Physics Letters A 383 (2019) 1915–1921

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Exact analytical results for density profile in Fourier space and elastic scattering function of a rotating harmonically confined ultra-cold Fermi gas

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ARTICLE INFO

Article history:
Received 7 December 2018
Received in revised form 11 March 2019
Accepted 15 March 2019
Available online 20 March 2019
Communicated by V.A. Markel

Keywords:
Density profile in Fourier space
Rotating Fermi ultra-cold gas
Bloch density matrix
Elastic scattering form factor

ABSTRACT

In this paper, the system dealt with consisting of an ultra-cold neutral spin-polarized Fermi gas undergoing rotation (or in the so-called synthetic magnetic field) trapped by an anisotropic harmonic potential in a two and three-dimensional space at zero temperature. Using the so-called Bloch propagator as a tool, we derive exact closed-form expressions for particle density in Fourier space which are valid for an arbitrary particle number confined by a two and three-dimensional rotating anisotropic harmonic trap. Numerical illustrations and discussions are presented. The results can be easily generalized at finite temperatures. The crossover from two-dimensional to the one-dimensional regime is shown to be reflected in the shape of the density distribution in Fourier space at very fast rotating velocity (or at strong synthetic magnetic field). In addition, an exact analytical expression of the elastic scattering factor is found, a quantity of interest used to probe the spatial distribution of the quantum gases.

1. Introduction

In the past twenty years, a considerable attention has been devoted to the field of quantum cold atom gases and the physics of ultra-cold quantum gases has been one of the most exciting and rapidly growing fields of physics. Thanks to advances in cooling, where temperatures of the order of a fraction of the Fermi temperatures were reached, and trapping techniques, led to experimental realization of trapped ultra-cold gases in various spatial geometries [1]. In addition, advances in high resolution imaging cold cloud gases allow to probe these quantum systems. Nowadays, experimentalists can emulate properties of real materials through the use of ultra-cold gases by engineering a desired hamiltonian [2]. An interesting issue in this field is to examine the response to rapid rotation of bosons or fermions cold gases. Starting with the experimental work on rotating Bose-Einstein condensates [3], tremendous investigations have been made. As the frequency of rotation increases, a Bose Einstein condensate (BEC) develops a vortex, with number of vortices that increases as the rotation becomes faster and in high-rotation limit the system exhibits atomic quantum Hall states and the BEC can be described with the lowest Landau-level (LLL) set of single-particle states [1], [4–9]. Theoretical studies on rotating Fermi gases reveal that Landau-level-like energy spectrum appears in the fast rotating regime [10] and this structure is depicted in the density profiles [9–14]. An atom-light interaction has been used to simulate artificial gauge fields acting on neutral matter in a similar way as particles in a rotating frame [15].

Experiments on cold rotating gases generally involves anisotropic harmonic trap potentials, which constitutes a motivation of the present study. The system we consider is an ultra-cold non-interacting spin-polarized Fermi atom gas undergoing rotation (or in a synthetic magnetic field) and trapped by an anisotropic harmonic potential in a two and three-dimensional space at zero temperature. Knowledge of the one-body density matrix is a fundamental quantity for studying the properties of the system. To obtain this density matrix we shall use the Bloch propagator $\exp(-\xi H)$ and, as we shall show further, the latter propagator written in coordinate representation and called the Bloch density matrix can be, through an appropriate inverse Laplace transform, related to the one-body density matrix for fixed particle number [16–20]. Explicit evaluation of the Bloch density matrix has been carried out for various systems [16], [21–25]. Among these works, M. A. Habeeb in [23] treated a general case of an anisotropic har-

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https://doi.org/10.1016/j.physleta.2019.03.021
0375-9601/

monic confinement of a charged particle subjected to a magnetic field by using the canonical transformations to transform the original problem to that of a particle moving in an anisotropic oscillator potential for which the Bloch density matrix is already known [24]. As pointed out by K. Yonei, the derived Bloch density matrix of Habeeb contains unfortunate misprints that have been corrected by Yonei in [25], who developed, for the same physical system treated by Habeeb, an alternative method based on a simple factorization of the Bloch operator $\exp(-\xi H)$. However, the derived Bloch density matrix of Yonei contains also unfortunate misprints. In this paper, following the same method as in [25], we will give the correct expression of the Bloch density matrix for our rotating system. This latter is, evidently, valid for the equivalent magnetic system if one uses the appropriate correspondences between the two systems. To our knowledge, the derived closed analytical expression has not been applied before in any practical situation. With the advent of the ultra-cold physics, we propose to use such expression and exploit it to obtain physical quantities of rotating ultra-cold gas or a gas in a synthetic magnetic field.

In the present work, we are interested in obtaining a closed-form of the density distribution in Fourier space. This density should not be confused with momentum density. The density distribution in Fourier space has been studied previously for some cases [26–28]. Starting from the corrected expression of the Bloch density matrix, we derive a closed-form of density distribution in Fourier space, which is valid for anisotropic harmonic trap under an arbitrary rotating angular velocity (or magnetic field strength) and for an arbitrary particle number. In addition, we study in Fourier space the effects of fast rotations or the effect of tight anisotropic trap, where one can observe the crossover from the two-dimensional to the one-dimensional regime and how it is reflected in the shape of this density distribution.

Light scattered from a confined spin-polarized ultra-cold non-interacting Fermi gas can be used to explore quantum statistics effects such as the shell structure or Pauli blocking effect in the density distribution of the atomic gas [29–32]. If the light frequency used is far off resonance from the single particle states of the gas, the scattering of the light beam will be characterized by the elastic structure factor. Then the quantum effects are depicted by studying the angular distribution of the elastic scattering light determined by this elastic structure factor, which is nothing but the square of the Fourier transform of the density distribution of the gas. The study of this factor was performed in [12] and the results were restricted to systems containing a few numbers of particles. In the present work, we give a general analytical result for this factor valid for an arbitrary particle number and for an arbitrary angular velocity. For sake of definiteness, the study presented here concern the rotating system, but by exploiting the analogy between the motion of charged particles in a magnetic field and neutral atoms in a rotating frame, one can easily, using the appropriate correspondences, transpose the results found here to the charged gas in a magnetic field system.

The paper is organized as follows. In Sec. 2, we introduce the Bloch density matrix as the main tool to calculate the corresponding single particle density matrix for the case of rotating two dimensional anisotropic harmonic oscillator and for arbitrary fixed particle number and then we obtain the single particle energy spectrum without solving the associated Schrödinger equation. In Sec. 3, we derive a closed-expression of the density distribution in Fourier space and then we study the behavior of the rotating system for various regimes of rotation. We, in particular, focus on fast rotating regime and strong anisotropic confinement. In Sec. 4, we extend the study to the three dimensional case and we apply in sec. 5 to obtain analytical expression of the elastic structure factor. The paper ends with a summary and outlook.

2. The Bloch density matrix and the single particle density for an anisotropic harmonic trap in rotating frame

The main theoretical tool used in our analysis is the propagator operator $U = \exp(-\xi H)$ associated to the single particle hamiltonian H. In the subsequent analysis, we consider ξ as a general complex mathematical parameter with positive real part and we denote by $C(\mathbf{r}, \mathbf{r}'; \xi) = \langle \mathbf{r} | \; e^{-\xi H} \; | \mathbf{r}' \rangle$ the matrix elements in configuration space of $\exp(-\xi H)$ operator. In two spatial dimensions, the exact eigenfunctions of H obey $H\varphi_{nm}(\mathbf{r}) = E_{nm}\varphi_{nm}(\mathbf{r})$ with E_{nm} are the corresponding eigenvalues, and in terms of the φ_{nm}'s, the matrix element $C(\mathbf{r}, \mathbf{r}'; \xi)$, called also Bloch density matrix, reads [16]

$$C(\mathbf{r}, \mathbf{r}'; \xi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varphi^{*}_{nm}(\mathbf{r}')\,\varphi_{nm}(\mathbf{r}) \exp(-\xi E_{nm}) \tag{1}$$

Using the following inverse Laplace transform identity [36]

$$\Theta(\mu - H) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\xi(\mu-H)}}{\xi} d\xi, \quad c > 0 \tag{2}$$

where $\Theta(x)$ is the unit step function $\Theta(x) = 1$ or 0 if $x > 0$ or < 0 respectively. The zero temperature one-body density matrix for a system of N non-interacting particles, defined as $\rho(\mathbf{r}, \mathbf{r}'; \mu) = \sum_n \sum_m \varphi^{*}_{nm}(\mathbf{r}')\,\varphi_{nm}(\mathbf{r})\,\Theta(\mu - E_{nm})$, can be written as an appropriate inverse Laplace transform, so that

$$\rho(\mathbf{r}, \mathbf{r}'; \mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\xi\mu}}{\xi} C(\mathbf{r}, \mathbf{r}'; \xi)\, d\xi \tag{3}$$

where the Fermi energy μ is obtained through the normalization condition of the local density to the total particle number N, that is

$$\int \rho(\mathbf{r}; \mu)\, d\mathbf{r} = N \tag{4}$$

In above, $\rho(\mathbf{r}; \mu) = \rho(\mathbf{r}, \mathbf{r}' = \mathbf{r}; \mu)$ stands for the local particle density, and according to Eq. (3), we have

$$\rho(\mathbf{r}; \mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\xi\mu}}{\xi} C(\mathbf{r}; \xi)\, d\xi \tag{5}$$

Let us suppose that our system consists of N non-interacting neutral spin polarized atoms (a one-component Fermi gas) each of mass m^*, in the xy-plane, trapped by a two-dimensional (2D) anisotropic harmonic potential $V(\mathbf{r}) = m^*\left(\omega_x^2 x^2 + \omega_y^2 y^2\right)/2$, where ω_x and ω_y are the trapping frequencies along the x and y directions. If we define the anisotropy parameter $\delta = (\omega_y/\omega_x)$ and we choose $\delta \geq 1$, then the x and y axis will be qualified respectively as the weakly and strongly (tightly) confined directions. The confinement in the z direction is assumed to be very tight so that the motion in that direction is frozen (quasi-2D regime). When the trap is set in rotation around the z axis and in the positive sense (counter clockwise) with angular velocity $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$, the quantum one-body Hamiltonian in the rotating frame takes the form [35]

$$H = \frac{1}{2m^*}\mathbf{p}^2 + \frac{1}{2}m^*\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) - \boldsymbol{\Omega}\cdot\mathbf{L} \tag{6}$$

here $\mathbf{p}$ and $\mathbf{L}$ denote respectively the momentum and the angular momentum operators of the particle. To describe the effects of rotation on the properties of the system, it is convenient to introduce, in addition to the anisotropy parameter δ, a new parameter denoted by η and given by $\eta = \Omega/\omega_x$.

To exhibit the analogy between the above hamiltonian and the one of a charged particle in a magnetic field, we can rewrite the hamiltonian in Eq. (6) in the following form

$$\begin{aligned} H &= \frac{1}{2m^*}\left(\mathbf{p} - m^*\boldsymbol{\Omega}\times\mathbf{r}\right)^2 + \frac{1}{2}m^*\left[\left(\omega_x^2-\Omega^2\right)x^2 + \left(\omega_y^2-\Omega^2\right)y^2\right] \\ &= \frac{1}{2m^*}\left(\mathbf{p} - q\mathbf{A}\right)^2 + \frac{1}{2}m^*\left[\left(\omega_x^2-\Omega^2\right)x^2 + \left(\omega_y^2-\Omega^2\right)y^2\right] \end{aligned} \tag{7}$$

where in the second form, $\mathbf{A} = (m^*/q)\,\boldsymbol{\Omega}\times\mathbf{r}$ is an equivalent vector potential giving rise to an equivalent magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$, exhibiting the equivalence between the Coriolis force and the Lorentz force. The second form in Eq. (7) describes the motion of a charged particle under a magnetic field and in an anisotropic weakened harmonic trap. Nowadays and thanks to lasers, a hamiltonian like in the last form of Eq. (7) can experimentally be engineered for neutral atoms in the field of ultra-cold atoms where a synthetic magnetic field is generated with a spatially dependent vector potential [15]. We shall refer throughout the paper to a neutral spin polarized Fermi gas trapped by a rotating anisotropic harmonic potential, since the system can be easily mapped into a charged system.

Closed analytical form of the Bloch density matrix given in Eq. (1) associated to hamiltonian similar to that in Eqs. (6) or (7) for the case of rotating anisotropic confining harmonic potential is known for some time. Calculations of this density matrix were carried out using different techniques. As stated in the introduction, in ref. [23], Habeeb used canonical transformations and later on Yonei [25] used a method based on the decomposition of the propagator $\exp(-\xi H)$ to a product of several factors of simple structure. Let us start with the analytical expression of the Bloch density matrix for hamiltonian in Eq. (6) as given in [25] and using our notations it reads

$$\begin{aligned} C\left(\mathbf{r},\mathbf{r}';\xi\right) &= g(\xi)\exp\left[-\left(iA/\hbar B\right)\left(xy - x'y'\right)\right] \\ &\quad\times\exp\Big[a\left(x+x'\right)^2 + a'\left(x-x'\right)^2 + b\left(y+y'\right)^2 \\ &\quad + b'\left(y-y'\right)^2 + c\left(x+x'\right)\left(y-y'\right) \\ &\quad + c'\left(x-x'\right)\left(y+y'\right)\Big] \end{aligned} \tag{8}$$

where A and B are respectively given by

$$A = \frac{\omega_y^2-\omega_x^2+S}{2S}; \quad B = \frac{2\Omega}{m^*S} \tag{9}$$

with

$$S = \sqrt{\left(\omega_y^2-\omega_x^2\right)^2 + 8\Omega^2\left(\omega_x^2+\omega_y^2\right)} \tag{10}$$

In Eq. (8), the function $g(\xi)$ is defined as

$$g(\xi) = \frac{1}{2\pi\hbar}\left(\frac{\mu_+\Omega_+}{\mu_-\Omega_-}\right)^{1/2}\left[\gamma_1\gamma_1'\sinh\left(\xi\hbar\Omega_+\right)\sinh\left(\xi\hbar\Omega_-\right)\right]^{-1/2} \tag{11}$$

with the masses $\mu_\pm$ and frequencies (modes) $\Omega_\pm$ are given as follows

$$\mu_\pm = \frac{2m^*S}{S + \left(\omega_y^2-\omega_x^2\right) \pm 4\Omega^2} \tag{12}$$

$$\Omega_\pm^2 = \frac{1}{2}\left(\omega_x^2+\omega_y^2+2\Omega^2 \pm \sqrt{\left(\omega_y^2-\omega_x^2\right)^2 + 8\Omega^2\left(\omega_x^2+\omega_y^2\right)}\right) \tag{13}$$

where the coefficients γ_1 and γ_1' read

$$\gamma_1 = \frac{\coth\left(\xi\hbar\Omega_-/2\right)}{\mu_-\Omega_-} + \mu_+\Omega_+B^2\coth\left(\xi\hbar\Omega_+/2\right) \tag{14}$$

$$\gamma_1' = \frac{\tanh\left(\xi\hbar\Omega_-/2\right)}{\mu_-\Omega_-} + \mu_+\Omega_+B^2\tanh\left(\xi\hbar\Omega_+/2\right) \tag{15}$$

Moreover, the coefficients a, a', b, b', c and c' in Eq. (8), which are functions of the trapping frequencies $\left(\omega_x,\omega_y\right)$, of the angular velocity Ω, and of the complex parameter ξ, are given in appendix. It is worth to mention that the expression in Eq. (8) has been obtained without using the explicit form of the single particle wave functions of the considered system.

It should be noted that Yonei corrected misprints appearing in the expression of the Bloch density matrix of an earlier work in [23]. However, in rederiving Eq. (8) we have realized that, for the general case of anisotropic trap, there are misprints in Yonei's paper [25] concerning the coefficients γ_1, γ_1', γ_2, γ_2' and c'. It is easy to check that the Bloch density given in [25], for the case of rotating two dimensional isotropic harmonic trap $\left(\omega_x = \omega_y\right)$, does not reduce to the well known expression derived earlier [18], [22]. We have given the correct expressions of γ_1 and γ_1' in Eqs. (14) and (15) respectively, and the correct expressions for the remaining coefficients, namely γ_2, γ_2' and c' can be found in the appendix. Obviously, after our corrections, Eq. (8) reduces for the isotropic case to the correct expression. We believe that, after the above-mentioned corrections, the expression of the Bloch density matrix in Eq. (8) is the correct one and we can now use it safely and go beyond to calculate the so-called Fourier transform of the particle density, which is our main interest in this work.

2.1. Single particle energy spectrum of the system

The single particle energy spectrum of the system can be easily obtained through the use of the Bloch density matrix and without recourse to the resolution of the Schrödinger equation. In fact, if $C(\mathbf{r};\xi)$ denotes the diagonal elements of the Bloch density matrix, then using Eq. (8) with coefficients in Eqs. (10)-(15) we obtain

$$C(\mathbf{r};\xi) = g(\xi)\exp\left(4ax^2 + 4by^2\right) \tag{16}$$

and after integration over $\mathbf{r}$, we get

$$\int C(\mathbf{r};\xi)\,d\mathbf{r} = \left[4\sinh\left(\frac{\xi\hbar\Omega_+}{2}\right)\sinh\left(\frac{\xi\hbar\Omega_-}{2}\right)\right]^{-1} \tag{17}$$

Using Taylor series expansion of the hyperbolic functions in Eq. (17), we obtain

$$\int C(\mathbf{r};\xi)\,d\mathbf{r} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\exp\left\{-\xi\left[\hbar\Omega_+\left(n+\frac{1}{2}\right) + \hbar\Omega_-\left(m+\frac{1}{2}\right)\right]\right\} \tag{18}$$

On the other hand, returning to Eq. (1), since the single particle wave functions are normalized, we can immediately write

$$\int C(\mathbf{r};\xi)\,d\mathbf{r} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\exp\left(-\xi E_{nm}\right) \tag{19}$$

Now, comparing the two Eqs. (18) and (19), we find the single particle energy spectrum for the rotating anisotropic harmonic trap

$$E_{nm} = \hbar\Omega_+\left(n+1/2\right) + \hbar\Omega_-\left(m+1/2\right), \quad n,m = 0,1,2\ldots \tag{20}$$

and recall that $\Omega_\pm$ are given in Eq. (13). Using the appropriate correspondences between rotating and magnetic cases, one can verify that the expression of the energy levels given by Eq. (20) is identical to those obtained in [33,34].

3. Density profile in Fourier space

The Fourier transform of the single particle density is defined as

$$n(\mathbf{k}) = \int \exp(-i\mathbf{k}\cdot\mathbf{r})\,\rho(\mathbf{r})\,d\mathbf{r} \tag{21}$$

The normalization condition for the local density implies that $n(\mathbf{k}=\mathbf{0}) = N$. The inverse Fourier transform reads $\rho(\mathbf{r}) = \left(1/4\pi^2\right)\int exp\,(i\mathbf{k}\cdot\mathbf{r})\,n(\mathbf{k})\,d\mathbf{k}$. Substituting Eq. (5) into (21), we find

$$n(\mathbf{k}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\xi\mu}}{\xi}\tilde{C}(\mathbf{k};\xi)\,d\xi \tag{22}$$

where $\tilde{C}(\mathbf{k};\xi)$ is the Fourier transform of $C(\mathbf{r};\xi)$, so that

$$\tilde{C}(\mathbf{k};\xi) = \int \exp(-i\mathbf{k}\cdot\mathbf{r})\,C(\mathbf{r};\xi)\,d\mathbf{r} \tag{23}$$

Using Eq. (8), we obtain the diagonal element in the form, $C(\mathbf{r};\xi) = g(\xi)\exp\left(4ax^2+4by^2\right)$. Carrying out the spatial double integration, equation (23) becomes

$$\begin{aligned}\tilde{C}(\mathbf{k};\xi) = \exp&\left[-\frac{1}{4}\left(1+\frac{\hbar^2B^2}{\lambda_+^2\lambda_-^2}\right)\left(\lambda_-^2k_x^2+\lambda_+^2k_y^2\right)\right]\\ &\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{L_n\left(\frac{\hbar^2B^2k_x^2}{2\lambda_+^2}+\frac{\lambda_+^2k_y^2}{2}\right)L_m\left(\frac{\lambda_-^2k_x^2}{2}+\frac{\hbar^2B^2k_y^2}{2\lambda_-^2}\right)\right.\\ &\left.\times\exp(-\xi E_{nm})\vphantom{\frac{1}{1}}\right\}\end{aligned} \tag{24}$$

where $\left(k_x,k_y\right)$ are the two components of the vector $\mathbf{k}$, and the lengths $\lambda_\pm = \sqrt{\hbar/\mu_\pm\Omega_\pm}$ have been introduced. In above, L_n and L_m are the Laguerre polynomials of degree n and m respectively. Now, inserting Eq. (24) into Eq. (22) and upon using the identity in Eq. (2), we end with the result for the density distribution in the Fourier space

$$\begin{aligned}n(\mathbf{k}) = \exp&\left[-\frac{1}{4}\left(1+\frac{\hbar^2B^2}{\lambda_+^2\lambda_-^2}\right)\left(\lambda_-^2k_x^2+\lambda_+^2k_y^2\right)\right]\\ &\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{L_n\left(\frac{\hbar^2B^2k_x^2}{2\lambda_+^2}+\frac{\lambda_+^2k_y^2}{2}\right)L_m\left(\frac{\lambda_-^2k_x^2}{2}+\frac{\hbar^2B^2k_y^2}{2\lambda_-^2}\right)\right.\\ &\left.\times\Theta(\mu-E_{nm})\vphantom{\frac{1}{1}}\right\}\end{aligned} \tag{25}$$

Due to the presence of the step function in above expression, the infinite sums are in fact limited by the maximum allowed values for both quantum numbers n and m. The result given in Eq. (25) is an exact analytical expression for the density distribution in Fourier space valid for an arbitrary particle number. The effect of the anisotropy of the trapping potential can be seen in the arguments of the exponential and also in Laguerre polynomials. Notice that performing the inverse Fourier transform can yield to the particle density in the physical space.

As a test of our expression in Eq. (25), let us consider the case of rotation of isotropic potential where one has $\left(\omega_x=\omega_y=\omega_0\right)$. In this case, the various coefficients appearing in Eq. (25) reduce to [see Eqs. (9), (10), (12) and (13)]

$$\begin{array}{ll}\Omega_\pm = \omega_0\pm\Omega; & \mu_\pm = 2m^*\omega_0/(\omega_0\pm\Omega)\\ B = 1/2m^*\omega_0; & \lambda_\pm = \sqrt{\hbar/2m^*\omega_0} = a_{ho}/\sqrt{2}\end{array} \tag{26}$$

where $a_{ho} = \sqrt{\hbar/m^*\omega_0}$ is the oscillator length. Then the density distribution in Fourier space in Eq. (25) becomes

$$\begin{aligned}n(\mathbf{k}) = \exp\left(-\frac{a_{ho}^2\mathbf{k}^2}{4}\right)&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{L_n\left(\frac{a_{ho}^2\mathbf{k}^2}{4}\right)L_m\left(\frac{a_{ho}^2\mathbf{k}^2}{4}\right)\right.\\ &\left.\times\Theta(\mu-E_{nm})\vphantom{\frac{1}{1}}\right\}\end{aligned} \tag{27}$$

which is identical to the result found in [28], where E_{nm} are now given by

$$\begin{aligned}&E_{nm} = \hbar(n+1/2)(\omega_0+\Omega)+\hbar(m+1/2)(\omega_0-\Omega),\\ &n,m = 0,1,2\ldots\end{aligned} \tag{28}$$

We can now study the evolution of the behavior of the system using the shapes of the particle density in Fourier space, particularly in the limit situations, i.e. tight anisotropy and very fast rotations. In Figs. 1(a-f) displayed below, we observe the effect of the angular velocity on the form of the system that evolves from the 2D regime until entering to the 1D regime. This crossover is reflected in the shapes of the density profile. In fact, for a very fast rotation $\eta = 0.999$ and an anisotropy $\delta = 2$, the system appears as elongated along the k_y axis in the Fourier space as shown in Fig. 1(e). Also for the case of tight anisotropy $\delta = 10$ and fast rotation $\eta = 0.99$ as shown in Fig. 1(f) where the density is smoothed due to the strong confinement. Then the particles of the system are distributed along the x axis in the real physical space taking the form of an elliptical shape associated with the formation of occupied Landau levels.

4. Generalization to the 3D case

Let us suppose now that our system is in a 3D space and trapped by an anisotropic harmonic potential $V(\mathbf{r}) = m^*\left(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2\right)/2$, where $\left(\omega_x\neq\omega_y\neq\omega_z\right)$. When the trap is set in rotation around the z axis and in the positive sense with angular velocity $\mathbf{\Omega} = \Omega\mathbf{e}_z$, the quantum one-body hamiltonian in the rotating frame takes the form

$$H = \frac{1}{2m^*}\mathbf{p}^2+\frac{1}{2}m^*\left(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2\right)-\mathbf{\Omega}\cdot\mathbf{L} \tag{29}$$

The parameter that control the influence of the rotation angular velocity on the physical properties of the system is the same as in the 2D case, that is $\eta = \Omega/\omega_x$. The form of the Bloch density matrix of the system is in this case

$$\begin{aligned}C\left(\mathbf{r},\mathbf{r}';\xi\right) = G(\xi)&\exp\left[-(iA/\hbar B)\left(xy-x'y'\right)\right]\\ &\times\exp\left[a\left(x+x'\right)^2+a'\left(x-x'\right)^2+b\left(y+y'\right)^2\right.\\ &+b'\left(y-y'\right)^2+c\left(x+x'\right)\left(y-y'\right)\\ &\left.+c'\left(x-x'\right)\left(y+y'\right)+d\left(z+z'\right)^2+d'\left(z-z'\right)^2\right]\end{aligned} \tag{30}$$

where

$$G(\xi) = \left(\frac{m^*\omega_z}{2\pi\hbar\sinh(\xi\hbar\omega_z)}\right)^{1/2}g(\xi) \tag{31}$$

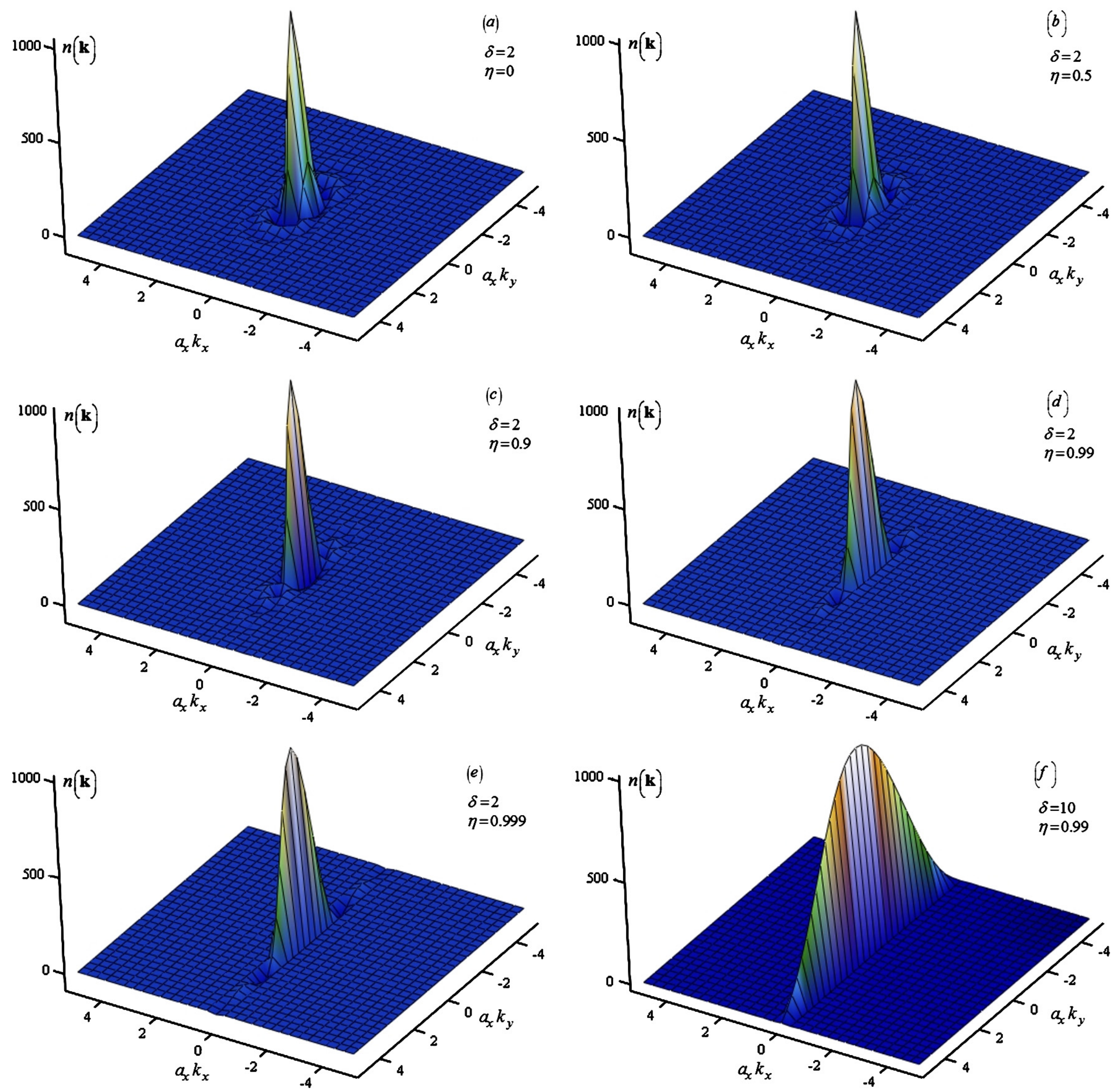

Fig. 1. Particle density in Fourier space for a 2D system containing $N = 1000$ non-interacting fermions in a rotating anisotropic harmonic trap for different values of the rotation ratio η (from non-rotating to ultra-fast rotating regime) at fixed anisotropy (*figures a-e*) and for tight anisotropy δ with fast rotation (*figure f*). The two components k_x and k_y of the momentum $\mathbf{k}$ are expressed in units of inverse of length $1/a_x$ where $a_x = \sqrt{\hbar/m^*\omega_x}$ is the oscillator length along the x axis.

and the two new coefficients d and d' are given in appendix. The energy levels of the Hamiltonian (29) are given as

$$E_{nm\ell} = \hbar\Omega_+ (n + 1/2) + \hbar\Omega_- (m + 1/2) + \hbar\omega_z (\ell + 1/2), \quad n, m, \ell = 0, 1, 2 \ldots \tag{32}$$

The Fourier transform of the local density is then extended to the 3D space. Following the same steps as was done in the precedent section, we arrive to the result

$$n(\mathbf{k}) = \exp\left\{-\frac{1}{4}\left[\left(1 + \frac{\hbar^2 B^2}{\lambda_+^2 \lambda_-^2}\right)\left(\lambda_-^2 k_x^2 + \lambda_+^2 k_y^2\right) + \lambda_z^2 k_z^2\right]\right\} \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\left\{L_n\left(\frac{\hbar^2 B^2 k_x^2}{2\lambda_+^2} + \frac{\lambda_+^2 k_y^2}{2}\right) \times L_m\left(\frac{\lambda_-^2 k_x^2}{2} + \frac{\hbar^2 B^2 k_y^2}{2\lambda_-^2}\right) L_\ell\left(\frac{\lambda_z^2 k_z^2}{2}\right)\Theta\left(\mu - E_{nm\ell}\right)\right\} \tag{33}$$

where, beside the lengths $\lambda_\pm$ given in Section 3, we have introduced the new length $\lambda_z = \sqrt{\hbar/2m^*\omega_z}$. In the next section, we will exploit the expression (33) for studying the elastic structure factor or the elastic scattering of the light from the fermion's cloud.

5. Elastic scattering function

Information on confined ultra-cold atomic systems can be obtained by imaging the gas during the free expansion of the gas which consists of switch-off, at certain initial time, the external potentials. Assuming a ballistic expansion, i.e.; neglecting the interactions during the free expansion, one can show that for long times, the properties of the expanded gas can be related to those of the initially confined gas [see for instance ref. [37]]. Another possibility to probe the confined gas is to study the scattering light from atoms of such gas (in situ). If the light frequency used to probe the cloud is off resonant, the scattering process involves two terms. The first is the diffraction of light by the finite size of the density of the gas, which does not result in excitations in the gas, and hence leaves it unchanged, this is the elastic scattering. The second is the inelastic part that is due to the excitations in the gas where transfer of energies between the light and particles occurs (the atom scatters between two different quasi-momentum states). If the light used is weak and far off resonance from the excitation of the internal atomic states of the gas, the elastic scattering entirely determines the intensity angular distribution, especially for the small values of the scattering angles [12], [31]. In fact, since we work under assumption that the temperature of the system is zero, the Pauli blocking effect, which is due to the exclusion principle, prohibits transition in a final states that are occupied, and will appear as a suppression of the photon scattering, predominantly at small angles where the small momentum transfer is less than the Fermi energy level. Therefore, we shall be concerned only by the elastic scattering contributions. An interesting quantity in this case is the static structure factor given by

$$S_{el}(\mathbf{k}) = \left| \int \exp(-i\mathbf{k}\cdot\mathbf{r})\, \rho(\mathbf{r})\, d\mathbf{r} \right|^2 = |n(\mathbf{k})|^2 \tag{34}$$

where $\mathbf{k}$ is the momentum transfer (recoil momentum) and will be defined further. Previous attempt to evaluate this function, for the system under study, has been made [12]. For this purpose, the authors of [12] have used the explicit form of the wave functions, found by direct resolution of the Schrödinger's equation through a factorization of the hamiltonian of the system, and performed a numerical calculation of this quantity. As a result of their approach, practical applications were limited to systems with a few atom number. In the present work, we give an exact analytical expression of the elastic structure factor $S_{el}(\mathbf{k})$ valid for an arbitrary number of atoms.

To study the angular dependence, let us suppose that the incident light beam is along the Oz axis, then $\mathbf{k}_i = k_i \mathbf{e}_z$. The light photons will be scattered from the gas in another direction defined by the spherical angles (θ, φ) with the final wave vector $\mathbf{k}_f$ resulting in a momentum change, that is $\mathbf{k} = \mathbf{k}_i - \mathbf{k}_f$. For an elastic diffusion, we have $\|\mathbf{k}_i\| = \|\mathbf{k}_f\| = k_0$, then the components of $\mathbf{k}$ along the x, y and z axis are respectively

$$k_x = -k_0 \cos\varphi \sin\theta; \quad k_y = -k_0 \sin\varphi \sin\theta; \quad k_z = k_0(1-\cos\theta) \tag{35}$$

Under these considerations, the expression of $S_{el}(\mathbf{k})$ is

$$\begin{aligned} S_{el}(\mathbf{k}) = \Bigg| \exp\Bigg\{ -\frac{1}{4}\Bigg[\left(1 + \frac{\hbar^2 B^2}{\lambda_+^2 \lambda_-^2}\right)\left(\lambda_-^2 \cos^2\varphi + \lambda_+^2 \sin^2\varphi\right)\sin^2\theta \\ + \lambda_z^2 (1-\cos\theta)^2 \Bigg] k_0^2 \Bigg\} \\ \times \sum_{n=0}^{n_{max}} \sum_{m=0}^{m_{max}} \sum_{\ell=0}^{\ell_{max}} \Bigg\{ L_n \left[\left(\frac{\hbar^2 B^2 \cos^2\varphi}{2\lambda_+^2} + \frac{\lambda_+^2 \sin^2\varphi}{2} \right) k_0^2 \sin^2\theta \right] \\ \times L_m \left[\left(\frac{\lambda_-^2 \cos^2\varphi}{2} + \frac{\hbar^2 B^2 \sin^2\varphi}{2\lambda_-^2} \right) k_0^2 \sin^2\theta \right] \\ \times L_\ell \left[\frac{\lambda_z^2 k_0^2 (1-\cos\theta)^2}{2} \right] \Bigg\} \Bigg|^2 \end{aligned} \tag{36}$$

where n_{max}, m_{max} and ℓ_{max} are the maximum allowed values for the quantum numbers n, m and ℓ. The angles (θ, φ) define the detector position relative to the incident beam. With different values of these angles one can draw the plots for the scattering factor, and the study of the quantum statistics effect, such the Pauli blocking effect, on the density distribution of the atomic gas can be done from Eq. (36).

6. Conclusion

In conclusion in this paper, we have given the correct expression of the Bloch density matrix associated to a one particle Hamiltonian of rotating anisotropic harmonic trap. Due to the experimental realization of traps that are strongly confining in one or two dimensions yielding to anisotropic traps and for practical use in ultra-cold quantum gases where atoms are confined in traps undergoing rotations or in a synthetic magnetic fields, we used this result of the Bloch density matrix to derive closed expression of the density distribution in Fourier space. Our expression in two dimensions is valid for arbitrary particle number and generalize previous results obtained for rotating isotropic traps. In ultra-fast rotating regime, a crossover from the two to the one-dimensional configuration was examined and depicted in the shape of the density profile in Fourier space corresponding to situations where the Lowest Landau Levels like appear as response of the gas to ultra-fast rotations [10], [13]. We have extended the study to the case of three dimensional rotating anisotropic quantum gas. We used the expression of the density in Fourier space in the context of light scattered by rotating gas to obtain the so-called elastic scattering factor.

A possible extension of the present work is to examine the intrinsic current distributions generated by the rotation of the anisotropic trap in a similar way as was done in ref. [14] for rotating isotropic trap. Due to the isotropic structure of the harmonic trap, the single particle wave functions used in [14] were expressed in polar coordinates to calculate the current distributions. In the case of anisotropic harmonic traps, the direct use of single particle wave functions is not suitable and we believe that the method used in the present work, based on the Bloch density matrix, is more appropriate in particular when the particle number is not small. Another possible extension is to obtain closed expression of the collective moment of inertia of rotating gas.

Appendix A

In this appendix, we display the coefficients appearing in the expressions of the Bloch density matrix (8) and (30)

$$a = -(4\hbar\gamma_1)^{-1}; \quad a' = -\left(4\hbar\gamma_1'\right)^{-1} \tag{A.1}$$

$$b = -(4\hbar\gamma_2)^{-1}; \quad b' = -\left(4\hbar\gamma_2'\right)^{-1} \tag{A.2}$$

$$c = \frac{i\coth(\xi\hbar\Omega_-/2)}{2\mu_-\Omega_- B\hbar\gamma_1}; \quad c' = \frac{i\tanh(\xi\hbar\Omega_-/2)}{2\mu_-\Omega_- B\hbar\gamma_1'} \tag{A.3}$$

$$d = -\frac{m^*\omega_z}{4\hbar}\tanh(\xi\hbar\omega_z/2); \quad d' = -\frac{m^*\omega_z}{4\hbar}\coth(\xi\hbar\omega_z/2) \tag{A.4}$$

where the coefficients γ_1 and γ_1' are respectively given by Eqs. (14) and (15), and the remaining coefficients are

$$\gamma_2 = \frac{\coth(\xi\hbar\Omega_+/2)}{\mu_+\Omega_+} + \mu_-\Omega_- B^2 \coth(\xi\hbar\Omega_-/2) \tag{A.5}$$

$$\gamma_2' = \frac{\tanh(\xi\hbar\Omega_+/2)}{\mu_+\Omega_+} + \mu_-\Omega_- B^2 \tanh(\xi\hbar\Omega_-/2) \tag{A.6}$$

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