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# *Master of Mathematics*

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**Presented by :**

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# Topic

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## A critical point theorem for perturbed functionals and localization of critical point in bounded convex

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**in front of the jury :**

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# Dedication

*I humbly dedicate this piece of work*

*To my father Ameur, who was my support and source of strength, who protects and encourages me.*

*To my mother fatima, who always accompanied me with her blessed prayers and kind words.*

*To my brothers: Ahmed, Mohamed, Mabrouk.*

*To my sisters: Warda, Chahra, Leila, Aicha.*

*To my nephews, To my nieces.*

*To my family.*

*To my friends , To my colleagues.*

*To all those who have been supportive.*

*I dedicate this work.*

*Sarra*

# Contents

# Notations

| | |
|---|---|
| $\Omega$ | Is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ an integer. |
| $\Gamma = \partial\Omega$ | Is the smooth boundary of $\Omega$. |
| $L^p(\Omega)$ | The Lebesgue space, $1 < p < +\infty$. |
| $L^\infty(\Omega)$ | The Lebesgue space, for $p = +\infty$. |
| $W_0^{1,p}(\Omega)$ | Sobolev space. |
| $(W_0^{1,p}(\Omega))'$ | Dual space of $W_0^{1,p}(\Omega)$. |
| $\mathcal{D}(\Omega)$ | The space of indefinitely differentiable function with compact support in $\Omega$. |
| $\rightharpoonup$ | Weak convergence. |
| $\rightarrow$ | Strong convergence. |
| $\Rightarrow$ | implication. |
| l.s.c | Lower semicontinuous. |
| u.s.c | upper semicontinuous. |
| $A$ | Banach space. |
| $A'$ | Dual space of $A$. |
| $A''$ | Bidual space of $A$. |
| $\hookrightarrow$ | Embedded continuously. |
| $\hookrightarrow_c$ | Embedded compactly. |
| $C^1(A, \mathbb{R})$ | The space of continuously differentiable functions from $A$ in $\mathbb{R}$. |
| $B(0, \rho)$ | The open ball of radius $\rho$ centered at zero. |
| $\bar{B}(0, \rho)$ | The closure of $B(0, \rho)$. |
| $B^c(0, \rho)$ | The complement of $B(0, \rho)$. |
| $(u, v)_{L^2}$ | The inner product on $L^2(\Omega)$ |
| $F$ | Closed subset. |
| $\partial F$ | Is the smooth boundary of $F$. |
| $F^\circ$ | Interior of $F$. |
| $\mathcal{X}$ | Hilbert space. |

# Introduction

The critical point theory and variational methods have undergone significant development in their applications to differential equations over the past 20-25 years; see, for example [2, 4, 9, 13, 17, 18].

This theory has attained a spectacular success in the last thirty-or-so years, reaching a high level of complexity and refinement, and its applications are scattered throughout thousands of research papers. Perhaps among them are two references that we relied on to complete this work [15, 16], where we detailed the proofs and collected concepts related to these theories in a way that facilitates the reader's understanding of their content.

Our work is organized as follows.

In the first chapter, we give some reminders on functional spaces and notion of differentiability and their main properties, critical point and Palais-Smale condition. Finally, we conclude with Ekeland's variational principle.

In the second chapter, we establish a new critical point theorem for a class of perturbed functionals without satisfying the Palais-Smale condition, which asserts the existence of critical point of functionals of the type $I = I_1 + I_2$, provided that $I_1$ has at least one critical point. After that, we apply our abstract results are motivated by the existence of solution of the following nonhomogeneous nonlinear Problem:

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}.\nabla u) = |u|^{q-2}u + \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where $\lambda \in \mathbb{R}$, $\Omega$ is a bounded set of $\mathbb{R}^N$with smooth boundary $\Gamma$, $1 < q < p$ with $p > N$ and $g(\cdot,\cdot)$ is continuous on $\bar{\Omega} \times [0,\infty)$.

In the last chapter, we establish the localization of a critical point of minimum type of a smooth functional which is obtained in a bounded convex conical set defined by a norm and a concave upper semicontinuous functional. After that, we apply our abstract results are motivated by the existence of the positive solution of the following Periodic Problem:

$$\begin{cases} -u''(t) + a^2 u(t) = f(u(t)) \quad \text{on } (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}$$

where $a \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(\mathbb{R}_+) \subset \mathbb{R}_+$.

# Chapter 1

# Preliminaries and some basic tools

In this chapter, we introduce the fundamental notions to be used later.

Let $\Omega$ be an open bounded set of $\mathbb{R}^N$ with smooth boundary $\Gamma$.

## 1.1 Functional spaces

For further details on these concepts, refer to [1, 3, 5, 6].

### 1.1.1 Lebesgue spaces

**Definition 1.1.** *Let $1 < p < \infty$.We set*

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R}, u \textit{ is measurable and } \int_\Omega |u(x)|^p \, dx < +\infty \right\},$$

*with the norm*

$$\|u\|_{L^p} = \|u\|_p = \left( \int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

**Definition 1.2.** *We set*

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R}, \textit{measurable}, \exists c > 0 : |u(x)| \le c \;\; \textit{a.e on } \Omega \},$$

*with the norm*

$$\|u\|_{L^\infty} = \|u\|_\infty = \inf \{ c : |u(x)| \le c \;\; \textit{a.e on } \Omega \}.$$

**Theorem 1.1.** ***(Dominated convergence theorem,Lebesgue)*** *Let $(u_n)_n \subset L^1$ be a sequence that satisfy*

- *$u_n(x) \to u(x)$ a.e. on $\Omega$ as $n \to \infty$,*
- *there exists $h \in L^1$ such that for every $n$, $|u_n(x)| \le h(x)$ a.e. on $\Omega$.*
  *Then $u \in L^1$ and $\lim_{n\to\infty} \|u_n - u\|_1 = 0$.*

**Theorem 1.2.** ***(Inverse dominated convergence theorem)*** *Let $(u_n)_n \subset L^p$ be a sequence and let $u \in L^p$ such that $\|u_n - u\|_p \to 0$. Then there exists a subsequence $(u_{n_k})_k$ that satisfy*

- *$u_{n_k}(x) \to u(x)$ a.e. on $\Omega$ as $k \to \infty$,*
- *there exists $h \in L^p$ such that for all $k$,$|u_{n_k}(x)| \le h(x)$ a.e. on $\Omega$.*

**Theorem 1.3.** ***(Holder's Inequality)*** *Let $1 \le p \le +\infty$, and $q$ its conjugate exponent. Assume that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |u(x)v(x)|\, dx \le \|u\|_p \|v\|_q .$$

### 1.1.2 Sobolev Spaces

Let $p \in \mathbb{R}$ with $1 \le p \le \infty$.

**Definition 1.3.** *Let $W^{1,p}(\Omega)$ is a Sobolev space, defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega), \nabla u \in L^p(\Omega)\},$$

*where $\nabla u$ is the gradient in the sense of distributions. With the norm*

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p} .$$

*We set*

$$H^1(\Omega) = W^{1,2}(\Omega).$$

**Remark 1.1.** *$H^1(\Omega)$ is a Hilbert space with the inner product*

$$(u,v)_{H^1} = (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2} ,$$

*that is*

$$(u,v)_{H^1} = \int_\Omega u(x)v(x)dx + \int_\Omega \nabla u(x)\nabla v(x)dx.$$

**Definition 1.4.** *Let $1 \le p < \infty$, $W_0^{1,p}(\Omega)$ denoted the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ and*

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}, u = 0 \text{ on } \Gamma\right\}.$$

*With the norm*

$$\|u\|_{W_0^{1,p}} = \|\nabla u\|_{L^p} .$$

*We set*

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

**Proposition 1.1.**

1. *$W^{1,p}$ is a Banach space for $1 \le p \le \infty$.*
2. *$W^{1,p}$ is reflexive for $1 < p < \infty$.*
3. *$W^{1,p}$ is separable for $1 \le p < \infty$.*

**Theorem 1.4.** ***(Poincare's Inequality)*** *We assume that $\Omega$ is a bounded open set of $\mathbb{R}^n$ and $1 \le p < +\infty$. Then there is a positive constant $C$ (dependent on $\Omega$ and $p$ ) such that*

$$\|u\|_{L^P} \le C \|\nabla u\|_{L^p} \quad \forall u \in W_0^{1,p}(\Omega).$$

### 1.1.3 Sobolev Inequalities

**Definition 1.5.** *Let $A$ and $B$ be two Banach spaces*

- *We say that $A$ is* ***embedded continuously*** *in $B$ noted $A \hookrightarrow B$ if*

1. *$A \subseteq B$;*
2. *the canonical injection $i : A \to B$, $i(u) = u$ is a continuous operator. That is there exists a constant $C > 0$ such that $\|u\|_B \leq C \|u\|_A$ , for all $u \in A$.*

- *We say that $A$ is* ***embedded compactly*** *in $B$ noted $A \hookrightarrow_c B$ if $A \hookrightarrow B$ and the canonical injection $i$ is a compact operator.*

**Corollary 1.1.** *We Assume that $\Omega$ is an open set of class $C^1$ with $\Gamma$ bounded. Let $1 \leq p \leq \infty$ , then we have the next continuous injections:*

$$W^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{p^*}(\Omega) & \text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} & \text{if } p < N, \\ L^q(\Omega) & \forall q \geq 1 & \text{if } p = N, \\ L^\infty(\Omega) & & \text{if } p > N. \end{cases}$$

**Theorem 1.5.** *(**Rellich-Kondrachov**) Assume that $\Omega$ is bounded and of class $C^1$. Then we have the next compact injections:*

$$W^{1,p}(\Omega) \hookrightarrow_c \begin{cases} L^q(\Omega) & \forall 1 \leq q < p^* & \text{if } p < N, \\ L^q(\Omega) & \forall q \geq 1 & \text{if } p = N, \\ C(\bar{\Omega}) & & \text{if } p > N. \end{cases}$$

Where $p^* = \frac{Np}{N-p}$ is called the critical Sobolev exponent.

## 1.2 Operators on Banach spaces

For further details on these concepts, refer to [1, 6, 7, 8, 10].

**Definition 1.6.** *(**Dual space**) Let $A$ be a Banach space, we denote by $A'$ its dual, namely the space of continuous linear functionals from $A$ to $\mathbb{R}$. With the norm*

$$\|J\|_{A'} = \sup_{\substack{u \in A \\ \|u\|=1}} |\langle J, u \rangle| \quad \forall J \in A'.$$

**Definition 1.7.** *(**Reflexive Spaces**) Let $A$ be a Banach space and let $i$ be the canonical injection from $A$ into $A''$. We say that $A$ is reflexive if $i(A) = A''$, meaning that $i$ is surjective.*

**Definition 1.8.** ***(Strong and Weak convergence)*** *Let $(u_n)$ be a sequence in $A$*

***1.*** *We say that $u_n$ converges weakly to $u$ if*

$$\langle f, u_n \rangle \to \langle f, u \rangle \quad \forall f \in A'.$$

*We shall write*

$$u_n \rightharpoonup u.$$

***2.*** *We say that $u_n$ converges strongly to $u$ if $\|u_n - u\| \to 0$. We shall write*

$$u_n \to u.$$

**Proposition 1.2.** *If $u_n \rightharpoonup u$ weakly and if $f_n \to f$ strongly in $A'$, then*

$$\langle f_n, u_n \rangle \to \langle f, u \rangle .$$

**Definition 1.9.** ***(Continuous Operator)*** *Let $A$ and $B$ be two Banach spaces. An operator $T : A \to B$ is said to be continuous if for any sequence $(u_n) \subset A$ that converges to a point $u_0$ in $A$, the sequence $(T(u_n))$ converges to $T(u_0)$ in $B$.*

**Definition 1.10.** ***( Compact Operator)*** *Let $A$ and $B$ be two Banach spaces. An operator $T : A \to B$ is said to be compact if for every bounded sequence $(u_n)$ in $A$, the sequence $(T(u_n))$ in $B$ has a convergent subsequence.*

**Theorem 1.6.** *Let $A$ be a reflexive Banach space and let $(u_n)$ be a bounded sequence in $A$. Then there is a subsequence that converges weakly to $u$.*

**Definition 1.11.**

1. *We say that $I : A \to \mathbb{R}$ is lower semicontinuous $(l.s.c)$ if for every $(u_n) \subset A$, we have*

$$u_n \to u_0 \text{ in } A \Rightarrow I(u_0) \leq \liminf_{n \to +\infty} I(u_n).$$

2. *We say that $I : A \to \mathbb{R}$ is upper semicontinuous $(u.s.c)$ if for every $(u_n) \subset A$, we have*

$$u_n \to u_0 \text{ in } A \Rightarrow I(u_0) \geq \limsup_{n \to +\infty} I(u_n).$$

**Theorem 1.7.** ***(Mean Value Theorem)*** *Assume that $F(x)$ is a function that satisfies both of the following:*

1. *$F(x)$ is continuous on the closed interval $[a, b]$ .*

2. *$F(x)$ is differentiable on the open interval $]a, b[$. Then there exists a number $c$ in the interval $]a, b[$ such that*

$$F(b) - F(a) = F'(c)(b - a).$$

**Lemma 1.1.** *If $1 \leq p < \infty$ and $a, b \geq 0$, then*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p). \tag{1.1}$$

**Lemma 1.2.** *1. If* $p \geq 2$*, then*

$$\begin{aligned} \left\langle |b|^{p-2}\, b - |a|^{p-2}\, a, b-a \right\rangle &\geq 2^{-1} \left( |b|^{p-2} + |a|^{p-2} \right) |b-a|^2 \\ &\geq 2^{2-p}\, |b-a|^p\,. \end{aligned} \tag{1.2}$$

*2. If* $1 < p < 2$*, then*

$$\left\langle |b|^{p-2}\, b - |a|^{p-2}\, a, b-a \right\rangle \geq C_p \frac{|b-a|^2}{(|b|+|a|)^{2-p}}. \tag{1.3}$$

## 1.3 Functional Differentiability and Critical point

For further details on these concepts, refer to [1, 2, 4].

### 1.3.1 Fréchet Differentiability

**Definition 1.12.** *Let $A$ be a Banach Space, $\Omega$ an open subset of $A$ and let $I : A \to \mathbb{R}$ be a functional. We say that I is (Fréchet) differentiable at $u \in A$ if there exists $A_u \in A'$ such that*

$$\lim_{\|v\| \to 0} \frac{I(u+v) - I(u) - A_u(v)}{\|v\|} = 0.$$

*Where $I(u+v) - I(u) = A_u(v) + o(\|v\|)$, with $\lim\limits_{\|v\|\to 0} \frac{o(\|v\|)}{\|v\|} = 0$.*

- *The operator $A_u$ is called the (Fréchet) differential of $I$ at $u$, denoted by $I'(u)$. We get*

  $$I(u+v) - I(u) = \langle I'(u), v \rangle + o(\|v\|),$$

  *as $\|v\| \to 0$.*

**Example 1.1.** *Let $A$ be a Banach space and let*

$$y : A \times A \to \mathbb{R}$$

*be a continuous bilinear form. Let $I : A \to \mathbb{R}$ be an associated functional such that*

$$I(u) = y(u,u).$$

*Then, $I$ is differentiable on $A$ and*

$$\langle I'(u), v \rangle = y(u,v) + y(v,u),$$

*for every $u, v \in A$; Indeed by linearity of $y$, we get*

$$\begin{aligned} I\,(u+v) - I\,(u) &= y\,(u+v, u+v) - y(u,u) \\ &= \underbrace{y(u,v) + y(v,u)}_{A_u(v)} - \underbrace{y(v,v)}_{o(\|v\|)}. \end{aligned}$$

*Now, we show that* $\lim_{\|v\|\to 0} \frac{y(v,v)}{\|v\|} = 0$. *By continuity of* $y$, *then there exists a constant* $c > 0$ *such that*

$$|y(u,v)| \leq c\,\|u\|\,\|v\| \Rightarrow |y(v,v)| \leq c\,\|v\|^2\,,$$

*consequently,*

$$\lim_{\|v\|\to 0} \frac{y(v,v)}{\|v\|} = 0.$$

*Next, we prove that* $A_u \in A'$. *Since* $y$ *is linear, then* $A_u$ *is linear and we have*

$$\begin{aligned} |A_u(v)| &\leq |y(u,v)| + |y(v,u)| \\ &\leq c\,\|v\|\,, \end{aligned}$$

*where* $c$ *is a positive constant. So* $A_u$ *is continuous. Therefore*

$$\langle I'(u), v\rangle = y(u,v) + y(v,u).$$

*In addition, if* $y$ *is symmetric, then*

$$\langle I'(u), v\rangle = 2y(u,v).$$

### 1.3.2 Gâteaux differentiable

**Definition 1.13.** *Let* $A$ *be a Banach space,* $\Omega \subseteq A$ *an open set and let* $I : \Omega \to \mathbb{R}$ *be a functional. We say that* $I$ *is Gâteaux differentiable at* $u \in \Omega$ *if there exists* $T_u \in A'$ *such that*

$$\lim_{t\to 0} \frac{I(u+tv) - I(u)}{t} = T_u(v).$$

- *The operator* $T_u$ *is called Gâteaux differential of I at u and is denoted by* $I'_G(u)$.

**Remark 1.2.** *By definition of Fréchet differentiability, it is obvious that if* $I$ *is differentiable at* $u$, *then it is Gâteaux differentiable and* $I'(u) = I'_G(u)$.

*Proof.* Assume that $I$ is differentiable at $u$. Then we have

$$\lim_{\|v\|\to 0} \frac{I(u+v) - I(u) - A_u(v)}{\|v\|} = 0.$$

We replace $v$ by $tw$, where $\|w\| = 1$, i.e. $\|v\| = t$, we get

$$\lim_{t\to 0} \frac{I(u+tw) - I(u)}{t} = A_u(w).$$

Thus, $I$ is Gâteaux differentiable at $u$. Therefore, the proof is complete. □

**Example 1.2.** *Let* $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ *be a functional given by*

$$I(u) = \int_\Omega |\nabla u|^p \, dx.$$

*Then, $I$ is Gâteaux differentiable and*

$$\langle I'_G(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx.$$

*Indeed, let $u, v \in W_0^{1,p}$ and let $t > 0$ be fixed. We consider the function $g : [0, t] \to \mathbb{R}$ such that*

$$g(s) = |\nabla u + s\nabla v|^p$$

*be a continuous on $[0, t]$ and derivable on $]0, t[$. Then by the Mean Value Theorem, there exists a number $c_t \in ]0, t[$ such that*

$$\frac{g(t) - g(0)}{t} = g'(c_t),$$

*that is,*

$$\frac{|\nabla u + t\nabla v|^p - |\nabla u|^p}{t} = p\,|\nabla u + c_t\nabla v|^{p-2}(\nabla u + c_t\nabla v).\nabla v$$

*We have $t \to 0$ as $c_t \to 0$, we get*

$$\lim_{t\to 0}\frac{|\nabla u + t\nabla v|^p - |\nabla u|^p}{t} = \lim_{c_t\to 0} p\int_\Omega |\nabla u + c_t\nabla v|^{p-2}(\nabla u + c_t\nabla v).\nabla v\ \ dx.$$

*Using Dominated convergence theorem Lebesgue, we get*

$$\lim_{c_t\to 0} p\int_\Omega |\nabla u + c_t\nabla v|^{p-2}(\nabla u + c_t\nabla v).\nabla v\ \ dx = \ p\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\ \ dx. \tag{1.4}$$

*Indeed, we have*

$$|\nabla u + c_t\nabla v|^{p-2}(\nabla u + c_t\nabla v).\nabla v \underset{c_t\to 0}{\longrightarrow} |\nabla u|^{p-2}\nabla u.\nabla v\ \ \textit{a.e on}\ \Omega.$$

*Using (1.1), we get*

$$\begin{aligned}
\left||\nabla u + c_t\nabla v|^{p-2}(\nabla u + c_t\nabla v).\nabla v\right| &= |\nabla u + c_t\nabla v|^{p-1}|\nabla v| \\
&\le (|\nabla u| + \underbrace{|c_t|}_{\le 1}|\nabla v|)^{p-1}.\,|\nabla v| \\
&\le 2^{p-2}\left(|\nabla u|^{p-1} + |\nabla v|^{p-1}\right).\,|\nabla v| \\
&\le 2^{p-2}\underbrace{\left(|\nabla u|^{p-1}|\nabla v| + |\nabla v|^p\right)}_{h(x)}.
\end{aligned}$$

*It remains to verify that $h \in L^1(\Omega)$, we have:*

- *Since $\nabla v \in L^p(\Omega)$, then $|\nabla v|^p \in L^1(\Omega)$.*
- *Check that $|\nabla u|^{p-1}|\nabla v| \in L^1(\Omega)$. Using Holder's Inequality, we get*

$$\begin{aligned}
\int_\Omega |\nabla u|^{p-1}|\nabla v|\,dx &\le \left[\int_\Omega \left(|\nabla u|^{p-1}\right)^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}\left[\int_\Omega |\nabla v|^p\,dx\right]^{\frac{1}{p}} \\
&\le \left[\int_\Omega |\nabla u|^p\,dx\right]^{\frac{p-1}{p}}\left[\int_\Omega |\nabla v|^p\,dx\right]^{\frac{1}{p}} \\
&= \|\nabla u\|_{L^p}^{p-1}\|\nabla v\|_{L^p} < +\infty,
\end{aligned}$$

*this shows the claim. Therefore, the relation (1.4) is proved. Then*

$$\lim_{t\to 0}\frac{I(u+tv)-I(u)}{t}=p\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v\,dx.$$

*Let* $T_u : W_0^{1,p} \to \mathbb{R}$ *be a map, given by*

$$T_u(v) = \ p\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v\ \ dx.$$

*We verify that* $T_u \in (W_0^{1,p})'$*:*

- *We have* $T_u$ *is linear because of the linearity of the integral and the gradient.*
- *Check that* $T_u$ *is continuous. Using Holder's Inequality, we get*

$$\begin{aligned} |T_u(v)| &= \left|p\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v\,dx\right| \\ &\leq p\int_\Omega |\nabla u|^{p-1}\,|\nabla v|\ \ dx \\ &\leq p\,\|\nabla u\|_{L^p}^{p-1}\,\|\nabla v\|_{L^p}\,, \end{aligned}$$

*we can choose* $c = p\,\|\nabla u\|_{L^p}^{p-1}$*. So we have the continuity of* $T_u$*. Therefore,* $I$ *is Gâteaux differentiable and moreover*

$$\langle I'_G(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v\,dx.$$

**Remark 1.3.** *Gâteaux differentiability does not imply differentiability.*

**Example 1.3.** *Consider the function* $F : \mathbb{R}^2 \to \mathbb{R}$ *defined by:*

$$F(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

*This function is Gâteaux differentiable at the origin* $(0,0)$*, but it is not differentiable at that point.*

**Proposition 1.3.** *Let* $\Omega \subseteq A$ *be an open set. If* $I$ *is Gâteaux differentiable on* $\Omega$ *and that* $I'_G$ *is continuous from* $\Omega$ *to* $A'$ *at* $u \in \Omega$*. Then* $I$ *is also differentiable at* $u$*, and* $I'_G(u) = I'(u)$*.*

**Example 1.4.** *Let* $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ *be a functional given by*

$$I(u) = \int_\Omega |\nabla u|^p\,dx.$$

*Then,* $I$ *is Fréchet differentiable and*

$$\langle I'(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v dx.$$

*Indeed, we already prove that $I$ is Gâteaux differentiable and it remains to prove that $I'_G$ is continuous from $W_0^{1,p}$ to $(W_0^{1,p})'$. To do this; Let $(u_n) \subset W_0^{1,p}$ be a sequence such that*

$$u_n \to u \text{ in } W_0^{1,p} \text{ that is } \nabla u_n \to \nabla u \text{ in } L^p,$$

*then we have to show that*

$$I'_G(u_n) \to I'_G(u) \text{ in } (W_0^{1,p})'.$$

*Let $v \in W_0^{1,p}$ such that $\|v\|_{W_0^{1,p}} = \|\nabla v\|_p = 1$, we have*

$$\|I'_G(u_n) - I'_G(u)\|_{(W_0^{1,p})'} = \sup_{\|v\|_{W_0^{1,p}}=1} |\langle I'_G(u_n) - I'_G(u), v\rangle| .$$

*Using the Holder's Inequality, we get*

$$\begin{aligned} |\langle I'_G(u_n) - I'_G(u), v\rangle| &= p\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n.\nabla v dx - \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx\right| \\ &\le p\int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|.|\nabla v|\, dx \\ &\le \left[\int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} . \underbrace{\left[\int_\Omega |\nabla v|^p\right]^{\frac{1}{p}}}_{=1} . \end{aligned}$$

*Now, we will prove that*

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \quad \text{in } L^{\frac{p}{p-1}}.$$

*Indeed, using the inverse dominated convergence theorem, we get*

$$\begin{cases} \nabla u_n \to \nabla u \quad a.e \text{ on } \Omega \\ \exists h \in L^p \text{ such that, } |\nabla u_n|^{p-1} \le |h|^{p-1} \in L^{\frac{p}{p-1}}. \end{cases}$$

*Therefore, by Dominated convergence theorem, we get*

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \quad \text{in } L^{\frac{p}{p-1}}.$$

*Consequently,*

$$\|I'_G(u_n) - I'_G(u)\|_{(W_0^{1,p})'} \le \left[\int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \xrightarrow[n\to+\infty]{} 0.$$

*That is,*

$$I'_G(u_n) \to I'_G(u) \text{ in } (W_0^{1,p})'.$$

*So we have the continuity of $I'_G$. Therefore, $I$ is differentiable and $I'_G = I'$, moreover*

$$\langle I'(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx.$$

### 1.3.3 Critical points

**Definition 1.14.** *Let $f : I \to \mathbb{R}$ be a function defined on an interval $I$ such that $c$ is an element of $I$. Then*

1. *$f(c)$ is a **local maximum** value of $f$ if there exists an interval $J$ containing $c$ such that, for all $x \in J \cap I$*
$$f(x) \leq f(c).$$
2. *$f(c)$ is a **local minimum** value of $f$ if there exists an interval $J$ containing $c$ such that, for all $x \in J \cap I$*
$$f(x) \geq f(c).$$
3. *$f(c)$ is **local extreme value** of $f$ if it is either a local maximum or local minimum value.*

**Definition 1.15.** *Let $A$ be a Banach space, $\Omega \subseteq A$ an open set, and let $I : \Omega \to \mathbb{R}$ is differentiable.*

- *We say that $u \in \Omega$ is a **critical point** of $I$ if*
$$I'(u) = 0.$$
- *We say that $c \in \mathbb{R}$ is a **critical value** of $I$ if there is $u \in \Omega$ such that*
$$I(u) = c \text{ and } I'(u) = 0.$$

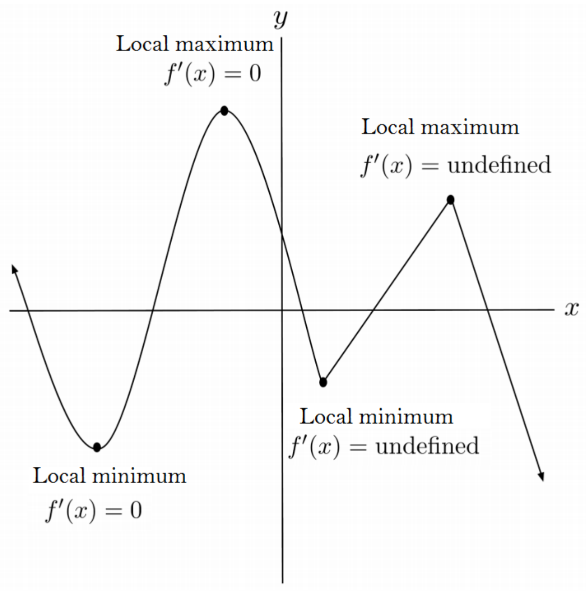


Figure 1.1: Local extreme of function f.

## 1.4 Variational Principal

For further details on these concepts, refer to [2, 12, 14].

**Definition 1.16.** ***(Palais-Smale sequence)*** *Let $A$ be a Banach space, let $I \in C^1(A, \mathbb{R})$ and let $(u_n) \subset A$ be a sequence such that*

$$\begin{cases} (I(u_n))_n \text{ is bounded} & \text{in } \mathbb{R}, \\ I'(u_n) \xrightarrow[n\to+\infty]{} 0 & \text{in } A'. \end{cases} \tag{1.5}$$

*Then, $(u_n)$ is called Palais-Smale sequence.*
*Let $c \in \mathbb{R}$, if*

$$\begin{cases} I(u_n) \xrightarrow[n\to+\infty]{} c & \text{in } \mathbb{R}, \\ I'(u_n) \xrightarrow[n\to+\infty]{} 0 & \text{in } A'. \end{cases} \tag{1.6}$$

*Then, $(u_n)$ is called Palais-Smale sequence at level $c$. In this case, $c$ is called Palais-Smale level.*

**Example 1.5.** *We consider the real function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = xe^{1-x},$$

*it is differentiable on $\mathbb{R}$ and its derivative function is*

$$f'(x) = (1 - x)e^{1-x}.$$

- *We take $u_n = 1 + \frac{1}{n}$, we get*

$$\begin{cases} f(u_n) = (1 + \frac{1}{n})e^{\frac{-1}{n}} \xrightarrow[n\to+\infty]{} 1, \\ f'(u_n) = \frac{-1}{n}e^{\frac{-1}{n}} \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

*Then, $(u_n)$ is a Palais-Smale sequence at level $c = 1$.*

- *We take $v_n = n$, we get*

$$\begin{cases} f(v_n) = ne^{1-n} \xrightarrow[n\to+\infty]{} 0, \\ f'(v_n) = (1 - n)e^{1-n} \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

*Then, $(v_n)$ is a Palais-Smale sequence at level $c = 0$.*

**Definition 1.17.** ***(Palais-Smale condition)*** *Let $A$ be a Banach space and let $I \in C^1(A, \mathbb{R})$.*

1. *We say that $I$ satisfies the Palais-Smale condition ($(PS)$ for short), if any sequence $(u_n) \subset A$ of Palais-Smale satisfy (1.5), has a convergent subsequence to some $u \in A$.*

2. *We say that $I$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ($(PS)_c$ for short), if any sequence $(u_n) \subset A$ of Palais-Smale satisfy (1.6), has a convergent subsequence to some $u \in A$.*

**Example 1.6.** *In example 1.5 we have $f$ satisfies the Palais-Smale condition at level $c = 1$ ($(PS)_1$), but $f$ does not satisfy the Palais-Smale condition at level $c = 0$ ($(PS)_0$).*

**Definition 1.18.** *Let $A$ be a real Banach space, let $c \in \mathbb{R}$ and let $F \subset A$ be a closed subset.We say that $I \in C^1(A, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ on $F$ ($(PS)_{F,c}$ for short), if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq F$ satisfy (1.6), has a convergent supsequence to some $u \in F$.*

In 1972, Ekeland [21] proved a theorem regarding the existence of an approximate minimizer for a function that is bounded below and lower semicontinuous. This theorem, known as Ekeland's Variational Principle, is widely used in nonlinear principle and serves as a valuable tool in studying Problems related to fixed point theory, optimization, nonlinear equations, and more. For further details, refer to [11, 19, 20].

**Theorem 1.8.** ***(Ekeland's Variational Principle)*** *Let $(A, d)$ be a complete metric space, and $I : A \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function, bounded from below. Then for all $\epsilon > 0$ there exists $u_\epsilon \in A$ such that*

$$\begin{cases} \inf\limits_{u\in A} I(u) \le I(u_\epsilon) \le \inf\limits_{u\in A} I(u) + \epsilon, \\ I(u) - I(u_\epsilon) + \epsilon\, d(u, u_\epsilon) > 0, \qquad \forall u \neq u_\epsilon. \end{cases}$$

**Corollary 1.2.** *Let $A$ be a Banach space and let $I \in C^1(A, \mathbb{R})$. We suppose that*

1. *$I$ is bounded from below.*

2. *$I$ satisfies the Palais-Smale condition at level $c = \inf\limits_{u\in A} I(u)$.*

*Then, $I$ reaches its minimum (i.e. $I$ has a critical point).*

# Chapter 2

# Critical Point Theory for Perturbed Functionals

In this chapter, we establish a new critical point theorem for a class of perturbed functionals without satisfying the Palais-Smale condition, which asserts the existence of critical point of functionals of the type $I = I_1 + I_2$, provided that $I_1$ has at least one critical point. Next, we apply our abstract result to nonhomogeneous nonlinear Problem.

## 2.1 Main abstract results

In this section we are going to prove the existence of at least one critical point of perturbed functionals without satisfying the Palais-Smale condition.

To show this, we need the following Theorem, which is a version of Ekeland's variational principle established by Ekeland [11] or Gonçalves and Miyagaki [12].

**Theorem 2.1.** *Let $A$ be a real Banach space. If $I \in C^1(A, \mathbb{R})$ satisfies $(PS)_{F,c}$ and bounded from below on a closed subset $F \subset A$ with a nonempty interior and if*

$$I(v) < 0 < \inf_{u \in \partial F} I(u) \quad \text{for some } v \in F^{\circ}, \tag{2.1}$$

*then*

$$c = \inf_{u \in F} I(u). \tag{2.2}$$

*is a critical value.*

*Proof.* Using the Ekeland's Variational Principle, for all $\epsilon > 0$ there is $u_\epsilon \in F$ such that

$$I(u_\epsilon) \leq \inf_{u \in F} I(u) + \epsilon, \tag{2.3}$$

and

$$I(u_\epsilon) < I(u) + \epsilon \left\| u - u_\epsilon \right\|, \qquad \forall u \neq u_\epsilon. \tag{2.4}$$

From (2.1), we get for all $\epsilon > 0$

$$0 < \epsilon < \inf_{u \in \partial F} I(u) - \inf_{v \in F^\circ} I(v). \tag{2.5}$$

Since $F^\circ \subset F$. Then

$$\inf_{u \in F} I(u) \leq \inf_{u \in F^\circ} I(u). \tag{2.6}$$

From (2.3), (2.5) and (2.6), We have

$$I(u_\epsilon) \leq \inf_{u \in F} I(u) + \epsilon \leq \inf_{u \in F^\circ} I(u) + \epsilon < \inf_{u \in \partial F} I(u).$$

Hence,

$$I(u_\epsilon) < \inf_{u \in \partial F} I(u). \tag{2.7}$$

Now, we will prove that $u_\epsilon \in F^\circ$. Using contradiction argument, assume that the claim is not satisfied, then $u_\epsilon \notin F^\circ$ moreover $u_\epsilon \in F$, thus $u_\epsilon \in \partial F$. So

$$I(u_\epsilon) \geq \inf_{u \in \partial F} I(u).$$

This is a contradiction to (2.7), then $u_\epsilon \in F^\circ$. After that replacing $u$ by $u_\epsilon + \lambda w$ in (2.4), for all $\lambda > 0$ small and $w$ in the unit hall $B \subset A$, we get:

$$\frac{I(u_\epsilon + \lambda w) - I(u_\epsilon)}{\lambda} + \epsilon \|w\| \geq 0,$$

which gives

$$\langle I'(u_\epsilon), w \rangle + \epsilon \|w\| \ \geq \ 0.$$

Then,

$$\langle I'(u_\epsilon), w \rangle \ \geq -\epsilon.$$

Replacing $w$ by $-w$, we get

$$\langle I'(u_\epsilon), w \rangle \ \leq \ \epsilon,$$

then,

$$|\langle I'(u_\epsilon), w \rangle| \leq \ \epsilon.$$

Therefore,

$$\|I'(u_\epsilon)\|_{A'} \leq \ \epsilon. \tag{2.8}$$

We choose $\epsilon = \frac{1}{n}$. From (2.3) and (2.8), we get

$$I(u_n) \to c \ \text{ in } \ \mathbb{R} \ \text{ and } \ I'(u_n) \to 0 \ \text{ in } \ A'. \tag{2.9}$$

This means that, $u_n$ is Palais-Smale sequence. However, since $I$ satisfies $(PS)_{F,c}$ there is a subsequence $(u_{n_k})$ (still denoted by $u_n$) such that

$$u_n \to u \in F.$$

Since $I$ and $I'$ are continuous , then

$$I(u_n) \to I(u) \text{ and } I'(u_n) \to I'(u). \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$I(u) = c \text{ and } I'(u) = 0.$$

Then $c$ is a critical value. Therefore, the proof is complete. □

We give now some definitions fundamental in this section.

**Definition 2.1.** *Let $A$ be a real Banach space and $I \in C^1(A, \mathbb{R})$.*

1. *We say that $u$ is a c-Ekeland solution of $I$ if*

$$I'(u) = 0 \text{ and } I(u) = c,$$

   *where $c = \inf_{u \in F} I(u)$.*

2. *We say that $I$ has the Ekeland geometry if $I$ satisfies (2.1).*

3. *We say that $I$ has a mountain pass geometry if there exists $u_1 \in A$ and constants $\rho, r > 0$ such that*

$$I(u_1) < 0, \quad \|u_1\| > \rho,$$

   *and*

$$I(u) \geq r \text{ when } \|u\| = \rho.$$

4. *We say that $u$ is a c-mountain pass solution of $I$, if it has a mountain pass geometry, $I'(u) = 0$ and $I(u) = c$, where*

$$c = \inf_{\gamma \in \Psi} \max_{0 \leq t \leq 1} I(\gamma(t)),$$

   *and*

$$\Psi = \{\gamma \in C([0,1], A) : \gamma(0) = 0, \ \gamma(1) = u\}.$$

*We denote by $\tilde{I}$ the functional of class $C^1(A, \mathbb{R})$ defined by*

$$\tilde{I}(u) = \begin{cases} I(u) & \text{for } \|u\| \leq 2M, \\ \alpha & \text{for } \|u\| \geq 4M, \end{cases} \tag{2.11}$$

*where $M$ is a positive constant and $\alpha \in \mathbb{R}$.*

The following theorem guarantees the existence of at least one critical point of functionals of the type $I = I_1 + I_2$.

**Theorem 2.2.** *Let $A$ be a real Banach space and let $I_\lambda$ be a real-valued functional on $A$ such that*

$$I_\lambda = I_1 + \lambda I_2,$$

*with $I_1, I_2 \in C^1(A, \mathbb{R})$ and $\lambda \in \mathbb{R}$. We assume that:*

*(i) $I_1$ has an Ekeland geometry, $I_1$ satisfies the $(PS)_{F,c}$-condition and $\tilde{I}_2$ as well as $\tilde{I'}_2$ are bounded, where $\tilde{I}_2$ and $\tilde{I'}_2$ are defined in (2.11).*

*(ii) There exists a positive constant $M$ such that*

$$\|u\| \leq M,$$

*where $M$ is given in (2.11) for every c-Ekeland solution $u$ of $I_1$.*

*(iii) For all $\lambda > 0$, $\tilde{I_\lambda} = I_1 + \lambda \tilde{I_2} \in C^1(A, \mathbb{R})$ and it satisfies the $(PS)_{F,c}$-condition.*

*Then there is $\lambda_0 > 0$ such that, for every $|\lambda| < \lambda_0$, $I_\lambda$ has a critical point.*

*Proof.* By condition $(i)$ along with Theorem 2.1, there is $c \in \mathbb{R}$, a closed subset $F \subset A$ and $u_1 \in A$ such that

$$c = \inf_{u \in F} I_1(u) = I_1(u_1).$$

From (2.1) and (2.6), we get

$$c = \inf_{u \in F} I_1(u) = I_1(u_1) < 0. \tag{2.12}$$

**Step1.** We have $\tilde{I_\lambda}$ admits a critical point $u_2 \in A$. Indeed, from assumption $(iii)$, and since $\tilde{I_2}$ is bounded, we get

$$I_1(u) - C\,|\lambda| \leq \tilde{I_\lambda}(u) \leq I_1(u) + C\,|\lambda|\,, \quad \forall u \in A, \tag{2.13}$$

where $C > 0$ is independent of $\lambda$ and $u$. From (2.1), (2.12) and (2.13), we get for $|\lambda|$ small enough,

$$-\infty < \inf_{u \in F} \tilde{I_\lambda}(u) < 0, \tag{2.14}$$

and

$$0 < \inf_{u \in \partial F} I_1(u) - C\,|\lambda| < \inf_{u \in \partial F} \tilde{I_\lambda}(u). \tag{2.15}$$

This implies that, for $|\lambda|$ small enough, $\tilde{I_\lambda}$ has the Ekeland geometry. Indeed, from (2.15), we have

$$0 < \inf_{u \in \partial F} \tilde{I_\lambda}(u).$$

It remains to prove that there exists $v \in F^\circ$ such that

$$\tilde{I_\lambda}(v) < 0.$$

By definition, for every $\epsilon > 0$, there exists $u_\epsilon \in F$ such that

$$\inf_{u\in F} \tilde{I}_\lambda(u) \le \tilde{I}_\lambda(u_\epsilon) < \inf_{u\in F} \tilde{I}_\lambda(u) + \epsilon.$$

From (2.14) and for $\epsilon$ small enough, we have

$$\tilde{I}_\lambda(u_\epsilon) < \inf_{u\in F} \tilde{I}_\lambda(u) + \epsilon < 0. \tag{2.16}$$

It is enough to verify that $u_\epsilon \in F^\circ$. For this, using contradiction argument. Assume that $u_\epsilon \notin F^\circ$, this means that $u_\epsilon \in \partial F$, but $\inf\limits_{u_\epsilon\in\partial F} \tilde{I}_\lambda(u_\epsilon) > 0$. This is contradiction to (2.16). Therefore, we can choose $u_\epsilon = v$. Consequently, $\tilde{I}_\lambda$ has the Ekeland geometry. And we have $\tilde{I}_\lambda$ satisfies the $(PS)_{F,c}$-condition. These facts in combination with Theorem 2.1 show that, for $|\lambda|$small enough, $\tilde{I}_\lambda$ admits a critical point $u_2 \in A$ such that

$$c_\lambda = \inf_{u\in F} \tilde{I}_\lambda(u) = \tilde{I}_\lambda(u_2). \tag{2.17}$$

**Step2.** We have $c_\lambda \to c$ as $\lambda \to 0$. Indeed, from (2.13), we get

$$\inf_{u\in F} I_1(u) - C\,|\lambda| \le \inf_{u\in F} \tilde{I}_\lambda(u) \le \inf_{u\in F} I_1(u) + C\,|\lambda|\,.$$

Using (2.12) and (2.17), we get

$$c - C\,|\lambda| \le c_\lambda \le c - C\,|\lambda|\,. \quad \text{for all } \lambda \in \mathbb{R}.$$

Then, $c_\lambda \to c$ as $\lambda \to 0$.

**Step3.** There is $\lambda_0 > 0$, for every $\lambda \in \mathbb{R}$ such that, for $|\lambda| < \lambda_0$, any $c_\lambda$-Ekeland solution $u$ of $\tilde{I}_\lambda$ satisfied

$$\|u\| \le 2M.$$

Using contradiction argument. Then there exists sequences $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\lambda_n \to 0$ and $(u_n)_{n\in\mathbb{N}} \subseteq F$ is a $c_{\lambda_n}$-Ekeland solution of $\tilde{I}_{\lambda_n}$ and

$$\|u_n\| > 2M. \tag{2.18}$$

Since $u_n$ is a $c_{\lambda_n}$-Ekeland solution of $\tilde{I}_{\lambda_n}$, then

$$\tilde{I}_{\lambda_n}(u_n) = c_{\lambda_n} \quad \text{and} \quad \tilde{I}'_{\lambda_n}(u_n) = 0.$$

On the other hand, by definition, we have

$$\begin{aligned} \tilde{I}_{\lambda_n}(u_n) &= I_1(u_n) + \lambda_n \tilde{I}_2(u_n) \\ \tilde{I}'_{\lambda_n}(u_n) &= I_1'(u_n) + \lambda_n \tilde{I}_2'(u_n). \end{aligned}$$

Hence, using Step2, the boundedness of $\tilde{I}_2$ and $\tilde{I}_2'$ and the fact that $\lambda_n \to 0$ leads to

$$I_1(u_n) \to c \quad \text{and} \quad I_1'(u_n) \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.19}$$

This means that, $u_n$ is a Palais-Smale sequence. However, since $I_1$ satisfies $(PS)_{F,c}$-condition there is a subsequence $(u_{n_k})$ (still denoted by $u_n$) and $u_0 \in F$ such that

$$u_n \to u_0 \text{ in } F.$$

Since $I_1$ and $I_1'$ are continuous , then

$$I_1(u_n) \to I_1(u_0) \text{ and } I_1'(u_n) \to I_1'(u_0) \quad \text{as } n \to +\infty. \tag{2.20}$$

From (2.19) and (2.20), we obtain

$$I_1(u_0) = c \text{ and } I_1'(u_0) = 0.$$

This shows that $u_0$ is a c-Ekeland solution of $I_1$. Therefore, by condition (ii), we get

$$\|u_n\| = \|u_n - u_0 + u_0\| \leq \|u_n - u_0\| + \|u_0\| < 2M$$

for each $n$ large enough. We have a contradiction to (2.18). Thus, step3 is proved.

Therefore, by the steps above, we conclude, for $|\lambda|$ small enough, $I_\lambda$ has a critical point and

$$\tilde{I}_\lambda(u_2) = I_\lambda(u_2) = c_\lambda \text{ and } \tilde{I}'_\lambda(u_2) = I_\lambda'(u_2) = 0.$$

The proof is achieved. □

**Remark 2.1.** *1. On the conditions of Theorem 2.2, it is indicated that the functional $I_\lambda$ does not have to satisfy the (PS)-condition.*

*2. Let $A$ be a Banach space of valued functions which has the next property: If $u_n \rightharpoonup u$ in $A$, then there exists a subsequence of $(u_n)$ (still denoted by $u_n$) such that $u_n \to u$ in $L^\infty(A)$. In this case we can replace $\|u\| \leq M$ in (2.11) by $\|u\|_\infty \leq M$. Moreover, we can replace condition $(ii)$ in Theorem 2.2 by*

$$\|u\|_\infty \leq M,$$

*where $M$ is given in (2.11) and $u$ being a c-Ekeland solution of $I_1$.*

As a direct consequence of Theorem 2.2, we can state the following result.

**Theorem 2.3.** *Let $A$ be a real Banach space and let $I_\lambda$ be a real-valued functional on $A$ such that*

$$I_\lambda = I_1 + \lambda I_2,$$

*with $\lambda \in \mathbb{R}$ and $I_1, I_2 \in C^1(A, \mathbb{R})$. We suppose that:*

*1. $I_\lambda$ has a mountain pass geometry, $I_1$ satisfies the $(PS)$-condition, $\tilde{I}_2$ as well as $\tilde{I}'_2$ are bounded.*

2. *For all c- mountain pass solution $u$ of $I_1$, there exists a positive constant $M$ such that*

$$\|u\| \leq M,$$

*where $M$ is given in (2.11) .*

3. *For all $\lambda > 0$, $\tilde{I_\lambda} = I_1 + \lambda \tilde{I_2}$ satisfies the $(PS)$-condition.*

*Then there is $\lambda_0 > 0$ such that, for each $|\lambda| < \lambda_0$, $I_\lambda$ has a critical point.*

*Proof.* It is enough to verify the following implication:

$$I_1 \text{ has a mountain pass geometry } \Rightarrow I_1 \text{ has an Ekeland geometry.}$$

Indeed, let $I_1$ has a mountain pass geometry, we set:

$$F = B^c(0,\rho) = \{u \in A \text{ such that } \|u\| \geq \rho\}.$$

For any $u \in \partial F$ ($\|u\| = \rho$), we have:

$$I_1(u) \geq r > 0.$$

Then

$$\inf_{u \in \partial F} I_1(u) > 0.$$

The mountain pass geometry implies that there exists $u_1 \in A$ such that

$$\|u_1\| > \rho \text{ and } I_1(u_1) < 0.$$

Since $u_1 \in F^\circ$ equivalent to $\|u_1\| > \rho$, then we can take $v = u_1 \in F^\circ$. In conclusion

$$I_1(v) < 0 < \inf_{u \in \partial F} I_1(u).$$

The proof is finished. □

## 2.2 Application: Nonlinear Problem in Sobolev Spaces

In this section, we are going to apply the abstract critical point results to nonhomogeneous nonlinear Problem, more exactly, we will study the existence of weak solutions of the following Problem:

$$\begin{cases} -\text{div}(|\nabla u|^{p-2} .\nabla u) = |u|^{q-2} u + \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{2.21}$$

where $\lambda \in \mathbb{R}$, $\Omega$ is a bounded set of $\mathbb{R}^N$ with smooth boundary $\Gamma$ and $(N \geq 3)$. We assume the following hypotheses:

(A) $1 < q < p$ and $p > N$.

(B) $g(\cdot,\cdot)$ is continuous on $\bar{\Omega}\times[0,\infty)$.

**Definition 2.2.** *We say that $u\in W_0^{1,p}$ is a weak solution of Problem (2.21) if and only if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\,dx-\int_\Omega |u|^{q-2}u.v\,dx-\lambda\int_\Omega g(x,u)v\,dx=0\quad \textit{for all } v\in W_0^{1,p}.$$

We denoted by $I_\lambda : W_0^{1,p}\to\mathbb{R}$ the energy function of Problem (2.21) defined by

$$I_\lambda(u)=\frac{1}{p}\int_\Omega|\nabla u|^p\,dx-\frac{1}{q}\int_\Omega|u|^q\,dx-\lambda\int_\Omega G(x,u)dx,$$

where $G(x,s)=\int_0^s g(x,t)dt$. Note that under conditions (A) and (B), the functional $I_\lambda$ : $W_0^{1,p}\to\mathbb{R}$ is well defined, of class $C^1(W_0^{1,p},\mathbb{R})$. In addition

$$\langle I_\lambda'(u),v\rangle=\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\,dx-\int_\Omega |u|^{q-2}u.v\,dx-\lambda\int_\Omega g(x,u)v\,dx\quad \text{for all } v\in W_0^{1,p}.$$

So, every critical point of $I_\lambda$ is a weak solution of Problem (2.21).

We introduce the functionals $I_1, I_2 : W_0^{1,p}\to\mathbb{R}$ defined by

$$\begin{aligned} I_1(u) &= \frac{1}{p}\int_\Omega|\nabla u|^p\,dx-\frac{1}{q}\int_\Omega|u|^q\,dx,\\ I_2(u) &= \int_\Omega G(x,u)dx. \end{aligned}$$

The following theorem guarantees the existence of a solution of Problem (2.21).

**Theorem 2.4.** *Assume that assumptions (A) and (B) are satisfied.Then there exists $\lambda_0>0$ such that for each $|\lambda|<\lambda_0$ Problem (2.21) has at least one weak solution.*

*Proof.* We will use Theorem 2.2. For this purpose, we need the following four steps:

**Step1.** $I_1$ has an Ekeland geometry.

By assumption (A), we know that $W_0^{1,p}$ is embedded continuously in $L^\infty$. Hence, there exists $\alpha>0$ such that

$$\|u\|_\infty\le\alpha\|u\|\quad\text{for all } u\in W_0^{1,p}. \tag{2.22}$$

First we prove that $0<\inf_{u\in\partial F} I_1(u)$. For this, we check that there exists $\rho, r>0$ such that $I_1(u)\ge r$ when $\|u\|=\rho$. We have

$$\begin{aligned} I_1(u) &= \frac{1}{p}\int_\Omega|\nabla u|^p\,dx-\frac{1}{q}\int_\Omega|u|^q\,dx,\\ &= \frac{1}{p}\|u\|^p-\frac{1}{q}\|u\|_\infty^q|\Omega|. \end{aligned}$$

Using (2.22), we get

$$I_1(u)\ge\|u\|^q\left(\frac{1}{p}\|u\|^{p-q}-\frac{\alpha^q}{q}|\Omega|\right). \tag{2.23}$$

We set $\rho > max\left(1, (\alpha^q |\Omega|)^{\frac{1}{p-q}}\right)$. Therefore, since $q < p$, we have

$$I_1(u) \geq \frac{1}{p}\rho^p - \frac{\alpha^q}{q}|\Omega|\rho^q = r > 0 \text{ for every } \|u\| = \rho. \tag{2.24}$$

We take

$$F = \bar{B}(0,\rho) = \left\{u \in W_0^{1,p} \text{ such that } \|u\| \leq \rho\right\}.$$

Since $u \in \partial F$ equivalent to $\|u\| = \rho$, then from (2.24) we get

$$\inf_{u\in\partial F} I_1(u) > 0. \tag{2.25}$$

Now, we will prove that $I_1(v) < 0$ for some $v \in F^\circ$. For this, let $\varphi \in W_0^{1,p} \setminus \{0\}$. Then

$$I_1(t\varphi) = \frac{t^p}{p}\|\varphi\|^p - \frac{t^q}{q}\int_\Omega |\varphi|^q\,dx.$$

Since $q < p$ we can choose $t > 0$ small enough so that

$$I_1(t\varphi) = \frac{t^p}{p}\|\varphi\|^p - \frac{t^q}{q}\int_\Omega |\varphi|^q\,dx < 0.$$

Since $t\varphi \in F^\circ$ equivalent to $\|t\varphi\| < \rho$, then we can take $v = t\varphi \in F^\circ$. Consequently

$$I_1(v) < 0. \tag{2.26}$$

Therefore, by (2.25) and (2.26) then $I_1$ has an Ekeland geometry.

**Step2.** $I_1$ satisfied the (PS)-condition.
Let $(u_n)_{n\in\mathbb{N}} \subseteq W_0^{1,p}$ be a (PS)-sequence of $I_1$, that is,

$$|I_1(u_n)| \leq C \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad I_1'(u_n) \to 0 \text{ in } (W_0^{1,p})' \quad \text{as } n \to \infty, \tag{2.27}$$

for some $C > 0$.
From (2.23)and (2.27), we get

$$C \geq I_1(u_n) \geq \frac{1}{p}\|u_n\|^p - \frac{\alpha^q}{q}|\Omega|\,\|u_n\|^q. \tag{2.28}$$

First we prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$. Using contradiction argument, suppose that the sequence $(u_n)_{n\in\mathbb{N}}$ is unbounded in $W_0^{1,p}$, then there exists a subsequence $(u_{n_k})$ ( still denoted by $u_n$) such that

$$\|u_n\| \to \infty \text{ as } n \to \infty.$$

Then,

$$\lim_{n\to\infty} \frac{1}{p}\|u_n\|^p - \frac{\alpha^q}{q}|\Omega|\,\|u_n\|^q = +\infty.$$

This is contradiction to (2.28). Thus, $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$.

Secondly, since $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$ which is reflexive then there exists a subsequence $(u_{n_k})$ ( still denoted by $u_n$) such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{in } W_0^{1,p}, \\ u_n \to u \quad \text{in } L^\infty \ \ (W_0^{1,p} \text{ embedded compactly in } L_\infty \text{ as } p > N). \end{cases}$$

Finally, we prove that $u_n \to u$ in $W_0^{1,p}$.
We will show that $\langle I_1'(u_n) - I_1'(u), u_n - u\rangle \xrightarrow[n\to\infty]{} 0$.

- We have

$$u_n \rightharpoonup u \text{ in } W_0^{1,p} \quad \text{and} \quad I_1'(u_n) \to 0 \text{ in } (W_0^{1,p})'.$$

By Proposition 1.2, we get

$$\langle I_1'(u_n), u_n - u\rangle \xrightarrow[n\to\infty]{} 0. \tag{2.29}$$

- Since $I_1'(u) \in (W_0^{1,p})'$ and $u_n \rightharpoonup u$ in $W_0^{1,p}$, we obtain

$$\langle I_1'(u), u_n - u\rangle \xrightarrow[n\to\infty]{} 0. \tag{2.30}$$

From (2.29) and (2.30), we have

$$\langle I_1'(u_n) - I_1'(u), u_n - u\rangle \xrightarrow[n\to\infty]{} 0. \tag{2.31}$$

On the other hand, we have

$$\begin{aligned} \langle I_1'(u_n) - I_1'(u), u_n - u\rangle &= \langle I_1'(u_n), u_n - u\rangle - \langle I_1'(u), u_n - u\rangle \\ &= \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).(\nabla u_n - \nabla u)\,dx \\ &\quad - \int_\Omega \left(|u_n|^{q-2}u_n - |u|^{q-2}u\right).(u_n - u)\,dx. \end{aligned}$$

Writing:

$$\int_\Omega \left(|u_n|^{q-2}u_n - |u|^{q-2}u\right).(u_n - u)\,dx = \int_\Omega |u_n|^q\,dx - \int_\Omega |u_n|^{q-2}u_n.udx - \int_\Omega u_n.\,|u|^{q-2}u\,dx + \int_\Omega |u|^q\,dx.$$

Since $u_n \to u$ in $L^q$, then

$$\int_\Omega |u_n|^q\,dx \to \int_\Omega |u|^q\,dx \ \text{ as } n \to +\infty.$$

Using Dominated convergence theorem Lebesgue, we get

$$\int_\Omega |u_n|^{q-2}u_n.u\,dx \to \int_\Omega |u|^q\,dx,$$

and

$$\int_\Omega u_n.\,|u|^{q-2}u\,dx \to \int_\Omega |u|^q\,dx.$$

Consequently,

$$\int_\Omega \left(|u_n|^{q-2}\, u_n - |u|^{q-2}\, u\right) . (u_n - u)\, dx \xrightarrow[n\to\infty]{} 0. \tag{2.32}$$

Combining (2.31) with (2.32), we deduce that

$$\int_\Omega \left(|\nabla u_n|^{p-2}\, \nabla u_n - |\nabla u|^{p-2}\, \nabla u\right) . (\nabla u_n - \nabla u)\, dx \xrightarrow[n\to\infty]{} 0. \tag{2.33}$$

If $p \geq 2$, using relation (1.2), we get

$$\begin{aligned}\int_\Omega \left(|\nabla u_n|^{p-2}\, \nabla u_n - |\nabla u|^{p-2}\, \nabla u\right) . (\nabla u_n - \nabla u)\, dx &\geq 2^{2-p} \int_\Omega |\nabla u_n - \nabla u|^p\, dx \\ &= 2^{2-p}\, \|u_n - u\|^p_{W_0^{1,p}} .\end{aligned}$$

Then by (2.33), we have

$$\|u_n - u\|^p_{W_0^{1,p}} \xrightarrow[n\to\infty]{} 0,$$

which implies

$$u_n \to u \;\text{ in } W_0^{1,p}.$$

If $1 < p < 2$. We have

$$\begin{aligned}\int_\Omega |\nabla u_n - \nabla u|^p\, dx &= \int_\Omega |\nabla u_n - \nabla u|^2 . |\nabla u_n - \nabla u|^{p-2}\, dx, \\ &\leq \int_\Omega |\nabla u_n - \nabla u|^2 . |\nabla u_n + \nabla u|^{p-2}\, dx, \\ &= \int_\Omega \frac{|\nabla u_n - \nabla u|^2}{|\nabla u_n + \nabla u|^{2-p}} dx.\end{aligned}$$

Using (1.3), we get

$$\int_\Omega |\nabla u_n - \nabla u|^p\, dx \leq \frac{1}{C_p} \int_\Omega \left(|\nabla u_n|^{p-2}\, \nabla u_n - |\nabla u|^{p-2}\, \nabla u\right) . (\nabla u_n - \nabla u)\, dx.$$

Then by (2.33), we obtain

$$\int_\Omega |\nabla u_n - \nabla u|^p\, dx \xrightarrow[n\to\infty]{} 0,$$

which implies

$$u_n \to u \;\text{ in } W_0^{1,p}.$$

Consequently, for every $p$, we have

$$u_n \to u \;\text{ in } W_0^{1,p}.$$

Therefore. $I_1$ satisfied the (PS)-condition.

**Step3.** For a c-Ekeland solution $u$ of $I_1$, there exists $M > 0$ such that

$$\|u\|_\infty \leq M.$$

Let $u \in W_0^{1,p}$ be a c-Ekeland solution of $I_1$. That is

$$I_1(u) = c \quad \text{and} \quad I_1'(u) = 0.$$

From (2.28), we have

$$C \geq \frac{1}{p} \|u\|^p - \frac{\alpha^q}{q} |\Omega| \|u\|^q .$$

It follows that there is a constant $\delta > 0$ independent of $u$ such that $\|u\| \leq \delta$, by assumption (A), we know that $W_0^{1,p}$ is embedded continuously in $L^\infty$. Hence, there is $\alpha > 0$ such that

$$\|u\|_\infty \leq \alpha \|u\| \quad \text{for all } u \in W_0^{1,p}.$$

Then, there exists $M > 0$ independent of $u$ such that

$$\|u\|_\infty \leq \alpha\delta = M.$$

Thus, step3 has been proved.

**Step4.** For every $\lambda \in \mathbb{R}$, $\tilde{I}_\lambda$ satisfies the (PS)-condition.
First we introduce the functionals $\tilde{I}_\lambda, \tilde{I}_2$. For this, we choose a function $K \in \mathcal{D}(\Omega)$ with $0 \leq K \leq 1$ in $\Omega$ such that

$$K(x) = \begin{cases} 1 & \text{for } |x| \leq 2M, \\ 0 & \text{for } |x| \geq 4M, \end{cases}$$

where $M$ is given in step3. Then the function

$$\overline{G}(t,u) = K(u)G(t,u) = \begin{cases} G(t,u) & \text{for } \|u\|_\infty \leq 2M, \\ 0 & \text{for } \|u\|_\infty > 4M, \end{cases}$$

is of class $C^1$ in $\Omega \times \mathbb{R}$ and by assumption (B) we know that $\overline{G}(t,u)$ and $\overline{G}_u(t,u)$ is bounded on $\Omega \times \mathbb{R}$.

Next, we define $\tilde{I}_\lambda, \tilde{I}_2 : W_0^{1,p} \to \mathbb{R}$ by

$$\begin{aligned} \tilde{I}_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \frac{1}{q}\int_\Omega |u|^q \, dx - \lambda \int_\Omega K(u)G(x,u)dx, \\ \tilde{I}_2(u) &= \int_\Omega K(u)G(x,u)dx. \end{aligned}$$

Now, let $(u_n)_{n\in\mathbb{N}} \subseteq W_0^{1,p}$ be a (PS)-sequence of $\tilde{I}_\lambda$, that is,

$$\left|\tilde{I}_\lambda(u_n)\right| \leq C \ \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \tilde{I}_\lambda'(u_n) \to 0 \ \text{ in } (W_0^{1,p})' \text{ as } n \to \infty,$$

for some $C > 0$.
By the boundedness of $\overline{G}$, we can suppose that $\|u_n\| > max(4M, 1)$ for all $n \geq 1$, then we get

$$C \geq \tilde{I}_\lambda(u_n) \geq \frac{1}{p} \|u_n\|^p - \frac{\alpha^q}{q} |\Omega| \|u_n\|^q . \tag{2.34}$$

First we prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$. Using contradiction argument, suppose that the sequence $(u_n)_{n\in\mathbb{N}}$ is unbounded in $W_0^{1,p}$, then there exists a subsequence $(u_{n_k})$ (still denoted by $u_n$) such that

$$\|u_n\| \to \infty \; as \; n \to \infty.$$

Then,

$$\lim_{n\to\infty} \frac{1}{p} \|u_n\|^p - \frac{\alpha^q}{q} |\Omega| \|u_n\|^q = +\infty.$$

This is contradiction to (2.34). Thus, $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$.

As before, by using the boundedness of $\overline{G}$ and $\overline{G}_u$, the rest of the proof is similar to that in Step2.

Therefore, by the boundedness of $\overline{G}$ and $\overline{G'}$ , we know that $\tilde{I}_2$ and $\tilde{I'}_2$ are bounded. Then, from Steps 1, 2, 3 and 4, we see that conditions of Theorem 2.2 are satisfied. Therefore, the Problem (2.21) has at least one weak solution. The proof is complete. □

# Chapter 3

# A critical point theorem in bounded convex sets

In this chapter, the localization of a critical point of minimum type of a smooth functional is obtained in a bounded convex conical set defined by a norm and a concave upper semicontinuous function. Finally, we conclude by applying our abstract result to a Periodic Problem.

## 3.1 A localization critical point theorem

Let $\mathcal{X}$ be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$ which is identified with its dual, let $K \subset \mathcal{X}$ be a convex, and $l : K \to \mathbb{R}_+$ be a concave upper semicontinuous function with $l(0) = 0$. Let $I \in C^1(\mathcal{X})$ be a functional and let $N : \mathcal{X} \to \mathcal{X}$ be the operator $N(u) = u - I'(u)$.

For every two numbers $r, R > 0$ we consider the conical set

$$K_{rR} = \{u \in K : r \leq l(u) \text{ and } \|u\| \leq R\}.$$

Suppose that $K_{rR} \neq \emptyset$ and

$$N(K_{rR}) \subset K.$$

In the following lemma, we gather some properties of the set $K_{rR}$.

**Lemma 3.1.** *$K_{rR}$ is a convex and closed set.*

*Proof.* Using concavity of $l$. Let $u, v \in K_{rR}$ and let $t \in [0, 1]$, we get

$$\begin{aligned} l(tu + (1-t)v) &\geq tl(u) + (1-t)l(v) \\ &\geq tr + (1-t)r = r, \end{aligned}$$

and

$$\begin{aligned} \|tu + (1-t)v\| &\leq t\|u\| + (1-t)\|v\|, \\ &\leq tR + (1-t)R = R. \end{aligned}$$

So we have the convexity. Then using $l$ is $u.s.c$, let $(u_n) \subset K_{rR}$ such that $u_n \to u$ in $\mathcal{X}$, then we will prove that $u \in K_{rR}$, we have

$$l(u) \geq \limsup_{n\to+\infty} l(u_n) \geq r,$$

and

$$\|u\| = \lim_{n\to\infty} \|u_n\| \leq R.$$

Therefore $u \in K_{rR}$ and so $K_{rR}$ is closed. Consequently, the proof is complete □

**Lemma 3.2.** *Let the next assumptions be fulfilled:*

$$m = \inf_{u\in K_{rR}} I(u) > -\infty; \tag{3.1}$$

$$\textit{there exists } \epsilon > 0 \textit{ such that } I(u) \geq m + \epsilon \textit{ for} \tag{3.2}$$

$$\textit{every } u \in K_{rR} \textit{ which concurrently satisfy } l(u) = r \textit{ and } \|u\| = R;$$

$$l(N(u)) \geq r \textit{ for all } u \in K_{rR}. \tag{3.3}$$

*Then there is a sequence* $(u_n) \subset K_{rR}$ *such that*

$$I(u_n) \leq m + \frac{1}{n} \tag{3.4}$$

*and*

$$\|I'(u_n) + \lambda_n u_n\| \leq \frac{1}{n}, \tag{3.5}$$

*where*

$$\lambda_n = \begin{cases} -\frac{\langle I'(u_n),u_n\rangle}{R^2} & \textit{if } \|u_n\| = R \textit{ and } \langle I'(u_n), u_n\rangle < 0 \\ 0 & \textit{otherwise}. \end{cases} \tag{3.6}$$

*Proof.* Using Ekeland's variational principle. there exists a sequence $(u_n) \subset K_{rR}$ such that

$$I(u_n) \leq m + \frac{1}{n}, \tag{3.7}$$

$$I(u_n) \leq I(v) + \frac{1}{n} \|v - u_n\| \tag{3.8}$$

for every $v \in K_{rR}$ and $n \geq 1$. Two cases are possible:

**Case(1).** There exists a subsequence of $(u_n)$ (still denoted by $u_n$) such that $r \leq l(u_n)$ and $\|u_n\| < R$ for all $n$. For a fixed but arbitrary $n$ and $t > 0$, consider the element

$$v = u_n - tI'(u_n). \tag{3.9}$$

We replacing $I'(u_n)$ by $u_n - N(u_n)$ in (3.9), we get

$$v = (1 - t)u_n + tN(u_n).$$

Now we will prove that $v \in K_{rR}$. Indeed, we have

- Both $u_n$ and $N(u_n)$ belong to $K$, one has that $v \in K$ for all $t \in (0,1)$.
- By the concavity of $l$ and (3.3), we get

$$l(v) \geq (1-t)l(u_n) + tl(N(u_n)) \geq r,$$

  for every $t \in (0,1)$.
- We have

$$\|v\| \leq (1-t)\|u_n\| + t\|N(u_n)\|.$$

  Since $\|N(u_n)\|$ is bounded and $t \to 0$. we get

$$\|v\| \leq R.$$

Hence $v \in K_{rR}$ for every sufficiently small $t > 0$. Replacing (3.9) in (3.8), we get

$$I(u_n) \leq I(u_n - tI'(u_n)) + \frac{t}{n}\|I'(u_n)\|.$$

Dividing by $t$, we find

$$\frac{I(u_n - tI'(u_n)) - I(u_n)}{t} \geq -\frac{1}{n}\|I'(u_n)\|,$$

letting $t$ go to 0, we get

$$\langle I'(u_n), -I'(u_n)\rangle \geq -\frac{1}{n}\|I'(u_n)\|,$$

that is,

$$\|I'(u_n)\|^2 \leq \frac{1}{n}\|I'(u_n)\|.$$

So

$$\|I'(u_n)\| \leq \frac{1}{n}.$$

Thus, relation (3.5) holds with $\lambda_n = 0$.

**Case(2).** There exists a subsequence of $(u_n)$ (still denoted by $u_n$) such that $\|u_n\| = R$ for all $n$. Passing eventually to a new subsequence, in vien of (3.2) and (3.7), we may suppose that $l(u_n) > r$ for every $n$. Two subcases are possible:

(a) Assume that $\langle I'(u_n), u_n\rangle > 0$ for every $n$. Then for any fixed index $n$, the relation (3.9) in (3.8) is still possible since

$$\begin{aligned}\|v\|^2 &= \|u_n - tI'(u_n)\|^2 = \|u_n\|^2 + t^2\|I'(u_n)\|^2 - 2t\langle I'(u_n), u_n\rangle \\ &= R^2 + t^2\|I'(u_n)\|^2 - 2t\langle I'(u_n), u_n\rangle \leq R^2,\end{aligned}$$

for $0 \leq t \leq 2\langle I'(u_n), u_n\rangle / \|I'(u_n)\|^2$.

(b) Assume that $\langle I'(u_n), u_n\rangle \leq 0$ for every $n$. For any fixed index $n$, consider the element

$$v = u_n - t(I'(u_n) + \lambda_n u_n + \epsilon u_n), \tag{3.10}$$

where $t, \epsilon > 0$ and $\lambda_n = -\langle I'(u_n), u_n\rangle / R^2 \geq 0$.

We replacing $I'(u_n)$ by $u_n - N(u_n)$ in (3.10), we get

$$v = (1-t)\frac{1 - t - t\lambda_n u_n - t\epsilon}{1-t}u_n + tN(u_n).$$

Now, we will prove that $v \in K_{rR}$. Indeed, we have

- Both $u_n$ and $N(u_n)$ belong to $K$, one has that $v \in K$ for every $t \in (0, 1)$ small enough that $1 - t - t\lambda_n u_n - t\epsilon > 0$. also,

$$\begin{aligned} \langle I'(u_n) + \lambda_n u_n + \epsilon u_n, u_n\rangle &= \langle I'(u_n), u_n\rangle + \lambda_n \|u_n\|^2 + \epsilon \|u_n\|^2 \\ &= \langle I'(u_n), u_n\rangle - \frac{\langle I'(u_n), u_n\rangle}{R^2}R^2 + \epsilon R^2 \\ &= \epsilon R^2 > 0. \end{aligned}$$

- As in case(a), we find that $\|v\| \leq R$ for sufficiently small $t > 0$.

- From $l(u_n) > r$, we have $\delta l(u_n) = r$ for some number $\delta \in (0, 1)$. Then, for every $\rho \in [\delta, 1]$, we have

$$\begin{aligned} l(\rho u_n) &= l(\rho u_n + (1-\rho)0) \geq \rho l(u_n) + (1-\rho)l(0) \\ &= \rho l(u_n) \geq \delta l(u_n) = r. \end{aligned}$$

  Consequently,

$$l(\rho u_n) \geq r. \tag{3.11}$$

  In particular, we may take

$$\rho = (1 - t - t\lambda_n u_n - t\epsilon)/(1-t),$$

  which belongs to $[\delta, 1]$ for sufficiently small $t$. By the concavity of $l$, the condition (3.3) and the relation (3.11), we get

$$\begin{aligned} l(v) &= l\left((1-t)\frac{1 - t - t\lambda_n u_n - t\epsilon}{1-t}u_n + tN(u_n)\right) \\ &= l((1-t)\rho u_n + tN(u_n)) \geq (1-t)l(\rho u_n) + tl(N(u_n)) \geq r. \end{aligned}$$

Hence $v \in K_{rR}$ for all sufficiently small $t > 0$. Replacing (3.10) in (3.8), we get

$$I(u_n) \leq I(u_n - t(I'(u_n) + \lambda_n u_n + \epsilon u_n)) + \frac{t}{n}\|I'(u_n) + \lambda_n u_n + \epsilon u_n\|.$$

Dividing by $t$, we find

$$\frac{I(u_n - t(I'(u_n) + \lambda_n u_n + \epsilon u_n)) - I(u_n)}{t} \geq -\frac{1}{n}\|I'(u_n) + \lambda_n u_n + \epsilon u_n\|.$$

letting $t$ tend to 0, we obtain

$$\langle I'(u_n), -(I'(u_n) + \lambda_n u_n + \epsilon u_n)\rangle \geq -\frac{1}{n}\|I'(u_n) + \lambda_n u_n + \epsilon u_n\|,$$

let $\epsilon \to 0$, we get

$$\langle I'(u_n), I'(u_n) + \lambda_n u_n \rangle \leq \frac{1}{n} \|I'(u_n) + \lambda_n u_n\|, \tag{3.12}$$

and use $\langle u_n, I'(u_n) + \lambda_n u_n \rangle = 0$ and (3.12) to deduce

$$\begin{aligned} \langle I'(u_n) + \lambda_n u_n, I'(u_n) + \lambda_n u_n \rangle &= \langle I'(u_n), I'(u_n) + \lambda_n u_n \rangle + \lambda_n \langle u_n, I'(u_n) + \lambda_n u_n \rangle \\ &\leq \frac{1}{n} \|I'(u_n) + \lambda_n u_n\|, \end{aligned}$$

that is,

$$\|I'(u_n) + \lambda_n u_n\|^2 \leq \frac{1}{n} \|I'(u_n) + \lambda_n u_n\|.$$

Consequently,

$$\|I'(u_n) + \lambda_n u_n\| \leq \frac{1}{n}.$$

Thus, relation (3.5) holds. Therefore, the proof is complete. □

Lemma 3.2 yields the next critical point theorem.

**Theorem 3.1.** *Suppose that the conditions of Lemma 3.2 are satisfied. In addition suppose that there exists a number $\nu$ such that*

$$\langle I'(u), u \rangle \geq \nu \ \textit{for all } u \in K_{rR} \ \textit{with } \|u\| = R, \tag{3.13}$$

$$I'(u) + \lambda u \neq 0 \ \textit{for every } u \in K_{rR} \ \textit{with } \|u\| = R \textit{ and } \lambda > 0, \tag{3.14}$$

*and a Palais-smale type condition holds, more precisely, any sequence as in the conclusion of Lemma 3.2 has a convergent subsequence. Then there exists $u \in K_{rR}$ such that*

$$I(u) = m \ \textit{ and } \ I'(u) = 0.$$

*Proof.* The sequence $(\lambda_n)$ given by (3.6) is bounded as a consequence of (3.13). Hence, there exists a subsequence of $(\lambda_n)$ (still denoted by $\lambda_n$) which converges to some number $\lambda$. Clearly $\lambda \geq 0$. Next using the Palais-smale type condition we may suppose that the sequence $(u_n)$ (still denoted by $u_n$) which converges to some element $u \in K_{rR}$. i.e. $I'(u_n)$ converges to $I'(u)$. Then letting $n \to 0$ in (3.4) and (3.5), we get

$$I(u) = m,$$

and

$$\begin{aligned} \|I'(u_n) + \lambda_n u_n\|^2 &= \langle I'(u_n) + \lambda_n u_n, I'(u_n) + \lambda_n u_n \rangle \\ &= \|I'(u_n)\|^2 + 2\lambda_n \langle I'(u_n), u_n \rangle + \lambda_n^2 \|u_n\|^2 \\ &\to \|I'(u)\|^2 + 2\lambda \langle I'(u), u \rangle + \lambda^2 \|u\|^2 = \|I'(u) + \lambda u\|^2. \end{aligned}$$

On the other hand, we have

$$\|I'(u_n) + \lambda_n u_n\| \to 0 \ \text{ as } n \to 0.$$

So,

$$I'(u) + \lambda u = 0.$$

From (3.6), we have that the case $\lambda > 0$ is possible only if $\|u\| = R$, which is excluded by condition (3.14). Consequently, $\lambda = 0$ and so

$$I'(u) = 0.$$

Therefore, the proof is complete. □

## 3.2 Application to periodic problem

In this section we are going to apply the critical point results to the Periodic Problem, more precisely, we will study the positive solution of the following Problem

$$\begin{cases} -u''(t) + a^2u(t) = f(u(t)) \quad on \;\; (0,T) \\ \quad u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{3.15}$$

where $a \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(\mathbb{R}_+) \subset \mathbb{R}_+$.

Let $\mathcal{X} = H^1_p(0,T)$ be the space of functions of the form

$$u(t) = \int_0^t v(s)ds + C,$$

with $u(0) = u(T)$, $C \in \mathbb{R}$ and $v \in L^2(0,T)$, with the inner product

$$\langle u, v\rangle = \int_0^T (u'v' + a^2uv)dt$$

and the corresponding norm

$$\|u\| = \left(\int_0^T (u'^2 + a^2u^2)dt\right)^{\frac{1}{2}}.$$

Let $K$ be a positive cone of $\mathcal{X}$, i.e. $K = \left\{u \in H^1_p(0,T) : \; u \geq 0 \text{ on } [0,T]\right\}$, and let $l : K \to \mathbb{R}_+$ be given by

$$l(u) = \min_{t\in[0,T]} u(t).$$

**Lemma 3.3.** *$l$ is a concave and upper semicontinuous function with l(0)=0.*

*Proof.*

- Check of $l$ is concave. Indeed, let $u, v \in K$ and let $\lambda \in [0,1]$, we get

$$\begin{aligned} l(\lambda u + (1-\lambda)v) &= \min_{t\in[0,T]} (\lambda u + (1-\lambda)v)(t), \\ &= \lambda \min_{t\in[0,T]} u(t) + (1-\lambda) \min_{t\in[0,T]} v(t), \\ &= \lambda l(u) + (1-\lambda)l(v). \end{aligned}$$

So we have the concavity.

- Check of $l$ is $u.s.c$. Indeed, let $(u_n) \subset K$ such that $u_n \to u$ in $K$, then we show that

$$l(u) \geq \limsup_{n\to+\infty} l(u_n).$$

Since $u_n \to u$, by definition we have

$$\min_{t\in[0,T]} u_n(t) \to \min_{t\in[0,T]} u(t)$$

Then,

$$\limsup_{n\to+\infty} \min_{t\in[0,T]} u_n(t) = \lim_{n\to+\infty} \min_{t\in[0,T]} u_n(t) = \min_{t\in[0,T]} u(t) = l(u).$$

Consequently,

$$l(u) = \limsup_{n\to+\infty} l(u_n).$$

- Check of $l(0) = 0$. Indeed, we have

$$l(0) = \min_{t\in[0,T]} 0 = 0.$$

Therefore, the proof is complete. □

Let $I : H_p^1(0,T) \to \mathbb{R}$ be the energy functional of Problem (3.15) defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_0^T F(u(t))dt, \tag{3.16}$$

where

$$F(\tau) = \int_0^\tau f(s)ds. \tag{3.17}$$

Then $I$ is differentiable on $H_p^1(0,T)$, and

$$\langle I'(u), \varphi\rangle = \int_0^T (u'\varphi' + a^2 u\varphi)dt - \int_0^T f(u)\varphi dt,$$

that is

$$\langle I'(u), \varphi\rangle = \langle u, \varphi\rangle - \int_0^T f(u)\varphi dt \quad \forall \varphi \in H_p^1(0,T). \tag{3.18}$$

Let $J$ be a map from the dual $(H_p^1(0,T))'$ to the space $H_p^1(0,T)$ given by $J(v) = w$, where $w$ is the weak solution of the Problem

$$\begin{cases} -w'' + a^2 w = v \ \ on\ (0,T) \\ w(0) - w(T) = w'(0) - w'(T) = 0. \end{cases} \tag{3.19}$$

**Lemma 3.4.** *We have $I'(u) = u - N(u)$, where*

$$N(u) = J(f(u(.))). \tag{3.20}$$

*Proof.* For this, assume that $N(u) = J(f(u(.))) = w_0$, where $w_0$ is the weak solution of the Problem

$$\begin{cases} -w_0'' + a^2 w_0 = f(u(t)) \ \ on \ \ (0,T), \\ w_0(0) - w_0(T) = w_0'(0) - w_0'(T) = 0. \end{cases} \tag{3.21}$$

We multiple the equation in the Problem (3.21) by $\varphi$ and we integrate over $[0,T]$, we get

$$\int_0^T (w_0'\varphi' + a^2 w_0 \varphi) dt = \int_0^T f(u)\varphi dt,$$

that is,

$$\langle w_0, \varphi \rangle = \int_0^T f(u)\varphi dt \quad \forall \varphi \in H_p^1(0,T).$$

On the other hand, we have $\langle w_0, \varphi \rangle = \langle N(u), \varphi \rangle$. Consequently,

$$\langle N(u), \varphi \rangle = \int_0^T f(u)\varphi dt.$$

Thus,

$$\begin{aligned} \langle I'(u), \varphi \rangle &= \langle u, \varphi \rangle - \int_0^T f(u)\varphi dt, \\ &= \langle u, \varphi \rangle - \langle N(u), \varphi \rangle, \\ &= \langle u - N(u), \varphi \rangle, \end{aligned}$$

that is,

$$I'(u) = u - N(u).$$

This shows the claim. □

**Lemma 3.5.** *We have* $N(K_{rR}) \subset K$.

*Proof.* To show this, we prove first that $N(K) \subset K$. Let $w_0 \in N(K)$ then we will proof that $w_0 \in K$ that is $w_0 \in H_p^1(0,T)$ and $w_0 \geq 0$. Then there exists $u \in K$ such that $N(u) = w_0 = J(f(u(.)))$, where $w_0$ is the weak solution of the Problem (3.21).
We pose $w_0 = w_0^+ - w_0^-$ then we multiple the equation in the Problem (3.21) by $w_0^-$ and we integrate over $[0,T]$, we get

$$\begin{aligned} \int_0^T \left[((w_0^+)' - (w_0^-)').(w_0^-)' + a^2(w_0^+ - w_0^-)(w_0^-)\right] dt &= \int_0^T f(u) w_0^- dt, \\ \int_0^T \left[(w_0^-)'\right]^2 + a^2 \left[w_0^-\right]^2 dt &= -\int_0^T f(u) w_0^- dt. \end{aligned}$$

We have $w_0^- \geq 0$ and since $f(\mathbb{R}_+) \subset \mathbb{R}_+$, then $f(u) \geq 0 \ \ \forall u \geq 0$. Therefore, we get

$$0 \leq \int_0^T \left[(w_0^-)'\right]^2 + a^2 \left[w_0^-\right]^2 dt = -\int_0^T f(u) w_0^- dt \leq 0,$$

so

$$\left[(w_0^-)'\right]^2 + a^2 \left[w_0^-\right]^2 = 0,$$

that is

$$\begin{cases} \left[(w_0^-)'\right]^2 = 0, \\ \left[w_0^-\right]^2 = 0. \end{cases}$$

Therefore $w_0^- = 0$ and so $w_0 = w^+ \geq 0$. In addition $w_0 \in H_p^1(0,T)$, and this is clear because $w_0$ is the weak solution of Problem (3.21). Thus, $w_0 \in K$. Consequently,

$$N(K) \subset K.$$

Note that $K_{rR} \subset K$ i.e. $N(K_{rR}) \subset N(K) \subset K$. That is

$$N(K_{rR}) \subset K.$$

Therefore, the proof is complete. □

**Lemma 3.6.** *We have $K_{rR} \neq \emptyset$.*

*Proof.* Let $c > 0$ be the embedding constant of the inclusion $H_p^1(0,T) \subset C[0,T]$, that is

$$\|u\|_{C[0,T]} \leq c\,\|u\|\,, \tag{3.22}$$

for every $u \in H_p^1(0,T)$. Note that $u \equiv 1$, then from (3.22), we get $1 \leq c\left(\int_0^T a^2\right)^{\frac{1}{2}}$. That is $1 \leq ac\sqrt{T}$, whence $a^2 \geq 1/(c^2T)$. Also, if $r$ and $R$ are positive numbers and $a\sqrt{T}r \leq R$, then the set $K_{rR} \neq \emptyset$. Indeed, since $r \leq R/(a\sqrt{T})$ then there exists $\lambda \in \left[r, R/(a\sqrt{T})\right]$ belongs to $K_{rR}$, and this is clear because

$$l(\lambda) = \min_{t\in[0,T]} \lambda = \lambda \geq r,$$

and

$$\|\lambda\| = \left(\int_0^T a^2\lambda^2 ds\right)^{\frac{1}{2}} = a\lambda\sqrt{T} \leq R.$$

Therefore, the set $K_{rR} \neq \emptyset$.

□

The following theorem guarantees the existence of a positive solution of Problem (3.15).

**Theorem 3.2.** *Let $r, R$ be positive constants such that $a\sqrt{T}r \leq R$. Suppose that $f$ is non-decreasing on the interval $[r, cR]$ and that the next assumptions holds:*

$$I(r) < \frac{R^2}{2} - TF(cR), \tag{3.23}$$

*and*

$$f(r) \geq a^2 r, \quad f(cR) \leq \frac{R}{cT}. \tag{3.24}$$

*Then Problem (3.15) has a positive solution $u$ with $r \leq u(t) \leq cR$ for every $t \in [0,T]$, which minimizes $I$ in the set $K_{rR}$.*

*Proof.* We apply Theorem 3.1. For this purpose, we need the following steps:

**Step1.** Check of assumption (3.1). Let $u \in K_{rR}$. Using $f(\mathbb{R}_+) \subset \mathbb{R}_+$ and (3.17), we get for every $\tau \geq 0$

$$F'(\tau) = f(\tau) \geq 0,$$

then $F$ is nondecreasing on $\mathbb{R}_+$. And the fact that $r \leq u(t) \leq cR$ for every $t \in [0, T]$. We have

$$F(u(t)) \leq F(cR),$$

thus

$$\int_0^T F(u(s))ds \leq \int_0^T F(cR)ds \leq TF(cR). \tag{3.25}$$

From (3.16) and (3.25), we obtain

$$I(u) \geq -\int_0^T F(u(s))ds \geq -TF(cR) > -\infty.$$

Consequently,

$$m = \inf_{u \in K_{rR}} I(u) > -\infty. \tag{3.26}$$

This shows the claim.

**Step2.** Check of assumption (3.2). Let $u \in K_{rR}$ which satisfy $l(u) = r$ and $\|u\| = R$. From (3.16) and (3.25), we get

$$I(u) = \frac{R^2}{2} - \int_0^T F(u(s))ds \geq \frac{R^2}{2} - TF(cR). \tag{3.27}$$

From the strict inequality (3.23), (3.26) and (3.27), we get

$$m = \inf_{u \in K_{rR}} I(u) < I(r) < \frac{R^2}{2} - TF(cR) \leq I(u),$$

where the constant function $r$ belongs to $K_{rR}$. Consequently,

$$I(u) > m,$$

that is there exists $\epsilon > 0$ such that

$$I(u) \geq m + \epsilon.$$

Which proves the claim.

**Step3.** Check of assumption (3.3). To show this, we will proof first that $J$ satisfies

$$J(\lambda v) = \lambda J(v).$$

Indeed, let $v \in (H_p^1(0, T))'$; we set $J(v) = w$ such that $w$ is a weak solution of Problem (3.19), then for every $\lambda \in \mathbb{R}$, we have

$$\begin{cases} -(\lambda w)'' + a(\lambda w) = \lambda v \;\; on \;\; (0, T), \\ \lambda w(0) - \lambda w(T) = (\lambda w)'(0) - (\lambda w)'(T) = 0. \end{cases} \tag{3.28}$$

So, $J(\lambda v) = \lambda w$ because $\lambda w$ is a weak solution of Problem(3.28). Consequently

$$J(\lambda v) = \lambda w = \lambda J(v).$$

Then for calculate $J(1)$ we salving the differential equation $-w'' + a^2 w = 1$.

- Find the homogeneous solution which corresponds to $-w'' + a^2 w = 0$. Assume the solution has the form $w_h = e^{rt}$. Substitute this into the differential equation, we obtain the characteristic equation $-r^2 + a^2 = 0$, which has the solution $r = \pm a$. Therefore, the homogeneous solution is given by:

$$w_h = c_1 e^{at} + c_2 e^{-at}.$$

- Find the particular solution. Assume the solution has the form $w_p = k$, where $k$ is a constant. Substituting this into the differential equation, we get $k = \frac{1}{a^2}$. Hence, the particular solution is:

$$w_p = \frac{1}{a^2}.$$

- Find the general solution. Combining the homogeneous and particular solutions, we get

$$w = w_h + w_p = c_1 e^{at} + c_2 e^{-at} + \frac{1}{a^2}.$$

- Applying boundary conditions. From the given boundary conditions $w(0) - w(T) = w'(0) - w'(T) = 0$, we can substitute them into the general solution, we get

$$w(0) - w(T) = c_1(1 - e^{aT}) + c_2(1 - e^{-aT}) = 0, \tag{3.29}$$

and we have

$$w'(t) = ac_1 e^{at} - ac_2 e^{-at},$$

so

$$w'(0) - w'(T) = ac_1(1 - e^{aT}) - ac_2(1 - e^{-aT}) = 0. \tag{3.30}$$

Solving (3.29) and (3.30), we obtain $c_1 = c_2 = 0$.

- Final solution. Substituting $c_1 = 0$ and $c_2 = 0$ into the general solution, we obtain:

$$w = \frac{1}{a^2}.$$

Therefore,

$$J(1) = w = \frac{1}{a^2}. \tag{3.31}$$

Finally, Using (3.20), the first inequality in (3.24) and (3.31) and the fact that $l$ is linear, we obtain:

$$\begin{aligned} l(N(u)) &= l(J(f(u))) \geq l(J(f(r))) = f(r)l(J(1)) \\ &= \frac{f(r)}{a^2} \geq r. \end{aligned}$$

Thus our claim holds.

**Step4.** Check of assumption (3.14). Using contradiction argument, suppose that $I'(u) + \lambda u = 0$ for some $u \in K_{rR}$ with $\|u\| = R$ and $\lambda > 0$, then for every $v \in H^1_p(0,T)$, we have

$$\langle I'(u) + \lambda u, v\rangle = 0,$$

that is,

$$\langle I'(u), v\rangle + \lambda \langle u, v\rangle = 0.$$

From (3.18), we obtain

$$\langle u, v\rangle - \int_0^T f(u)vdt + \lambda \langle u, v\rangle = 0.$$

Consequently,

$$(1+\lambda)\int_0^T (u'v' + a^2uv)dt = \int_0^T f(u)vdt,$$

we replace $v$ by $u$, we get

$$(1+\lambda)\|u\|^2 = \int_0^T f(u)udt,$$

and the fact that $u(t) \leq cR$ and $\|u\| = R$, we obtain

$$R^2 < (1+\lambda)R^2 = \int_0^T f(u)udt \leq \int_0^T f(cR)cRdt = Tf(cR)cR,$$

that is,

$$\frac{R}{cT} < f(cR),$$

this is contradiction to the second inequality in (3.24). This proves the claim.

**Step5.** Check of assumption (3.13). Let $u \in K_{rR}$ with $\|u\| = R$. Replacing $\varphi$ by $u$ in (3.18), we obtain

$$\langle I'(u), u\rangle = R^2 - \int_0^T f(u)udt.$$

Using the second inequality in (3.24) and the fact that $u \leq cR$, we get

$$\begin{aligned}\langle I'(u), u\rangle &\geq R^2 - Tf(cR)cR \\ &\geq R^2 - T\frac{R}{cT}cR = 0,\end{aligned}$$

it is enough we choose $\nu = 0$. Which proves the claim.

**Step6.** Check of a Palais-smale type condition. For show this, let $(u_n) \subset K_{rR}$ such that (3.4) and (3.5) are satisfied.

First we prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_p(0,T)$. Since $u_n \in K_{rR}$, then $\|u_n\| \leq R$. Consequently, $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_p(0,T)$.

Secondly, since $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_p(0,T)$ which is reflexive then there exists a subsequence $(u_{n_k})$ ( still denoted by $u_n$) such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } H^1_p(0,T), \\ u_n \to u & \text{in } C[0,T] \ \ (H^1_p(0,T) \text{ embedded compactly in } C[0,T]). \end{cases}$$

Since $K_{rR}$ is closed, then $u \in H_p^1(0,T)$.

Finally, we prove that $u_n \to u$ in $H_p^1(0,T)$.
We will show that $\langle I'(u_n) + \lambda_n u_n - I'(u) - \lambda u, u_n - u\rangle \xrightarrow[n\to\infty]{} 0$.

- We have

$$u_n \rightharpoonup u \text{ in } H_p^1(0,T) \quad \text{and} \quad I'(u_n) + \lambda_n u_n \to 0 \text{ in } (H_p^1(0,T))'.$$

By Proposition 1.2, we get

$$\langle I'(u_n) + \lambda_n u_n, u_n - u\rangle \xrightarrow[n\to\infty]{} 0. \tag{3.32}$$

- Since $I'(u) + \lambda u \in (H_p^1(0,T))'$ and $u_n \rightharpoonup u$ in $H_p^1(0,T)$, we obtain

$$\langle I'(u) + \lambda u, u_n - u\rangle \xrightarrow[n\to\infty]{} 0. \tag{3.33}$$

From (3.32) and (3.33), we have

$$\langle I'(u_n) + \lambda_n u_n - I'(u) - \lambda u, u_n - u\rangle \xrightarrow[n\to\infty]{} 0. \tag{3.34}$$

On the other hand, using (3.18), we obtain

$$\begin{aligned}
\langle I'(u_n) + \lambda_n u_n - I'(u) - \lambda u, u_n - u\rangle &= \langle I'(u_n) + \lambda_n u_n, u_n - u\rangle - \langle I'(u) + \lambda u, u_n - u\rangle, \\
&= \langle I'(u_n), u_n - u\rangle - \langle I'(u), u_n - u\rangle + \langle \lambda_n u_n - \lambda u, u_n - u\rangle, \\
&= \langle u_n - u, u_n - u\rangle - \int_0^T (f(u_n) - f(u))(u_n - u)dt \\
&\quad + \langle \lambda_n u_n - \lambda u, u_n - u\rangle .
\end{aligned}$$

Writing

$$\begin{aligned}
\langle \lambda_n u_n - \lambda u, u_n - u\rangle &= \langle \lambda_n u_n - \lambda u_n + \lambda u_n - \lambda u, u_n - u\rangle, \\
&= (\lambda_n - \lambda)\langle u_n, u_n - u\rangle + \lambda \langle u_n - u, u_n - u\rangle .
\end{aligned}$$

Since $\lambda_n$ converges to $\lambda$ and $\langle u_n, u_n - u\rangle$ is bounded, then

$$(\lambda_n - \lambda)\langle u_n, u_n - u\rangle \xrightarrow[n\to\infty]{} 0.$$

Consequently,

$$\langle \lambda_n u_n - \lambda u, u_n - u\rangle = \lambda \langle u_n - u, u_n - u\rangle .$$

Therefore,

$$\begin{aligned}
\langle I'(u_n) + \lambda_n u_n - I'(u) - \lambda u, u_n - u\rangle &= (1+\lambda)\langle u_n - u, u_n - u\rangle - \int_0^T (f(u_n) - f(u))(u_n - u)dt, \\
&= (1+\lambda)\|u_n - u\| - \int_0^T (f(u_n) - f(u))(u_n - u)dt.
\end{aligned}$$

Writing

$$\int_0^T (f(u_n)-f(u))(u_n-u)dt = \int_0^T f(u_n)u_n dt - \int_0^T f(u_n)u dt - \int_0^T f(u)u_n dt + \int_0^T f(u)u dt.$$

Since $(u_n)$ is bounded and $f$ is a continuous function, using Dominated convergence theorem Lebesgue, we get

$$\int_0^T f(u_n)u_n dt \to \int_0^T f(u)u dt,$$

and

$$\int_0^T f(u_n)u dt \to \int_0^T f(u)u dt,$$

and

$$\int_0^T f(u)u_n dt \to \int_0^T f(u)u dt.$$

Consequently,

$$\int_0^T (f(u_n)-f(u))(u_n-u)dt \xrightarrow[n\to\infty]{} 0. \tag{3.35}$$

Combining (3.34) and (3.35), we deduce that

$$(1+\lambda)\left\|u_n-u\right\| \xrightarrow[n\to\infty]{} 0.$$

Which implies that

$$u_n \to u \text{ in } H_p^1(0,T).$$

So, Step6 has been proved.

Consequently, thanks to Theorem 3.1 the Problem (3.15) has a positive solution. Therefore, the proof is complete. □

**Example 3.1.** *For every $\lambda > 0$, the equation $-u'' + a^2u = \lambda\sqrt{u}$ has a T-periodic solution satisfying $u(t) \geq \lambda^2/a^4$ for every $t \in [0,T]$.*

*To show this, we use Theorem 3.2. It is clear that $f(u) = \lambda\sqrt{u}$ is nondecreasing and if we take $r = \lambda^2/a^4$, we have*

$$f(r) = \lambda\sqrt{r} = \lambda^2/a^2 \geq a^2 r,$$

*then the first inequality in (3.24) is satisfied. And we have*

$$F(\tau) = \int_0^\tau f(s)ds = \lambda\int_0^\tau \sqrt{s}ds = \frac{2}{3}\lambda\tau^{\frac{3}{2}}.$$

*We choose $R$ large enough that $a\sqrt{T}r \leq R$ and (3.23) and the second inequality in (3.24) are satisfied, that is*

$$\begin{aligned} I(r) &= \frac{1}{2}\left\|r\right\|^2 - \int_0^T F(r)dt \\ &= \frac{1}{2}a^2r^2T - \frac{2}{3}T\lambda r^{\frac{3}{2}} \\ &< \frac{R^2}{2} - \frac{2}{3}T\lambda(cR)^{\frac{3}{2}}, \end{aligned}$$

*and*

$$f(cR) = \lambda\sqrt{cR} \leq \frac{R}{cT}.$$

*Consequently, the equation* $-u'' + a^2u = \lambda\sqrt{u}$ *has a positive solution* $u$ *with* $u(t) \geq \lambda^2/a^4$ *for all* $t \in [0, T]$.

# Conclusion

In this memory, we established a new critical point theorem for a class of perturbed functionals without satisfying the Palais-Smale condition.

Also, we have localized a critical point of minimum type of smooth functional which is obtained in a bounded convex conical set. This is a method used to determine positive solutions by restricting the search to a conical set.

This work has fascinating to me as I was able to explore new methods for studying boundary Problems in PDE. It also allowed me to review and solidify many concepts that were studied during the years of training. Additionally, I gained proficiency in mathematical text writing software and editing methodologies.

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# الملخّص:


في هذا العمل، قمنا بدراسة نظرية جديدة للنقطة الحرجة في الحالتين التاليتين:

في الأولى، قمنا بإثبات نظرية النقطة الحرجة لفئة من التوابع الدالية المضطربة التي لا تستوفي شرط بالي-سميل، والتي تضمن وجود نقطة حرجة للدوال المضطربة من النوع $I = I_1 + I_2$ ، بشرط أن يكون ل $I_1$ نقطه حرجة واحدة على الأقل. تم تطبيق النتيجة المجردة الرئيسية على المسألة الغير خطية التالية:

$$\begin{cases} -div(|\nabla u|^{p-2}.\nabla u) = |u|^{q-2}u + \lambda g(x,u) & dans\ \Omega, \\ u = 0 & sur\ \Gamma, \end{cases}$$

حيث $\lambda \in \mathbb{R}$ ،$\Omega$ هي مجموعة محدودة في $\mathbb{R}^N$ مع حدود سلسة $\Gamma$، $1 < q < p$ مع $p > N$ و $g(\cdot,\cdot)$ مستمرة على $\overline{\Omega} \times [0,\infty)$ .

في الحالة الأخيرة، قمنا بإثبات محلية نقطة حرجة تمثل الحد الأدنى لتابع دالي نظامية في مجموعة مخروطيه محدبة محدودة معرفة بواسطة نظيم ودالة نصف مستمرة علوية مقعرة. تم تطبيق نتيجتنا المجردة على المسألة الدورية التالية:

$$\begin{cases} -u''(t) + a^2 u(t) = f\big(u(t)\big) & sur(0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}$$

حيث $a \neq 0$ و $f : \mathbb{R} \to \mathbb{R}$ هي دالة مستمرة مع $f(\mathbb{R}_+) \subset \mathbb{R}_+$ .



**الكلمات المفتاحية:** نقطة حرجة، دالة مضطربة، حل ضعيف، مبدأ ايكلاند، مسألة دورية، حل موجب.

# Abstract

In this work, we have studied a new critical point theorem in the following two cases:

In the first case, we have established a new critical point theorem for a class of perturbed functionals without satisfying the Palais-Smale condition, which asserts the existence of critical point of functionals of the type $I = I_1 + I_2$, provided that $I_1$ has at least one critical point. The main abstract result is applied to the following nonhomogeneous Problem:

$$\begin{cases} -\text{div}(|\nabla u|^{p-2} .\nabla u) = |u|^{q-2} u + \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Where $\lambda \in \mathbb{R}$, $\Omega$ is a bounded set of $\mathbb{R}^N$with smooth boundary $\Gamma$, $1 < q < p$ with $p > N$ and $g(\cdot,\cdot)$ is continuous on $\bar{\Omega} \times [0,\infty)$.

The last case, we have established the localization of a critical point of minimum type of a smooth functional is obtained in a bounded convex conical set defined by a norm and a concave upper semicontinuous functional. Our abstract result is applied to the following Periodic Problem:

$$\begin{cases} -u''(t) + a^2 u(t) = f(u(t)) \quad \text{on } (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Where $a \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(\mathbb{R}_+) \subset \mathbb{R}_+$.

**Keywords:** Critical point, Perturbed Functional, Weak solution, Ekeland's Principle, Periodic Problem, Positive Solution.

# Résumé

Dans ce travail, nous avons étudié un nouveau théorème de point critique dans les deux cas suivants:

Dans le premier cas, nous avons établi un nouveau théorème de point critique pour une classe de fonctionnelles perturbées sans satisfaire la condition de palais-smale, qui affirme l'éxistence d'un point critique des fonctionnelles de type $I = I_1 + I_2$, à condition que $I_1$ possèd au moins un point critique. Le résultat abstrait principal est appliqué au Problème non homogène suivant:

$$\begin{cases} -\text{div}(|\nabla u|^{p-2} .\nabla u) = |u|^{q-2} u + \lambda g(x,u) & \text{dans } \Omega, \\ u = 0 & \text{sur } \Gamma. \end{cases}$$

Où $\lambda \in \mathbb{R}$, $\Omega$ est une ensemble borné de $\mathbb{R}^N$ avec une frontière lisse $\Gamma$, $1 < q < p$ avec $p > N$ et $g(\cdot,\cdot)$ est continue sur $\bar{\Omega} \times [0, \infty)$.

Dans le dernier cas, nous avons établi la localisation d'un point critique de type minimum d'une fonctionnelle régulière qui est obtenue dans un ensemble conique convexe borné défini par une norme et une fonctionnelle concave semi-continue supérieure. Notre résultat abstrait est appliqué au Problème Périodique suivant:

$$\begin{cases} -u''(t) + a^2 u(t) = f(u(t)) \ \text{ sur } \ (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Où $a \neq 0$ et $f : \mathbb{R} \to \mathbb{R}$ est une fonction continue avec $f(\mathbb{R}_+) \subset \mathbb{R}_+$.



**Mots clés:** Point Critique, Fonctoinnelle Perturbée, Solution faible, Principe d'Ekeland, Problème Périodique, Solution Positive.

# جدول المصطلحات

| English | Français | عربي |
| --- | --- | --- |
| Argument | Argument | عمدَة |
| Associate | Associer | أرفَقَ |
| Belong | Appartenir | انتمى |
| Bilinear | Bilinéaire | ثنائيّ الخطيّة |
| Boundary, Frontier | Frontière | حافَة |
| Bounded | Borné | محدود |
| Bounded below | Borné inférieurement, Minoré | محدود من الأدنى |
| Canonical | Canonique | قانونيّ |
| Case | Cas | حالَة |
| Centre | Centre | مركَزَ |
| Characteristic | Caractéristique | مُميِّزة |
| Class | Classe | صنْف، صفّ |
| Closed | Fermé | مغلق |
| Closure | Fermeture, Adhérence | غلْق، ملاصِقة |
| Compact | Compact | متراصٌ |
| Concave | Concave | مقَعَّر |
| Condition | Condition | شرط |
| Cone | Cône | مخروط |
| Conic | Conique | مخروطيّ |
| Conjugate | Conjugué | مرافق |
| Constant | Constant | ثابتٌ |

# جدول المصطلحات

| | | |
|---|---|---|
| Contain | Contenir | احتَوى |
| Continuity | Continuité | استمرار |
| Continuous | continu | مستمرّ |
| Contradiction | Contradiction | تناقض |
| Converge | Converger | تقاربَ |
| Convergence | Convergence | تقاربٌ |
| Convex | Convexe | محدَّب |
| Critical | Critique | حرِج |
| Complete | Complet | تامّ |
| Decreasing | Décroissant | متناقصٌ |
| Definition | Définition | تعريف |
| Dependent | Dépendant | مرتبط |
| Differentiable | Dérivable | قابل للاشتقاق، للمفاضلة |
| Differential | Différentielle | تفاضلية |
| Dominated | Dominé | مُهَيْمَن ، مسيطَر |
| Dual | Dual | ثُنْويٌّ |
| Element | Elément | عنصر |
| Empty | Vide | خالٍ |
| Equation | Equation | معادلة |
| Equivalence | Equivalence | تكافؤ |
| Existence | Existence | وجود |
| Exponent | Exposant | أسّ، قوة |
| Extreme (point ) | Extrême (point -) | قُصوِيَّة، متطرفة (نقطة-) |

# جدول المصطلحات

| | | |
|---|---|---|
| Example | Exemple | مثال |
| Fixed (point ) | Fixe (point -) | ثابتة، صامِدة (نقطة-) |
| Function | Fonction | دالة |
| Functional | Fonctionnel | دالّي |
| General | Général | عامّ |
| Geometry | Géométrie | هندسة |
| Gradient | Gradient | تدرُّج |
| Homogeneous | Homogène | مستمر، مُسْتشاكِل متجانس |
| Hypothesis | Hypothèse | فرَضِيّة |
| Implication | Implication | استلزام |
| Imply | Impliquer | استلزم، أدى الى، اقتضَى |
| Independent | Indépendant | مستقِلّ |
| Inequality | Inégalité | متباينة |
| Injection | Injection | تباين |
| Inner product | Produit interne | جداء داخليّ، جداء سلّمي |
| Integral | Intégrale | تكامل |
| Interior (of a set) | Intérieur (d'un ensemble) | داخلية (مجموعة) |
| Interval | Intervalle | مجال |
| Inverse | Inverse | مقلوب، عكسي |
| Lemma | Lemme | توطِئة |
| Linear | Linéaire | خطيّ |
| Local | Local | محلّيّ ، موضعيّ |
| Localization | Localisation | تمْحيل ، مؤْضعة |

# جدول المصطلحات

| | | |
|---|---|---|
| Maximum | Maximum | قيمة عظمى |
| Mean | Moyenne | متوسّط، معدّل |
| Measurable | Mesurable | قابل للقياس |
| Metric | Métrique | متري |
| Minimize | Minimiser | صَغَّر |
| Minimum | Minimum | قيمة صغرى |
| Norm | Norme | نظيم |
| Notation | Notation | ترميز، رمز |
| Open | Ouvert | مفتوح |
| Operator | Opérateur | مؤثِر |
| Particular | Particulier | خاص |
| Periodic | Périodique | دوريّ |
| Perturbation | Perturbation | تشويش، اضطراب |
| Positive | Positif | موجب |
| Principal | Principal | رئيسيّ |
| Problem | Problème | مسألة، مشكل |
| Proof, Demonstration | Démonstration | برهان، إثبات |
| Proposition | Proposition | قضية |
| Prove, Demonstrate | Démontrer | برْهن، أثبت |
| Real | Réel | حقيقيّ |
| Reflexive | Réflexif | إنعكاسيّ |
| Relation | Relation | علاقة |
| Result | Résultat | نتيجة |

# جدول المصطلحات

| | | |
|---|---|---|
| Satisfy | Satisfaire | حقّقَ |
| Semi-continuous | Semi-continu | نصف مستمر |
| Separable | Séparable | قابل للفصل، فَصول |
| Sequence | Suit | متتالية |
| Similar | Semblable | مشابهٌ |
| Set | Ensemble | مجموعة |
| Solution | Résolution, Solution | حلٌّ |
| Space | Espace | فضاء |
| Strong | Fort | قويّ |
| Subset | Sous-ensemble | مجموعة جزئية |
| Surjective, onto (map) | Surjective (application-) | غامِر(تطبيق-) |
| Symmetric | Symétrique | نظير |
| Theorem | Théorème | نظرية، مبرهَنة |
| Type | Type | نَمَطٌ |
| Unbounded | Illimité, Non borné | غير محدود |
| Upper | Supérieure | عليا |
| Variational | Variationnel | تغّيرِيّ |
| Value | Valeur | قيمة |
| Weak | Faible | ضعيف |