely continuous .

proof. We first prove that T is continuous .Let $u_0 \in C\left(\overline{\Omega}; \mathbb{R}^n\right)$ and choose any number $R \succ |u_0|_\infty$.Let $\epsilon \succ 0$. Since h is uniformly continuous on the compact set $\overline{\Omega}^2 \times \overline{B}_R(0; \mathbb{R}^n)$, there exists a $\delta_\epsilon \succ 0$ such that for every $u \in C\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying $|u - u_0|_\infty \leq \delta_\epsilon$ one has $u(y) \in \overline{B}_R(0; \mathbb{R}^n)$ and

$$|h(x, y, u(y)) - h(x, y, u_0(y))| \leq \epsilon$$

for all $x, y \in \overline{\Omega}$ .Then

$$\begin{aligned} |T(u)(x) - T(u_0)(x)| &\leq \int_{\Omega} |h(x, y, u(y)) - h(x, y, u_0(y))| \, dy \\ &\leq \epsilon \mu(\Omega) \end{aligned}$$

for every $x \in \overline{\Omega}$. Hence

$$|T(u) - T(u_0)|_\infty \leq \epsilon \mu(\Omega)$$

whenever $|u - u_0|_\infty \leq \delta_\epsilon$. Therefore T is continuous at u_0 .

Next, given a bounded subset Y of $C\left(\overline{\Omega}; \mathbb{R}^n\right)$, we shall prove that $T(Y)$ is relatively compact in $C\left(\overline{\Omega}; \mathbb{R}^n\right)$ .According to the Ascoli-Arzèla theorem, we have to show that $T(Y)$ is bounded and equicontinuous .

Inded, since Y is bounded there exists a constant $c \succ 0$ such that

$$|u|_\infty \leq c \qquad \text{for all} \quad u \in Y$$

It follows that for any $u \in Y$ we have

$$|T(u)|_\infty \leq M \mu(\Omega),$$

where

$$M = \max_{\overline{\Omega}^2 \times \overline{B}_c(0; \mathbb{R}^n)} |h(x, y, z)|.$$

Hence the set $T(Y)$ is bounded in $C\left(\overline{\Omega}; \mathbb{R}^n\right)$.

On the other hand , using the uniform continuity of h on the compact $\overline{\Omega}^2 \times \overline{B}_c(0; \mathbb{R}^n)$, for each $\epsilon \succ 0$ there exists a $\delta_\epsilon \succ 0$ such that

$$|h(x, y, u(y))| - |h(x', y, u(y))| \leq \epsilon$$

for all $x, x', y \in \overline{\Omega}$ with $|x - x'| \leq \delta_\epsilon$ and $u \in Y$. This immediately yields

$$|T(u)(x) - T(u)(x')| \leq \epsilon \mu(\Omega),$$

for all $x, x' \in \overline{\Omega}$ satisfying $|x - x'| \leq \delta_\epsilon$ and $u \in Y$. Thus $T(Y)$ is equicontinuous . ■

Theorem 3.1.2 [11] *Let $R \succ 0$ and $h : \overline{\Omega}^2 \times \overline{B}_R(0; \mathbb{R}^n) \to \mathbb{R}^n$ be a continuous mapping. Then the operator $T : \overline{B}_R\left(0; C\left(\overline{\Omega}; \mathbb{R}^n\right)\right) \to C\left(\overline{\Omega}; \mathbb{R}^n\right)$ given by* (3.1) *is completely continuous.*

proof. Essentially the same reasoning as in proof of Theorem 3.1.1 establishes the result. ■

Theorem 3.1.3 [11] *Let* $R \succ 0$ *and* $h : \overline{\Omega}^2 \times \overline{B}_R(0;\mathbb{R}^n) \to \mathbb{R}^n$ *be a continuous mapping .Assume*

$$M\mu(\Omega) \leq R \tag{3.2}$$

where

$$M = \max_{\overline{\Omega}^2 \times \overline{B}_R(0;\mathbb{R}^n)} |h(x,y,z)|.$$

then the Fredholm integral equation has at least one solution $u \in C\left(\overline{\Omega};\mathbb{R}^n\right)$ with $|u|_\infty \leq R$.

proof. According to Theorem 3.1.2 , the operator $T : \overline{B}_R\left(0; C\left(\overline{\Omega};\mathbb{R}^n\right)\right) \to C\left(\overline{\Omega};\mathbb{R}^n\right)$ given by (3.1) is completely continuous . On the other hand, (3.2) guarantees that

$$T\left(\overline{B}_R\left(0; C\left(\overline{\Omega};\mathbb{R}^n\right)\right)\right) \subset \overline{B}_R\left(0; C\left(\overline{\Omega};\mathbb{R}^n\right)\right).$$

Thus the conclusion folows Theorem 1.3.4 . ■

3.1.1 Fredholm Integral Equation in the Urysohn form

In this section we present general existence theorems for the Fredholm integral equation in $\mathbb{R}^n$

$$u(x) = \int_\Omega h(x,y,u(y))\,dy, x \in \overline{\Omega} \tag{3.3}$$

here $\Omega \subset \mathbb{R}^n$ is a bounded open set. We seek continuous solutions with values in a given ball

$$B = \{z \in \mathbb{R}^n : |z| \leq R\},$$

i.e.,$u \in C\left(\overline{\Omega}; B\right)$.

Theorem 1.3.6 yields the following existence principle which can be sum-marized as follows: boundedness yields existence.

Theorem 3.1.4 [11] *let* $h : \overline{\Omega}^2 \times B \longrightarrow \mathbb{R}^n$ *be continuous. Assume that*

$$|u|_\infty \prec R \tag{3.4}$$

for any solution $u \in C\left(\overline{\Omega}; B\right)$ to

$$u(x) = \lambda \int_\Omega h(x, y, u(y))\, dy, \quad x \in \overline{\Omega}, \tag{3.5}$$

for each $\lambda \in (0, 1)$. then (3.3) has a solution in $C\left(\overline{\Omega}; B\right)$.

proof. let $K = X = C\left(\overline{\Omega}; \mathbb{R}^n\right)$with norm $|.|_\infty$,

$$U = \left\{u \in C\left(\overline{\Omega}; \mathbb{R}^n\right) : |u|_\infty \prec R\right\},$$

u_0 be the null function and $T : \overline{U} \longrightarrow C\left(\overline{\Omega}; \mathbb{R}^n\right)$ be given by

$$T(u)(x) = \int_\Omega h(x, y, u(y))\, dy \quad \left(x \in \overline{\Omega}\right).$$

The result follows from the Leary-Schauder Theorems ■

Notice Theorem 3.1.3 is a corollary of Theorem 3.1.4. Indeed, if $u \in C\left(\overline{\Omega}; B\right)$ is any solution of (3.5) for some $\lambda \in (0.1)$, then for any $x \in \overline{\Omega}$, using (3.2) we obtain

$$\begin{aligned} |u(x)| &= \lambda \left| \int_\Omega h(x, y, u(y))\, dy \right| \\ &\leq \lambda \int_\Omega |h(x, y, u(y))|\, dy \\ &\leq \lambda M \mu(\Omega) \\ &\leq \lambda R \prec R. \end{aligned}$$

Hence u satisfies (3.4) .

3.1.2 Fredholm Integral Equation in the Hammerstrin form

An immediate consequence of Theorem 3.1.4 is the following existence result for the Hammerstrin integral equation in $\mathbb{R}^n$

$$u(x) = \int_{\Omega} k(x,y)\, f(y, u(y))\, dy, x \in \overline{\Omega} \tag{3.6}$$

Corollary 3.1.1 *Let* $k : \overline{\Omega}^2 \longrightarrow \mathbb{R}$ *and* $f : \overline{\Omega} \times B \to \mathbb{R}^n$ *be continuous functions . Assume*

$$|k(x,y)| \leq 1 \tag{3.7}$$

for all $x, y \in \overline{\Omega}$ *and that there exists a continuous nondecreasing function* ψ *:*$[0{,}R] \to \mathbb{R}_+$ *with* $\psi(R) \succ 0$*, and* $\phi \in C\left(\overline{\Omega}; \mathbb{R}_+\right)$ *, such that*

$$|f(y,z)| \leq \phi(y)\, \psi(|z|) \tag{3.8}$$

for all $y \in \overline{\Omega}\,, z \in B$ *,and*

$$|\phi|_{L^1(\Omega)} \leq \frac{R}{\psi(R)}. \tag{3.9}$$

then (3.6) *has a solution* $u \in C\left(\overline{\Omega}; \mathbb{R}^n\right)$ *with* $|u|_\infty \leq R$*.*

proof. *The result follows from Theorem* 3.1.4*, where*

$$h(x,y,z) = k(x,y)\, f(y,z)\,,$$

once we show that (3.4) *holds. For this, let* $u \in C\left(\overline{\Omega}; B\right)$ *be any solution of* (3.5) *for some* $\lambda \in (0,1)$*. Thene using* (3.7) $-$ (3.9) *we obtain*

$$\begin{aligned} |u(x)| &\leq \lambda \int_{\Omega} |k(x,y)|\, \phi(y)\, \psi(|u(y)|)\, dy \\ &\leq \lambda \psi(R) \int_{\Omega} \phi(y)\, dy \\ &= \lambda \psi(R)\, |\phi|_{L^1(\Omega)} \\ &\leq \lambda R \prec R. \end{aligned}$$

Hence $|u|_\infty \prec R$*. Therefore Theorem* 3.1.4 *applies.* ■

In particular, if k is the Green's function of a certain differential operator with respect to some boundary conditions, we can guarantee (3.4) by some other types of conditions on f.

Here are some examples for $\overline{\Omega} = [0,1]$, that is for the Hammerstien equation in $\mathbb{R}^n$

$$u(x) = \int_0^1 k(x,y) f(y, u(y)) \, dy, \ \ x \in [0,1] \cdot \tag{3.10}$$

Theorem 3.1.5 [11] *Let k be the Green's function of the difference operator $-u'' + u$ with respect to one set of the homogeneous Dirichelet, Neumann, or periodic boundary conditions*

$$u(0) = u(1) = 0 \ \ \text{(Dirichelet)}, \tag{3.11}$$

$$u'(0) = u'(1) = 0 \ \ \text{(Neumann)}, \tag{3.12}$$

$$u(0) - u(1) = u'(0) - u'(1) = 0 \qquad \text{(periodic conditions)} \cdot \tag{3.13}$$

Let $f : [0,1] \times B \to \mathbb{R}^n$ be continuous. Assume that for every $z \in B$ with $|z| = R$, one has

$$(z, f(y,z)) \leq R^2 \text{ for all } \ y \in [0,1] \cdot \tag{3.14}$$

Then (3.10) has a solution $u \in C([0,1]; \mathbb{R}^n)$ with $|u|_\infty \leq R$.

proof. To apply Theorem 3.1.4 we need to prove (3.4) for any solution $u \in C([0,1]; B)$ of (3.5) . Let $u \in C([0,1]; B)$ be a solution of (3.5) for somme $\lambda \in (0,1)$. Clearly, $|u|_\infty \leq R$. Assume

$$|u|_\infty = R.$$

Then there exists a point $x_0 \in [0,1]$ such that

$$|u(x_0)| = R.$$

On the other hand, since k is a Green's function $u \in C^2([0,1];\mathbb{R}^n)$ and solves the problem

$$\begin{cases} -u''(x) \quad + u(x) = \lambda f(x, u(x)), & x \in [0,1] \\ \qquad\qquad u \in B. \end{cases} \tag{3.15}$$

Here B stands for either homogeneous Dirichlet, Neumann, or periodic boundary conditions. We shall derive a contradiction to (3.14) once we show that

$$(u(x_0), f(x_{0,} u(x_0)) \succ R^2 \tag{3.16}$$

If $x_0 \in (0,1)$ then since $|u|^2$achieves its maximum at x_0, we must have

$$\left(|u|^2\right)''(x_0) \leq 0.$$

Direct computation shows that

$$\left(|u|^2\right)' = (u,u)' = 2(u,u')$$

and

$$\left(|u|^2\right)'' = 2|u'|^2 + 2(u,u''). \tag{3.17}$$

Consequently

$$(u(x_0), \ u''(x_0)) \leq 0.$$

This together with (3.15) yields

$$\begin{aligned} 0 \ &\geq \ (u(x_0), \ u(x_0) - \lambda f(x_0, u(x_0))) \\ &= \ R^2 - \lambda(u(x_0) \ , f(x_0, \ u(x_0))) \\ &\succ \ \lambda\left(R^2 - (u(x_0)), \ f(x_0, \ u(x_0))\right). \end{aligned}$$

Thus (3.16) holds·

Next we prove (3.16) in case that $x_0 = 0\cdot$ This case can not holds under conditions (3.11) ·

Case 1.Assume B means (3.12) · Assume

$$(u(0),\ f(0,u(0))) \leq R^2\cdot$$

Thene

$$\begin{aligned} R^2 &\succ \lambda R^2 \\ &\geq (u(0), \lambda f(0,u(0))) \\ &= (u(0), -u''(0) + u(0)) \\ &= R^2 - (u(0), u''(0)) \cdot \end{aligned}$$

It follows that $(u(0), u''(0)) \succ 0$. Now (3.17) implies $\left(|u|^2\right)''(0) \succ 0$ and so $\left(|u|^2\right)'$ is strictly increasing near 0. Then

$$\left(|u|^2\right)'(x) \succ \left(|u|^2\right)'(0) = 2(u(0),\ u'(0)) = 0$$

for $x \succ 0$ near 0. So $|u|^2$ is strictly increasing near 0, which is impossible since $|u|^2$ achieves its maximum at 0. Thus (3.16) holds.

Case 2. Assume B means (3.13) . If

$$(u(0), u'(0)) = (u(1), u'(1)) \neq 0,$$

it follows that $|u|^2$ can not achieve its maximum at $x_0 = 0$ (equivalently, at 1) . Hence

$$(u(0), u'(0)) = (u(1), u'(1)) = 0,$$

and we make the same reasoning as at Case 1.

Finally, we prove (3.16) similarly if $x_0 = 1$.

Therefore in all cases (3.16) holds, yielding a contradiction. ■

Remark 3.1.1 *Theorem* 3.1.5 *is in fact an existence result for the two−point boundary value problem*

$$\begin{cases} -u''(x) \quad + u(x) = f(x, u(x)) & x \in [0,1] \\ \qquad\qquad u \in B. \end{cases}$$

3.2 Existence Results for Volterra Integral Equations

3.2.1 Homogeneous Volterra Integral Equation

Theorem 3.2.1 [8] *Let* $h : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. Then the Volterra operator associated to* $h, T : C([a,b]; \mathbb{R}^n) \to C([a,b]; \mathbb{R}^n)$ *given by*

$$T(u)(t) = \int_a^t h(t, s, u(s))\, ds, \ \ (t \in [a,b])$$

is completely continuous.

proof. see[8] ■

This section presents general existence theorems for the Volterra integral equation via Leray-Schauder's theorem.We check continuous solution $u \in C([a,b]; B)$ for the volterra integral equation in the first kind

$$u(t) = \int_a^t h(t, s, u(s))\, ds, \qquad t \in [a,b] \tag{3.18}$$

where

$$B = \{z \in \mathbb{R}^n : |z| \le R\},$$

The Leray-Schauder's theorem the following existence principle which can be summarized as follows'boundedness yields existence' as we well see in the next theorem

Theorem 3.2.2 [8] *Let* $h : [a,b]^2 \times B \to \mathbb{R}^n$ *be continuous. Assume that*

$$|u|_\infty \prec \mathbb{R} \tag{3.19}$$

for any solution $u \in C([a,b]; B)$ to

$$u(t) = \lambda \int_a^t h(t, s, u(s))\, ds, \quad t \in [a, b] \tag{3.20}$$

for each $\lambda \in (0, 1)$. then (3.18) has a solution in $C([a, b]; B)$.

proof. Let $K = X = C([a, b]; \mathbb{R}^n)$ with norm $|\cdot|_\infty$

$$U = \{u \in C([a, b]; \mathbb{R}^n) : |u|_\infty \prec \mathbb{R}\}$$

u_0 be the null function and $T : \overline{U} \to C([a, b]; \mathbb{R}^n)$ be given by

$$T(u(x)) = \int_a^t h(t; s; u(s))\, ds, \quad t \in [a, b]$$

The result follows from Leray-Shauder's Theorem ■

Next we give a sufficient condition for (3.19) in the case of the equation in $\mathbb{R}^n$.

$$u(t) = \int_a^t k(t, s) f(s, u(s))\, ds, \quad t \in [a, b]. \tag{3.21}$$

Corollary 3.2.1 *Let $k : [a, b]^2 \to \mathbb{R}$ and $f : [a, b] \times B \to \mathbb{R}^n$ be continuous functions. Assume that*

$$|k(t, s)| \leq 1 \tag{3.22}$$

for all $t, s \in [a, b]$, there exists a continuous nondecreasing function $\psi : (0, R] \to (0, \infty)$ and a $\phi \in C([a, b]; \mathbb{R}_+)$ such that

$$|f(s, z)| \leq \phi(s)\, \psi(|z|) \tag{3.23}$$

for all $s \in [a, b]$, $z \in B$, and

$$|\phi|_{L^1[a,b]} \leq \int_0^R \frac{1}{\psi(\sigma)} d\sigma. \tag{3.24}$$

then (3.21) has a solution $u \in C([a; b]; \mathbb{R}^n)$ with $|u|_\infty \leq R$.

proof. let $u \in C([a, b]; B)$ be any solution of (3.20) for some $\lambda \in (0, 1)$. Here

$$h(t, s, z) = k(t, s) f(s, z),$$

then

$$|u(t)| \leq \lambda \int_a^t |k(t,s) f(s,u(s))| \, ds \leq \lambda \int_a^t \phi(s) \psi(|u(s)|) \, ds \qquad (3.25)$$

for all $t \in [a,b]$. Here we have understand that $\psi(0) = \lim\limits_{t \downarrow 0} \psi(t)$. Let

$$c(t) = \min \left\{ R, \lambda \int_a^t \phi(s) \psi(|u(s)|) \, ds \right\}.$$

clearly c is nondecreasing. We claim that $c(b) \prec R$. Assume the contrary.
Then, since $c(a) = 0$, there exists a subinterval $\left[a', b'\right] \subset [a,b]$ with

$$c(a') = 0, c(b') = R \text{ and } c(t) \in (0,R) \text{ for } t \in (a', b').$$

since by (3.25),

$$|u(t)| \leq c(t) \leq R \qquad \text{on} \quad [a,b],$$

and ψ is nondecreasing on $[0,R]$, we have

$$c'(s) = \lambda \phi(s) \psi(|u(s)|) \leq \lambda \phi(s) \psi(c(s))$$

for all $s \in [a', b']$. Now integration from a' to b' yields

$$\begin{aligned}
\int_{a'}^{b'} \frac{c'(s)}{\psi(c(s))} ds &= \int_0^R \frac{1}{\psi(\sigma)} d\sigma \\
&\leq \lambda \int_{a'}^{b'} \phi(s) \, ds \\
&\leq \lambda \int_a^b \phi(s) \, ds \\
&\prec \int_a^b \phi(s) \, ds,
\end{aligned}$$

a contradition . Notice we may assume $|\phi|_{L^1[a,b]} \succ 0$ since otherwise we have nothing to prove. Hence $c(b) \prec R$ and so, by (3.25), $|u(t)| \prec R$ for all $t \in [a,b]$.Therefor $|u|_\infty \prec R$ and theorem 3.2.2 applies. ■

Remark 3.2.1 *Notice Corollary* 3.2.1 *can be directly derived from Schauder's fixed point theorem if we observe that* $T(D) \subset D$, *where*

$$\begin{aligned} D &= \{u \in C([a,b];\mathbb{R}^n) : |u(t)| \leq \gamma(t) \ \ on \ \ [a,b]\}, \\ \gamma(t) &= I^{-1}\left(\int_a^t \phi(s)\,ds\right), \\ I(t) &= \int_0^t \frac{1}{\psi(\sigma)}d\sigma, \end{aligned}$$

and T is the Volterra integral operator associated to the right hand side of the equation (3.21). We note that $\gamma(t) \leq R$ for all $t \in [a,b]$ because of (3.24).

Corollary 3.2.1 yields a global existence result for the initial value problem

$$\begin{cases} u'(t) = f(t,u(t)), \ t \in [0,t_1], \\ \qquad u(0) = 0. \end{cases} \tag{3.26}$$

Theorem 3.2.3 [8] *Let* $f \in C([0,t_1]\times\mathbb{R}^n;\mathbb{R}^n)$. *Assume that there exists a continuous nondecreasing function* $\psi : (0,\infty) \to (0,\infty)$ *suche that*

$$|f(s,z)| \leq \psi(|z|)$$

for all $s \in [0,t]$, $z \in \mathbb{R}^n$, *and*

$$t_1 \prec \int_0^\infty \frac{1}{\psi(\sigma)}d\sigma\cdot$$

then (3.26) *has a solution* $u \in C^1([0,t_1];\mathbb{R}^n)\cdot$

proof. problem (3.26) is equivalent to (3.21). Here $[a,b] = [0,t_1]$ and $k \equiv 1$. Choose any $R \succ 0$ such that

$$t_1 \leq \int_0^R \frac{1}{\psi(\sigma)}d\sigma\cdot$$

Thene (3.24) holds and Corollary 3.2.1 applies. ■

3.2.2 Non-Homogeneous Volterra Integral Equation

Theorem 3.2.4 [6] *Consider the nonlinear Volterra integral equation*

$$u(t) = f(t) + \int_a^t h(t,s,u(s))\,ds \qquad -\infty \prec a \leq t \leq b \prec +\infty \tag{3.27}$$

where f is continuous over $[a,b]$. Assume that $h(t,s,u)$ satisfies the following conditions
:

$$|h(t,s,u)| \le V_1(t) V_2(s) \phi(|u|), \qquad \left|\frac{\partial h}{\partial u}(t,s,u)\right| \le V_1(t) V_2(s) \psi(|u|), \tag{3.28}$$

where $V_1(\cdot) \in C([a,b])$ and positive, $V_2(\cdot) \in L^1([a,b])$ and positive and where $\psi(\cdot)$ is apositive and continuous function over $[0,+\infty[$.Finally, we assume that the function $\phi(\cdot)$ is positive, continuous and satisfies the condition $\lim_{y\to+\infty}(\phi(y)/y) = L \prec +\infty$.Under the above condition , (3.27) has a continuous solution over $[a,b]$.

proof. Let $X = (C([a,b]), \|\cdot\|_\infty)$ denotes the Banach space of continnuous functions over $[a,b]$ and define the operator T over X by

$$Tu(t) = T(u)(t) = f(t) + \int_a^t h(t,s,u(s))\,ds\ .$$

By using the conditions of the theorem, it is easy to check that $TX \subset X$ and T is compact. From Leray-Schauder principle, to prove the result of the theorem , it suffices to prove that T is continuous over X and any solution of $u = \lambda Tu, 0 \le \lambda \le 1$ is bounded by the same constant $R \succ 0$. To prove the continuity of the T over $C([a,b])$, it suffices to replace $\mu(t)$ by 1 in the proof of the continuity of the operator T of the previous theorem and follow the different steps of this proof . Next, we note that the condition $\lim_{y\to+\infty}(\phi(y)/y) = L \prec +\infty$ implies the existence of a positive real number $A \succ 0$ such that $|\phi(x)| \le (3/2)L = L'$, for all $x \ge A$.

Let $u \in C([a,b])$ be a solution of $u = \lambda Tu, 0 \le \lambda \le 1$, then we have

$$\begin{aligned}
|u(t)| &\le |\lambda| |f(t)| + |\lambda| \int_a^t |h(t,s,u(s))\,ds| \le \|f\|_\infty + \int_a^t V_1(t) V_2(s) \phi(|u(s)|)\,ds \\
&\le \|f\|_\infty + \|V_1\|_\infty \int_a^b V_2(s) \sup_{x\in[0,A]} \phi(x)\,ds + \|V_1\|_\infty \int_a^b V_2(s) L' |u(s)|\,ds \qquad (3.29) \\
&\le \left[\|f\|_\infty + \sup_{x\in[0,A]} \phi(x) \|V_2\|_1\right] + \int_a^b (L' \|V_1\|_\infty V_2(s)) |u(s)|\,ds \\
&\le R_1 + \int_a^b R_2 V_2(s) |u(s)|\,ds.
\end{aligned}$$

By using the general version of Gronwall's together with the previous inequality, one conclude that$|u(t)| \leq R_1 \exp(R_2 \|V_2\|_1) = R$. Since R_1 and R_2 do not depent on the solution u, then one conclude that the solution of $u = \lambda Tu, 0 \leq \lambda \leq 1$ are uniformily bounded by the same constant R.Finally, by using the Leary-schauder principle, one concludes that T has a fixed point in $X = C([a,b])$ or equivalently, the nonlinear Volterra equation (3.27) has a continuous solution over $[a,b]$. ■

The uniqueness of the solution of the nonlinear Volterra equation (3.27) is given by the following proposition.

Proposition 3.2.1 [6] *consider the nonlinear Volterra equation* (3.27) *and assume that* $h(t,s,u)$ *satisfies the condition of the theorem* 3.2.4 *with* $V_2(.) \in (L^1 \cap L^p)([a,b])$ *for some* $p \succ 1$. *Then* (3.27) *has a unique solution* .

proof. The existence of a solution is ensured by Theorem 3.2.4 .Next, note that in the proof of Theorem 3.2.4, we have shown that the continuous solutions of $u = Tu$ are uniformly bounded by the same constant R and consequently they are contained in a closed ball B_R given by

$$B_R = \{u \in C([a,b]) ; \|u\|_\infty \leq R\}. \tag{3.30}$$

Hence, to prove the uniqueness of the solution of (3.27) , it suffices to check that there exists $n_0 \in \mathbb{N}$ such that T^{n_0} is a contraction in B_R . By using the notation of the proof of Theorem 3.2.4,one can easily check that for all $u, v \in C([a,b])$, we have

$$\begin{aligned} |Tv(t) - Tu(t)| &\leq \|v-u\|_\infty \|V_1\|_\infty \|V_2\|_p (t-a)^{\frac{1}{q}} \sup_{x \in B_R} \psi(|x|) \\ &\leq C \|v-u\|_\infty (t-a)^{\frac{1}{q}}. \end{aligned} \tag{3.31}$$

Similarly, one shows that

$$\left|T^2 v(t) - T^2 u(t)\right| \leq C^2 \|v-u\|_\infty \frac{1}{q+1} (t-a)^{1+\frac{1}{q}}. \tag{3.32}$$

Continuing in this manner, one can easily show that

$$|T^n v - T^n u| \leq C^n \|v-u\|_\infty \prod_{i=1}^{n-1} \frac{1}{q+i} (t-a)^{n-1+\frac{1}{q}} \tag{3.33}$$

Hence

$$\|T^n v(t) - T^n u(t)\|_\infty \leq C^n \|v-u\|_\infty \left[\prod_{i=1}^{n-1} \frac{1}{q+i}\right] (b-a)^{n-1+\frac{1}{q}}. \tag{3.34}$$

Since $\lim_{n\to+\infty} \left[\prod_{i=1}^{n-1} \frac{1}{q+i}\right] C^n (b-a)^{n-1+\frac{1}{q}} = 0$,then there exist $n_0 \in \mathbb{N}$ such that T^{n_0} is contraction over B_R.Consequently, the fixed point of T^{n_0} is unique. Since a fixed point of T is also a fixed point of T^{n_0}, then one concludes that the fixed point of T is also unique and consequently , the solution of (3.27) is unique. ■

3.3 The Cauchy Problem for an Integral Equation with Delay

Theorem 3.3.1 *Assume* $f \in C([-\tau, t_1] \times \mathbb{R}^n; \mathbb{R}^n)$, $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$ *and that* $\varphi(0) = \int_{-\tau}^{0} f(s, \varphi(s))\, ds$ *holds. Then the delay integral operator*

$T : D(T) \to C([0, t_1]; \mathbb{R}^n)$

$$T(u)(t) = \int_{t-\tau}^{t} f(s, u^{\sim}(s))\, ds \qquad (t \in [0, t_1])$$

where

$$D(t) = \{u \in C([0, t_1]; \mathbb{R}^n) : u(0) = \varphi(0)\}$$

and

$$\widetilde{u}(t) = \begin{cases} \varphi(t) & \text{for } t \in [-\tau, 0], \\ u(t) & \text{for } t \in [0, t_1] \end{cases},$$

is completely continuous .

proof. Use the Ascoli-Arzéla theorem and folow the same steps as in the proof of Theorem 3.1.1 We omit the details. ■

In this section we turn back to the intial value problem

$$\begin{cases} u(t) = \int_{t-\tau}^{t} f(s, u(s))\, ds, & 0 \leq t \leq t_1, \\ u(t) = \varphi(t), & -\tau \leq t \leq 0. \end{cases} \tag{3.35}$$

The technique we use here is based upon the Leray-Schauder principle. In addition we seek positive solution u with $u(t) \geq a$ for all $t \in [0, t_1]$, where $a \in \mathbb{R}_+$. Here, for $z, z' \in \mathbb{R}^n$, by $z \geq z'$ we mean $z_i \geq z'_i$, $i = 1, 2, \ldots, n$. Also $z \geq a$ stands for $z_i \geq a$, $i = 1, 2, \ldots, n$.

Our assumption are as follows:

(i) $f \in C\left([-\tau, t_1] \times [a, \infty)^n ; \mathbb{R}^n_+\right)$;

(ii) $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$, satisfies $\varphi(0) = \int_{-\tau}^{0} f(s, \varphi(s))\, ds$ and $\varphi(t) \geq a$ on $[-\tau, 0]$;

(iii) there exists a function $g \in C([-\tau, t_1]; \mathbb{R}^n)$ such that

$$f(t, s) \geq g(t)$$

for all $t \in [-\tau, t_1]$, $z \in [a, \infty)^n$, and

$$\int_{t-\tau}^{t} g(s)\, ds \geq a$$

for all $t \in [0, t_1]$;

(iv) there exists a continuous nondecreasing function $\psi : (a, \infty) \rightarrow (0, \infty)$ such that

$$|f(t, z)| \leq \psi(|z|)$$

for all $t \in [0, t_1]$, $z \in [a, \infty)^n$, and

$$t_1 \prec \int_{b}^{\infty} \frac{1}{\psi(\sigma)} d\sigma,$$

where

$$b = \int_{-\tau}^{0} |f(s, \varphi(s))|\, ds.$$

Theorem 3.3.2 [11] *Assume that the assumptions $(i) - (iv)$ are satisfied. Then (3.35) has at least one solution $u \in C([-\tau, t_1]; \mathbb{R}^n)$ with $u(t) \geq a$ for all $t \in [-\tau, t_1]$.*

proof. We first note with $b \geq |\varphi(0)|$. Indeed

$$b = \int_{-\tau}^{0} |f(s, \varphi(s))|\, ds \geq \left| \int_{-\tau}^{0} f(s, \varphi(s))\, ds \right| = |\varphi(0)|.$$

We apply The Leray-Schauder Theorem with $X = C([0, t_1]; \mathbb{R}^n)$,

$$K = \{u \in X : u(0) = \varphi(0), u(t) \geq a \ \text{ on } \ [0, t_1]\ \}$$

and $T : K \to K$ given by

$$T(u)(t) = \int_{t-\tau}^{t} f(s, \widetilde{u}(s))\, ds \qquad (t \in [0, t_1])$$

where

$$\widetilde{u}(t) = \begin{cases} \varphi(t), t \in [-\tau, 0] \\ u(t), t \in [0, t_1]. \end{cases}$$

According to $(i) - (iii)$, the operator T is well defined.In addition , T is completely continuous (see Theorem 3.3.1). In what follows we shall establish the boundedness of all solution to

$$u = (1-\lambda) u_0 + \lambda T(u) \tag{3.36}$$

for $\lambda \in (0,1)$, where u_0 is the constant function $\varphi(0)$. To this end , let u be any solution of (3.36) for some $\lambda \in (0,1)$. Then

$$|u(t)| \leq (1-\lambda) b + \lambda \int_{t-\tau}^{t} |f(s, \widetilde{u}(s))|\, ds$$

Let $c : [0, t_1] \to \mathbb{R}$ be given by

$$c(t) = (1-\lambda) b + \lambda \int_{t-\tau}^{t} |f(s, \widetilde{u}(s))|\, ds\ .$$

Notice $c(0) = b$. Furthermore, for $t \in [0, t_1]$ one has

$$\begin{aligned} c'(t) &= \lambda\left(|f(t, u(t))| - |f(t-\tau, \widetilde{u}(t-\tau))|\right) \\ &\leq \lambda |f(t, u(t))| \\ &\leq \lambda \psi(|u(t)|), \end{aligned}$$

and, since $|u(t)| \leq c(t)$ and ψ is nondecreasing,

$$c'(t) \leq \lambda \psi(c(t)).$$

Therefore

$$\frac{c'(s)}{\psi(c(s))} \leq \lambda \prec 1.$$

Integration from 0 to t yields

$$\int_b^{c(t)} \frac{1}{\psi(\sigma)} d\sigma = \int_0^t \frac{c'(s)}{\psi(c(s))} ds \prec t_1.$$

Hence $c(t) \prec R$, where R is any fixed number satisfying

$$t_1 \leq \int_b^R \frac{1}{\psi(\sigma)} d\sigma.$$

Therefore $|u(t)| \prec R$ for all $t \in [0, t_1]$. ■

3.4 Periodic Solutions of an Integral Equation with Delay

In this section we establish the existence of periodic solutions of a given period $\omega \succ 0$ for the integral equation in $\mathbb{R}^n$

$$u(t) = \int_{t-\tau}^{t} f(s, u(s)) ds \tag{3.37}$$

The ideas in this section were adapted from the paper Precup [12] .

Our assumption are:

(1) $f \in C\left(\mathbb{R} \times \mathbb{R}_+^n; \mathbb{R}_+^n\right)$

(2) $f(t+\omega, z) = f(t, z)$ for every $t \in \mathbb{R}, z \in \mathbb{R}_+^n$;

(3) there exists a number $a \succ 0$ and an $\omega-$periodic function $g \in C(\mathbb{R}; \mathbb{R}^n)$ such that

$$f(t, z) \geq g(t)$$

for all $t \in \mathbb{R}, z \in [a, \infty)^n$, and

$$\int_{t-\tau}^{t} g(s) ds \geq a$$

for all $t \in \mathbb{R}$;

(4) there exist two numbers b, R with

$$a\sqrt{n} \prec b \prec R$$

and a function $\psi \in C([0, R]; \mathbb{R}_+)$with $\psi(t) \succ 0$ on $[b, R]$, such that

$$|f(t, z)| \leq \psi(|z|)$$

for all $t \in \mathbb{R}$ and $z \in [a, \infty)^n$ with $|z| \leq R$,

$$\omega \leq \int_b^R \frac{1}{\psi(\sigma)} d\sigma \tag{3.38}$$

and

$$|f(t, z)| \leq \frac{b}{\tau} \tag{3.39}$$

for all $t \in \mathbb{R}$ and $z \in [a, \infty)^n$ satisfying $b \leq |z| \leq R$.

Theorem 3.4.1 [11] *Assume that* $(1) - (4)$ *are satisfied . Then* (3.37) *has an* $\omega -$*periodic continuous solution* u *such that*

$$u(t) \geq a \quad on \quad [0, \omega],$$

$$\min_{t \in [0,\omega]} |u(t)| \prec b \quad \text{and} \quad \max_{t \in [0,\omega]} |u(t)| \leq R.$$

proof. We shall apply the Leray-Schauder principle. Here X is the space of all continuous $\omega-$periodic functions u with the norm $|u|_\infty = \max_{t \in [0,\omega]} |u(t)|$,

$$K = \{u \in X : u(t) \geq a \text{ on } [0, \omega]\}$$

and

$$U = \left\{ u \in K : \min_{t \in [0,\omega]} |u(t)| \prec b, |u|_\infty \prec R \right\}$$

Also, u_0 is the constant vector-valued function $(a, a, \ldots, a)$ simply denoted by a,and $T : K \to K$ is given by

$$T(u)(t) = \int_{t-\tau}^{t} f(s, u(s)) ds \qquad (t \in \mathbb{R})$$

It is easy to show that (1) − (3)guarantee that T is well definded and completely continuous . We now claim that the Leray-Schauder boundary

condition (1.1) satisfied. Assume by contradiction that there are $u \in \partial U$ and $\lambda \in (0,1)$ such that

$$u = (1-\lambda)\,u_0 + \lambda T\,(u)\,,$$

that is

$$u\,(t) = (1-\lambda)\,a + \lambda \int_{t-\tau}^{t} f\,(s, u\,(s))\,ds\ , \qquad t \in \mathbb{R} \tag{3.40}$$

Since $u \in \partial U$ we have either

$$|u|_\infty = R \quad \text{and} \quad \min_{t\in[0,\omega]} |u\,(t)| \prec b, \tag{3.41}$$

or

$$|u|_\infty \leq R \quad \text{and} \quad \min_{t\in[0,\omega]} |u\,(t)| = b. \tag{3.42}$$

first assume (3.41) . By differentiating (3.40) we obtain

$$u'\,(t) = \lambda f\,(t, u\,(t)) - \lambda f\,(t-\tau, u\,(t-\tau))\,.$$

Then

$$(u\,(t)\,, u'\,(t)) = \lambda\,(u\,(t)\,, f\,(t, u\,(t))) - \lambda\,(u\,(t)\,, f\,(t-\tau, u\,(t-\tau)))\,.$$

Since all the components of the vectors $u\,(t)$ and $f\,(t-\tau, u\,(t-\tau))$ are nonnegative, we have that

$$(u\,(t)\,, f\,(t-\tau, u\,(t-\tau))) \geq 0.$$

Hence

$$(u\,(t)\,, u'\,(t)) \leq \lambda\,(u\,(t)\,, f\,(t, u\,(t)))\,.$$

Furthermore, using (4) we obtain

$$(u\,(t)\,, u'\,(t)) \leq \lambda\,|u\,(t)|\,|f\,(t, u\,(t))| \leq \lambda\,|u\,(t)|\,\psi\,(|u\,(t)|)\,.$$

since

$$|u|' = \left(\sqrt{|u|^2}\right)' = \frac{(u, u')}{|u|}$$

we obtain

$$|u(t)|' \leq \lambda \psi(|u(t)|). \tag{3.43}$$

Let $t_0 \in [0, \omega]$ be such that

$$|u(t_0)| = \min_{t \in [0,\omega]} |u(t)|$$

and let $t_1 \in (t_0, t_0 + \omega]$ be such that $|u(t_1)| = R$. From(3.43), by integration from t_0 to t_1,we obtain

$$\int_{|u(t_0)|}^{|u(t_1)|} \frac{1}{\psi(\sigma)} d\sigma \leq (t_1 - t_0) \prec \omega.$$

This, according to (3.38) , is impossible since $|u(t_0)| \prec b$ and $|u(t_1)| = R$.

Thus (3.41) can not hold.

Now assume (3.42) . Let $t_0 \in [0, \omega]$ be such that

$$|u(t_0)| = \min_{t \in [0,\omega]} |u(t)| = b$$

Then form (3.40) and (3.39) we obtain

$$b = |u(t_0)| \leq (1 - \lambda) a\sqrt{n} + \lambda b \prec b$$

a contradiction .

Thus the Leray-Schauder boundary condition holds and Theorem 1.3.6 applies ■

Example 3.4.1 *Let $n = 1, \tau = \omega = 1$ and*

$$\psi(z) = \begin{cases} 5z, & z \in [0, 1] \\ -4z + 9, & z \in [1, 2] \\ 1, & z \in [2, 3] \\ 3z - 8, & z \in [3, 5] \\ z + 2, & z \in [5, \infty). \end{cases}$$

and let

$$f(t, z) = g_1(t) \psi_k(z), \qquad t \in \mathbb{R}, z \in [4(k-1), 4k],$$

$k = 1, 2, \ldots,$ where

$$\psi_k(t) = 4(k-1) + \psi_1(z - 4(k-1))$$

and g_1 is any nonnegative continuous function with a period $\omega \succ 0$, such that

$$\int_{t-1}^{t} g_1(s)\,ds \geq 1, \qquad t \in [0,\omega].$$

it is easy to see that if

$$\omega \max_{t\in[0,\omega]} g_1(t) \leq (4(k-1)+1)^{-1},$$

then all the assumptions of Theorem are satisfied

$$\begin{aligned} a &= 4(k-1)+1, \quad b = 4(k-1)+2, \qquad R = 4(k-1)+3, \\ g(t) &= (4(k-1)+1)\, g_1(t), \\ \psi(z) &= \psi_k(z) \max_{t\in[0,\omega]} g_1(t). \end{aligned}$$

There for each $k \in \mathbb{N} \setminus \{0\}$ satisfying

$$4(k-1)+1 \leq \left(\omega \max_{t\in[0,\omega]} g_1(t)\right)^{-1},$$

the equation (3.37) has at least one continuous $\omega-$periodic solution u_k such that

$$4(k-1)+1 \leq \inf_{t\in\mathbb{R}} u_k(t) \prec 4(k-1)+2$$

and

$$\sup_{t\in\mathbb{R}} u_k(t) \leq 4(k-1)+3$$

In particular, when $g_1(t) \equiv 1$ such solution are the constant functions

$$u_k(t) = 4(k-1) + \frac{9}{5}$$

for $k = 1, 2, \ldots, .$

Conclusion

In this work, we had studied the nonlinear integral equations, we proved the existence of solution using the Leray-Schauder principle.

The applications of the Leray-Schauder principle makes possible to avoid some difficult conditions given in other fixed point theorems such that the invariance condition $T(D) \subset D$ which has to be guaranteed for a bounded closed convex subset D of a Banach space for Schauder's fixed point theorem and contraction principle for Banach's fixed point theorem, and requires instead that a'boundary condition'is satisfied.

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ملخص

الهدف من هذه المذكرة دراسة مبدأ ليراي شودر و تطبيقاته في برهان وجود حل للمعادلات التكاملية الغير خطية ذات الشكل العام

$$u(x) = f(x) + \int_{\Omega} K\big(x, y, u(y)\big)\, dy \qquad x \in \Omega$$

حيث$K(x,y)$ تسمى نواة المعادلة التكاملية ، $f(x)$ ، $K(x,y)$ دوال معطاة .

و نخص بالذكر معادلات فولتيرا التكاملية ، معادلات فريدهولم التكاملية و معادلات تكاملية مع تأخير.

الكلمات المفتاحية : مبدأ ليراي شودر، معادلة تكاملية، معادلة فولتيرا، معادلة فريدهولم ، معادلة تكاملية مع تأخير وشرط الحدود.

abstract

The aim of this memoir is to study the Leray-Schauder principle and its applications in proving the existence of a solution to nonlinear integral equations, which have the general form

$$u(x) = f(x) + \int_{\Omega} K\big(x, y, u(y)\big)\, dy \qquad x \in \Omega$$

Where K is called the kernel of the integral equation , and both the Kernel $K(x,y)$ and the function $f(x)$ in the integral equations functions.

In particular, we mention the integral equation of volterra, the integral equation of fredholm and integral equations with Delay.

Keywords : the Leray-Schauder principle, integral equation, volterra equations, fredholm equations, integral quation with Delay and the boundray condition .

Résumé

L'objectif de cette mémoire est d'étudier le principe de Leray- Schauder et ses applications pour prouver l'existence d'une solution des équations intégrales non linéaires, qui ont la forme générale suivante

$$u(x) = f(x) + \int_{\Omega} K\big(x, y, u(y)\big)\, dy \qquad x \in \Omega$$

ou K est appelée le noyau de l'équation intégrale, et le noyau $K(x,y)$ et la fonction $f(x)$ dans l'équation intégrale sont des fonctions données.

En particulier, nous mentionnons les équations intégrales de Volterra, les équations intégrales de Fredholm et les équations intégrales avec retard.

Mots clés : principe de Leray-Schauder, équation intégrale, équations de Volterra, équations de Fredholm, équations intégrales avec retard et la condition au limites.