PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
UNIVERSITY OF MOHAMED BOUDIAF /M'SILA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCES
DEPARTMENT OF MATHEMATICS

N° d'ordre:.....

THESIS

Presented for obtaining the diploma
of doctoral degree 3rd cycle LMD

Speciality
Mathematics

Option
Numerical Analysis, Partial Differential Equations and Applications

Presented by:

Imane Aouina

Theme

Existence and Uniqueness of Some Cauchy Type Problems in Fractional q-Difference Calculus

Thesis defended publicly on: July 19, 2026, before the jury composed of:

Benhamidouch Noureddine,	Prof.,	University of M'sila,	**Chair**
Arioua Yacine,	Prof.,	University of M'sila,	**Supervisor**
Basti Bilal,	MCA.,	University of M'sila,	**Co-Supervisor**
Saadi Abderachid,	Prof.,	NHSM,	**Examiner**
Gagui Bachir,	MCA.,	University of M'sila,	**Examiner**
Abdelkbir Saad,	MCA.,	University of M'sila,	**Examiner**

University Year: 2025/2026

Existence and Uniqueness of Some Cauchy Type Problems in Fractional q-Difference Calculus

Imane Aouina

Dedication

I dedicate this modest scientific work with all my gratitude

To my father and my mother.
To my brothers and sisters.
To my supervisors for providing me with valuable and attention in order to complete this research.

Imane Aouina

Acknowledgements

All praise be to Allah, Lord of the worlds, blessed and exalted; to him belongs perfection alone, and peace and blessings be upon our master Muhammad, his prophet, and his trustworthy Messenger.

I thank almighty Allah, who blessed me with completing this research.
And I extend my immense thanks and sincere gratitude and appreciation.
I would like to express my greatude
to the supervisor of my thesis, *Professor Dr*. ARIOUA YACINE.
I also extend my sincere thanks and appreciation to *Dr.* BILAL BASTI for the valuable advice and guidance he provided me during the preparation of this study.
I also extend thanks, respect, and appreciation to the
Honorable members of the defense committee:
Professor Dr. SAADI ABDERACHID, *Dr.* GAGUI BACHIR, *Dr.* ABDELKBIR SAAD
and the president of the committee, *Professor Dr.* BENHAMIDOUCHE NOUREDDINE.
I also thank my family for their support and assistance to me,
and I do not forget to thank you to all my honorable teachers who taught me during my university journey.

Table of contents

Introduction **1**

1 Preliminaries and Background Materials **7**
1.1 Basic Quantum Calculus Theory 8
1.1.1 Classical q-Calculus 8
1.1.2 Fractional q-Calculus 11
1.2 Some Fixed Point Theorems 22

2 ON n-NONLINEAR CAPUTO FRACTIONAL q-DIFFERENTIAL SYSTEMS **24**
2.1 Fundamental Lemma 25
2.2 Existence and Uniqueness Results 26
2.3 Ulam-Hyers Stability 31
2.4 Illustrative Examples 34

3 STABILITY ANALYSIS OF q-FRACTIONAL ORDER SYSTEMS VIA LYAPUNOV FUNCTION **37**
3.1 Fundamental Lemmas 37
3.2 Existence and Uniqueness Results 44
3.3 Applications 44

4 BOUNDARY VALUE PROBLEM FOR NONLINEAR FRACTIONAL q-DIFFERENTIAL EQUATIONS **47**
4.1 Fundamental Lemmas 48
4.2 Existence and Uniqueness Results 57
4.3 Ulam-Hyers Stability 59
4.4 Illustrative Examples 62

5 NONLINEAR FRACTIONAL q-DIFFERENTIAL EQUATIONS INVOLVING H-K q-DERIVATIVES **64**
5.1 Fundamental Lemmas 65
5.2 Existence and Uniqueness Results 69
5.3 Ulam-Hyers Stability 74
5.4 Illustrative Examples 77

Conclusion **81**

Bibliography **85**

Introduction

In recent years, with the end of the twentieth century and the beginning of the twenty first century, quantum mechanics and the revolution of quantum computers have-brought about a significant transformation in physics. These developments have evolved to include quantum dots, which has led mathematicians to develop quantum calculus along the traditional lines of classical calculus [1], also known as q-calculus, a branch of mathematics. It studies calculus without the notion of limits for a real parameter q. It was first introduced by EULER. Additionally, JACKSON was the first to introduce a new concept of the q-derivative and the definite q-integral in quantum calculus (see [2,4]), which later became influential in various disciplines such as mechanics, physics, and mathematics, including orthogonal polynomials, hypergeometric functions, number theory, and combinatorics (for more details, see [5, 6]).

Based on the q-extension of Cauchy's formula [7], AL–SALAM started the introduction of the concept of fractional q-calculus. Thereafter, AL–SALAM (see [8, 9]) and AGARWAL [10] developed the concept of certain fractional q-integrals and q-derivatives, where they proved the semigroup property for left and right RIEMANN-type fractional q-integrals. The concepts of the RIEMANN-LIOUVILLE and CAPUTO derivatives have been generalized to the q-form by means of the JACKSON derivative and fractional integrals. The CAPUTO q-derivative is considered the natural extension of the classical CAPUTO derivative, as it allows the imposition of initial conditions in terms of integer-order derivatives. On the other hand, the RIEMANN-LIOUVILLE derivative is mainly used in theoretical studies and functional analysis, particularly for proving fundamental properties such as existence, uniqueness, and semigroup properties. However, it is less suitable from the viewpoint of physical formulation of initial conditions.

Fractional calculus has attracted considerable attention from researchers due to its emerging applications in various fields of science and engineering, such as viscoelasticity, fluid me-

chanics, electrochemistry, and control. It is considered a branch of mathematics that studies derivatives and integrals of non-integer orders. Conversely, recent studies have confirmed that certain physical systems can achieve more accurate modelling using formulations of-fractional derivatives. For more details, one can refer to some of the books (see [11–14]).

Fractional-order differential equations have witnessed rapid development in many fields due to their intriguing results, particularly those related to the study of the existence, uniqueness, and stability properties of solutions. Analytical and numerical methods have encouraged researchers to explore numerous concepts and results, as well as to study applications in modelling real-world phenomena, such as control systems, fluid dynamics, and biological systems.

The theory of existence and uniqueness of solutions to fractional differential equations for initial and boundary value problems of fractional order in the CAPUTO or RIEMANN–LIOUVILLE sense and other fractional operators has been investigated recently (see [16–18]).

Lyapunov functions are functional tools used to prove the stability of fractional order systems, the two prominent works (see [19, 20]) can be cited, however the relation between the Lyapunov functions and fractional differential equation is neither elementary nor simple. Some Lyapunov functions have been proposed in works related to fractional sliding mode control [21]. Those Lyapunov functions have been used to prove the stability of resulting fractional order system. However. the analysis have possible, using the classical Lyapunov direct method.

In this context, fractional q-difference equations have received increasing attention in recent years. It has been shown that this class of equations has diverse applications in modelling complex systems that combine memory effects with discrete structures, as such features naturally arise in quantum physics. On the other hand, quantum calculus is considered the most suitable framework for describing discrete quantum phenomena, while fractional-calculus is well-suited for incorporating hereditary properties and modelling biological systems, which include the study of viscoelastic materials, anomalous diffusion, and biological systems. Several papers address the existence and uniqueness of solutions for q-fractional initial and boundary value problems (Refer to [22–28]).

This thesis aims to investigate several problems related to nonlinear fractional q-differential equations. It primarily focuses on the analysis of the existence, uniqueness, and stability of solutions to initial and boundary value problems within the framework of q-fractional derivatives of CAPUTO, KATUGAMPOLA, and HILFER-KATUGAMPOLA types. The adopted methodology consists of reformulating the considered problems as fixed-point problems by means of a suitable q-integral operator, utilizing classical fixed-point theorems such as BANACH, SCHAUDER and LERAY-SCHAUDER alternative [29]. Furthermore, stability is investigated using the concept of Ulam-Hyers stability (see [30, 31]). In addition, a Lemma for CAPUTO-type fractional q-derivatives has been investigated when $0 < \alpha < 1$, to establish the stability of many q-fractional order systems, which can be nonlinear and time-varying.

This thesis is organized as follows:

In the first chapter, we present some basic concepts of q-calculus and provide definitions and lemmas related to the theory of q-calculus and fractional q-calculus that are needed in this work, such as q-Gamma and q-Beta functions, which play an important role in the theory of fractional q-differential equations. We also recall classical the fixed point theorems, including those of BANACH, SCHAUDER AND LERAY-SCHAUDER alternative theorems. Following this, the CAPUTO, KATUGAMPOLA, and HILFER-KATUGAMPOLA, approaches to the generalization of the q-derivative concepts are examined.

In the second chapter, we study the existence, uniqueness and stability for a system of n-nonlinear fractional q-differential equations (q-FDEs)

$$\begin{cases} {}^{C}\mathcal{D}_{q}^{\alpha_1} u_1(t) = f_1(t, u(t)), \ t \in [0, T], \\ {}^{C}\mathcal{D}_{q}^{\alpha_2} u_2(t) = f_2(t, u(t)), \ t \in [0, T], \\ \vdots \\ {}^{C}\mathcal{D}_{q}^{\alpha_n} u_n(t) = f_n(t, u(t)), \ t \in [0, T], \end{cases}$$

with the initial conditions:

$$u_i\left(0^{+}\right) = \zeta_i \in \mathbb{R}, \text{ for each } i \in \{1, \ldots, n\},$$

where ${}^{C}\mathcal{D}_{q}^{\alpha_i}$ represents the CAPUTO fractional q-derivative of order α_i, for each $i \in \{1, \ldots, n\}$ with $0 < \alpha_i \leq 1$ and $u = (u_1, u_2, \ldots, u_n)$, for $n \in \mathbb{N}$. In addition $f_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for each $i \in \{1, \ldots, n\}$, with $T > 0$ and $0 < q < 1$.

In the third chapter, we focus on the study of the stability of q-fractional order systems (q-FOS) with initial condition.

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{q,t_0} u(t) = f(t, u(t)), & t \in T_q, \\ u(t_0) = u_0, & t_0 \geq 0, \end{cases}$$

where the operator ${}^{C}\mathcal{D}^{\alpha}_{q,t_0}$ represents the CAPUTO fractional q-derivative of order $0 < \alpha < 1$, with $0 < q < 1$ and $u, u_0 \in \mathbb{R}^n$. In addition $f : T_q \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous functions, for $t \geq t_0$, and $t_0 \in T_q$, where $T_q = \{q^n t_0,\ n \in \mathbb{N}\} \cup \{0\}$. which represents a q-geometric time scale.

In the fourth chapter, we address the study of boundary value problem for a nonlinear fractional q-differential equation involving the KATUGAMPOLA q-derivative.

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha}_{q,a}\, u(t) + f(t, u(t)) = 0, & t \in [a, b], b > a > 0 \\ u(a) = 0, \quad {}^{\rho}\mathcal{J}^{\eta}_{q,a} u(b) = 0, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{\alpha}_{q,a}$, ${}^{\rho}\mathcal{J}^{\eta}_{q,a}$ represent the KATUGAMPOLA fractional q-derivative and q-integral of order α and η respectively, with $1 < \alpha \leq 2$, $0 < \eta \leq 1$, $\rho > 0$ and $0 < q < 1$. In addition $f : [a, b] \times \mathbb{C} \to \mathbb{C}$ be a continuous function.

In the fifth chapter, it is devoted to the study of nonlinear fractional q-differential equations involving HILFER-KATUGAMPOLA q-derivatives of moving orders in the Banach space $\mathcal{L}^1_{q,\rho}([a, b], \mathbb{C})$ with initial conditions.

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[u(t) - h(t)] = f\left(t, u(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[u(t) - h(t)]\right), & t \in \Omega = [a, b], b > a > 0, \\ {}^{\rho}\mathcal{D}^{\gamma-1}_{q,a^+} u(a) = \omega \in \mathbb{C}, \text{ and } {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+} u(a) = 0, & \text{for } k = 2, 3, \ldots, n, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}$ and ${}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}$ represent the q-analogue of HILFER-KATUGAMPOLA fractional derivatives of order α and β respectively and the operator ${}^{\rho}\mathcal{D}^{\alpha}_{q,a^+}$ represents the KATUGAMPOLA fractional q-derivative of order α, with $0 \leq \eta \leq 1$, $m - 1 < \beta \leq m \in \mathbb{N}$, $\theta = \beta + \eta(m - \beta)$, $\max\{\theta, n - 1\} < \alpha \leq n$ for $n \in \mathbb{N} - \{1\}$, and $\gamma = \alpha + \eta(n - \alpha)$, and $f : \Omega \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear complex-valued function.

List of abbreviations and symbols

List of symbols of q-calculus and q-fractional calculus.

Sets

$\mathbb{N}$	Nonzero natural numbers $\{1, 2, 3, \ldots\}$.
$\mathbb{N}_0$	Natural numbers $\mathbb{N} \cup \{0\}$.
$\mathbb{Z}$	Integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
$\mathbb{Z}_0^-$	Nonpositive integers $\{\ldots, -3, -2, -1\} \cup \{0\}$.
$\mathbb{R}$	Real numbers $(-\infty, \infty)$.
$\mathbb{R}_+$	Nonnegative real numbers $(0, \infty)$.
$\mathbb{R}^*$	Nonzero real numbers $(-\infty, 0) \cup (0, \infty)$.
$\mathbb{C}$	Complex numbers, $z \in \mathbb{C}$, then $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.

Functions Spaces

$C(\Omega)$	BANACH space of all continuous functions u on $\Omega \subset \mathbb{R}$, with the norm $\|u\| = \sup_{t \in \Omega} \|u(t)\|$.
$\mathcal{L}_{q,\rho}^p([a,b])$	BANACH space of all the functions u defined on $[a,b]$, for $\rho \in \mathbb{R}$, and $1 \leq p < \infty$, with the norm, $\|u\|_{\mathcal{L}_{q,\rho}^p} = \sup_{t \in [a,x]} \left(\int_a^x t^{\rho-1} \|u(t)\|^p \, d_q t \right)^{\frac{1}{p}} < \infty$.
$\mathcal{L}_{q,\rho}^1([a,b])$	BANACH space of all the functions u defined on $[a,b]$, for $\rho \in \mathbb{R}$, with the norm $\|u\|_{\mathcal{L}_{q,\rho}^1} = \int_a^b t^{\rho-1} \|u(t)\| \, d_q t < \infty$.

Concepts and Spacial Functions

$[n]_q$	The q-analogue of n or (q-bracket of n).
$[n]_q!$	The q-factorial of n.
$(a;q)_k$	The q-POCHHAMMER symbol (q-shifted factorial), $k \in \mathbb{N}$.
$\Gamma_q(\cdot)$	The q-analogue of the EULER gamma function.
$B_q(\cdot,\cdot)$	The q-analogue of the EULER beta function.

Operators

D_q	JACKSON q-derivative or q-difference operators.
D_q^n	JACKSON q-derivative of higher order $n,\ n \in \mathbb{N}$.
$\mathcal{I}_q$	JACKSON q-integral operator.
$\mathcal{I}_q^n$	q-Cauchy formula for the n^{th} integrals, $n \in \mathbb{N}$.
$\mathcal{I}_q^\alpha u$	RIEMANN-LIOUVILLE fractional q-integral of order α.
$\mathcal{D}_q^\alpha u$	RIEMANN-LIOUVILLE fractional q-derivative of order α.
${}^C\mathcal{D}_q^\alpha u$	CAPUTO fractional q-derivative of order α.
${}^\rho\mathcal{I}_q^\alpha u$	KATUGAMPOLA fractional q-integral of order α.
${}^\rho\mathcal{D}_q^\alpha u$	KATUGAMPOLA fractional q-derivative of order α.
$\mathcal{D}_q^{\alpha,\eta} u$	HILFER fractional q-derivative of oder α and η.
${}^\rho\mathcal{D}_q^{\alpha,\eta} u$	HILFER-KATUGAMPOLA fractional q-derivatives of order α and η.

H-K	HILFER-KATUGAMPOLA.
q-FDEs	q-Fractional Differential Equations.
q-FOS	q-Fractional Order Systems.

CHAPTER 1

PRELIMINARIES AND BACKGROUND MATERIALS

This chapter deals with the presentation of fundamental definitions and basic concepts related to q-calculus and q-fractional calculus theory such as the (q-integers, q-factorial, q-shifted, JACKSON q-difference and q-integral operators). In addition, we will define the special functions such as Euler q-Gamma and q-Beta functions, including the fractional q-derivation, fractional q-integration, relative definitions of operators of q-fractional order. Furthermore, it will introduce several elements of functional analysis that will be necessary to this work.

Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$. For a real number ρ, and a positive number p, we define the space $\mathcal{L}^p_{q,\rho}([a, b])$ to be the space of all the functions u defined on $[a, b]$ satisfying

$$\|u\|_{\mathcal{L}^p_{q,\rho}} = \sup_{x\in[a,b]} \left(\int_a^x t^\rho |u(t)|^p \, d_q t \right)^{\frac{1}{p}} < \infty.$$

For $p = 1$ and $\rho \in \mathbb{R}$, $0 < q < 1$, we denote the Banach space by $\mathcal{L}^1_{q,\rho}([a, b])$, for which

$$\mathcal{L}^1_{q,\rho}([a, b]) = \left\{ u; \int_a^b t^\rho |u(t)| \, d_q t < \infty \right\},$$

associated with th enorm

$$\|u\|_{\mathcal{L}^1_{q,\rho}} = \int_a^b t^\rho |u(t)| \, d_q t.$$

Let $C(\Omega)$ is a Banach space of all continuous functions on Ω, associated with the norm

$$\|u\| = \sup_{t\in\Omega} |u(t)|.$$

Let $C_q^n(\Omega)$ be the space of all continuous functions with continuous q-derivatives up to order $n-1$ on the interval Ω.
The space $C_q^n(\Omega)$ is a Banach space associated with the norm

$$\|u\| = \sum_{k=0}^{n-1} \max_{t\in\Omega} \left|D_q^k u(t)\right|,\ u \in C_q^n(\Omega).$$

1.1 Basic Quantum Calculus Theory

1.1.1 Classical q-Calculus

In this section, we present some important definitions of q-calculus for a real parameter q, which include the q-integer, q-factorial, q-shifted, JACKSON q-difference and q-integral operators, that will be used in this work.

Definition 1.1 ([6]). For $n \in \mathbb{N}$ and $q > 0$. The q-integer of n is defined by

$$[n]_q = \begin{cases} n, & q = 1, \\ \frac{1-q^n}{1-q}, & q \neq 1, \end{cases}$$

where

$$\frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},$$

for any positive integer n, this is called the q-analogue of n.
We can give this definition for any q-real or q-complex number.

Example 1.1. For $n \in \mathbb{N}$, we present the values of q-integer as follows:

1. $[1]_q = \frac{1-q}{1-q}$.

2. $[2]_q = \frac{1-q^2}{1-q} = \frac{(1-q)(1+q)}{(1-q)} = 1 + q$.

3. $[3]_q = \frac{1-q^3}{1-q} = \frac{(1-q)\left(1+q+q^2\right)}{(1-q)} = (1 + q + q^2)$.

Definition 1.2 ([6]). The q-factorial of n is defined by

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [n]_q [n-1]_q \dots [1]_q, & \text{if } n \geq 1, \end{cases}$$

where n is any positive integer and $q > 0$.

Example 1.2. For $n \in \mathbb{N}$, we present the values of q-factorial as follows:

1. $[1]_q! = [1]_q = \frac{1-q}{1-q} = 1.$

2. $[2]_q! = [2]_q [1]_q = \frac{1-q^2}{1-q} = 1 + q = 2, \text{ if } q = 1.$

3. $[3]_q! = [3]_q [2]_q [1]_q = \frac{1-q^3}{1-q} \frac{1-q^2}{1-q} = \frac{(1-q)^2\left(1+q+q^2\right)(1+q)}{(1-q)^2} = (1 + q + q^2)(1 + q) = 6, \text{ if } q = 1.$

Definition 1.3 ([32]). The q-analogue of the POCHHAMMER symbol (q-shifted factorial), is defined by

$$(a;q)_k = \begin{cases} 1, & k = 0, \\ \prod_{i=0}^{k-1} (1 - q^i a), & k \in \mathbb{N} \cup \{\infty\}. \end{cases}$$

Definition 1.4 ([32]). The q-analogue of $(a-b)^k$, is defined by

$$(a-b)_q^{(k)} = \begin{cases} 1, & k = 0, \\ \prod_{i=0}^{k-1} (a - q^i b), & k \in \mathbb{N}. \end{cases}$$

Definition 1.5 ([32]). For α is a real number, the q-shifted factorial is defined by

$$(a-b)_q^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{1 - q^i \frac{b}{a}}{1 - q^{\alpha+i} \frac{b}{a}}, \ a, b \in \mathbb{R}.$$

We can give this definition for α is a complex number.

We now present the JACKSON q-difference and q-integral operators.

Definition 1.6 ([4]). The q-derivatives or (q-difference operator) of a function u, where $0 < q < 1$, is defined by

$$D_q u(t) = \begin{cases} \frac{u(t)-u(qt)}{(1-q)t}, & t \neq 0, \\ \grave{u}(0), & t = 0. \end{cases}$$

Note that

$$\lim_{q\to 1} D_q u(t) = \lim_{q\to 1} \frac{u(t) - u(qt)}{(1-q)t} = \frac{du(t)}{dt},$$

if u is differentiable at t.

Example 1.3. Consider the function $u(t) = t^n, n \in \mathbb{N}$, then the q-derivative of the function $u(t)$ as follows

$$\begin{aligned} D_q(t^n) &= \frac{d_q(t^n)}{d_q t} = \frac{t^n - (qt)^n}{(1-q)t} \\ &= \frac{(1-q^n)t^n}{(1-q)t} \\ &= \frac{1-q^n}{1-q} t^{n-1} \\ &= [n]_q t^{n-1}. \end{aligned}$$

Property 1.1. The following relations of q-derivative operators is valid

1. The q-derivative for any arbitrary functions $u(t)$ and $v(t)$ is linear operator and for any constant a and b

$$D_q(au(t) \pm bv(t)) = aD_q u(t) \pm bD_q v(t).$$

2. The q-derivative of a product at $t \neq 0$

$$D_q(u(t) \cdot v(t)) = v(t) D_q u(t) + u(qt) D_q v(t).$$

3. The q-derivative of a quotient at $t \neq 0$

$$D_q\left(\frac{u(t)}{v(t)}\right) = \frac{v(qt) D_q u(t) - u(qt) D_q v(t)}{v(t) v(qt)}, \; v(t) v(qt) \neq 0.$$

4. The q-analogue of the rule of integration by parts is defined by

$$\int_a^b u(t) (D_q v)(t) \, d_q t = u(t) v(t)]_a^b - \int_a^b v(qt) (D_q u)(t) \, d_q t.$$

Corollary 1.1 ([3]). The q-derivatives of the q-factorial function with respect to t is

$$\begin{aligned} D_q\left[(t-b)_q^{(\alpha)}\right] &= [\alpha]_q (t-b)_q^{(\alpha-1)}, \\ D_q\left[(a-t)_q^{(\alpha)}\right] &= -[\alpha]_q (a-qt)_q^{(\alpha-1)}. \end{aligned}$$

Definition 1.7 ([2]). The q-integral (JACKSON integral) of a function u is defined by

$$\int_0^t u(s)\, d_q s = (1-q)\, t \sum_{i=0}^{\infty} q^i u\left(tq^i\right),\ t \in \mathbb{R},$$

as $t \to \infty$, is given by

$$\int_0^{\infty} u(s)\, d_q s = (1-q) \sum_{i=-\infty}^{\infty} q^i u\left(q^i\right),$$

and the q-integral or (JACKSON integral) of a function u on $[a,b]$, is defined by

$$\int_a^b u(s)\, d_q s = \int_0^b u(s)\, d_q s - \int_0^a u(s)\, d_q s. \tag{1.1}$$

Definition 1.8 ([2]). The n^{th} q-integral operator is defined by

$$\mathcal{I}_{q,a}^n u = \begin{cases} u, & n = 0, \\ \mathcal{I}_{q,a}\mathcal{I}_{q,a}^{n-1} u, & n \in \mathbb{N}. \end{cases}$$

Definition 1.9 ([4]). The q-derivatives of higher order, is defined by

$$D_q^n u = \begin{cases} u, & n = 0, \\ D_q^{n-1} D_q u, & n \in \mathbb{N}. \end{cases}$$

Property 1.2. The following rules concerning the relation between q-integral and q-derivative.

1. $\left(D_q \mathcal{I}_{q,a} u\right)(t) = u(t)$.
2. $\left(\mathcal{I}_{q,a} D_q u\right)(t) = \int_a^b D_q u(s)\, d_q s = u(b) - u(a)$.
3. $\left(D_q^n \mathcal{I}_{q,a}^n u\right)(t) = u(t),\ n \in \mathbb{N}$.
4. $\left(\mathcal{I}_{q,a}^n D_q^n u\right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{\left(D_q^k u\right)(a)}{[k]_q!} (t-a)_q^k,\ n \in \mathbb{N}$.

1.1.2 Fractional q-Calculus

In this section, we present the important concepts in the context of q-fractional calculus, which is an extension of the classical fractional calculus. We give some definitions concerning fractional q-integrals and q-derivatives operators and give certain properties of these concepts. We first begin by introducing the special functions of q-fractional calculus.

Special Functions

In what follows, we present the q-analogue of the EULER Gamma and Beta functions. The EULER q-Gamma function was introduced by THOMAE [33] and later by JACKSON [34], which is an extension of classical EULER Gamma function, which generalizes the q-factorial $[n]_q!$ and allows $[n]_q$ to take also non-integer and even complex values.

Definition 1.10 (Euler q-Gamma function [35])**.** The EULER q-Gamma function $\Gamma_q(\varsigma)$ defined by

$$\Gamma_q(\varsigma) = \frac{(q;q)_\infty}{(q^\varsigma;q)_\infty}(1-q)^{1-\varsigma},\ 0<|q|<1.$$

where $\varsigma \in \mathbb{C}\backslash\{-n,\ n\in\mathbb{N}_0\}$.

Here, we take the principal values of q^ς and $(1-q)^{1-\varsigma}$, then $\Gamma_q(\varsigma)$ is a meromorphic function with poles at $\varsigma=-n$, $n\in\mathbb{N}_0$.

Since $\Gamma_q(\varsigma)$ has nonzeros, then $\frac{1}{\Gamma_q(\varsigma)}$ is an entire function with zeros at $\varsigma=-n$, $n\in\mathbb{N}_0$.

Definition 1.11 (Euler q-Gamma Function [35])**.** The EULER q-Gamma function $\Gamma_q(\varsigma)$ defined by

$$\Gamma_q(\varsigma) = \int_0^\infty t^{\varsigma-1}E_q^{-qt}d_qt, \tag{1.2}$$

for any $\varsigma>0$.

The basic properties of the q-Gamma function is satisfies

1. From definition of the q-Gamma function (1.10), and use the q-Pochhammer symbol, we get another expression of the q-Gamma function. We have

$$\Gamma_q(\varsigma) = \frac{(q;q)_\infty}{(q^\varsigma;q)_\infty}(1-q)^{1-\varsigma},\ 0<|q|<1,$$

where $(q;q)_\infty$ is called the q-Pochhammer symbol satisfies $(q;q)_\infty = (q;q)_k\left(q^{k+1};q\right)_\infty$, with taking $k=\varsigma-1$, obviously

$$\Gamma_q(\varsigma) = \frac{(q;q)_{\varsigma-1}}{(1-q)^{\varsigma-1}} = (1-q)^{(\varsigma-1)}(1-q)^{1-\varsigma}. \tag{1.3}$$

For any nonnegative integer n, we take $\varsigma=n$, we have

$$\Gamma_q(n) = \frac{(q;q)_{n-1}}{(1-q)^{n-1}},\ n\in\mathbb{N}.$$

2. For $\varsigma > 0$, the q-Gamma function is the unique logarithmically convex function satisfies the function equation. Using (1.2) and q-integration by parts, we have

$$\begin{aligned}\Gamma_q(\varsigma+1) &= \int_0^\infty t^\varsigma E_q^{-qt} d_q t = -\int_0^\infty t^\varsigma d_q E_q^{-t}, \\ &= -\left[t^\varsigma E_q^{-t}\right]_0^\infty + \int_0^\infty D_q\left(t^\varsigma\right) E_q^{-qt}, \\ &= [\varsigma]_q \int_0^\infty t^{\varsigma-1} E_q^{-qt} d_q t, \\ &= [\varsigma]_q \Gamma_q(\varsigma).\end{aligned}$$

Since

$$\Gamma_q(1) = \int_0^\infty E_q^{-qt} d_q t = -\int_0^\infty d_q E_q^{-t} = E_q^0 - E_q^{-\infty} = 1,$$

for any nonnegative integer n, we have

$$\Gamma_q(n+1) = [n]_q!,$$

it is one of the basic functions of fractional calculus. This function plays an important role when combined with the Gamma function.

Example 1.4. We present the values of the q-Gamma function as follows:

1. $\Gamma_q(1) = \frac{(q;q)_0}{(1-q)^0} = \frac{(1-q)^{(0)}}{(1-q)^0} = \frac{1}{1} = 1.$

2. $\Gamma_q(2) = \frac{(q;q)_1}{(1-q)^1} = \frac{\prod_{i=0}^{0}\left(1-q^{i+1}\right)}{(1-q)} = \frac{1-q}{1-q} = 1.$

3. $\Gamma_q(3) = \frac{(q;q)_2}{(1-q)^2} = \frac{\prod_{i=0}^{1}\left(1-q^{i+1}\right)}{(1-q)^2} = \frac{(1-q)\left(1-q^2\right)}{(1-q)^2} = \frac{1-q^2}{1-q} = 1+q.$

From this, we conclude that the q-Gamma $\Gamma_q(n)$ generalizes the q-analogue of the factorial, which mean $\Gamma_q(n) = [n-1]_q!$.

Definition 1.12 (EULER q-Beta Function [35])**.** For a positive values of the two parameters ς and υ, the q-Beta function of EULER q-integral is defined by

$$B_q(\varsigma,\upsilon) = \int_0^1 t^{\varsigma-1}(1-qt)_q^{\upsilon-1} d_q t, \ \left(\operatorname{Re}(\varsigma) > 0, \ \operatorname{Re}(\upsilon) > 0\right).$$

The q-Gamma and q-Beta functions are related to each other by the following relation

$$B_q(\varsigma, \upsilon) = \frac{\Gamma_q(\varsigma)\,\Gamma_q(\upsilon)}{\Gamma_q(\varsigma+\upsilon)}.$$

Fractional Quantum Operators

The definition of fractional integrals is motivated by beginning with an n-fold iterated integral and demonstrating that it can be rewritten as a single integral involving the parameter n, and generalize by replacing the integer n with a positive real order $\alpha > 0$.

A similar idea holds in q-calculus, using the JACKSON integral instead of the classical integral. AL-SALAM [7] defined this q-analogue as an extension of the following Cauchy formula.

Let $u : [a,b] \to \mathbb{R}$, be a continuous function on $[a,b]$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$.

We first begin by noting the JACKSON integral by

$$\mathcal{I}_a^1 u(t) = \int_a^t u(s)\, d_q s = (t-a)(1-q)\sum_{j=0}^{\infty} q^j u\left(tq^j\right).$$

The iterated of JACKSON integral allows to obtain the multiple q-integral by

$$\begin{aligned}\int_a^t \int_a^\tau u(s)\, d_q\tau d_q s &= \int_a^t \int_{qs}^t u(s)\, d_q\tau d_q s \\ &= \int_a^t u(s)\left(\int_{qs}^t d_q\tau\right) d_q s \\ &= \int_a^t (t-qs)\, u(s)\, d_q s \\ &= (t-a)(1-q)\sum_{j=0}^{\infty} q^j u\left(tq^j\right)\left(t-q^j a\right).\end{aligned} \tag{1.4}$$

If $n \geq 1$, is a given integer

$$\begin{aligned}\mathcal{I}_a^n u(t) &= \int_a^t \int_a^{t_{n-1}} \cdots \int_a^{t_1} u(s)\, d_q s d_q t_1 \cdots d_q t_{n-1} \\ &= \frac{1}{[n-1]_q!}\int_a^t (t-qs)_{n-1}\, u(s)\, d_q s,\ n \in \mathbb{N}.\end{aligned} \tag{1.5}$$

In quantum fractional calculus, an q-integral of arbitrary order $\alpha > 0$, is obtained by generalizing the q-analogue of the Cauchy formula (1.5), is replaced by a real number α, and used the EULER q-Gamma function instead of the q-factorial.

We now define the q-analogue of the left sided RIEMANN-LIOUVILLE integral by [36] following [10] and derivative of the following definitions

Definition 1.13 (The R-L fractional q-integral [10])**.** The left-sided RIEMANN-LIOUVILLE fractional q-integral of order $\alpha \geq 0$ and u be a function defined on $[a, b]$ is defined by

$$\mathcal{I}_{q,a}^{\alpha} u(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{(\alpha-1)} u(s) \, d_q s, \ \alpha > 0,$$

and $\mathcal{I}_{q,a}^{0} u(t) = u(t)$.

Definition 1.14 (The R-L fractional q-derivative [10])**.** The left-sided RIEMANN-LIOUVILLE fractional q-derivative of order $\alpha > 0$ and u be a function defined on $[a, b]$ is defined by

$$\mathcal{D}_{q,a}^{\alpha} u(t) = D_q^m \mathcal{I}_{q,a}^{m-\alpha} u(t),$$

and $\mathcal{D}_{q,a}^{0} u(t) = u(t)$. Where m is the smallest integer greater than or equal to α.

Property 1.3 ([10, 37])**.** Let $\alpha, \beta > 0$, and u be a function defined on $[a, b]$, then the following identity holds

1. $\mathcal{I}_{q,a}^{\alpha} \mathcal{I}_{q,a}^{\beta} u(t) = \mathcal{I}_{q,a}^{\alpha+\beta} u(t)$.
2. $\mathcal{D}_{q,a}^{\alpha} \mathcal{I}_{q,a}^{\alpha} u(t) = u(t)$.

Property 1.4 ([15])**.** For $\alpha, \beta \in \mathbb{R}^+$ and $(0 < a < t < b)$, consider the function $u(t) = (t - a)^{(\beta-1)}$, we have

$$\begin{aligned} \left(\mathcal{I}_{q,a}^{\alpha} u\right)(t) &= \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)} (t - a)^{(\alpha+\beta-1)}, \\ \left(\mathcal{D}_{q,a}^{\alpha} u\right)(t) &= \frac{\Gamma_q(\beta)}{\Gamma_q(\beta - \alpha)} (t - a)^{(\beta-\alpha-1)}. \end{aligned}$$

Proof. For $\beta > 0$, according to the Definition of the formula (1.1), we have

$$\begin{aligned}\mathcal{I}_{q,a}^{\alpha}\left[(t-a)^{(\beta-1)}\right] &= \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{(\alpha-1)}(s-a)^{(\beta-1)}\,d_qs \\ &= \frac{1}{\Gamma_q(\alpha)}\left[\int_0^t (t-qs)_q^{(\alpha-1)}(s-a)^{(\beta-1)}\,d_qs\right. \\ &\quad \left.-\int_0^a (t-qs)_q^{(\alpha-1)}(s-a)^{(\beta-1)}\,d_qs\right].\end{aligned} \tag{1.6}$$

Also, the following equality is valid

$$\int_0^a (t-qs)_q^{(\alpha-1)}(s-a)^{(\beta-1)}\,d_qs = a^{\beta}(1-q)\sum_{k=0}^{\infty}\left(t-aq^{k+1}\right)^{(\alpha-1)}\left(q^k-1\right)^{(\beta-1)}q^k = 0,$$

and we have,

$$\int_0^t (t-qs)_q^{(\alpha-1)}(s-a)^{(\beta-1)}\,d_qs = t^{\alpha+\beta-1}(1-q)\sum_{k=0}^{\infty}\left(1-q^{k+1}\right)^{(\alpha-1)}\left(1-\frac{a}{qt}q^{1-k}\right)^{(\beta-1)}q^{\beta k},$$

by using (Lemma 16, see [15]), we get

$$\int_0^t (t-qs)_q^{(\alpha-1)}(s-a)^{(\beta-1)}\,d_qs = (1-q)\frac{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}}{(1-q)^{(\alpha+\beta-1)}}(t-a)^{(\alpha+\beta-1)},$$

applying (1.3), we obtain

$$\begin{aligned}\int_0^t (t-qs)_q^{(\alpha-1)}(s-a)^{(\beta-1)}\,d_qs &= (1-q)\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{(1-q)^{2-\alpha-\beta}}\frac{(1-q)^{1-\alpha-\beta}}{\Gamma_q(\alpha+\beta)}(t-a)^{(\alpha+\beta-1)} \\ &= \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}(t-a)^{(\alpha+\beta-1)},\end{aligned}$$

by substituting in (1.6), the required formula is thus obtained.
For fractional q-derivative, we have

$$\begin{aligned}\mathcal{D}_{q,a}^{\alpha}\left[(t-a)^{(\beta-1)}\right] &= D_q\left(\mathcal{I}_{q,a}^{1-\alpha}(t-a)^{(\beta-1)}\right) \\ &= D_q\left[\frac{1}{\Gamma_q(1-\alpha)}\int_a^t (t-qs)_q^{(-\alpha)}(s-a)^{(\beta-1)}\,d_qs\right] \\ &= D_q\left[\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{(\beta-\alpha)}\right] \\ &= \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)}(t-a)^{(\beta-\alpha-1)}.\end{aligned}$$

□

Particularly, for $\beta = 1$, and $\alpha > 1$, by using a q-integration by parts, then RIEMANN-LIOUVILLE'S fractional q-integral and q-derivative of a constant is

$$\begin{aligned}
\mathcal{I}_{q,a}^{\alpha}(1) &= \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{(\alpha-1)}\, d_q s \\
&= \frac{-1}{[\alpha]_q \Gamma_q(\alpha)}\int_a^t {}_sD_q\,(t-s)_q^{(\alpha)}\, d_q s \\
&= \frac{-1}{\Gamma_q(\alpha+1)}\left[(t-s)_q^{(\alpha)}\right]_a^t + \int_a^t (t-qs)_q^{\alpha} D_q(1)\, d_q s \\
&= \frac{1}{\Gamma_q(\alpha+1)}(t-a)_q^{(\alpha)}.
\end{aligned}$$

And

$$\mathcal{D}_{q,a}^{\alpha}(1) = \frac{1}{\Gamma_q(1-\alpha)}(t-a)_q^{(-\alpha)}.$$

The q-analogue of the Caputo Fractional Derivative

The CAPUTO derivative is one of the most widely used Definitions, based in the JACKSON derivative and q-fractional integrals, the CAPUTO q-derivative can be regarded as a natural extension of the classical CAPUTO derivative, which is given by

Definition 1.15 (The Caputo fractional q-derivative [38])**.** The q-analogue of the left-sided CAPUTO fractional derivative of order α of a function $u \in C^n([a,b],\mathbb{R})$, is defined by

$$\begin{aligned}
{}^{C}\mathcal{D}_q^{\alpha}u(t) &= \mathcal{I}_{q,a}^{n-\alpha}D_q^n u(t), \\
&= \frac{1}{\Gamma_q(n-\alpha)}\int_a^t (t-qs)_q^{(n-\alpha-1)} D_q^n u(s)\, d_q s,
\end{aligned}$$

and $n = \lceil \alpha \rceil$.

Lemma 1.1 ([38])**.** *Let α be a positive number and let $n = \lceil \alpha \rceil$, then the following equality holds*

$$\mathcal{I}_{q,a}^{\alpha}\,{}^{C}\mathcal{D}_{q,a}^{\alpha}u(t) = u(t) - \sum_{k=0}^{n-1}\frac{D_q^k u(a^+)}{\Gamma_q(k+1)}(t-a)^k,\ n-1 < \alpha \le n \in \mathbb{N}.$$

If $0 < \alpha \le 1$, we have

$$\mathcal{I}_{q,a}^{\alpha}\,{}^{C}\mathcal{D}_{q,a}^{\alpha}u(t) = u(t) - u(a).$$

Lemma 1.2 ([39]). *Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. Assume that $u \in C^n[a,b]$ and $D_q^n u \in C[a,b]$, then the following relations between* RIEMANN-LIOUVILLE *fractional q-derivative and* CAPUTO *fractional q-derivative, is given by*

$$ {}^{C}\mathcal{D}_{q,a}^{\alpha} u(t) = \mathcal{D}_{q,a}^{\alpha} \left(u(t) - \sum_{k=0}^{n-1} \frac{D_q^k u(a^+)}{\Gamma_q(k+1)} (t-a)^k \right), \ t \neq 0. $$

Also,

$$ {}^{C}\mathcal{D}_{q,a}^{\alpha} u(t) = \mathcal{D}_{q,a}^{\alpha} u(t), \text{ if } D_q^k u\left(a^+\right) = 0, \ (k = 0, 1, \ldots, n-1). $$

The q-analogue of Katugampola Fractional Integral and Derivative Operators

In the context of quantum fractional calculus. The operators KATUGAMPOLA fractional q-integral ${}^{\rho}\mathcal{J}_{q,a}^{\alpha}$ and q-derivative ${}^{\rho}\mathcal{D}_{q,a}^{\alpha}$ are q-extension of calssical generalized fractional integral and derivative operators. The KATUGAMPOLA fractional q-integral combines the q-analogue of RIEMANN-LIOUVILLE and HADAMARD fractional integral into a single form. It is presented in the following Definition.

Definition 1.16 (The Katugampola Fractional q-Integral [41, 43])**.** Let $\alpha > 0$, $\rho > 0$ and $0 < q < 1$. The left-sided KATUGAMPOLA fractional q-integral of order α of the function $u \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C})$ is defined by

$$ {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha} u(t) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \int_a^t s^{\rho-1} \left(t^\rho - (qs)^\rho\right)_{q^\rho}^{(\alpha-1)} u(s) \, d_q s. \tag{1.7} $$

Definition 1.17 (The Katugampola Fractional q-Derivative [42, 43])**.** Let $n-1 < \alpha \leq n \in \mathbb{R}$, $\rho > 0$ and $0 < q < 1$. The left-sided KATUGAMPOLA fractional q-derivative of the order α of the function $u \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C})$ is defined by

$$ \begin{aligned} {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha} u(t) &= {}^{\rho}\delta_q^n \, {}^{\rho}\mathcal{J}_q^{n-\alpha} u(t) \\ &= \left(t^{1-\rho} D_q\right)^n {}^{\rho}\mathcal{J}_q^{n-\alpha} u(t). \end{aligned} \tag{1.8} $$

It can also be written as follows

$$ {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha} u(t) = \frac{[\rho]_q^{1-n+\alpha}}{\Gamma_{q^\rho}(n-\alpha)} \left(t^{1-\rho} D_q\right)^n \int_a^t s^{\rho-1} \left(t^\rho - (qs)^\rho\right)_{q^\rho}^{(n-\alpha-1)} u(s) \, d_q s. \tag{1.9} $$

The KATUGAMPOLA fractional q-derivative is its reduction to the identity operator when

$\alpha = 0$, i.e., $\left({}^{\rho}\mathcal{D}^{0}_{q,a^{+}}u\right)(t) = u(t)$.

Remark 1.1. The q-analogue of KATUGAMPOLA fractional integral and derivative operators becomes

1. For $a = 0$, (1.7) and (1.9) gives generalized fractional q-integral and q-derivative operators introduced by MOMENZADEH [42].
2. For $\rho \to 1$, (1.7) and (1.9) reduce the RIEMANN-LIOUVILLE fractional q-integral and q-derivative operators introduced by [42].
3. For $\rho \to 0^{+}$ and $q \to 1^{-}$, (1.7) and (1.9) and use L'Hopital rule, become HADAMARD fractional integral and derivative operators [13].
4. For $q \to 1$, (1.7) and (1.9) become KATUGAMPOLA fractional integral and derivative operators introduced by UDITA KATUGAMPOLA [40].

The main properties of Katugampola fractional q-integral and q-derivative

In this section, we investigate the main properties of the KATUGAMPOLA fractional q-integral and q-derivative, given in the following Lemmas.

Lemma 1.3 ([43]). *Let $\alpha, \beta > 0$, then the semi-group property for the* KATUGAMPOLA *fractional q-integral is given by*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{q}\ {}^{\rho}\mathcal{J}^{\beta}_{q}u\right)(t) = \left({}^{\rho}\mathcal{J}^{\alpha+\beta}_{q}u\right)(t), \forall u \in \mathcal{L}^{1}_{q,\rho}(\Omega, \mathbb{C}).$$

Lemma 1.4 ([43]). *For $n-1 < \alpha \leq n \in \mathbb{N}$, and $u \in \mathcal{L}^{1}_{q,\rho}(\Omega, \mathbb{C})$, we get*

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{q,a^{+}}\ {}^{\rho}\mathcal{J}^{\alpha}_{q,a^{+}}u\right)(t) = u(t), \forall t \in \Omega.$$

Lemma 1.5 ([43]). *Let $n-1 < \alpha \leq n \in \mathbb{N}$, be such that $u \in \mathcal{L}^{1}_{q,\rho}(\Omega, \mathbb{C})$, then*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{q,a^{+}}\ {}^{\rho}\mathcal{D}^{\alpha}_{q,a^{+}}u\right)(t) = u(t) - \sum_{k=1}^{n} \frac{[\rho]_{q}^{k-\alpha}\ {}^{\rho}\mathcal{D}^{\alpha-k}_{q,a^{+}}u(a)}{\Gamma_{q^{\rho}}(\alpha-k+1)} (t^{\rho}-a^{\rho})^{(\alpha-k)}_{q^{\rho}},$$

if $0 < \alpha \leq 1$

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{q,a^{+}}\ {}^{\rho}\mathcal{D}^{\alpha}_{q,a^{+}}u\right)(t) = u(t) - \frac{[\rho]_{q}^{1-\alpha}\ {}^{\rho}\mathcal{D}^{\alpha-1}_{q,a^{+}}u(a)}{\Gamma_{q^{\rho}}(\alpha)} (t^{\rho}-a^{\rho})^{(\alpha-1)}_{q^{\rho}}.$$

Lemma 1.6 ([43]). *Assume $\beta \geq \alpha \geq 0$, if $u \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$, then*

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{q,a^+}\ {}^{\rho}\mathcal{J}^{\beta}_{q,a^+}u\right)(t) = {}^{\rho}\mathcal{J}^{\beta-\alpha}_{q,a^+}u(t),\ \forall t \in \Omega.$$

Moreover, for all $t \in \Omega$, we obtain

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{q,a^+}\ {}^{\rho}\mathcal{J}^{\beta}_{q,a^+}u\right)(t) = {}^{\rho}\mathcal{D}^{\alpha-\beta}_{q,a^+}u(t),$$

if ${}^{\rho}\mathcal{D}^{\alpha-\beta}_{q,a^+}u(t)$ exists and $\alpha \geq \beta \geq 0$.

Lemma 1.7 ([41]). *The following Jackson for real positive α and $\lambda > -1$, holds*

$$\int_a^t s^{\rho-1}\left(t^{\rho} - (qs)^{\rho}\right)^{(\alpha-1)}_{q^{\rho}}\left(s^{\rho} - a^{\rho}\right)^{(\lambda)}_{q^{\rho}} d_q s = \frac{1}{[\rho]_q}\left(\frac{\Gamma_{q^{\rho}}(\alpha)\,\Gamma_{q^{\rho}}(\lambda+1)}{\Gamma_{q^{\rho}}(\alpha+\lambda+1)}\right)\left(t^{\rho} - a^{\rho}\right)^{(\alpha+\lambda)}_{q^{\rho}}.$$

Lemma 1.8 ([43]). *Let $\lambda \in \mathbb{C}$ be such that $Re(\lambda) > -1$, then*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\left(t^{\rho} - a^{\rho}\right)^{(\lambda)}_{q^{\rho}}\right)(t) = \frac{\Gamma_{q^{\rho}}(\lambda+1)}{[\rho]^{\alpha}_q\,\Gamma_{q^{\rho}}(\alpha+\lambda+1)}\left(t^{\rho} - a^{\rho}\right)^{(\alpha+\lambda)}_{q^{\rho}},$$

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{q,a^+}\left(t^{\rho} - a^{\rho}\right)^{(\lambda)}_{q^{\rho}}\right)(t) = \frac{[\rho]^{\alpha}_q\,\Gamma_{q^{\rho}}(\lambda+1)}{\Gamma_{q^{\rho}}(\lambda-\alpha+1)}\left(t^{\rho} - a^{\rho}\right)^{(\lambda-\alpha)}_{q^{\rho}},$$

hold for every $\alpha, \rho > 0$ and $0 < q < 1$.
In particular, $\forall \alpha, \rho > 0$, if we put $\lambda = \alpha - k$, we conclude that

$${}^{\rho}\mathcal{D}^{\alpha}_{q,a}\left(t^{\rho} - a^{\rho}\right)^{(\alpha-k)}_{q^{\rho}} = 0, \text{ for each } k \in \{1, 2, \ldots, n\}. \tag{1.10}$$

Corollary 1.2. The following formulations with respect to t and s have q-derivatives that can be expressed as

$${}_tD_q\left[\left(t^{\rho} - s^{\rho}\right)^{(\alpha)}_{q^{\rho}}\right] = t^{\rho-1}\left[\rho\alpha\right]_q\left(t^{\rho} - s^{\rho}\right)^{(\alpha-1)}_{q^{\rho}}, \tag{1.11}$$

and

$${}_sD_q\left[\left(t^{\rho} - s^{\rho}\right)^{(\alpha)}_{q^{\rho}}\right] = -s^{\rho-1}\left[\rho\alpha\right]_q\left(t^{\rho} - (qs)^{\rho}\right)^{(\alpha-1)}_{q^{\rho}}. \tag{1.12}$$

Also, we have

$$\left[\rho\alpha\right]_q = \frac{1-q^{\rho\alpha}}{1-q} = \frac{1-q^{\rho\alpha}}{1-q^{\rho}}\frac{1-q^{\rho}}{1-q} = \left[\rho\right]_q\left[\alpha\right]_{q^{\rho}}. \tag{1.13}$$

The q-analogue of Hilfer-Katugampola Fractional Derivative

Definition 1.18 (The H-K Fractional q-Derivative [44]). Let $n-1<\alpha\leq n\in\mathbb{N}$ and $0\leq\eta\leq 1$. The q-analogue of left-sided HILFER-KATUGAMPOLA fractional derivative ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^{\pm}}$ is defined by

$$ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^{+}}u\left(t\right)=\left({}^{\rho}\mathcal{J}^{\eta(n-\alpha)}_{q,a^{+}}\left(t^{1-\rho}D_{q}\right)^{n}\ {}^{\rho}\mathcal{J}^{(1-\eta)(n-\alpha)}_{q,a^{+}}u\right)\left(t\right).$$

The derivative ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^{+}}$ can be expressed in terms of the KATUGAMPOLA fractional q-integral ${}^{\rho}\mathcal{J}^{\alpha}_{q,a^{+}}$ and q-derivative ${}^{\rho}\mathcal{D}^{\alpha}_{q,a^{+}}$ as follows

$$\begin{aligned} {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^{+}} &= {}^{\rho}\mathcal{J}^{\eta(n-\alpha)}_{q,a^{+}}\left(t^{1-\rho}D_{q}\right)^{n}\ {}^{\rho}\mathcal{J}^{n-\gamma}_{q,a^{+}} \\ &= {}^{\rho}\mathcal{J}^{\eta(n-\alpha)}_{q,a^{\pm}}\ {}^{\rho}\mathcal{D}^{\gamma}_{q,a^{+}}\ , \end{aligned}$$

where $\gamma=\alpha+\eta\left(n-\alpha\right)$. Consequently, ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^{+}}$ is well defined in $\mathcal{L}^{1}_{q,\rho}\left(\Omega,\mathbb{C}\right)$.

Remark 1.2. The q-analogue of HILFER-KATUGAMPOLA fractional derivative ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^{\pm}}$ becomes

1. For $\rho\to 1$, gives the q-analogue of HILFER fractional derivative.
2. For $\rho\to 0^{+}$, gives the q-analogue of HILFER-HADAMARD fractional derivative.
3. The q-analogue of KATUGAMPOLA fractional derivative (1.9) for $\eta=0$, which generalizes
   - **(i)** RIEMANN-LIOUVILLE type fractional q-derivative when $\rho\to 1$.
   - **(ii)** HADAMARD type fractional q-derivative when $\rho\to 0^{+}$.
4. The q-analogue of CAPUTO-KATUGAMPOLA fractional derivative [42] for $\eta=1$, which generalizes
   - **(i)** CAPUTO type fractional q-derivative when $\rho\to 1$.
   - **(ii)** CAPUTO-HADAMARD type fractional q-derivative when $\rho\to 0^{+}$.

Lemma 1.9 ([44]). *Let $n-1<\alpha\leq n\in\mathbb{N}$, $0\leq\eta\leq 1$, $0<q<1$, and $\rho>0$, be such that $\gamma=\alpha+\eta\left(n-\alpha\right)$. If $u\in\mathcal{L}^{1}_{q,\rho}\left(\Omega,\mathbb{C}\right)$, then*

$$\begin{aligned} \left({}^{\rho}\mathcal{J}^{\alpha}_{q,a^{+}}\ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^{+}}u\right)\left(t\right) &= \left({}^{\rho}\mathcal{J}^{\gamma}_{q,a^{+}}\ {}^{\rho}\mathcal{D}^{\gamma}_{q,a^{+}}u\right)\left(t\right) \\ &= u\left(t\right)-\sum_{k=1}^{n}\frac{\left[\rho\right]_{q}^{k-\gamma}\ {}^{\rho}\mathcal{D}^{\gamma-k}_{q}u\left(a\right)}{\Gamma_{q^{\rho}}\left(\gamma-k+1\right)}\left(t^{\rho}-a^{\rho}\right)^{(\gamma-k)}_{q^{\rho}}. \end{aligned}$$

Also, if ${}^{\rho}\mathcal{D}^{\eta(n-\alpha)}_{q,a^+}u$ *exists. Then*

$$ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}u = {}^{\rho}\mathcal{J}^{\eta(n-\alpha)}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\eta(n-\alpha)}_{q,a^+}u. \tag{1.14} $$

Property 1.5 ([44]). Let $\delta \in \mathbb{C}$ be such that $\mathrm{Re}(\delta) > -1$, then the following identity hold for every $\alpha, \rho > 0$ and $0 < q < 1$.

$$ \begin{aligned} \left({}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\left(t^{\rho}-a^{\rho}\right)^{(\delta)}_{q^{\rho}}\right)(t) &= \frac{\Gamma_{q^{\rho}}(\delta+1)}{[\rho]^{\alpha}_{q}\,\Gamma_{q^{\rho}}(\alpha+\delta+1)}\left(t^{\rho}-a^{\rho}\right)^{(\alpha+\delta)}_{q^{\rho}}. \\ \left({}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\left(t^{\rho}-a^{\rho}\right)^{(\delta)}_{q^{\rho}}\right)(t) &= \left({}^{\rho}\mathcal{D}^{\alpha}_{q,a^+}\left(t^{\rho}-a^{\rho}\right)^{(\delta)}_{q^{\rho}}\right)(t) \\ &= \frac{[\rho]^{\alpha}_{q}\,\Gamma_{q^{\rho}}(\delta+1)}{\Gamma_{q^{\rho}}(\delta-\alpha+1)}\left(t^{\rho}-a^{\rho}\right)^{(\delta-\alpha)}_{q^{\rho}}. \end{aligned} \tag{1.15} $$

1.2 Some Fixed Point Theorems

In the sequel of this section, we will present the essential Definitions and Theorems necessary that we need for this study

Definition 1.19 (Equicontinuous [45]). Let E be a BANACH space. A part P in $C(E)$ is called equicontinuous if

$$ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall t_1, t_2 \in E,\ \forall \mathcal{A} \in P,\ \ \|t_1 - t_2\| < \delta \Rightarrow \|\mathcal{A}(t_1) - \mathcal{A}(t_2)\| < \varepsilon. $$

Definition 1.20 (Relatively compact subset). A subset P of a topological space E is a relatively compact if and only if any sequence in P has a sub-sequence convergent in E.

Definition 1.21 (Completely continuous). Let E be a Banach space, we say that $\mathcal{A} : E \to E$ is completely continuous if $\mathcal{A} : E \to E$ is continuous and for any bounded subset P of E, the set $\mathcal{A}(P)$ is relatively compact.

Theorem 1.1 (Ascoli-Arzela [29]). *Let E be a compact space and P in $C(E)$. If $\mathcal{A}(P)$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{A}$ is relatively compact.*

Definition 1.22 ([29]). Let E be a space and $\mathcal{A}$ a map of E, or of a subset of E, into E.

- The map $\mathcal{A}$ is called a contraction mapping if there exists $k \in (0,1)$ such that

$$\forall x, y \in E, \ \|\mathcal{A}x - \mathcal{A}y\| \leq k \|x - y\| .$$

- A point $x \in E$ is called a fixed point for $\mathcal{A}$ if $\mathcal{A}x = x$.

Theorem 1.2 (Banach's fixed point [29]). *Let P be a non-empty closed subset of a* BANACH *space E, then any contraction mapping $\mathcal{A}$ of P into itself has a unique fixed point.*

Theorem 1.3 (Schauder's fixed point [29]). *Let E be a* BANACH *space, and P be a closed, convex and nonempty subset of E. Let $\mathcal{A} : P \to P$ be a continuous mapping such that $\mathcal{A}(P)$ is a relatively compact subset of E. Then $\mathcal{A}$ has at least one fixed point in P.*

Theorem 1.4 (Leray-Schauder Nonlinear Alternative [29]). *Let E be a Banach space, and Ω a bounded open subset of E with $0 \in \Omega$. Then, every completely continuous map $\mathcal{A} : \bar{\Omega} \to E$ has at least one of the following two properties*

1. $\mathcal{A}$ has a fixed point in $\bar{\Omega}$.
2. There is an $x \in \partial\Omega$ and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}x$.

CHAPTER 2

ON n-NONLINEAR CAPUTO FRACTIONAL q-DIFFERENTIAL SYSTEMS

The scientific paper corresponding to this chapter has been published in: Journal of Filomat 39:21 (2025), (see [47]).

This chapter aims to study the existence, uniqueness and Ulam-Hyers stability of solutions for a system of n-nonlinear fractional q-differential equations (q-FDEs)

$$\begin{cases} {}^{C}\mathcal{D}_{q}^{\alpha_1} u_1(t) = f_1(t, u(t)),\ t \in [0,T], \\ {}^{C}\mathcal{D}_{q}^{\alpha_2} u_2(t) = f_2(t, u(t)),\ t \in [0,T], \\ \vdots \\ {}^{C}\mathcal{D}_{q}^{\alpha_n} u_n(t) = f_n(t, u(t)),\ t \in [0,T], \end{cases} \tag{2.1}$$

with the initial conditions:

$$u_i\left(0^{+}\right) = \zeta_i \in \mathbb{R}, \text{ for each } i \in \{1, \ldots, n\}, \tag{2.2}$$

where ${}^{C}\mathcal{D}_{q}^{\alpha_i}$ represents the Caputo fractional q-derivative of order $0 < \alpha_i \leq 1$ for each $i \in \{1, \ldots, n\}$ and $u = (u_1, u_2, \ldots, u_n)$ for $n \in \mathbb{N}$. In addition $f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions, with $T > 0$ and $0 < q < 1$.

The following are the primary novelties of the current work:

• Considering the diverse conditions we applied to system (2.1), our work can be seen as a continuation of the studies mentioned [48–50].

• Our work continues upon the obtained results in [48, 49] by extending the fractional equations to n-nonlinear q-FDEs. This will necessitate the use of more complex approaches

with different spaces and norms.

• Our findings expand upon those in [50] by considering n-nonlinear with q-fractional derivatives.

• We also address the qualitative aspect by investigating the Ulam-Hyers stability of our system.

2.1 Fundamental Lemma

In what follows, we present the following Lemma to demonstrate our main results

Lemma 2.1 (The Equivalent q-Integral Equations). *Let $0 < \alpha_i \leq 1$, for any $u_i \in C([0,T],\mathbb{R})$ and continuous functions $(f_i)_{i=\overline{1,n}} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, the solutions of problem (2.1)–(2.2) is equivalent to the n-fractional q-integral equations*

$$\begin{cases} u_1(t) = \zeta_1 + \int_0^t G_{\alpha_1}(t,qs) f_1(s,u(s)) d_q s, \\ u_2(t) = \zeta_2 + \int_0^t G_{\alpha_2}(t,qs) f_2(s,u(s)) d_q s, \\ \vdots \\ u_n(t) = \zeta_n + \int_0^t G_{\alpha_n}(t,qs) f_n(s,u(s)) d_q s, \end{cases} \tag{2.3}$$

where G_{α_i} are continuous functions, also called the Green's functions written as follows

$$G_{\alpha_i}(t,qs) = \frac{(t-qs)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)}, \tag{2.4}$$

for each $i \in \{1,\dots,n\}$, $s \in (0,t]$ and $t \in [0,T]$

Proof. Let $0 < \alpha_i \leq 1$, $0 < q < 1$, and $u_i \in C([0,T],\mathbb{R})$ for each $i \in \{1,\dots,n\}$. Starting, applying $\mathcal{I}_q^{\alpha_i}$ on both sides of system (2.1), we obtain

$$\left(\mathcal{I}_q^{\alpha_i}\ {}^C\mathcal{D}_q^{\alpha_i} u_i(t)\right) = \mathcal{I}_q^{\alpha_i} f_i(s,u(s)),$$

from Lemma 1.1, by using conditions (2.2), we get

$$u_i(t) = \zeta_i + \mathcal{I}_q^{\alpha_i} f_i(t,u(t)). \tag{2.5}$$

Otherwise, by applying $\mathcal{D}_q^{\alpha_i}$ on both sides of the equations (2.5), utilizing Lemma 1.2, we find that

$$\mathcal{D}_q^{\alpha_i}(u_i(t) - \zeta_i) = \mathcal{D}_q^{\alpha_i}\mathcal{I}_q^{\alpha_i} f_i(t,u(t)).$$

Then

$$ {}^{C}\mathcal{D}_{q}^{\alpha_i} u_i(t) = f_i(t, u(t)). $$

Furthermore, from (2.5), we find

$$ u_i\left(0^{+}\right) = \zeta_i, \text{ for each } i \in \{1, \ldots, n\}. $$

The proof is complete. □

2.2 Existence and Uniqueness Results

In this section, by applying the properties of Schauder's and Banach's fixed point theorems we investigate the existence and uniqueness of solutions to the problem (2.1)-(2.2). Furthermore, we impose the following assumptions

($\mathcal{H}_1$) For any $u, \tilde{u} \in \mathbb{R}^n$, there exist $n \times n$ nonnegative constants $(\lambda_{i,k})_{i,k=\overline{1,n}}$ such that the functions f_i satisfy

$$ |f_i(t,u) - f_i(t,\tilde{u})| \leq \sum_{k=1}^{n} \lambda_{i,k} |u_k - \tilde{u}_k|, \quad \text{for all } t \in [0,T]. $$

($\mathcal{H}_2$) For any $u \in \mathbb{R}^n$, there exist two families of nonnegative continuous functions $(\psi_i)_{i=\overline{1,n}}$ and $(\omega_{i,k})_{i,k=\overline{1,n}}$ such that

$$ |f(t,u)| \leq \psi_i(t) + \sum_{k=1}^{n} \omega_{i,k}(t) |u_k|, \quad \text{for all } t \in [0,T]. $$

For each $i \in \{1, \ldots, n\}$, we denote

$$ \lambda_i = \max_{1 \leq k \leq n} \{\lambda_{i,k}\}, \quad \text{and} \quad \zeta = \max_{1 \leq i \leq n} \{|\zeta_i|\}, $$

and

$$ \psi_i^* = \sup_{0 \leq t \leq T} \psi_i(t), \quad \omega_i^* = \sup_{0 \leq t \leq T} \left\{ \max_{1 \leq k \leq n} \{\omega_{i,k}(t)\} \right\}. $$

Let us define $E = X_1 \times X_2 \times \cdots \times X_n$, where $X_i = \{u_i, \quad u_i \in C([0,T], \mathbb{R})\}_{i=\overline{1,n}}$, with the norm

$$ \|u\|_E = \max_{1 \leq i \leq n} \|u_i\|. $$

Under this norm $(E, \|\cdot\|_E)$ is a Banach space.

Let us define the integral operators

$$\mathcal{T}_i u(t) = \zeta_i + \int_0^t G_{\alpha_i}(t, qs) f_i(s, u(s)) d_q s. \tag{2.6}$$

Since $f_i(t,u)$ and $G_{\alpha_i}(t,qs)$ are continuous functions for each $i \in \{1,\dots,n\}$, then $\mathcal{T}_i$ are continuous, i.e., $\mathcal{T}_i(E) \subset X_i$.

Let $\mathcal{T} : E \to E$ be a system of n-operators that defined by

$$\mathcal{T}u(t) = \begin{pmatrix} \mathcal{T}_1 u(t) \\ \mathcal{T}_2 u(t) \\ \vdots \\ \mathcal{T}_n u(t) \end{pmatrix}, \tag{2.7}$$

where $(\mathcal{T}_i)_{i=\overline{1,n}}$ are the integral operators given by (2.6), characterized by the norm

$$\|\mathcal{T}u\|_E = \max_{1\le i\le n} \|\mathcal{T}_i u\|, \text{ with } \|\mathcal{T}_i u\| = \sup_{0\le t\le T} |\mathcal{T}_i u(t)|.$$

Theorem 2.1. *Assume the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold, for each $i \in \{1,\dots,n\}$, if*

$$\frac{n\omega_i^* T^{\sigma_i}}{\Gamma_q(\alpha_i+1)} < 1, \tag{2.8}$$

then problem (2.1)–(2.2) has at least one solution on $[0,T]$.

Proof. To begin the proof, we transform problem (2.1)–(2.2) into a fixed point problem. Let the operator $\mathcal{T} : E \to E$ defined by (2.7), with $[\mathcal{T}_i(u)(t)]_{i=\overline{1,n}}$ being the integral operators given by

$$\mathcal{T}_i(u)(t) = \zeta_i + \int_0^t G_{\alpha_i}(t, qs) f_i(s, u(s)) d_q s, \quad t \in [0,T],$$

where $G_{\alpha_i}(t,qs)$ defined in (2.4), for each $i \in \{1,\dots,n\}$.
We demonstrate that $\mathcal{T}$ satisfies the assumptions of Schauder's fixed point theorem. This proof will be completed in three steps.

Step 1: $\mathcal{T}$ is a continuous operator.

Consider the sequences $(u_m)_{m\in\mathbb{N}} = (u_1^m, u_2^m, \dots, u_n^m)$, such that $\lim_{m\to\infty} u_m \to u$ in E.

For each $i \in \{1, \ldots, n\}$ and $t \in [0, T]$, we obtain

$$\begin{aligned} |\mathcal{T}_i u_m(t) - \mathcal{T}_i u(t)| &\leq \int_0^t G_{\alpha_i}(t, qs) \left| f_i(s, u_m(s)) - f_i(s, u(s)) \right| d_q s \\ &\leq \int_0^t G_{\alpha_i}(t, qs) \sum_{k=1}^{n} \lambda_{i,k} \left| u_k^m(s) - u_k(s) \right| d_q s \\ &\leq \frac{n \lambda_i T^{\alpha_i}}{\Gamma_q(\alpha_i + 1)} \left\| u_m - u \right\|_E. \end{aligned}$$

Then, $\mathcal{T}$ is continuous.

Step 2: $\mathcal{T}$ is defined from a bounded, closed and convex subset into itself.

For each $i \in \{1, \ldots, n\}$, using (2.8), we consider r satisfies

$$r \geq \left(\zeta + \frac{\psi_i^* T^{\alpha_i}}{\Gamma_q(\alpha_i + 1)} \right) \left(\frac{\Gamma_q(\alpha_i + 1)}{\Gamma_q(\alpha_i + 1) - n\omega_i^* T^{\alpha_i}} \right).$$

Thus the subset ϑ_r defined by

$$\vartheta_r = \{ u \in E, \quad \|u\|_E \leq r \},$$

is clearly bounded, closed and convex in E.

Let $\mathcal{T} : \vartheta_r \to E$ be the integral operator defined by (2.7). Then $\mathcal{T}(\vartheta_r) \subset \vartheta_r$. Indeed, by using $(\mathcal{H}_2)$, for each $i \in \{1, \ldots, n\}$ and $t \in [0, T]$, we get

$$\begin{aligned} |f_i(t, u(t))| &\leq \psi_i(t) + \sum_{k=1}^{n} \omega_{i,k}(t) |u_k|, \\ &\leq \psi_i^* + n\omega_i^* \|u\|_E. \end{aligned}$$

Thus

$$\begin{aligned}
|\mathcal{T}_i u(t)| &\leq \left|\zeta_i + \int_0^t G_{\alpha_i}(t,qs)\, f_i(s,u(s))\, d_q s\right| \\
&\leq \max_{1\leq i\leq n}\{|\zeta_i|\} + \int_0^t G_{\alpha_i}(t,qs)\,|f_i(s,u(s))|\, d_q s \\
&\leq \zeta + \frac{(\psi_i^* + n\omega_i^* \|u\|_E)\, T^{\alpha_i}}{\Gamma_q(\alpha_i+1)} \\
&\leq \frac{\left(\zeta + \frac{\psi_i^* T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\right)\left(\frac{\Gamma_q(\alpha_i+1)}{\Gamma_q(\alpha_i+1)-n\omega_i^* T^{\alpha_i}}\right)}{\left(\frac{\Gamma_q(\alpha_i+1)}{\Gamma_q(\alpha_i+1)-n\omega_i^* T^{\alpha_i}}\right)} + \frac{n\omega_i^* T^{\alpha_i}}{\Gamma_q(\alpha_i+1)} r \\
&\leq r.
\end{aligned}$$

Hence, $\|\mathcal{T}_i u\| \leq r$ for each $i \in \{1,\ldots,n\}$, which implies that $\|\mathcal{T}u\|_E \leq r$. Consequently $\mathcal{T}(\vartheta_r) \subset \vartheta_r$.

Step 3: $\mathcal{T}(\vartheta_r)$ **is an equicontinuous subset**.

Let $t_1, t_2 \in [0,T]$, $t_1 < t_2$ and $u \in \vartheta_r$. Then, for each $i \in \{1,\ldots,n\}$, we have

$$\begin{aligned}
&|\mathcal{T}_i u(t_2) - \mathcal{T}_i u(t_1)| \\
&= \left|\int_0^{t_2} G_{\alpha_i}(t_2,qs)\, f_i(s,u(s))\, d_q s - \int_0^{t_1} G_{\alpha_i}(t_1,qs)\, f_i(s,u(s))\, d_q s\right| \\
&\leq \int_0^{t_1} |[G_{\alpha_i}(t_2,qs) - G_{\alpha_i}(t_1,qs)]\, f_i(s,u(s))|\, d_q s + \int_{t_1}^{t_2} G_{\alpha_i}(t_2,qs)\,|f_i(s,u(s))|\, d_q s \\
&\leq (\psi_i^* + n\omega_i^* r)\left(\int_0^{t_1} |G_{\alpha_i}(t_2,qs) - G_{\alpha_i}(t_1,qs)|\, d_q s + \int_{t_1}^{t_2} G_{\alpha_i}(t_2,qs)\, d_q s\right). \qquad (2.9)
\end{aligned}$$

As $t_2 > t_1$, we obtain

$$\begin{aligned}
|G_{\alpha_i}(t_2,qs) - G_{\alpha_i}(t_1,qs)| &= \frac{1}{\Gamma_q(\alpha_i)}\left[(t_1-qs)^{(\alpha_i-1)} - (t_2-qs)^{(\alpha_i-1)}\right] \\
&= \frac{1}{\Gamma_q(\alpha_i+1)}\, {}_sD_q\left[(t_2-s)^{(\alpha_i)} - (t_1-s)^{(\alpha_i)}\right].
\end{aligned}$$

Then

$$\int_0^{t_1} |G_{\alpha_i}(t_2,qs) - G_{\alpha_i}(t_1,qs)|\, d_q s \leq \frac{1}{\Gamma_q(\alpha_i+1)}\left[(t_2-t_1)^{(\alpha_i)} + (t_2^{\alpha_i} - t_1^{\alpha_i})\right],$$

and

$$\int_{t_1}^{t_2} G_{\alpha_i}(t_2, qs)\, d_q s = \frac{-1}{\Gamma_q(\alpha_i+1)} \left[(t_2-s)^{(\alpha_i)}\right]_{t_1}^{t_2} = \frac{1}{\Gamma_q(\alpha_i+1)} (t_2-t_1)^{(\alpha_i)}.$$

Thus, (2.9) gives

$$|\mathcal{T}_i u(t_2) - \mathcal{T}_i u(t_1)| \leq \frac{(\psi_i^* + n\omega_i^* r)}{\Gamma_q(\alpha_i+1)} \left[2(t_2-t_1)^{(\alpha_i)} + (t_2^{\alpha_i} - t_1^{\alpha_i})\right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero for each $i \in \{1,\ldots,n\}$. Combining steps 1 to 3 and applying the Ascoli-Arzela Theorem, we conclude that $\mathcal{T} : \vartheta_r \to \vartheta_r$ is continuous and compact, satisfies the assumptions of Schauder's fixed point Theorem [29]. Therefore, $\mathcal{T}$ has at least a fixed point which is the solution of problem (2.1)–(2.2) on $[0,T]$.

□

Theorem 2.2. *Assume that the hypothesis $(\mathcal{H}_1)$ holds. If*

$$\max_{1\leq i\leq n} \left\{ \frac{n\lambda_i T^{\alpha_i}}{\Gamma_q(\alpha_i+1)} \right\} < 1, \tag{2.10}$$

then problem (2.1)–(2.2) admits a unique solution on $[0,T]$.

Proof. Let $u, \tilde{u} \in E$. By using $(\mathcal{H}_1)$, we obtain for all $t \in [0,T]$ that

$$|\mathcal{T}_i u(t) - \mathcal{T}_i \tilde{u}(t)| \leq \frac{n\lambda_i T^{\alpha_i}}{\Gamma_q(\alpha_i+1)} \|u-\tilde{u}\|_E, \quad \text{for each } i \in \{1,\ldots,n\}.$$

Consequently

$$\|\mathcal{T}u - \mathcal{T}\tilde{u}\|_E \leq \max_{1\leq i\leq n} \left\{ \frac{n\lambda_i T^{\alpha_i}}{\Gamma_q(\alpha_i+1)} \right\} \|u-\tilde{u}\|_E.$$

Based on statement (2.10), it follows that $\mathcal{T}$ is a contraction operator.

The Banach contraction principle [29] leads us to conclude that $\mathcal{T}$ has a unique fixed point. This fixed point is the unique solution of problem (2.1)–(2.2) on $[0,T]$. □

2.3 Ulam-Hyers Stability

In 1940, Ulam [30] posed the challenge of functional equations, which was further developed by Hyers [31] in 1941 using Banach spaces. As a result, this issue led to the development of the Ulam–Hyers stability. In this section, we investigate the Ulam-Hyers stability by applying the important Definitions (2.1) and (2.2) and employing the equivalent q-integral equation

Definition 2.1. The system (2.1) is Ulam-Hyers stable if there exists a real number $\mu > 0$, such that for all $\varepsilon = \max\limits_{1\leq i\leq n} \{\varepsilon_i\} > 0$ and for any solution $v \in E$ satisfying the inequality

$$\left| {}^{C}\mathcal{D}^{\alpha_i}_{q,0^+} v_i(t) - f_i(t, v(t)) \right| \leq \varepsilon_i,\ t \in [0,T],\ i \in \{1,\ldots,n\}, \tag{2.11}$$

there exists a solution $u \in E$ of system (2.1) satisfying

$$\|u - v\|_E \leq \mu\varepsilon.$$

Definition 2.2. The system (2.1) is generalized Ulam-Hyers stable if there exists a function $\theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying $\theta(0) = 0$, such that for any solution $v \in E$ of inequality (2.11), there exists a solution $u \in E$ of (2.11) satisfying the inequality

$$\|u - v\|_E \leq \theta(\varepsilon).$$

Remark 2.1. A function $v \in E$ is a solution of inequality (2.11) if and only if there exists a function $g \in E$ and $(\varepsilon_i)_{i=\overline{1,n}} > 0$, such that for all $t \in [0,T]$ and $i \in \{1,\ldots,n\}$,

1. $|g_i(t)| \leq \varepsilon_i$,
2. ${}^{C}\mathcal{D}^{\alpha_i}_{q,0^+} v_i(t) = f_i(t, v(t)) + g_i(t)$.

Before proving the Theorem, we introduce the following Lemma

Lemma 2.2. *If a function $v \in E$ is the solution of inequality (2.11), then there exists $(\varepsilon_i)_{i=\overline{1,n}} > 0$, ensuring that v remains the solution of the integral inequality*

$$\left| v_i(t) - v_i(0) - \frac{1}{\Gamma_q(\alpha_i)} \int_0^t G_{\alpha_i}(t, qs) f_i(s, v(s)) d_q s \right| \leq \frac{\varepsilon_i T^{\alpha_i}}{\Gamma_q(\alpha_i + 1)}.$$

Proof. If v is a solution of (2.11), by Remark 2.1, we have

$$v_i(t) = v_i(0) + \frac{1}{\Gamma_q(\alpha_i)} \int_0^t G_{\alpha_i}(t, qs) [f_i(s, v(s)) + g_i(s)] d_q s,$$

and

$$\begin{aligned}
&\left| v_i(t) - v_i(0) - \frac{1}{\Gamma_q(\alpha_i)} \int_0^t G_{\alpha_i}(t, qs) f_i(s, v(s)) d_q s \right| \\
&= \left| v_i(0) + \frac{1}{\Gamma_q(\alpha_i)} \int_0^t G_{\alpha_i}(t, qs) [f_i(s, v(s)) + g_i(s)] d_q s \right. \\
&\quad \left. - v_i(0) - \frac{1}{\Gamma_q(\alpha_i)} \int_0^t G_{\alpha_i}(t, qs) f_i(s, v(s)) d_q s \right| \\
&\leq \frac{1}{\Gamma_q(\alpha_i)} \int_0^t G_{\alpha_i}(t, qs) |g_i(s)| d_q s \\
&\leq \frac{\varepsilon_i T^{\alpha_i}}{\Gamma_q(\alpha_i + 1)}, \text{ for each } i \in \{1, \ldots, n\}.
\end{aligned}$$

□

Theorem 2.3. *Assume that $(\mathcal{H}_1)$ and (2.10) hold, then system (2.1) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon = \max_{1 \leq i \leq n} \{\varepsilon_i\} > 0$, assume that $v \in E$ be a solution of inequality (2.11), and $u \in E$ is the unique solution of system (2.1), with the conditions

$$y_i(0) = v_i(0), \quad \text{for each } i \in \{1, \ldots, n\},$$

then

$$u_i(t) = v_i(0) + \frac{1}{\Gamma_q(\alpha_i)} \int_0^t G_{\alpha_i}(t, qs) f_i(s, u(s)) d_q s,$$

and

$$\begin{aligned}|v_i(t)-u_i(t)| &= \left|v_i(t)-v_i(0)-\frac{1}{\Gamma_q(\alpha_i)}\int_0^t G_{\alpha_i}(t,qs)\,f_i(s,u(s))\,d_qs\right| \\ &\leq \left|v_i(t)-v_i(0)-\frac{1}{\Gamma_q(\alpha_i)}\int_0^t G_{\alpha_i}(t,qs)\,f_i(s,v(s))\,d_qs\right| \\ &\quad +\frac{1}{\Gamma_q(\alpha_i)}\int_0^t G_{\alpha_i}(t,qs)\,|f_i(s,v(s))-f_i(s,u(s))|\,d_qs.\end{aligned}$$

Using $(\mathcal{H}_1)$ and Lemma 2.2, for all $t\in[0,T]$ and $i\in\{1,\ldots,n\}$, we get

$$\|v_i-u_i\|\leq\frac{T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\left(\varepsilon_i+n\lambda_i\|u-v\|_E\right),$$

then

$$\|u-v\|_E\leq\max_{1\leq i\leq n}\left\{\frac{T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\left(\varepsilon_i+n\lambda_i\|u-v\|_E\right)\right\}.$$

Since $\varepsilon=\max\limits_{1\leq i\leq n}\{\varepsilon_i\}>0$, we have

$$\|u-v\|_E\leq\max_{1\leq i\leq n}\left\{\frac{T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\right\}\varepsilon+\Lambda\|u-v\|_E,$$

with $\Lambda=\max\limits_{1\leq i\leq n}\left\{\frac{n\lambda_iT^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\right\}<1$. Then

$$\|u-v\|_E\leq\mu\varepsilon,$$

where $\mu=\max\limits_{1\leq i\leq n}\left\{\frac{T^{\alpha_i}}{(1-\Lambda)\Gamma_q(\alpha_i+1)}\right\}$.
We deduce that system (2.1) is Ulam-Hyers stable and is consequently generalized Ulam-Hyers stable if we set $\theta(\epsilon)=\mu\epsilon$.

□

2.4 Illustrative Examples

Example 2.1. Consider the following problem:

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{1}{2}}_{\frac{1}{4},0^+}\left(u_1\right)(t) = \frac{\pi t}{2\sqrt{e}} + \frac{\sin(t)(|u_1(t)|+|u_2(t)|)}{\sqrt{1+e}(1+q+|u_1(t)|+|u_2(t)|)}, & t \in \left[0, \frac{\pi}{6}\right], \\ {}^{C}\mathcal{D}^{\frac{2}{3}}_{\frac{1}{4},0^+}\left(u_2\right)(t) = \exp\left(-\pi^2 t\right)\left(\frac{1}{100\sqrt{e}} + \frac{\tan(t)}{\pi}\left(|u_1(t)| + |u_2(t)|\right)\right), & t \in \left[0, \frac{\pi}{6}\right], \end{cases} \tag{2.12}$$

and

$$u_i(0) = \zeta_i \in \mathbb{R}, \quad \text{for each } i \in \{1,2\}.$$

We set

$$\begin{aligned} f_1(t,u_1,u_2) &= \frac{\pi t}{2\sqrt{e}} + \frac{\sin(t)\left(|u_1|+|u_2|\right)}{\sqrt{1+e}\,(1+q+|u_1|+|u_2|)}, \\ f_2(t,u_1,u_2) &= \exp\left(-\pi^2 t\right)\left(\frac{1}{100\sqrt{e}} + \frac{\tan(t)}{\pi}\left(|u_1|+|u_2|\right)\right). \end{aligned}$$

The functions f_i are continuous for any $u \in \mathbb{R}^2$. Also

$$\begin{aligned} |f_1(t,u_1,u_2) - f_1(t,\tilde{u}_1,\tilde{u}_2)| &\leq \sum_{k=1}^{2} \frac{1+q}{2\sqrt{1+e}}|u_k - \tilde{u}_k|, \\ |f_2(t,u_1,u_2) - f_2(t,\tilde{u}_1,\tilde{u}_2)| &\leq \sum_{k=1}^{2} \frac{\sqrt{3}}{3\pi}|u_k - \tilde{u}_k|, \end{aligned}$$

hence, $(\mathcal{H}_1)$ is satisfied for each $i \in \{1,2\}$, with

$$\lambda_1 = \max_{1\leq k\leq 2}\left(\lambda_{1,1},\lambda_{1,2}\right) = \frac{5}{8\sqrt{1+e}} \quad \text{and} \quad \lambda_2 = \max_{1\leq k\leq 2}\left(\lambda_{2,1},\lambda_{2,2}\right) = \frac{\sqrt{3}}{3\pi}.$$

For each $i \in \{1,2\}$ and $t \in \left[0, \frac{\pi}{6}\right]$, we have

$$\begin{aligned} |f_1(t,u_1,u_2)| &\leq \frac{\pi t}{2\sqrt{e}} + \frac{\sin(t)}{\sqrt{1+e}}\left(|u_1|+|u_2|\right), \\ |f_2(t,u_1,u_2)| &\leq \frac{\exp\left(-\pi^2 t\right)}{100\sqrt{e}} + \frac{\exp\left(-\pi^2 t\right)\tan(t)}{\pi}\left(|u_1|+|u_2|\right). \end{aligned}$$

Thus, hypothesis $(\mathcal{H}_2)$ is satisfied, with

$$\psi_1(t) = \frac{\pi t}{2\sqrt{e}}, \quad \omega_{1,1}(t) = \omega_{1,2}(t) = \frac{\sin(t)}{\sqrt{1+e}},$$

and

$$\psi_2(t) = \frac{\exp(-\pi^2 t)}{100\sqrt{e}}, \quad \omega_{2,1}(t) = \omega_{2,2}(t) = \frac{\exp(-\pi^2 t)\tan(t)}{\pi}.$$

For each $i \in \{1,2\}$, we obtain

$$\psi_1^* = \frac{\pi^2}{12\sqrt{e}}, \quad \omega_1^* = \frac{1}{2\sqrt{1+e}} \quad \text{and} \quad \psi_2^* = \frac{1}{100\sqrt{e}}, \quad \omega_2^* = \frac{\sqrt{3}}{3\pi}.$$

Now, we show that

$$\begin{aligned} \frac{n\omega_1^* T^{\sigma_1}}{\Gamma_q(\alpha_1+1)} &= \frac{2\times\frac{1}{2\sqrt{1+e}}\times\left(\frac{\pi}{6}\right)^{\frac{1}{2}}}{\Gamma_{\frac{1}{4}}\left(\frac{1}{2}+1\right)} \simeq 0,395925 < 1, \\ \frac{n\omega_2^* T^{\sigma_2}}{\Gamma_q(\alpha_2+1)} &= \frac{2\times\frac{\sqrt{3}}{3\pi}\times\left(\frac{\pi}{6}\right)^{\frac{2}{3}}}{\Gamma_{\frac{1}{4}}\left(\frac{2}{3}+1\right)} \simeq 0,249391 < 1. \end{aligned}$$

It follows from Theorem 2.1, that problem (2.12) has at least one solution.

Example 2.2. Consider the following problem:

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{1}{2}}_{\frac{1}{2},0^+}(u_1)(t) = \frac{\arcsin(t)(|u_1(t)|+|u_2(t)|+|u_3(t)|)}{\pi\sqrt{1+qt}}, & t \in \left[0,\frac{1}{2}\right], \\ {}^{C}\mathcal{D}^{\frac{5}{6}}_{\frac{1}{2},0^+}(u_2)(t) = \frac{\exp^{-\pi^2 t}}{2\pi(1-q)(1+|u_1(t)|+|u_2(t)|+|u_3(t)|)}, & t \in \left[0,\frac{1}{2}\right], \\ {}^{C}\mathcal{D}^{\frac{6}{7}}_{\frac{1}{2},0^+}(u_3)(t) = \frac{\arccos(t)(2+|u_1(t)|+|u_2(t)|+|u_3(t)|)}{\pi(t^2+1)(1+|u_1(t)|+|u_2(t)|+|u_3(t)|)}, & t \in \left[0,\frac{1}{2}\right], \end{cases} \tag{2.13}$$

and

$$u_i(0) = \zeta_i \in \mathbb{R}, \text{ for each } i \in \{1,2,3\}.$$

We set

$$\begin{aligned} f_1(t,u_1,u_2,u_3) &= \frac{\arcsin(t)\left(|u_1|+|u_2|+|u_3|\right)}{\pi\sqrt{1+qt}}, \\ f_2(t,u_1,u_2,u_3) &= \frac{\exp^{-\pi^2 t}}{2\pi(1-q)\left(1+|u_1|+|u_2|+|u_3|\right)}, \\ f_3(t,u_1,u_2,u_3) &= \frac{\arccos(t)\left(2+|u_1|+|u_2|+|u_3|\right)}{\pi\left(t^2+1\right)\left(1+|u_1|+|u_2|+|u_3|\right)}. \end{aligned}$$

The functions f_i are continuous for any $u,\,\tilde{u}\in\mathbb{R}^3$. Also

$$\begin{aligned} |f_1(t,u_1,u_2,u_3)-f_1(t,\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)| &\leq \sum_{k=1}^{3}\frac{1}{6}|u_k-\tilde{u}_k|, \\ |f_2(t,u_1,u_2,u_3)-f_2(t,\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)| &\leq \sum_{k=1}^{3}\frac{1}{2\pi(1-q)}|u_k-\tilde{u}_k|, \\ |f_3(t,u_1,u_2,u_3)-f_3(t,\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)| &\leq \sum_{k=1}^{3}\frac{1}{3}|u_k-\tilde{u}_k|. \end{aligned}$$

Hence, the hypothesis $(\mathcal{H}_1)$ is satisfied with $\lambda_1=\frac{1}{6}$, $\lambda_2=\frac{1}{\pi}$, and $\lambda_3=\frac{1}{3}$.
For each $i\in\{1,2,3\}$ we obtain

$$\begin{aligned} \frac{n\lambda_1 T^{\alpha_1}}{\Gamma_q(\alpha_1+1)} &= \frac{3\times\frac{1}{6}\times\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma_{\frac{1}{2}}\left(\frac{1}{2}+1\right)}\simeq 0,383932<1, \\ \frac{n\lambda_2 T^{\alpha_2}}{\Gamma_q(\alpha_2+1)} &= \frac{3\times\frac{1}{\pi}\times\left(\frac{1}{2}\right)^{\frac{5}{6}}}{\Gamma_{\frac{1}{2}}\left(\frac{5}{6}+1\right)}\simeq 0,557904<1, \\ \frac{n\lambda_3 T^{\alpha 3}}{\Gamma_q(\alpha_3+1)} &= \frac{3\times\frac{1}{3}\times\left(\frac{1}{2}\right)^{\frac{6}{7}}}{\Gamma_{\frac{1}{2}}\left(\frac{6}{7}+1\right)}\simeq 0,571772<1. \end{aligned}$$

It remains to show that the condition (2.10)

$$\max_{1\leq i\leq 3}\left\{\frac{n\lambda_i T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\right\}\simeq 0,571772<1,$$

is satisfied. It follows from Theorem 2.2, that problem (2.13) admits a unique solution.

CHAPTER 3

STABILITY ANALYSIS OF q-FRACTIONAL ORDER SYSTEMS VIA LYAPUNOV FUNCTION

This chapter generalizes several results in both classical and fractional aspects. These generalizations serves as an analogue to assist in identifying of appropriate Lyapunov candidate functions for establishing the stability of q-fractional order systems (q-FOS). The analysis is conducted by applying Lyapunov's direct method within the q-fractional framework.

The primary focus of this chapter is to establish the stability of q-FOS involving the Caputo-type fractional q-derivatives

$$ {}^{C}\mathcal{D}^{\alpha}_{q,t_0} u(t) = f(t, u(t)), \quad t \in T_q, \tag{3.1} $$

with the initial condition

$$ u(t_0) = u_0, \quad t_0 \geq 0, \tag{3.2} $$

where ${}^{C}\mathcal{D}^{\alpha}_{q,t_0}$ represents the Caputo fractional q-derivative of order $0 < \alpha < 1$, with $0 < q < 1$ and $u, u_0 \in \mathbb{R}^n$. In addition $f : T_q \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, and $t \geq t_0$, $t_0 \in T_q$, where the geometric time scale defined by $T_q = \{q^n t_0,\ n \in \mathbb{N}\} \cup \{0\}$.

3.1 Fundamental Lemmas

In this section, we present the space $\mathcal{C}(T_q, \mathbb{R})$ be the Banach space of all continuous functions associated with the norm

$$ \|u\| = \sup_{t \in T_q} |u(t)|. $$

For all continuous functions with continuous q-derivative up to order $n-1$, we give

$$\mathcal{C}_q^n(T_q,\mathbb{R}) = \left\{u(t),\ D_q^m u(t) \in \mathcal{C}(T_q,\mathbb{R}),\ \forall\, m = 0,1,\ldots,n\right\}.$$

We mention the necessary Definitions and Theorems was proved in [46].

Definition 3.1. ([46])The function θ is referred to as being of class $\mathcal{K}$ iff $\theta \in \mathcal{C}([0,\gamma),\mathbb{R}_+)$, where $\gamma \in \mathbb{R}_+$, $\theta(0) = 0$ and θ increases strictly monotonically in u.

Definition 3.2. ([46])A scalar function $V : T_q \times \vartheta_\gamma \to \mathbb{R}_+$, where $\vartheta_\gamma = \{u \in \mathbb{R}^n : \ \|u\| < \gamma\}$, is called to be positive definite iff

$$V(t,0) = 0,\ \forall t \in T_q,$$

and $\theta \in \mathcal{K}$ such that

$$\theta(\|u\|) \le V(t,u), \forall (t,u) \in T_q \times \vartheta_\gamma.$$

Theorem 3.1. *([46])The stationary point of (3.1) is stable, if there is a positive definite scalar function, $V \in \mathcal{C}(T_q \times \vartheta_\gamma, \mathbb{R}_+)$, where*

$${}^{C}\mathcal{D}_{q,t_0}^{\alpha} V(t,u) \le 0,\ \forall (t,u) \in T_q \times \vartheta_\gamma,\ \ t \ge t_0. \tag{3.3}$$

Theorem 3.2. *([46])The stationary point of (3.1) is asymptotically stable, if there is a positive definite function, $V \in \mathcal{C}(T_q \times \vartheta_\gamma, \mathbb{R}_+)$, where*

$${}^{C}\mathcal{D}_{q,t_0}^{\alpha} V(t,u) \le -\sigma V(t,u), \forall (t,u) \in T_q \times \vartheta_\gamma,\ \ t \ge t_0, \tag{3.4}$$

for all $\sigma \in \mathcal{K}$.

In the following, we establish the significant Lemmas, which will be principal role in this work.

Lemma 3.1. *Let $u \in \mathcal{C}(T_q,\mathbb{R})$ be a continuous and q-derivative function, then for any time $t \ge t_0$.*

$$\frac{1}{[2]_q}\, {}^{C}\mathcal{D}_{q,t_0}^{\alpha} u^2(t) \le u(t)\, {}^{C}\mathcal{D}_{q,t_0}^{\alpha} u(t), \quad \textit{for all } \alpha \in (0,1). \tag{3.5}$$

Proof. Proving that statement (3.5) is valid, is equivalent to prove that

$$u(t)\,{}^{C}\mathcal{D}^{\alpha}_{q,t_0}u(t)-\frac{1}{[2]_q}\,{}^{C}\mathcal{D}^{\alpha}_{q,t_0}u^2(t)\geq 0,\qquad \text{for all } \alpha\in(0,1)\,, \tag{3.6}$$

using Definition of the CAPUTO fractional q-derivative, given by

$${}^{C}\mathcal{D}^{\alpha}_{q,t_0}u(t)=\frac{1}{\Gamma_q(1-\alpha)}\int_{t_0}^{t}(t-qs)_q^{-\alpha}\,D_q(u(s))\,d_qs, \tag{3.7}$$

in the same way, we have

$$\begin{aligned}\frac{1}{[2]_q}\,{}^{C}\mathcal{D}^{\alpha}_{q,t_0}u^2(t) &= \frac{1}{[2]_q\,\Gamma_q(1-\alpha)}\int_{t_0}^{t}(t-qs)_q^{-\alpha}\,D_q\left(u^2(s)\right)d_qs\\ &= \frac{{}^{[2]_q}}{[2]_q\,\Gamma_q(1-\alpha)}\int_{t_0}^{t}(t-qs)_q^{-\alpha}\,u(s)\,D_q(u(s))\,d_qs\\ &= \frac{1}{\Gamma_q(1-\alpha)}\int_{t_0}^{t}(t-qs)_q^{-\alpha}\,u(s)\,D_q(u(s))\,d_qs.\end{aligned}$$

Then, the statement (3.6) takes the form

$$\frac{1}{\Gamma_q(1-\alpha)}\int_{t_0}^{t}(t-qs)_q^{-\alpha}\,(u(t)-u(s))\,D_q(u(s))\,d_qs\geq 0, \tag{3.8}$$

we put $w(s)=u(t)-u(s)$, thus ${}_sD_q(w(s))=-D_q(u(s))$, then statement (3.8), can be written as

$$\frac{-1}{\Gamma_q(1-\alpha)}\int_{t_0}^{t}(t-qs)_q^{-\alpha}\,w(s)\;{}_sD_qw(s)\;d_qs\geq 0. \tag{3.9}$$

We know that ${}_sD_qw^2(s)=[2]_q\,w(s)\;{}_sD_qw(s)$, thus

$$w(s)\;{}_sD_qw(s)=\frac{1}{[2]_q}\;{}_sD_qw^2(s)\,,$$

then, the statement (3.9) it is written as follows

$$\frac{-1}{[2]_q\,\Gamma_q(1-\alpha)}\int_{t_0}^{t}(t-qs)_q^{-\alpha}\;{}_sD_qw^2(s)\;d_qs\geq 0, \tag{3.10}$$

by using q-integration by part, we have

$$\begin{aligned} D_q\left[(t-s)_q^{-\alpha} w^2(s)\right] &= (t-qs)_q^{-\alpha}\, {}_sD_q w^2(s) + w^2(s)\, {}_sD_q (t-s)_q^{-\alpha}, \\ &= (t-qs)_q^{-\alpha}\, {}_sD_q w^2(s) + [\alpha]_q (t-qs)_q^{-\alpha-1} w^2(s), \end{aligned}$$

this gives

$$(t-qs)_q^{-\alpha}\, {}_sD_q w^2(s) = {}_sD_q\left[(t-s)_q^{-\alpha} w^2(s)\right] - [\alpha]_q (t-qs)_q^{-\alpha-1} w^2(s),$$

then substituting in (3.10), we find

$$\begin{aligned} & u(t)\, {}^C\mathcal{D}^{\alpha}_{q,t_0} u(t) - \frac{1}{[2]_q}\, {}^C\mathcal{D}^{\alpha}_{q,t_0} u^2(t) \\ = & \frac{-1}{[2]_q \Gamma_q(1-\alpha)} \int_{t_0}^{t} {}_sD_q\left[(t-s)_q^{-\alpha} w^2(s)\right] - [\alpha]_q (t-qs)_q^{-\alpha-1} w^2(s)\, d_q s \\ = & \frac{-1}{[2]_q \Gamma_q(1-\alpha)} \int_{t_0}^{t} {}_sD_q\left[(t-s)_q^{-\alpha} w^2(s)\right] d_q s + \frac{[\alpha]_q}{[2]_q \Gamma_q(1-\alpha)} \int_{t_0}^{t} (t-qs)_q^{-\alpha-1} w^2(s)\, d_q s \\ = & \frac{(t-t_0)_q^{-\alpha}}{[2]_q \Gamma_q(1-\alpha)} w^2(t_0) + \frac{[\alpha]_q}{[2]_q \Gamma_q(1-\alpha)} \int_{t_0}^{t} (t-qs)_q^{-\alpha-1} w^2(s)\, d_q s \geq 0. \end{aligned} \tag{3.11}$$

The statement (3.11) is evidently true. Good, the proof is correct. □

Corollary 3.1. When $u \in [\mathcal{C}(T_q, \mathbb{R})]^n$, Lemma 3.1 is still valid. For all $\alpha \in (0,1)$, $t \in T_q$

$$\frac{1}{[2]_q}\, {}^C\mathcal{D}^{\alpha}_{q,t_0}\left(u^{\top}(t)\, u(t)\right) \leq u^{\top}(t)\, {}^C\mathcal{D}^{\alpha}_{q,t_0} u(t). \tag{3.12}$$

The result follows directly and is obtained by decomposing expression (3.12) into a sum of scalar products, followed by the application of Lemma 3.1.

Remark 3.1.

1. When the function $u(t)$ is constant, the inequality (3.5) is satisfied.
2. When $\alpha = 1$, we obtain the product rule for integer order q-derivatives. In this case, the statement can be written as follows $\frac{1}{[2]_q} D_q u^2(t) \leq u(t)\, D_q u(t)$.

Hence, this case can be regarded as a specific application of Lemma 3.1.

Lemma 3.2 (q-Fractional Comparison Principle)**.** *The comparison Lemma for q-FOS. If*

$$ {}^{C}\mathcal{D}^{\alpha}_{q,t_0}\omega(t) \geq {}^{C}\mathcal{D}^{\alpha}_{q,t_0}\varphi(t), \text{ and } \omega(t_0)=\varphi(t_0), \quad \text{then } \omega(t)\geq\varphi(t), $$

for $\alpha\in(0,1)$ *and* $0<q<1$.

Proof. Assume that ${}^{C}\mathcal{D}^{\alpha}_{q,t_0}\omega(t)=f_1(t,\omega)$ and ${}^{C}\mathcal{D}^{\alpha}_{q,t_0}\varphi(t)=f_2(t,\varphi)$, then we have $f_1(t,\omega)\geq f_2(t,\varphi)$. Using the equivalent Volterra q-integral

$$ \begin{aligned} \omega(t) &= \omega(t_0)+\int_{t_0}^{t} G_\alpha(t,qs)\, f_1(s,\omega)\, d_qs, \\ \varphi(t) &= \varphi(t_0)+\int_{t_0}^{t} G_\alpha(t,qs)\, f_2(s,\varphi)\, d_qs, \end{aligned} $$

hence

$$ \varphi(t)=\omega(t)+\int_{t_0}^{t} G_\alpha(t,qs)\left(f_2(s,\varphi)-f_1(s,\omega)\right) d_qs, \tag{3.13} $$

where $G_\alpha(t,qs)=\frac{(t-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)}$.
For $0<qs<s<t$, we have $t-qs>0$ this implies $(t-qs)_q^{\alpha-1}>0$, and $0<\alpha<1$ then $\Gamma_q(\alpha)>0$, thus $G_\alpha(t,qs)>0$ and $f_1(t,\omega)\geq f_2(t,\varphi)$, this implies that $f_2(s,\varphi)-f_1(s,\omega)\leq 0$. From (3.13) conclude that $\varphi(t)-\omega(t)\leq 0$ then $\omega(t)\geq\varphi(t)$. □

Theorem 3.3. *Let* $u=0$ *be an stationary point of the following q-FOS*

$$ {}^{C}\mathcal{D}^{\alpha}_{q,t_0}u(t)=f(t,u), \tag{3.14} $$

where $u\in\mathcal{C}(T_q,\mathbb{R})$*, and* $\alpha\in(0,1)$, $0<q<1$, *if the following conditions is verified.*

1. The stationary point of the system (3.14) is stable, if $u(t)\,f(t,u)\leq 0, \forall u$.
2. The stationary point of the system (3.14) is asymptotically stable, if $u(t)\,f(t,u)<0$, $\forall u\neq 0$.

Proof. To prove the stationary point of the system (3.14) is stable. We present the Lyapunov function, which is positive definite

$$ V(u(t))=\frac{1}{[2]_q}u^2(t), \tag{3.15} $$

using Lemma 3.1, we get

$$ {}^{C}\mathcal{D}^{\alpha}_{q,t_0} V\left(u\left(t\right)\right) \leq u\left(t\right) {}^{C}\mathcal{D}^{\alpha}_{q,t_0} u\left(t\right) = u\left(t\right) f\left(t,u\right). \tag{3.16} $$

If

$$ u\left(t\right) f\left(t,u\right) \leq 0, \text{ then } u\left(t\right) {}^{C}\mathcal{D}^{\alpha}_{q,t_0} u\left(t\right) \leq 0, \ \forall u, $$

hence, from inequality (3.16), we get ${}^{C}\mathcal{D}^{\alpha}_{q,t_0} V\left(u\left(t\right)\right) \leq 0$, then the q-fractional derivative of Lyapunov function is negative semi definite. Therefore, using the q-fractional comparison principle of Lemma 3.2. We set $V\left(t\right) = V\left(u\left(t\right)\right)$ and assume that,

$$ \varphi\left(t\right) = V\left(t\right), $$

by applying ${}^{C}\mathcal{D}^{\alpha}_{q,t_0}$ on both sides, we get

$$ {}^{C}\mathcal{D}^{\alpha}_{q,t_0} \varphi\left(t\right) = {}^{C}\mathcal{D}^{\alpha}_{q,t_0} V\left(t\right) \leq 0, \ \forall t \geq t_0. $$

Let us assume that there exists constant function $\omega\left(t\right) = V\left(t_0\right)$, satisfies

$$ {}^{C}\mathcal{D}^{\alpha}_{q,t_0} \omega\left(t\right) = {}^{C}\mathcal{D}^{\alpha}_{q,t_0} V\left(t_0\right), \text{ then } {}^{C}\mathcal{D}^{\alpha}_{q,t_0} \omega\left(t\right) = 0, \text{ because } {}^{C}\mathcal{D}^{\alpha}_{q,t_0} \left(\text{Const}\right) = 0, $$

this implies that,

$$ {}^{C}\mathcal{D}^{\alpha}_{q,t_0} \omega\left(t\right) \geq {}^{C}\mathcal{D}^{\alpha}_{q,t_0} V\left(t\right). $$

Since $\omega\left(t_0\right) = V\left(t_0\right)$ at $t = t_0$, that is mean $\omega\left(t_0\right) = \varphi\left(t_0\right)$. By q-fractional comparison principle, we conclude that

$$ V\left(t\right) \leq \omega\left(t\right), \text{ then } V\left(t\right) \leq V\left(t_0\right), \forall t \geq t_0, $$

hence, $V\left(t\right)$ is decreasing, from (3.15), it follows that

$$ V\left(u\left(t\right)\right) \leq V\left(u\left(t_0\right)\right), \text{ then } \frac{1}{\left[2\right]_q} u^2\left(t\right) \leq \frac{1}{\left[2\right]_q} u^2\left(t_0\right), \forall t \geq t_0. \tag{3.17} $$

Based on the Definition of stability in the sense of Lyapunov [46].
For any $\varepsilon > 0$, choose $\vartheta = \varepsilon$, such that $V\left(u\left(t\right)\right) \leq V\left(u\left(t_0\right)\right), \forall t \geq t_0$, then $V\left(u\left(t\right)\right)$ is decreasing.

Hence, if

$$|u(t_0)| \leq \vartheta = \varepsilon, \text{ then } |u(t)| \leq \varepsilon,$$

then, the statement (3.17) allows concluding that the origin of the system (3.14), is stable in the sense of Lyapunov.
We now turn to the second proof. If

$$u(t) f(t,u) < 0, \text{ then } u(t) \, {}^{C}\mathcal{D}^{\alpha}_{q,t_0} u(t) < 0, \ \forall u \neq 0,$$

from inequality (3.16), we get ${}^{C}\mathcal{D}^{\alpha}_{q,t_0} V(u(t)) < 0$, then the q-fractional derivative of Lyapunov function is negative definite. Since $V(u(t))$ is positive definite and strictly decreasing, then there exists V_0, where $V_0 = \lim_{t\to\infty} V(u(t)) \geq 0$.
Currently, $V_0 = 0$. If that is not case, suppose that $V_0 > 0$ since $V(u(t)) \geq V_0$.
Hence, from Theorem 3.2, we have

$$\sigma V(u(t)) \geq \sigma V_0. \text{ Consequently } -\sigma V(u(t)) < -\sigma V_0.$$

As a result, we get

$${}^{C}\mathcal{D}^{\alpha}_{q,t_0} V(u(t)) \leq -\sigma V_0,$$

using Lemma (1.1) for $0 < \alpha < 1$, we have

$$V(u(t)) \leq V(u(t_0)) - \sigma V_0 \frac{(t-t_0)^{\alpha}_{q}}{\Gamma_q(\alpha+1)}.$$

Therefore, we obtain $\lim_{t\to\infty} V(u(t)) = -\infty$, because $\lim_{t\to\infty} (t-t_0)^{\alpha}_{q} = \infty$, which contradicts the hypothesis stating that $V(u(t)) \geq 0$, then $V_0 > 0$ is not possible.
Consequently, we take $V_0 = 0$, then

$$\lim_{t\to\infty} V(u(t)) = V_0 = 0, \text{ we concloud that } |u(t)| \to 0,$$

then, the origin of the system (3.14) is asymptotically stable. □

Remark 3.2. If $u \in [\mathcal{C}(T_q, \mathbb{R})]^n$, that is the system (3.14) is vectorial. Theorem 3.3 remains valid. The proof can be easily verified, by relying on a Lyapunov candidate function defined by $V(u(t)) = \frac{1}{[2]_q} u^{\top}(t) u(t)$ and amploying Lemma 3.1.

3.2 Existence and Uniqueness Results

To investigate the existence and uniqueness of solutions to this problem (3.1)-(3.2), one may refer to second chapter.

3.3 Applications

In the following, we present some examples to illustrate the stability of q-FOS.

Example 3.1. We propose the following q-fractional order nonlinear system, where $\alpha \in (0,1)$ and $0 < q < 1$

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{q,0} u(t) = v(t) - u(t)\left(u^2(t) + v^2(t)\right), \\ {}^{C}\mathcal{D}^{\alpha}_{q,0} v(t) = -u(t) - v(t)\left(u^2(t) + v^2(t)\right), \end{cases} \tag{3.18}$$

with initial conditions

$$u(0) = u_0, \text{ and } v(0) = v_0,$$

To prove the stability of system (3.18), proposing the following quadratic function as a Lyapunov candidate, which is positive definite

$$V(u,v)(t) = \frac{1}{[2]_q} u^2(t) + \frac{1}{[2]_q} v^2(t). \tag{3.19}$$

From Lemma 3.1, it can be found that

$$\begin{aligned} {}^{C}\mathcal{D}^{\alpha}_{q,0} V(u,v)(t) &= \frac{1}{[2]_q}\, {}^{C}\mathcal{D}^{\alpha}_{q,0} u^2(t) + \frac{1}{[2]_q}\, {}^{C}\mathcal{D}^{\alpha}_{q,0} v^2(t) \qquad (3.20) \\ &\leq u(t)\, {}^{C}_{0}\mathcal{D}^{\alpha}_{q,t} u(t) + v(t)\, {}^{C}_{0}\mathcal{D}^{\alpha}_{q,t} v(t) \\ &= u(t)\left[v(t) - u(t)\left(u^2(t) + v^2(t)\right)\right] \\ &\quad + v(t)\left[-u(t) - v(t)\left(u^2(t) + v^2(t)\right)\right] \\ &= -\left[u^2(t) + v^2(t)\right]^2 \leq 0. \qquad (3.21) \end{aligned}$$

From item 1, the statement (3.21) of q-fractional derivative of the Lyapunov function ${}^{C}\mathcal{D}^{\alpha}_{q,0}V\left(u,v\right)\left(t\right)\leq 0$, is negative semi-definite.
Now, using the comparison principle of Lemma 3.2.
We set $V\left(t\right)=V\left(u,v\right)\left(t\right)$ and assume that

$$\varphi\left(t\right)=V\left(t\right),$$

then

$${}^{C}\mathcal{D}^{\alpha}_{q,0}\varphi\left(t\right)={}^{C}\mathcal{D}^{\alpha}_{q,0}V\left(t\right)\leq 0\ ,\ \forall t\geq 0.$$

So, consider the constant function $\omega\left(t\right)=V\left(0\right)$, and we have

$${}^{C}\mathcal{D}^{\alpha}_{q,0}\omega\left(t\right)={}^{C}\mathcal{D}^{\alpha}_{q,0}V\left(0\right)=0,\ \text{then}\ {}^{C}\mathcal{D}^{\alpha}_{q,0}\omega\left(t\right)=0\ ,$$

hence

$${}^{C}\mathcal{D}^{\alpha}_{q,0}\omega\left(t\right)\geq{}^{C}\mathcal{D}^{\alpha}_{q,0}V\left(t\right),$$

since $\varphi\left(t\right)=V\left(t\right)$ and we have $\omega\left(0\right)=V\left(0\right)$ at $t=0$, then $\omega\left(0\right)=\varphi\left(0\right)$ at $t=0$.
We applied the comparison, we conclude that

$$V\left(t\right)\leq\omega\left(t\right)\ \text{then}\ V\left(t\right)\leq V\left(0\right),\ \text{this implies that}\ V\left(u,v\right)\left(t\right)\leq V\left(u,v\right)\left(0\right),\forall u,v.$$

From (3.19), then

$$\frac{1}{[2]_q}u^2\left(t\right)+\frac{1}{[2]_q}v^2\left(t\right)\leq\frac{1}{[2]_q}u^2\left(0\right)+\frac{1}{[2]_q}v^2\left(0\right),\forall t\geq 0, \tag{3.22}$$

according to the Definition of the stability in the sense of Lyapunov [46], the statement (3.22), show that the origin of system (3.18) is stable in the sense of Lyapunov.

Example 3.2. We propose the following q-fractional-order nonlinear system, where $\alpha\in(0,1)$ and $0<q<1$

$$\begin{cases}{}^{C}\mathcal{D}^{\alpha}_{q,0}u\left(t\right)=-u\left(t\right)\left(u^2\left(t\right)+v^2\left(t\right)+[2]_q\right)+[4]_q\,u\left(t\right)v^2\left(t\right),\\ {}^{C}\mathcal{D}^{\alpha}_{q,0}v\left(t\right)=-[4]_q\,u^2\left(t\right)v\left(t\right)-v\left(t\right)\left(u^2\left(t\right)+v^2\left(t\right)+[2]_q\right),\end{cases} \tag{3.23}$$

with initial conditions

$$u(0) = u_0 \text{ , and } v(0) = v_0. \tag{3.24}$$

Similarly, we propose Lyapunov's direct method, let the quadratic function as a Lyapunov candidate, which is positive definite

$$V(u,v)(t) = \frac{1}{[2]_q} u^2(t) + \frac{1}{[2]_q} v^2(t). \tag{3.25}$$

From Lemma 3.1, using the Lyapunov candidate function (3.25), it is evident that

$$\begin{aligned}
{}^{C}\mathcal{D}^{\alpha}_{q,0} V(u,v)(t) &= \frac{1}{[2]_q} {}^{C}\mathcal{D}^{\alpha}_{q,0} u^2(t) + \frac{1}{[2]_q} {}^{C}\mathcal{D}^{\alpha}_{q,0} v^2(t) \\
&\leq u(t) {}^{C}\mathcal{D}^{\alpha}_{q,0} u(t) + v(t) {}^{C}\mathcal{D}^{\alpha}_{q,0} v(t) \\
&= u(t)\left[-u(t)\left(u^2(t) + v^2(t) + [2]_q\right) + [4]_q u(t) v^2(t)\right] \\
&\quad + v(t)\left[-[4]_q u^2(t) v(t) - v(t)\left(u^2(t) + v^2(t) + [2]_q\right)\right] \\
&= -[2]_q \left(u^2(t) + v^2(t)\right) < 0.
\end{aligned} \tag{3.26}$$

From item 2, the statement (3.26) show that the q-fractional derivative of the Lyapunov function ${}^{C}\mathcal{D}^{\alpha}_{q,0} V(u,v)(t) < 0$, is negative definite.
Now, by applying the inequality (3.4) of Theorem 3.2, we have

$$V(u,v)(t) = \frac{1}{[2]_q} u^2(t) + \frac{1}{[2]_q} v^2(t) \leq u^2(t) + v^2(t),$$

therefore

$$u^2(t) + v^2(t) \geq V(u,v)(t),$$

using (3.26), we obtain

$${}^{C}\mathcal{D}^{\alpha}_{q,0} V(u,v)(t) \leq -[2]_q V(u,v)(t).$$

Therefore, it can be concluded that the origin of system (3.23) is asymptotically stable in the sense of Lyapunov.

CHAPTER 4

BOUNDARY VALUE PROBLEM FOR NONLINEAR FRACTIONAL q-DIFFERENTIAL EQUATIONS

This chapter discusses and theoretically studies the properties of the q-Green's function and then proving the existence, uniqueness and Ulam-Hyers stability of boundary value problem (BVP) solutions for nonlinear fractional q-differential equation involving the Katugampola q-derivative.

$$ {}^{\rho}\mathcal{D}_{q,a}^{\alpha}\, u\left(t\right) + f\left(t, u\left(t\right)\right) = 0, \quad t \in \left[a, b\right], b > a > 0, \tag{4.1} $$

with the boundary conditions

$$ u\left(a\right) = 0, \quad {}^{\rho}\mathcal{J}_{q,a}^{\eta} u\left(b\right) = 0, \tag{4.2} $$

where ${}^{\rho}\mathcal{D}_{q,a}^{\alpha}$ and ${}^{\rho}\mathcal{J}_{q,a}^{\eta}$ represent the Katugampola fractional q-derivative and q-integral of order $1 < \alpha \leq 2$ and $0 < \eta \leq 1$ respectively, with $\rho > 0$, $0 < q < 1$, and $f : [a, b] \times \mathbb{C} \to \mathbb{C}$ be a continuous function.

4.1 Fundamental Lemmas

In this section, we define the space $\mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$ to study the BVP (4.1)-(4.2), which is denoted by

$$\mathcal{L}^1_{q,\rho}([a,b],\mathbb{C}) = \left\{u : [a,b] \to \mathbb{C},\ \|u\|_{\mathcal{L}^1_{q,\rho}} < \infty\right\},$$

$\mathcal{L}^1_{q,\rho}$ is a Banach space endowed with the norm

$$\|u\|_{\mathcal{L}^1_{q,\rho}} = \int_a^b t^{\rho-1} |u(t)|\, d_q t < \infty.$$

In the following, we first present some significant Lemmas to illustrate our main results.

Lemma 4.1. *For any $\alpha, \rho > 0$, if $u \in \mathcal{L}^1_{q,\rho}[a,b]$. Then the fractional q-differential equation of the type Katugampola*

$$ {}^{\rho}\mathcal{D}^{\alpha}_{q,a} u(t) = 0, \text{ for } t \in [a,b], \tag{4.3}$$

has a unique solution

$$u(t) = C_1 (t^\rho - a^\rho)^{(\alpha-1)}_{q^\rho} + C_2 (t^\rho - a^\rho)^{(\alpha-2)}_{q^\rho} + \cdots + C_n (t^\rho - a^\rho)^{(\alpha-n)}_{q^\rho}, \tag{4.4}$$

where $C_k \in \mathbb{R}$, with $k = 1, 2, \ldots, n$.

Proof. Let $\alpha > 0$, from Lemma 1.8, we have

$$ {}^{\rho}\mathcal{D}^{\alpha}_{q,a} (t^\rho - a^\rho)^{(\alpha-k)}_{q^\rho} = 0, \text{ for each } k = 1, 2, \ldots, n.$$

Consequently, the fractional q-differential equation (4.3), admits the following particular solution

$$u(t) = C_k (t^\rho - a^\rho)^{(\alpha-k)}_{q^\rho},\ C_k \in \mathbb{R}, \text{ for each } k = 1, 2, \ldots, n. \tag{4.5}$$

Thus, the general solution of (4.3), is a sum particular solution (4.5), given by the formula (4.4). □

We now proceed to presenting the q-Green's integral equation of the BVP (4.1)-(4.2).

Lemma 4.2 (The q-Green's Integral Equation)**.** *Let $1 < \alpha \leq 2$, $0 < \eta \leq 1$, and $\alpha, \rho > 0$, $q \in (0,1)$, we give $u \in \mathcal{L}^1_{q,\rho}([a,b])$ and $f(t,u)$ is a continuous functions. Then the BVP (4.1)-(4.2) has a unique solution as follows*

$$u(t) = \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, u(s))\, d_q s, \text{ for } t \in [a,b], \tag{4.6}$$

where $G_{q^\rho}(t,s)$ represents the q-Green's function, given by

$$G_{q^\rho}(t,s) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \begin{cases} \left[\frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} (b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} - (t^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \right], & a \leq s \leq t \leq b, \\ \frac{(b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)}, & a \leq t \leq s \leq b. \end{cases} \tag{4.7}$$

Proof. Let $u \in \mathcal{L}^1_{q,\rho}([a,b], \mathbb{C})$, satisfies the BVP (4.1)-(4.2). Firstly, by applying ${}^\rho\mathcal{J}^\alpha_{q,a}$ on both sides of Eq (4.1)

$$\left({}^\rho\mathcal{J}^\alpha_{q,a}\, {}^\rho\mathcal{D}^\alpha_{q,a} u\right)(t) + {}^\rho\mathcal{J}^\alpha_{q,a} f(t, u(t)) = 0, \tag{4.8}$$

from Lemma (1.5), and for $1 < \alpha \leq 2$, we get

$$\left({}^\rho\mathcal{J}^\alpha_{q,a}\, {}^\rho\mathcal{D}^\alpha_{q,a} u\right)(t) = u(t) + C_1 (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} + C_2 (t^\rho - a^\rho)_{q^\rho}^{(\alpha-2)},$$

for some $C_1, C_2 \in \mathbb{R}$, thus (4.8), gives

$$u(t) = -C_1 (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} - C_2 (t^\rho - a^\rho)_{q^\rho}^{(\alpha-2)} + \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \int_a^t s^{\rho-1} (t^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} f(s, u(s))\, d_q s.$$

On the other hand, using the conditions (4.2), then

$$u(a) = 0, \text{ this implies, } 0 = 0 - 0 - C_2 \lim_{t \to a} (t^\rho - a^\rho)_{q^\rho}^{(\alpha-2)}, \text{ then } C_2 = 0.$$

we easily obtain

$$u(t) = -C_1 (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} - {}^\rho\mathcal{J}^\alpha_{q,a} f(t, u(t)), \tag{4.9}$$

we proceed to the second condition, we have

$$\begin{aligned} {}^{\rho}\mathcal{J}_{q,a}^{\eta} u(t) &= -C_1\, {}^{\rho}\mathcal{J}_{q,a}^{\eta}\left(t^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\alpha-1)} - {}^{\rho}\mathcal{J}_{q,a}^{\eta}\, {}^{\rho}\mathcal{J}_{q,a}^{\alpha} f(t,u(t)) \\ &= -C_1\left[\frac{\Gamma_{q^{\rho}}(\alpha)\left(t^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)}}{[\rho]_q^{\eta}\,\Gamma_{q^{\rho}}(\eta+\alpha)}\right] - {}^{\rho}\mathcal{J}_{q,a}^{\eta+\alpha} f(t,u(t)), \end{aligned}$$

then

$$\begin{aligned} {}^{\rho}\mathcal{J}_{q,a}^{\eta} u(b) &= -C_1\left[\frac{\Gamma_{q^{\rho}}(\alpha)\left(b^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)}}{[\rho]_q^{\eta}\,\Gamma_{q^{\rho}}(\eta+\alpha)}\right] \\ &\quad -\frac{[\rho]_q^{1-(\eta+\alpha)}}{\Gamma_{q^{\rho}}(\eta+\alpha)}\int_a^b s^{\rho-1}\left(b^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)} f(s,u(s))\, d_q s, \end{aligned}$$

we have ${}^{\rho}\mathcal{J}_{q,a}^{\eta} u(b)=0$, satisfies

$$0 = -C_1\left[\frac{\Gamma_{q^{\rho}}(\alpha)\left(b^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)}}{[\rho]_q^{\eta}\,\Gamma_{q^{\rho}}(\eta+\alpha)}\right] - \frac{[\rho]_q^{1-(\eta+\alpha)}}{\Gamma_{q^{\rho}}(\eta+\alpha)}\int_a^b s^{\rho-1}\left(b^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)} f(s,u(s))\, d_q s.$$

Hence, C_1 given as follows

$$C_1 = -\frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^{\rho}}(\alpha)\left(b^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)}}\int_a^b s^{\rho-1}\left(b^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)} f(s,u(s))\, d_q s,$$

by substituting in (4.9), we get

$$\begin{aligned} u(t) &= \frac{[\rho]_q^{1-\alpha}\left(t^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\alpha-1)}}{\Gamma_{q^{\rho}}(\alpha)\left(b^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)}}\int_a^b s^{\rho-1}\left(b^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)} f(s,u(s))\, d_q s \\ &\quad -\frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^{\rho}}(\alpha)}\int_a^t s^{\rho-1}\left(t^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\alpha-1)} f(s,u(s))\, d_q s \\ &= \frac{[\rho]_q^{1-\alpha}\left(t^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\alpha-1)}}{\Gamma_{q^{\rho}}(\alpha)\left(b^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)}}\left[\int_a^t s^{\rho-1}\left(b^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)} f(s,u(s))\, d_q s\right. \\ &\quad \left.+\int_t^b s^{\rho-1}\left(b^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\eta+\alpha-1)} f(s,u(s))\, d_q s\right] - \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^{\rho}}(\alpha)}\int_a^t s^{\rho-1}\left(t^{\rho}-(qs)^{\rho}\right)_{q^{\rho}}^{(\alpha-1)} f(s,u(s))\, d_q s, \end{aligned}$$

we easily obtain

$$\begin{aligned} u(t) &= \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \int_a^t s^{\rho-1} \left[\frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} (b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} - (t^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \right] f(s, u(s)) \, d_q s \\ &\quad + \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \int_t^b s^{\rho-1} \frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} (b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} f(s, u(s)) \, d_q s. \end{aligned}$$

Hence

$$u(t) = \int_a^b G_{q^\rho}(t, s) \, s^{\rho-1} f(s, u(s)) \, d_q s,$$

where $G_{q^\rho}(t, s)$ given by (4.7). The proof is complete.

□

In the following Lemma, we study the essential properties of the q-Green's function.

Lemma 4.3. *Let $1 < \alpha \leq 2$, $0 < \eta \leq 1$, $\rho > 0$, and $q \in (0, 1)$. The q-Green's function $G_{q^\rho}(t, s)$ defined by (4.7), satisfies the following conditions*

1. $G_{q^\rho}(t, s) \geq 0, \forall (t, s) \in [a, b] \times [a, b]$.
2. $G_{q^\rho}(t, s) \leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} (b^\rho - a^\rho)_{q^\rho}^{(\alpha-1)}, \forall (t, s) \in [a, b] \times [a, b]$.

Proof. 1. Let $1 < \alpha \leq 2$, $0 < \eta \leq 1$, $\eta + \alpha \in (1, 3]$, $\rho > 0$, and $q \in (0, 1)$.
In the case $a \leq t \leq s \leq b$, we have

$$G_{q^\rho}(t, s) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[\frac{(b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \right].$$

Note that $[\rho]_q^{1-\alpha} > 0$, and $\Gamma_{q^\rho}(\alpha) > 0$ for $\alpha > 0$ then $\frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} > 0$, $(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)} > 0$, $(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \geq 0$ for $t \geq a$ and $(b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)} \geq 0$, for $s \leq b$, (as $qs \leq s$, $0 < q < 1$ and $\eta + \alpha \in (1, 3]$). It follows that

$$\frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[\frac{(b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \right] \geq 0.$$

Hence, we get $G_{q^\rho}(t, s) \geq 0$.

In the case $a \leq s \leq t \leq b$, we have

$$G_{q^\rho}(t,s) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[\frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} (b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} - (t^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \right].$$

Since $s \leq t$, the q-power function is monotonic, then $(t^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \leq (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)}$. Furthermore, as $a \leq qs \leq s \leq t \leq b$, we have $(b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)} \leq (b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}$, thus

$$\frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} (b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} \leq (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)},$$

we get

$$G_{q^\rho}(t,s) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} - (t^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \right] \geq 0.$$

Consequently, we obtain $G_{q^\rho}(t,s) \geq 0$. For the both cases, we conclude that $G_{q^\rho}(t,s) \geq 0$, $\forall (t,s) \in [a,b] \times [a,b]$.

2. we proceed to the second prove, we know that $G_{q^\rho}(t,s) \geq 0$, then $|G_{q^\rho}(t,s)| = G_{q^\rho}(t,s)$. In the case $a \leq t \leq s \leq b$, we have

$$G_{q^\rho}(t,s) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[\frac{(b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \right],$$

for $a \leq t \leq b$, we have

$$t^\rho - a^\rho \leq b^\rho - a^\rho \text{ then } (t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \leq (b^\rho - a^\rho)_{q^\rho}^{(\alpha-1)},$$

also

$$(b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)} \leq (b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)},$$

then

$$G_{q^\rho}(t,s) \leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} (b^\rho - a^\rho)_{q^\rho}^{(\alpha-1)}.$$

In the case $a \leq s \leq t \leq b$, we have

$$G_{q^\rho}(t,s) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[\frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} (b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} - (t^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \right],$$

since $G_{q^\rho}(t,s) \geq 0$, then

$$\begin{aligned} G_{q^\rho}(t,s) &\leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[\frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} (b^\rho - (qs)^\rho)_{q^\rho}^{(\eta+\alpha-1)}}{(b^\rho - a^\rho)_{q^\rho}^{(\eta+\alpha-1)}} \right] \\ &\leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} (b^\rho - a^\rho)_{q^\rho}^{(\alpha-1)}, \end{aligned}$$

we conclude that the q-Green's functions is bounded in $[a,b] \times [a,b]$.
The proof is complete. □

We now state some hypotheses that will be needed throughout this work:
$(\mathcal{H}_1)$ $f : [a,b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function such that

$$f(\cdot, u(\cdot)) \in \mathcal{L}_{q,\rho}^1([a,b],\mathbb{C}), \quad u \in \mathcal{L}_{q,\rho}^1([a,b],\mathbb{C}).$$

$(\mathcal{H}_2)$ For any $u, \tilde{u} \in \mathbb{C}$, there exist nonnegative constant ϖ, such that the function f satisfies

$$|f(t,u) - f(t,\tilde{u})| \leq \varpi |u - \tilde{u}|.$$

$(\mathcal{H}_3)$ There exist two nonnegative functions $\varphi \in \mathcal{L}_{q,\rho}^1([a,b],\mathbb{R}_+)$ and $\omega \in \mathcal{C}([a,b],\mathbb{R}_+)$, such that the function f satisfies

$$|f(t,u)| \leq \varphi(t) + \omega(t)|u(t)|,$$

for any $u \in \mathbb{C}$ and for each $t \in [a,b]$. Additionally, we introduce the notation

$$\varphi^* = \|\varphi\|_{\mathcal{L}_{q,\rho}^1}, \quad \omega^* = \sup_{t\in[a,b]} |\omega(t)|.$$

Let us define the integral operator $\mathcal{T} : \mathcal{L}_{q,\rho}^1([a,b],\mathbb{C}) \to \mathcal{L}_{q,\rho}^1([a,b],\mathbb{C})$ by

$$\mathcal{T}u(t) = \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s,u(s))\, d_q s,$$

with $G_{q^\rho}(t,s)$ is defined by (4.7). Therefore $\mathcal{T}u$ is an element of $\mathcal{L}_{q,\rho}^1([a,b],\mathbb{C})$ endowed with the norm

$$\|\mathcal{T}u\|_{\mathcal{L}_{q,\rho}^1} = \int_a^b t^{\rho-1} |\mathcal{T}u(t)|\, d_q t.$$

We now present the following Lemma, which is essential for the existence and uniqueness results.

Lemma 4.4. *If $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold. Then $\mathcal{T}$ is a completly continuous*

Proof. Let $u \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$, we have

$$\begin{aligned}
\|\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}} &= \int_a^b t^{\rho-1}\left|\mathcal{T}u(t)\right| d_qt \\
&= \int_a^b t^{\rho-1}\left|\int_a^t G_{q^\rho}(t,s)\, s^{\rho-1} f(s,u(s))\, d_qs\right| d_qt \\
&\leq \int_a^b s^{\rho-1}\left|f(s,u(s))\right|\left[\int_{qs}^b t^{\rho-1}\left|G_{q^\rho}(t,s)\right| d_qt\right] d_qs.
\end{aligned}$$

By Lemma (4.3), the q-Green's function $G_{q^\rho}(t,s) \geq 0$, implying that $|G_{q^\rho}(t,s)| = G_{q^\rho}(t,s)$. Moreover, the q-Green's function satisfies a boundedness property

$$G_{q^\rho}(t,s) \leq \mathcal{M} = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)}\left(b^\rho - a^\rho\right)^{(\alpha-1)}_{q^\rho},$$

we get

$$\begin{aligned}
\|\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}} &\leq \mathcal{M}\int_a^b s^{\rho-1}\left|f(s,u(s))\right|\left[\int_{qs}^b t^{\rho-1} d_qt\right] d_qs \\
&\leq \mathcal{M}\int_a^b s^{\rho-1}\left|f(s,u(s))\right|\left[\int_{qs}^b \frac{{}_tD_q\left[t^\rho\right]}{[\rho]_q} d_qt\right] d_qs \\
&\leq \frac{\mathcal{M}}{[\rho]_q}\int_a^b s^{\rho-1}\left|f(s,u(s))\right|\left(b^\rho - (qs)^\rho\right) d_qs \\
&\leq \frac{\mathcal{M}q^\rho\left(b^\rho - a^\rho\right)}{[\rho]_q}\int_a^b s^{\rho-1}\left|f(s,u(s))\right| d_qs \\
&\leq \frac{\mathcal{M}q^\rho\left(b^\rho - a^\rho\right)}{[\rho]_q}\left\|f(t,u)\right\|_{\mathcal{L}^1_{q,\rho}}.
\end{aligned}$$

By hypothesis $(\mathcal{H}_3)$, satisfies

$$\begin{aligned}
\|f(t,u)\|_{\mathcal{L}^1_{q,\rho}} &\leq \|\varphi\|_{\mathcal{L}^1_{q,\rho}} + \sup_{t\in[a,b]}|\omega(t)|\,\|u\|_{\mathcal{L}^1_{q,\rho}} \\
&\leq \varphi^* + \omega^*\|u\|_{\mathcal{L}^1_{q,\rho}}.
\end{aligned}$$

Consequently

$$\|\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{\mathcal{M}q^\rho\left(b^\rho-a^\rho\right)}{[\rho]_q}\left(\varphi^*+\omega^*\|u\|_{\mathcal{L}^1_{q,\rho}}\right)<\infty. \tag{4.10}$$

Hence, we deduce $\mathcal{T}:\mathcal{L}^1_{q,\rho}\left([a,b],\mathbb{C}\right)\to\mathcal{L}^1_{q,\rho}\left([a,b],\mathbb{C}\right)$ is well defined.

In what follows, the proof is completed in several steps.

Step 1: $\mathcal{T}$ is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence to u in $\mathcal{L}^1_{q,\rho}\left([a,b],\mathbb{C}\right)$, i,e,. $\lim\limits_{n\to\infty}u_n=u$, then

$$\begin{aligned}
\|\mathcal{T}u_n-\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}} &= \int_a^b t^{\rho-1}\left|\mathcal{T}u_n(t)-\mathcal{T}u(t)\right|d_qt\\
&= \int_a^b t^{\rho-1}\left|\int_a^t G_{q^\rho}(t,s)\,s^{\rho-1}\left(f(s,u_n(s))-f(s,u(s))\right)d_qs\right|d_qt\\
&\leq \int_a^b s^{\rho-1}\left|f(s,u_n(s))-f(s,u(s))\right|\left[\int_{qs}^b t^{\rho-1}G_{q^\rho}(t,s)\,d_qt\right]d_qs\\
&\leq \frac{\mathcal{M}q^\rho\left(b^\rho-a^\rho\right)}{[\rho]_q}\int_a^b s^{\rho-1}\left|f(s,u_n(s))-f(s,u(s))\right|d_qs\\
&\leq \frac{\mathcal{M}q^\rho\left(b^\rho-a^\rho\right)}{[\rho]_q}\left\|f(t,u_n)-f(t,u)\right\|_{\mathcal{L}^1_{q,\rho}}.
\end{aligned}$$

Hypothesis $(\mathcal{H}_2)$, satisfies

$$\|\mathcal{T}u_n-\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}}\leq\frac{\varpi\mathcal{M}q^\rho\left(b^\rho-a^\rho\right)}{[\rho]_q}\|u_n-u\|_{\mathcal{L}^1_{q,\rho}}.$$

Since $u_n\to u$ as $n\to\infty$, then $\|\mathcal{T}u_n-\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}}\to 0$ as $n\to\infty$.

Consequently, $\lim\limits_{n\to\infty}\|\mathcal{T}u_n-\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}}=0$, this gives the continuity of $\mathcal{T}$.

Step 2: $\mathcal{T}(\mathcal{P})$ is relatively compact.

We first begin show that $\mathcal{T}(\mathcal{P})$ is bounded, let $\mathcal{P}=\left\{u\in\mathcal{L}^1_{q,\rho}\left([a,b],\mathbb{C}\right),\ \|u\|_{\mathcal{L}^1_{q,\rho}}\leq r\right\}$ be a bounded subset of $\mathcal{L}^1_{q,\rho}\left([a,b],\mathbb{C}\right)$.

Let $u \in \mathcal{P}$, by hypothesis $(\mathcal{H}_3)$, we easily get

$$\begin{aligned}\|\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}} &\leq \frac{\mathcal{M}q^\rho\left(b^\rho-a^\rho\right)}{[\rho]_q}\|f\left(t,u\right)\|_{\mathcal{L}^1_{q,\rho}}\\ &\leq \frac{\mathcal{M}q^\rho\left(b^\rho-a^\rho\right)}{[\rho]_q}\left(\varphi^*+\omega^*\|u\|_{\mathcal{L}^1_{q,\rho}}\right)\\ &\leq \frac{\mathcal{M}q^\rho\left(b^\rho-a^\rho\right)}{[\rho]_q}\left(\varphi^*+\omega^* r\right),\end{aligned}$$

for all $u \in \mathcal{P}$, we deduce that $\mathcal{T}\left(\mathcal{P}\right)$ is bounded. In the sequel, we demonstrate that $\mathcal{T}$ $\left(\mathcal{P}\right)$ is equicontinuous in the next step.

Let $u \in \mathcal{P}$, and $t_1, t_2 \in [a,b]$, be such that $t_1 \leq t_2$, we have

$$\begin{aligned}\left|\mathcal{T}u\left(t_2\right)-\mathcal{T}u\left(t_1\right)\right| &\leq \int_a^b \left|G_{q^\rho}\left(t_2,s\right)-G_{q^\rho}\left(t_1,s\right)\right| s^{\rho-1}\left|f\left(s,u\left(s\right)\right)\right| d_q s\\ &\leq \int_a^{t_1} \left|G_{q^\rho}\left(t_2,s\right)-G_{q^\rho}\left(t_1,s\right)\right| s^{\rho-1}\left|f\left(s,u\left(s\right)\right)\right| d_q s\\ &\quad+\int_{t_1}^{t_2} \left|G_{q^\rho}\left(t_2,s\right)-G_{q^\rho}\left(t_1,s\right)\right| s^{\rho-1}\left|f\left(s,u\left(s\right)\right)\right| d_q s\\ &\quad+\int_{t_2}^{b} \left|G_{q^\rho}\left(t_2,s\right)-G_{q^\rho}\left(t_1,s\right)\right| s^{\rho-1}\left|f\left(s,u\left(s\right)\right)\right| d_q s.\end{aligned}$$

For $a \leq s \leq t_1 \leq t_2 \leq b$, we have

$$\begin{aligned}G_{q^\rho}\left(t_2,s\right)-G_{q^\rho}\left(t_1,s\right) &= \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}\left(\alpha\right)}\left[\left(\frac{\left(t_2^\rho-a^\rho\right)_{q^\rho}^{(\alpha-1)}\left(b^\rho-(qs)^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}{\left(b^\rho-a^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}-\left(t_2^\rho-(qs)^\rho\right)_{q^\rho}^{(\alpha-1)}\right)\right.\\ &\quad\left.-\left(\frac{\left(t_1^\rho-a^\rho\right)_{q^\rho}^{(\alpha-1)}\left(b^\rho-(qs)^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}{\left(b^\rho-a^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}-\left(t_1^\rho-(qs)^\rho\right)_{q^\rho}^{(\alpha-1)}\right)\right]\\ &= \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}\left(\alpha\right)}\left[\frac{\left(b^\rho-(qs)^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}{\left(b^\rho-a^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}\left(\left(t_2^\rho-a^\rho\right)_{q^\rho}^{(\alpha-1)}-\left(t_1^\rho-a^\rho\right)_{q^\rho}^{(\alpha-1)}\right)\right.\\ &\quad\left.-\left(\left(t_2^\rho-(qs)^\rho\right)_{q^\rho}^{(\alpha-1)}-\left(t_1^\rho-(qs)^\rho\right)_{q^\rho}^{(\alpha-1)}\right)\right]\\ &\leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}\left(\alpha\right)}\frac{\left(b^\rho-(qs)^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}{\left(b^\rho-a^\rho\right)_{q^\rho}^{(\eta+\alpha-1)}}\left(\left(t_2^\rho-a^\rho\right)_{q^\rho}^{(\alpha-1)}-\left(t_1^\rho-a^\rho\right)_{q^\rho}^{(\alpha-1)}\right).\end{aligned}$$

Since $a \leq qs \leq s \leq b$, we get

$$G_{q^\rho}(t_2, s) - G_{q^\rho}(t_1, s) \leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[(t_2^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} - (t_1^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \right]. \tag{4.11}$$

For $a \leq t_1 \leq s \leq t_2 \leq b$ and $a \leq t_1 \leq t_2 \leq s \leq b$, we find the same result of inequality (4.11). Consequently,

$$\begin{aligned} |\mathcal{T}u(t_2) - \mathcal{T}u(t_1)| &\leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left[(t_2^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} - (t_1^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \right] \int_a^b s^{\rho-1} |f(s, u(s))| \, d_q s \\ &\leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} (\varphi^* + \omega^* r) \left[(t_2^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} - (t_1^\rho - a^\rho)_{q^\rho}^{(\alpha-1)} \right]. \end{aligned} \tag{4.12}$$

For all $u \in \mathcal{P}$, as t_1 tends to t_2, the right hand side of the inequality (4.12) approaches zeros. Hence $\lim_{t_1 \to t_2} \|\mathcal{T}u_{t_2} - \mathcal{T}u_{t_1}\|_{\mathcal{L}^1_{q,\rho}} = 0$.

As a result, the operator $\mathcal{T}$ is continuous and maps bounded sets into relatively compact sets. Therefore, $\mathcal{T}$ is a compact operator, according to the Arzela–Ascoli Theorem. Consequently, the operator $\mathcal{T} : \mathcal{L}^1_{q,\rho}([a,b], \mathbb{C}) \to \mathcal{L}^1_{q,\rho}([a,b], \mathbb{C})$ is completely continuous. The proof is complete. □

4.2 Existence and Uniqueness Results

This section will demonstrate the existence and uniqueness of solutions for the BVP (4.1)-(4.2) using the properties of the Leray-Schauder's and Banach's fixed point Theorems.

Theorem 4.1. *Assume the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold, there exists $r > 0$, such that*

$$r > \frac{l_1}{1 - l_2}, \tag{4.13}$$

then, the q-fractional BVP (4.1)-(4.2) has at least one solution $u \in \mathcal{P}$.

Proof. By the previous Lemma 4.4, we have $\mathcal{T}$ is completely continuous operator, by an appropriate choice of r, the Leray-Schauder Theorem can be applied.
Let $u \in \partial\mathcal{P}$, and $\mu \in (0,1)$ with $u = \mu\mathcal{T}(u)$, we obtain

$$\|u\|_{\mathcal{L}^1_{q,\rho}} = \|\mu\mathcal{T}(u)\|_{\mathcal{L}^1_{q,\rho}} \leq \|\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}}, \text{ for all } \mu \in (0,1).$$

From (4.10), we have

$$\|\mathcal{T}u\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{\mathcal{M}q^\rho\left(b^\rho - a^\rho\right)}{[\rho]_q}\left(\varphi^* + \omega^* \|u\|_{\mathcal{L}^1_{q,\rho}}\right),$$

then

$$\begin{aligned}\|u\|_{\mathcal{L}^1_{q,\rho}} &\leq \frac{\mathcal{M}q^\rho\left(b^\rho - a^\rho\right)}{[\rho]_q}\left(\varphi^* + \omega^* \|u\|_{\mathcal{L}^1_{q,\rho}}\right)\\ &\leq l_1 + l_2 \|u\|_{\mathcal{L}^1_{q,\rho}}.\end{aligned}$$

If $l_2 < 1$, we get

$$\|u\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{l_1}{1-l_2},$$

by choosing $r > \frac{l_1}{1-l_2}$, we ensure that no solution on $\partial\mathcal{P}$ and from Lemma 4.4. By the Leary shauder's Theorem (see [29]), there exists a fixed point in $\mathcal{P}$. Hence, the BVP (4.1)-(4.2) has at least one solution $u \in \mathcal{P}$. The proof is complete. □

Theorem 4.2. *Assume the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold, If*

$$\frac{\varpi\mathcal{M}q^\rho\left(b^\rho - a^\rho\right)}{[\rho]_q} < 1, \tag{4.14}$$

then, the BVP (4.1)-(4.2) has a unique solution.

Proof. Let $u_1, u_2 \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$, then

$$\begin{aligned}\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{\mathcal{L}^1_{q,\rho}} &= \int_a^b t^{\rho-1}\left|\mathcal{T}u_1(t) - \mathcal{T}u_2(t)\right| d_qt\\ &= \int_a^b t^{\rho-1}\left|\int_a^t G_{q^\rho}(t,s)\, s^{\rho-1}\left(f(s,u_1(s)) - f(s,u_2(s))\right) d_qs\right| d_qt\\ &\leq \int_a^b s^{\rho-1}\left|f(s,u_1(s)) - f(s,u_2(s))\right|\left[\int_{qs}^b t^{\rho-1}G_{q^\rho}(t,s)\, d_qt\right] d_qs\\ &\leq \frac{\mathcal{M}q^\rho\left(b^\rho - a^\rho\right)}{[\rho]_q}\int_a^b s^{\rho-1}\left|f(s,u_n(s)) - f(s,u(s))\right| d_qs\\ &\leq \frac{\mathcal{M}q^\rho\left(b^\rho - a^\rho\right)}{[\rho]_q}\left\|f(t,u_1) - f(t,u_2)\right\|_{\mathcal{L}^1_{q,\rho}}.\end{aligned}$$

by hypothesis $(\mathcal{H}_2)$, we have

$$\|f(t,u_1)-f(t,u_2)\|_{\mathcal{L}^1_{q,\rho}} \leq \varpi \|u_1-u_2\|_{\mathcal{L}^1_{q,\rho}}.$$

Therefore

$$\|\mathcal{T}u_1-\mathcal{T}u_2\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{\varpi \mathcal{M} q^\rho (b^\rho-a^\rho)}{[\rho]_q} \|u_1-u_2\|_{\mathcal{L}^1_{q,\rho}}.$$

Thus, according to statement (4.14), the operator $\mathcal{T}$ is a contraction. By the Banach's contraction principle (see [29]) we conclude that $\mathcal{T}$ admits a fixed point, which corresponds to the unique solution of problem (4.1)–(4.2). The proof is complete. □

4.3 Ulam-Hyers Stability

In this section, we investigate devoted the Ulam–Hyers stability of the following equation

$$ {}^{\rho}\mathcal{D}^{\alpha}_{q,a} u(t) + f(t,u(t)) = 0, \quad t \in [a,b], \tag{4.15}$$

By using Definitions 4.1and 4.2 to verify the stability of solutions.

Definition 4.1. The Eq (4.15) is Ulam-Hyers stable if there exists a real number $\mu > 0$ such that for each $\varepsilon > 0$ and each solution $v \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$ of the inequality

$$\left| {}^{\rho}\mathcal{D}^{\alpha}_{q,a} v(t) + f(t,v(t)) \right| \leq \varepsilon, \tag{4.16}$$

there exists a solution $u \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$ of the Eq (4.15), such that

$$\|v-u\|_{\mathcal{L}^1_{q,\rho}} \leq \mu\varepsilon.$$

Definition 4.2. The Eq (4.15) is generalized Ulam-Hyers stable if there exists $\theta \in \mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$, with $\theta(0)=0$, such that for each solution v, which is in $\mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$ of the inequality (4.16), there exists a solution u of the Eq (4.15), with

$$\|v-u\|_{\mathcal{L}^1_{q,\rho}} \leq \theta(\varepsilon).$$

We present the following Lemma aimed to simplify subsequent calculations from Lemma.

Lemma 4.5. *If $v \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$ be an approximate solution of (4.16), then there exists $\varepsilon > 0$, such that v will be the solution of the inequality:*

$$\left| v(t) - \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s \right| \leq \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^\alpha_q\, \Gamma_{q^\rho}(\alpha+1)} \varepsilon,$$

Proof. If $v \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$ be an approximate solution. Then there exists $g \in \mathcal{C}([a,b],\mathbb{C})$, such that

$$ {}^\rho\mathcal{D}^\alpha_{q,a} v(t) + f(t, v(t)) = g(t), \tag{4.17}$$

and $|g(t)| \leq \varepsilon$, for all $t \in [a,b]$. Applying ${}^\rho\mathcal{J}^\alpha_{q,a}$ on both sides of equation (4.17), we get

$$v(t) = \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s + {}^\rho\mathcal{J}^\alpha_{q,a} g(t).$$

Then, for all $t \in [a,b]$, we obtain

$$\begin{aligned}
\left| v(t) - \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s \right| &= \left| \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s + {}^\rho\mathcal{J}^\alpha_{q,a} g(t) \right. \\
&\quad \left. - \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s \right| \\
&= \left| {}^\rho\mathcal{J}^\alpha_{q,a} g(t) \right| \\
&= \frac{[\rho]^{1-\alpha}_q}{\Gamma_{q^\rho}(\alpha)} \int_a^b s^{\rho-1} \left(b^\rho - (qs)^\rho\right)^{(\alpha-1)}_{q^\rho} |g(s)|\, d_q s \\
&\leq \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^\alpha_q\, \Gamma_{q^\rho}(\alpha+1)} \varepsilon.
\end{aligned}$$

The proof is complete. □

Theorem 4.3. *Assume that the assumption $(\mathcal{H}_1) - (\mathcal{H}and2)$ hold, and $\mathcal{M}\varpi < 1$, then the Eq (4.15) is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$, we define $v \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$ be an approximate solution and $u \in \mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$

is the exact solution of the Eq (4.15), with the conditions

$$u(a) = v(a) = 0, \;\; {}^{\rho}\mathcal{J}^{\eta}_{q,a} u(b) = \;\; {}^{\rho}\mathcal{J}^{\eta}_{q,a} v(b) = 0.$$

Satisfies

$$u(t) = \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, u(s))\, d_q s.$$

Then, we have

$$\begin{aligned} |v(t) - u(t)| &\leq \left| v(t) - \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s + \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s \right. \\ &\quad \left. - \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, u(s))\, d_q s \right| \\ &\leq \left| v(t) - \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} f(s, v(s))\, d_q s \right| \\ &\quad + \left| \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} \left(f(s, v(s)) - f(s, u(s)) \right) d_q s \right|. \end{aligned}$$

Using Lemma 5.4, we get

$$\leq \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^\alpha_q \, \Gamma_{q^\rho}(\alpha+1)} \varepsilon + \int_a^b G_{q^\rho}(t,s)\, s^{\rho-1} \left| f(s, v(s)) - f(s, u(s)) \right| d_q s,$$

the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied and the q-Green's function is bound, we get

$$\begin{aligned} \|v - u\|_{\mathcal{L}^1_{q,\rho}} &\leq \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^\alpha_q \, \Gamma_{q^\rho}(\alpha+1)} \varepsilon + \mathcal{M} \int_a^b s^{\rho-1} \left| f(s, v(s)) - f(s, u(s)) \right| d_q s \\ &\leq \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^\alpha_q \, \Gamma_{q^\rho}(\alpha+1)} \varepsilon + \mathcal{M} \left\| f(t, v) - f(t, u) \right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^\alpha_q \, \Gamma_{q^\rho}(\alpha+1)} \varepsilon + \mathcal{M}\varpi \left\| v - u \right\|_{\mathcal{L}^1_{q,\rho}}, \end{aligned}$$

it follows that

$$\|v-u\|_{\mathcal{L}^1_{q,\rho}} \leq \mu\varepsilon.$$

Set $\mu = \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{(1-\mathcal{M}\varpi)[\rho]_q^\alpha \Gamma_{q^\rho}(\alpha+1)}$. Therefore, the Eq (4.15) is Ulam-Hyers stable on $[a,b]$, and consequently generalized Ulam-Hyers stable if we set $\theta(\epsilon) = \mu\epsilon$.
The proof is complete.

□

4.4 Illustrative Examples

In the following, we present two examples to illustrate the main results established in this work.

Example 4.1. Consider the following problem

$$\begin{cases} {}^\rho\mathcal{D}^\alpha_{q,a}(u)(t) = \frac{\ln\left(2\sqrt{t}\right)}{3}\left(1+\frac{3\cos(t)}{45\ln(t+1)}|u(t)|\right), \quad t\in\left[\frac{1}{2},\frac{\pi}{2}\right], \\ u\left(\frac{1}{2}\right)=0, \text{ and } \left({}^\rho\mathcal{I}^\eta_{q,a}u\right)\left(\frac{\pi}{2}\right)=0. \end{cases}$$

We set

$$f(t,u) = \frac{\ln\left(2\sqrt{t}\right)}{3}\left(1+\frac{3\cos(t)}{45\ln(t+1)}|u|\right),$$

where $\alpha=\frac{3}{2}$, $\eta=\frac{1}{2}$, $q=\frac{1}{4}$, $\rho=1$ and $n=2$.
The function f is continuous for any $u\in\mathbb{C}$, and $\forall t\in\left[\frac{1}{2},\frac{\pi}{2}\right]$, we have

$$f(t,u) = \frac{\ln\left(2\sqrt{t}\right)}{3} + \frac{\ln\left(2\sqrt{t}\right)\cos(t)}{45\ln(t+1)}|u|,$$

by taking $\varphi(t) = \frac{\ln\left(2\sqrt{t}\right)}{3}$ and $\omega(t) = \frac{\ln\left(2\sqrt{t}\right)\cos(t)}{45\ln(t+1)}$, where $\varphi\in\mathcal{L}^1_{q,\rho}([a,b],\mathbb{R}_+)$ and $\omega\in\mathcal{C}([a,b],\mathbb{R}_+)$.
Hence, under both hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ are satisfied, then

$$|f(t,u)| \leq \varphi(t) + \omega(t)|u|.$$

We have

$$\varphi^* = \frac{\ln\left(2\sqrt{\frac{\pi}{2}}\right)}{3} \simeq 0.3063128444, \text{ and } \omega^* = \frac{\ln\left(2\sqrt{\frac{\pi}{2}}\right)\cos\left(\frac{\pi}{2}\right)}{45\ln\left(\frac{\pi}{2}+1\right)} \simeq 0.02161916355.$$

Also

$$\begin{aligned}\mathcal{M} &= \frac{1}{\Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right)}\left(\frac{\pi}{2}-\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}_{\frac{1}{4}} \simeq 0.7332977366,\\ l_1 &= \frac{\mathcal{M}\left(\frac{1}{4}\right)\left(\frac{\pi}{2}-\frac{1}{2}\right)}{[1]_{\frac{1}{4}}}\varphi^* \simeq 0.06013017034,\\ l_2 &= \frac{\mathcal{M}\left(\frac{1}{4}\right)\left(\frac{\pi}{2}-\frac{1}{2}\right)}{[1]_{\frac{1}{4}}}\omega^* \simeq 0.004243909489.\end{aligned}$$

If we select $r=\frac{1}{2}$. Consequently, the condition (4.13) is fulfilled

$$\frac{l_1}{1-l_2}=0,06038615298<r.$$

According to the Theorem 4.1, that the BVP has at least one solution.

Example 4.2. Consider the following problem

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha}_{q,a}\left(u\right)(t)=\frac{1}{\sqrt{1+t}}+\frac{2\sin^2(t)|u(t)|}{\pi}, \quad t\in\left[\frac{1}{2},\frac{\pi}{2}\right], \\ u\left(\frac{1}{2}\right)=0, \text{ and } \left({}^{\rho}\mathcal{I}^{\eta}_{q,a}u\right)\left(\frac{\pi}{2}\right)=0. \end{cases}$$

We set

$$f\left(t,u\right)=\frac{1}{\sqrt{1+t}}+\frac{2\sin^2\left(t\right)|u\left(t\right)|}{\pi}, \text{ for } u\in\mathbb{C},$$

where $\alpha=\frac{3}{2}$, $\eta=\frac{1}{2}$, $q=\frac{1}{4}$, $\rho=1$ and $n=2$.

The function f is continuous and for any $u,\tilde{u}\in\mathbb{C}$, and $t\in\left[\frac{1}{2},\frac{\pi}{2}\right]$, we have

$$\begin{aligned}\left|f\left(t,u\right)-f\left(t,\tilde{u}\right)\right| &= \left|\frac{2\sin^2\left(t\right)|u|}{\pi}-\frac{2\sin^2\left(t\right)|\tilde{u}|}{\pi}\right|,\\ &\leq \frac{2}{\pi}\left|u-\tilde{u}\right|.\end{aligned}$$

Consequently, the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ is satisfied, with $\varpi=\frac{2}{\pi}\simeq 0.6366197724$.

Therefore, we have $\mathcal{M}\simeq 0.7332977366$. Then, the condition (4.14)

$$\frac{\varpi\mathcal{M}q^{\rho}\left(b^{\rho}-a^{\rho}\right)}{[\rho]_q}=\frac{\frac{1}{2\pi}\times 0.7332977366\times\left(\frac{\pi}{2}-\frac{1}{2}\right)}{[1]_{\frac{1}{4}}}\simeq 0,1249704544<1,$$

is satisfied. Based on the Theorem 4.2 that the BVP has a unique solution.

CHAPTER 5

NONLINEAR FRACTIONAL q-DIFFERENTIAL EQUATIONS INVOLVING H-K q-DERIVATIVES

The scientific paper corresponding to this chapter has been published in: Journal of Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 32 (2025), (see [51]).

This chapter investigates the problem of existence, uniqueness and Ulam-Hyers stability of solutions of nonlinear fractional q-differential equations involving HILFER-KATUGAMPOLA q-derivatives of moving orders in the Banach space $\mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$.

$$ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\left[u(t)-h(t)\right]=f\left(t,u(t),{}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}\left[u(t)-h(t)\right]\right), \tag{5.1} $$

for $b>a>0$, $t\in\Omega=[a,b]$, with the initial conditions

$$ {}^{\rho}\mathcal{D}^{\gamma-1}_{q,a^+}u(a)=\omega\in\mathbb{C},\quad \text{and}\quad {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+}u(a)=0,\ \text{for } k=2,\ 3,\ldots,\ n, \tag{5.2} $$

where ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}$ and ${}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}$ represent the q-analogue of HILFER-KATUGAMPOLA fractional derivatives of order α and β, with $0\leq\eta\leq 1$, $m-1<\beta\leq m\in\mathbb{N}$, $\theta=\beta+\eta(m-\beta)$, $\max\{\theta,n-1\}<\alpha\leq n$, for $n\in\mathbb{N}-\{1\}$, and $\gamma=\alpha+\eta(n-\alpha)$. While $f:\Omega\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is a nonlinear complex-valued function. In addition, $0<q<1$, $\rho>0$, and $h\in\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$ is a complex-valued function given by

$$ h(t)=\frac{\omega\left(t^{\rho}-a^{\rho}\right)^{(\gamma-1)}_{q^{\rho}}}{[\rho]^{\gamma-1}_{q}\,\Gamma_{q^{\rho}}(\gamma)}. \tag{5.3} $$

5.1 Fundamental Lemmas

Throughout this section, we established the significant Lemmas to illustrate our main results. we use the space $\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$ to study the problem (5.1)-(5.2) which is denoted by

$$\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})=\left\{u:\Omega\to\mathbb{C},\ \|u\|_{\mathcal{L}^1_{q,\rho}}<\infty\right\}, \text{for } \rho\in\mathbb{R},\ 0<q<1,$$

which is a Banach space with the norm

$$\|u\|_{\mathcal{L}^1_{q,\rho}}=\int_\Omega t^{\rho-1}\left|u\left(t\right)\right|d_qt. \tag{5.4}$$

Lemma 5.1. *The* KATUGAMPOLA *fractional q-integral* ${}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}$ *is well defined and bounded in* $\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$, *with*

$$\left\|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}u\right\|_{\mathcal{L}^1_{q,\rho}}\leq\lambda\|u\|_{\mathcal{L}^1_{q,\rho}},$$

for any $\alpha>0$, $0<q<1$, *and* $\rho>0$.

Proof. Let $u\in\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$. Employing (1.4) and (1.11), gives us

$$\begin{aligned}
\left\|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}u\right\|_{\mathcal{L}^1_{q,\rho}} &= \int_a^b t^{\rho-1}\left|\frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)}\int_a^t s^{\rho-1}\left(t^\rho-(qs)^\rho\right)^{(\alpha-1)}_{q^\rho}u\left(s\right)d_qs\right|d_qt\\
&\leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)}\int_a^b s^{\rho-1}\left|u\left(s\right)\right|\left[\int_{qs}^b t^{\rho-1}\left(t^\rho-(qs)^\rho\right)^{(\alpha-1)}_{q^\rho}d_qt\right]d_qs\\
&\leq \frac{[\rho]_q^{1-\alpha}}{[\rho\alpha]_q\,\Gamma_{q^\rho}(\alpha)}\int_a^b s^{\rho-1}\left|u\left(s\right)\right|\left[\int_{q\tau}^b {}_tD_q\left(t^\rho-(qs)^\rho\right)^{(\alpha)}_{q^\rho}d_qt\right]d_qs\\
&\leq \frac{1}{[\rho]_q^{\alpha}\,\Gamma_{q^\rho}(\alpha+1)}\int_a^b s^{\rho-1}\left|u\left(s\right)\right|\left(b^\rho-(qs)^\rho\right)^{(\alpha)}_{q^\rho}d_qs\\
&\leq \frac{\left(b^\rho-(qa)^\rho\right)^{(\alpha)}_{q^\rho}}{[\rho]_q^{\alpha}\,\Gamma_{q^\rho}(\alpha+1)}\int_a^b s^{\rho-1}\left|u\left(s\right)\right|d_qs\\
&\leq \lambda\|u\|_{\mathcal{L}^1_{q,\rho}}.
\end{aligned}$$

The proof is complete. □

Definition q-integral solution

By introducing the following Lemma, we put $0 \leq \eta \leq 1$, $m-1 < \beta \leq m \in \mathbb{N}$, $\theta = \beta + \eta(m-\beta)$, and $\max\{\theta, n-1\} < \alpha \leq n \in \mathbb{N} - \{1\}$. Also $\gamma = \alpha + \eta(n-\alpha)$, $0 < q < 1$, and $\rho > 0$.

Lemma 5.2. *We have*

1. *The complex-valued function h given by (5.3) satisfies ${}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1} h(t) = \omega$, and can be expressed by*
$$h(t) = \sum_{k=1}^{n} \frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}_q^{\gamma-k} h(a)}{\Gamma_{q^\rho}(\gamma-k+1)} (t^\rho - a^\rho)_{q^\rho}^{(\gamma-k)}.$$

2. *We have*
$$\left({}^{\rho}\mathcal{J}_{q,a^+}^{\gamma}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma} h\right)(t) = 0,\ \forall t \in \Omega. \tag{5.5}$$

Proof. We apply (1.15) for $\alpha = \gamma - k$ and $\delta = \gamma - 1$, we get

$$\begin{aligned}
{}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k} h(t) &= \frac{\omega}{[\rho]_q^{\gamma-1}\Gamma_{q^\rho}(\gamma)}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}(t^\rho - a^\rho)_{q^\rho}^{(\gamma-1)} \\
&= \frac{\omega}{[\rho]_q^{\gamma-1}\Gamma_{q^\rho}(\gamma)} \left(\frac{[\rho]_q^{\gamma-k}\Gamma_{q^\rho}(\gamma-1+1)}{\Gamma_{q^\rho}(\gamma-1-(\gamma-k)+1)} (t^\rho - a^\rho)_{q^\rho}^{(\gamma-1-(\gamma-k))} \right) \\
&= \frac{\omega}{[\rho]_q^{k-1}\Gamma_{q^\rho}(k)} (t^\rho - a^\rho)_{q^\rho}^{(k-1)}.
\end{aligned}$$

1. For $k=1$, we obtain ${}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1} h(t) = \omega$, and ${}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k} h(a) = 0$ for each $k = 2, 3, \ldots, n$. Then
$$h(t) = \sum_{k=1}^{n} \frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}_q^{\gamma-k} h(a)}{\Gamma_{q^\rho}(\gamma-k+1)} (t^\rho - a^\rho)_{q^\rho}^{(\gamma-k)}.$$

2. Similarly, Lemma 1.5 implies
$$\begin{aligned}
\left({}^{\rho}\mathcal{J}_{q,a^+}^{\gamma}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma} h\right)(t) &= h(t) - \sum_{k=1}^{n} \frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}_q^{\gamma-k} h(a)}{\Gamma_{q^\rho}(\gamma-k+1)} (t^\rho - a^\rho)_{q^\rho}^{(\gamma-k)} \\
&= h(t) - h(t) \\
&= 0.
\end{aligned}$$

The proof is complete.

□

In what follows, we proceed to define the q-integral solution of the problem (5.1)-(5.2).

Lemma 5.3. *Let $u \in \mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$, then problem (5.1)–(5.2) is equivalent to the q-integral equation*

$$u(t) = h(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t), \quad t \in \Omega, \tag{5.6}$$

where $\varphi \in \mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$, satisfies

$$\varphi(t) = f\left(t, h(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t), {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi(t)\right).$$

Proof. Let $u \in \mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$ satisfies problem (5.1)–(5.2). Starting applying ${}^{\rho}\mathcal{J}^{\alpha}_{q,a}$ on both sides (5.1), we obtain

$${}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\,{}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[u(t)-h(t)] = {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f\left(t, u(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[u(t)-h(t)]\right). \tag{5.7}$$

As

$$\begin{aligned} {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\,{}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[u(t)-h(t)] &= {}^{\rho}\mathcal{J}^{\gamma}_{q,a^+}\,{}^{\rho}\mathcal{D}^{\gamma}_{q,a^+}[u(t)-h(t)] \\ &= {}^{\rho}\mathcal{J}^{\gamma}_{q,a^+}\,{}^{\rho}\mathcal{D}^{\gamma}_{q,a^+}u(t) - {}^{\rho}\mathcal{J}^{\gamma}_{q,a^+}\,{}^{\rho}\mathcal{D}^{\gamma}_{q,a^+}h(t). \end{aligned}$$

After using (5.5) from Lemma 5.2, and employing Lemma 1.9, we get

$${}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\,{}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[u(t)-h(t)] = u(t) - \sum_{k=1}^{n} \frac{[\rho]_q^{k-\gamma}\,{}^{\rho}\mathcal{D}^{\gamma-k}_{q}u(a)}{\Gamma_{q^\rho}(\gamma-k+1)}(t^\rho - a^\rho)^{(\gamma-k)}_{q^\rho}.$$

Substituting in (5.7) and using conditions (5.2), we obtain

$$u(t) = h(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t),$$

where $\varphi(t) = f\left(t, u(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[u(t)-h(t)]\right)$. In addition,

$${}^{\rho}\mathcal{D}^{\gamma-1}_{q,a^+}u(t) = \omega + {}^{\rho}\mathcal{D}^{\eta(n-\alpha)-1}_{q,a^+}\varphi(t), \quad \text{for every } t \in \Omega.$$

As ${}^{\rho}\mathcal{D}^{\gamma-1}_{q,a^+}u(a) = \omega$, we deduce that

$${}^{\rho}\mathcal{D}^{\eta(n-\alpha)-1}_{q,a^+}\varphi(a) = 0. \tag{5.8}$$

Since $\beta+\eta\left(m-\beta\right)=\theta<\alpha$, employing Lemma 1.3 allows us to write

$$\begin{aligned}
{}^{\rho}\mathcal{D}_{q,a^{+}}^{\beta,\eta}\left[u\left(t\right)-h\left(t\right)\right] &= {}^{\rho}\mathcal{D}_{q,a^{+}}^{\beta,\eta}\left[h\left(t\right)+{}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi\left(t\right)-h\left(t\right)\right] \\
&= {}^{\rho}\mathcal{D}_{q,a^{+}}^{\beta,\eta}\ {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi\left(t\right) \\
&= {}^{\rho}\mathcal{J}_{q,a^{+}}^{\eta(m-\beta)}\ {}^{\rho}\mathcal{D}_{q,a^{+}}^{\theta}\ {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi\left(t\right) \\
&= {}^{\rho}\mathcal{J}_{q,a^{+}}^{\eta(m-\beta)}\ {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha-\beta-\eta(m-\beta)}\varphi\left(t\right) \\
&= {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha-\beta}\varphi\left(t\right),
\end{aligned}$$

then we can define $\varphi\in\mathcal{L}_{q,\rho}^{1}\left(\Omega,\mathbb{C}\right)$ as a nonlinear complex-valued function that is satisfying the functional equation

$$\varphi\left(t\right)=f(t,\omega+{}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi\left(t\right),{}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha-\beta}\varphi\left(t\right)).$$

Otherwise, assume that $u\in\mathcal{L}_{q,\rho}^{1}\left(\Omega,\mathbb{C}\right)$ satisfies equation (5.6), next, we prove that u satisfies problem (5.1)–(5.2). We apply the operator ${}^{\rho}\mathcal{D}_{q,a^{+}}^{\alpha,\eta}$ Hilfer-Katugampola q-derivatives on both sides of equation (5.6), we obtain

$${}^{\rho}\mathcal{D}_{q,a^{+}}^{\alpha,\eta}\left[u\left(t\right)-h\left(t\right)\right]={}^{\rho}\mathcal{D}_{q,a^{+}}^{\alpha,\eta}\ {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi\left(t\right),$$

from (1.14), Lemma 1.5, and (5.8), we have

$$\begin{aligned}
{}^{\rho}\mathcal{D}_{q,a^{+}}^{\alpha,\eta}\ {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi\left(t\right) &= {}^{\rho}\mathcal{J}_{q,a^{+}}^{\eta(n-\alpha)}\ {}^{\rho}\mathcal{D}_{q,a^{+}}^{\eta(n-\alpha)}\varphi\left(t\right) \\
&= \varphi\left(t\right)-\frac{\left[\rho\right]_{q}^{1-\eta(n-\alpha)}\ {}^{\rho}\mathcal{D}_{q,a^{+}}^{\eta(n-\alpha)-1}\varphi\left(a\right)}{\Gamma_{q^{\rho}}\left(\eta\left(n-\alpha\right)\right)}\left(t^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\eta(n-\alpha)-1)} \\
&= \varphi\left(t\right),
\end{aligned}$$

then we get equation (5.1).

We now show that conditions (5.2) hold. Applying ${}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}$ in both sides of (5.6), we get

$$\begin{aligned}
{}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}u\left(t\right) &= {}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}\left[h\left(t\right)+{}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi\left(t\right)\right] \\
&= {}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}h\left(t\right)+{}^{\rho}\mathcal{J}_{q,a^{+}}^{k-\eta(n-\alpha)}\varphi\left(t\right), \quad \forall k\in\overline{1,n}.
\end{aligned}$$

For $t=a$, we arrive to conditions (5.2). The proof is complete. □

5.2 Existence and Uniqueness Results

In this section, by applying the properties of Schauder's and Banach's fixed point theorems, we investigate the existence and uniqueness of solutions for the problem (5.1)-(5.2) under the following assumptions:

$(\mathcal{H}_1)$ $f : \Omega \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear complex-valued function such that

$$f\left(\cdot, u\left(\cdot\right), {}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}\left[u\left(\cdot\right) - h\left(\cdot\right)\right]\right) \in \mathcal{L}_{q,\rho}^{1}\left(\Omega, \mathbb{C}\right), \quad u, h \in \mathcal{L}_{q,\rho}^{1}\left(\Omega, \mathbb{C}\right),$$

and there exist two nonnegative constants c_1 and c_2 such that the function f satisfies

$$\left|f\left(t, u, v\right) - f\left(t, \tilde{u}, \tilde{v}\right)\right| \leq c_1 \left|u - \tilde{u}\right| + c_2 \left|v - \tilde{v}\right|,$$

for any $u, v, \tilde{u}, \tilde{v} \in \mathbb{C}$.

$(\mathcal{H}_2)$ There exist three nonnegative functions $y_1 \in \mathcal{L}_{q,\rho}^{1}\left(\Omega, \mathbb{R}_+\right)$ and $\left(y_i\right)_{i=2,3} \in \mathcal{C}\left(\Omega, \mathbb{R}_+\right)$, such that the function f satisfies

$$\left|f\left(t, u, v\right)\right| \leq y_1\left(t\right) + y_2\left(t\right)\left|u\right| + y_3\left(t\right)\left|v\right|,$$

for any $u, v \in \mathbb{C}$ and for each $t \in \Omega$.

We denote

$$y_1^* = \left\|y_1\right\|_{\mathcal{L}_{q,\rho}^{1}}, \quad y_2^* = \sup_{t\in\Omega}\left|y_2\left(t\right)\right|, \quad y_3^* = \sup_{t\in\Omega}\left|y_3\left(t\right)\right|, \quad \text{and } \lambda_\alpha = \frac{\left(b^\rho - \left(qa\right)^\rho\right)_{q^\rho}^{(\alpha)}}{\left[\rho\right]_q^\alpha \Gamma_{q^\rho}\left(\alpha + 1\right)}.$$

For simplifying the writing, we substitute λ with λ_α, and we use κ instead of $\lambda_{\alpha-\beta}$.

Theorem 5.1. *Assume the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. If we put $\kappa \in \left(0, \min\left(\frac{1}{c_2}, \frac{1}{u_3^*}\right)\right)$ and*

$$\frac{\lambda y_2^*}{1 - \kappa y_3^*} < 1, \tag{5.9}$$

then problem (5.1)–(5.2) has a least one solution in $\mathcal{L}_{q,\rho}^{1}\left(\Omega, \mathbb{C}\right)$.

Proof. First, we will transform problem (5.1)–(5.2) into a fixed point problem, we define the operator

$$\mathcal{B}u\left(t\right) = h\left(t\right) + \frac{\left[\rho\right]_q^{1-\alpha}}{\Gamma_{q^\rho}\left(\alpha\right)} \int_a^t s^{\rho-1}\left(t^\rho - \left(qs\right)^\rho\right)_{q^\rho}^{(\alpha-1)} \varphi\left(s\right) d_q s, \tag{5.10}$$

where $\varphi = f\left(t, u, {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi\right) \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$. Therefore $\mathcal{B}u$ is an element of $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ equipped with the norm

$$\|\mathcal{B}u\|_{\mathcal{L}^1_{q,\rho}} = \int_a^b t^{\rho-1} |\mathcal{B}u(t)| \, d_q t.$$

As $(\mathcal{H}_1)$ holds, we notice that if $u \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$, then $\mathcal{B}$ is a continuous operator as demonstrated in step1.

Next, we demonstrate that $\mathcal{B}$ satisfies the assumptions of Schauder's fixed point theorem, this could be proved through three steps.

1. $\mathcal{B}$ is a continuous operator

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} u_n = u$ in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$. After using Lemma 5.1, we get

$$\begin{aligned} \|\mathcal{B}u_n - \mathcal{B}u\|_{\mathcal{L}^1_{q,\rho}} &= \left\|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi_n - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi\right\|_{\mathcal{L}^1_{q,\rho}} \\ &= \left\|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}(\varphi_n - \varphi)\right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \lambda \|\varphi_n - \varphi\|_{\mathcal{L}^1_{q,\rho}}, \end{aligned}$$

where

$$\begin{cases} \varphi_n(t) = f\left(t, u_n(t), {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi_n(t)\right), \\ \varphi(t) = f\left(t, u(t), {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi(t)\right). \end{cases}$$

Hypothesis $(\mathcal{H}_1)$ implies

$$\begin{aligned} \|\varphi_n - \varphi\|_{\mathcal{L}^1_{q,\rho}} &= \left\|f(t, u_n, {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi_n) - f(t, u, {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi)\right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq c_1 \|u_n - u\|_{\mathcal{L}^1_{q,\rho}} + c_2 \left\|{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}(\varphi_n - \varphi)\right\|_{\mathcal{L}^1_{q,\rho}}, \end{aligned}$$

then

$$\|\varphi_n - \varphi\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{c_1}{1 - \kappa c_2} \|u_n - u\|_{\mathcal{L}^1_{q,\rho}}.$$

Then

$$\|\mathcal{B}u_n - \mathcal{B}u\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{\lambda c_1}{1 - \kappa c_2} \|u_n - u\|_{\mathcal{L}^1_{q,\rho}}.$$

Since $u_n \to u$, when $n \to \infty$, then $\|\mathcal{B}u_n - \mathcal{B}u\|_{\mathcal{L}^1_{q,\rho}} \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \|\mathcal{B}u_n - \mathcal{B}u\|_{\mathcal{L}^1_{q,\rho}} = 0,$$

this implies the continuity of $\mathcal{B}$.

2. $\mathcal{B}$ is defined from a bounded, closed and convex subset into itself

Using (5.9), we define

$$r \geq \left(\frac{|\omega| \left(b^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\gamma)}}{[\rho]_q^{\gamma} \Gamma_{q^{\rho}} (\gamma + 1)} + \frac{\lambda y_1^*}{1 - \kappa y_3^*} \right) \left(\frac{1 - \kappa y_3^*}{1 - \kappa y_3^* - \lambda y_2^*} \right)$$

and define the subset P_r as follows

$$P_r = \left\{ u \in \mathcal{L}_{q,\rho}^1 (\Omega,\ \mathbb{C}), \quad \|u\|_{\mathcal{L}_{q,\rho}^1} \leq r \right\}.$$

It is clear that P_r is bounded closed and convex subset of $\mathcal{L}_{q,\rho}^1 (\Omega, \mathbb{C})$.

Let $\mathcal{B} : P_r \to \mathcal{L}_{q,\rho}^1 (\Omega, \mathbb{C})$ be the integral operator defined by (5.10), then $\mathcal{B}(P_r) \subset P_r$.

Indeed, by employing $(\mathcal{H}_1)$, we get

$$\begin{aligned} \|\varphi\|_{\mathcal{L}_{q,\rho}^1} &= \left\| f(t, u, {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha-\beta} \varphi) \right\|_{\mathcal{L}_{q,\rho}^1} \\ &\leq \|y_1\|_{\mathcal{L}_{q,\rho}^1} + \sup_{t \in \Omega} |y_2(t)| \, \|u\|_{\mathcal{L}_{q,\rho}^1} + \sup_{t \in \Omega} |y_3(t)| \left\| {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha-\beta} \varphi \right\|_{\mathcal{L}_{q,\rho}^1}. \end{aligned}$$

Therefore

$$\|\varphi\|_{\mathcal{L}_{q,\rho}^1} \leq y_1^* + y_2^* \|u\|_{\mathcal{L}_{q,\rho}^1} + \kappa y_3^* \|\varphi\|_{\mathcal{L}_{q,\rho}^1},$$

next

$$\|\varphi\|_{\mathcal{L}_{q,\rho}^1} \leq \frac{y_1^* + y_2^* r}{1 - \kappa y_3^*}. \tag{5.11}$$

Consequently

$$\begin{aligned} \|\mathcal{B}u\|_{\mathcal{L}_{q,\rho}^1} &= \left\| h + {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha} \varphi \right\|_{\mathcal{L}_{q,\rho}^1} \\ &\leq \|h\|_{\mathcal{L}_{q,\rho}^1} + \left\| {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha} \varphi \right\|_{\mathcal{L}_{q,\rho}^1} \\ &\leq \frac{|\omega| \left(b^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\gamma)}}{[\rho]_q^{\gamma} \Gamma_{q^{\rho}} (\gamma + 1)} + \lambda \|\varphi\|_{\mathcal{L}_{q,\rho}^1} \\ &\leq \frac{|\omega| \left(b^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\gamma)}}{[\rho]_q^{\gamma} \Gamma_{q^{\rho}} (\gamma + 1)} + \lambda \left(\frac{y_1^* + y_2^* r}{1 - \kappa y_3^*} \right) \\ &\leq \frac{\left(\frac{|\omega| (b^{\rho} - a^{\rho})_{q^{\rho}}^{(\gamma)}}{[\rho]_q^{\gamma} \Gamma_{q^{\rho}} (\gamma+1)} + \frac{\lambda y_1^*}{1 - \kappa y_3^*} \right) \left(\frac{1 - \kappa y_3^*}{1 - \kappa y_3^* - \lambda y_2^*} \right)}{\left(\frac{1 - \kappa y_3^*}{1 - \kappa y_3^* - \lambda y_2^*} \right)} + \frac{\lambda y_2^* r}{1 - \kappa y_3^*} \\ &\leq r, \end{aligned}$$

then $\mathcal{B}(P_r) \subset P_r$.

3. $\mathcal{B}(P_r)$ **is an equicontinuous subset**

Let $t_1, t_2 \in \Omega$, be such that $t_1 \leq t_2$, and $u \in P_r$. By using (5.4) and (5.11), we can obtain

$$\max_{t\in\Omega} |\varphi(t)| \leq \varphi^* = \frac{[\rho]_q (y_1^* + y_2^* r)}{(b^\rho - a^\rho)(1 - \kappa y_3^*)}.$$

On the other hand

$$\begin{aligned}
& |\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| \\
\leq\ & |h(t_2) - h(t_1)| + \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \left| \int_a^{t_2} s^{\rho-1} (t_2^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \varphi(s)\, d_q s \right. \\
& \left. - \int_a^{t_1} s^{\rho-1} (t_1^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \varphi(s)\, d_q s \right| \\
\leq\ & \frac{|\omega| \left((t_2^\rho - a^\rho)_{q^\rho}^{(\gamma-1)} - (t_1^\rho - a^\rho)_{q^\rho}^{(\gamma-1)} \right)}{[\rho]_q^{\gamma-1} \Gamma_{q^\rho}(\gamma)} + \frac{\varphi^* [\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \times \\
& \left[\int_a^{t_1} s^{\rho-1} \left((t_2^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} - (t_1^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \right) d_q s + \right. \\
& \left. \int_{t_1}^{t_2} s^{\rho-1} (t_2^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} d_q s \right]. \qquad (5.12)
\end{aligned}$$

By applying (1.12), we get

$$s^{\rho-1} \left((t_2^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} - (t_1^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} \right) = \frac{-1}{[\rho\alpha]_q} {}_sD_q \left((t_2^\rho - s^\rho)_{q^\rho}^{(\alpha)} - (t_1^\rho - s^\rho)_{q^\rho}^{(\alpha)} \right),$$

then

$$\int_a^{t_1} \frac{(t_2^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} - (t_1^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)}}{\tau^{1-\rho}} d_q s \leq \frac{(t_2^\rho - a^\rho)_{q^\rho}^{(\alpha)} - (t_1^\rho - a^\rho)_{q^\rho}^{(\alpha)} - (t_2^\rho - t_1^\rho)_{q^\rho}^{(\alpha)}}{[\rho\alpha]_q}.$$

Also

$$s^{\rho-1} (t_2^\rho - (q\tau)^\rho)_{q^\rho}^{(\alpha-1)} d_q s = \frac{-1}{[\rho\alpha]_q} {}_sD_q (t_2^\rho - s^\rho)_{q^\rho}^{(\alpha)}$$

then

$$\int_{t_1}^{t_2} s^{\rho-1} (t_2^\rho - (qs)^\rho)_{q^\rho}^{(\alpha-1)} d_q s \leq \frac{(t_2^\rho - t_1^\rho)_{q^\rho}^{(\alpha)}}{[\rho\alpha]_q}.$$

Thus, (5.12) gives us

$$\begin{aligned}|\mathcal{B}u(t_2)-\mathcal{B}u(t_1)| &\leq \frac{|\omega|\left((t_2^\rho-a^\rho)_{q^\rho}^{(\gamma-1)}-(t_1^\rho-a^\rho)_{q^\rho}^{(\gamma-1)}\right)}{[\rho]_q^{\gamma-1}\Gamma_{q^\rho}(\gamma)}\\ &\quad+\frac{\varphi^*\left((t_2^\rho-a^\rho)_{q^\rho}^{(\alpha)}-(t_1^\rho-a^\rho)_{q^\rho}^{(\alpha)}\right)}{[\rho]_q^{\alpha}\Gamma_{q^\rho}(\alpha+1)}.\end{aligned}$$

The right-hand side of the latter inequality approaches zero as t_1 tends to t_2. Hence

$$\lim_{t_1\to t_2}\|\mathcal{B}u_{t_2}-\mathcal{B}u_{t_1}\|_{\mathcal{L}^1_{q,\rho}}=0.$$

As a consequence of steps ***1.***, ***2.***, and ***3.***, and with the aid of the Ascoli-Arzela theorem, we deduce the continuity of $\mathcal{B}$, its compactness, and its satisfaction of the assumption required by Schauder's fixed point theorem [29]. Consequently, $\mathcal{B}$ possesses a fixed point in $\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$ that solves problem (5.1)–(5.2).

□

Now, we applied the Banach's fixed point theorem to find the uniqueness of the problem (5.1)-(5.2)

Theorem 5.2. *Assume the hypothesis $(\mathcal{H}_1)$ holds. If we put $\kappa\in\left(0,\frac{1}{c_2}\right)$ and*

$$\frac{\lambda c_1}{1-\kappa c_2}<1, \tag{5.13}$$

then problem (5.1)–(5.2) admits a unique solution in $\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$.

Proof. Let $u_1,u_2\in\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$, then

$$\|\mathcal{B}u_1-\mathcal{B}u_2\|_{\mathcal{L}^1_{q,\rho}}\leq\lambda\|\varphi_1-\varphi_2\|_{\mathcal{L}^1_{q,\rho}}$$

where

$$\varphi_i=f\left(t,u_i,{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi_i\right)\in\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C}),\quad\forall i=1,2.$$

We have

$$\|\varphi_1-\varphi_2\|_{\mathcal{L}^1_{q,\rho}}\leq\frac{c_1}{1-\kappa c_2}\|u_1-u_2\|_{\mathcal{L}^1_{q,\rho}}.$$

Therefore

$$\|\mathcal{B}u_1 - \mathcal{B}u_2\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{\lambda c_1}{1-\kappa c_2} \|u_1 - u_2\|_{\mathcal{L}^1_{q,\rho}}.$$

Thus, according to (5.13), $\mathcal{B}$ is considered a contraction operator.

Banach's contraction principle [29] helps us infer that $\mathcal{B}$ has only one fixed point which is the unique solution of problem (5.1)–(5.2). □

5.3 Ulam-Hyers Stability

In this section, by using the Definitions 5.1 and 5.2 to verify the stability of solutions to the equation (5.1) in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$.

Definition 5.1. Equation (5.1) is Ulam-Hyers stable if there exists a real number $\mu > 0$ such that for each $\varepsilon > 0$ and each solution $v \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ of the inequality

$$\left| {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+} [v(t) - h(t)] - f\left(t, v(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+} [v(t) - h(t)]\right) \right| \leq \varepsilon, \tag{5.14}$$

there exists a solution $u \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ of (5.1), with

$$\|v - u\|_{\mathcal{L}^1_{q,\rho}} \leq \mu\varepsilon.$$

Definition 5.2. Equation (5.1) is generalized Ulam-Hyers stable if there exists $\theta \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, with $\theta(0) = 0$, such that for each solution v, which is in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ of the inequality (5.14), there exists a solution u of (5.1) with

$$\|v - u\|_{\mathcal{L}^1_{q,\rho}} \leq \theta(\varepsilon).$$

Before proceeding, we present the following remark introduced in [**?**], followed by a lemma aimed to simplify subsequent calculations.

Remark 5.1. If $v \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ is a solution of the inequality (5.14), then there exists $g \in \mathcal{C}(\Omega, \mathbb{C})$, such that

1. ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+} [v(t) - h(t)] = f\left(t, v(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+} [v(t) - h(t)]\right) + g(t)$, for any $t \in \Omega$,
2. $|g(t)| \leq \varepsilon$, for all $t \in \Omega$.

Lemma 5.4. *If $v \in \mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$ is the solution of the inequality (5.14), then there exists $\varepsilon > 0$ such that v will be the solution of the inequality:*

$$\left| v(t) - \varpi(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f\left(t, v(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[v(t) - h(t)]\right)\right| \leq \ell\varepsilon,$$

with

$$\ell = \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^{\alpha}_{q}\,\Gamma_{q^\rho}(\alpha+1)} \quad \textit{and} \quad \varpi(t) = \sum_{k=1}^{n} \frac{[\rho]^{k-\gamma}_{q}\ {}^{\rho}\mathcal{D}^{\gamma-k}_{q} v(a)}{\Gamma_{q^\rho}(\gamma-k+1)} (t^\rho - a^\rho)^{(\gamma-k)}_{q^\rho}.$$

Proof. If $v \in \mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$ is a solution of (5.14). Then from Remark 5.1, we have

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[v(t) - h(t)] = f_v(t) + g(t), & t \in \Omega, \\ |g(t)| \leq \varepsilon, & \varepsilon > 0, \end{cases}$$

where $f_v(t)$ is a simple notation for $f\left(t, v(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[v(t) - h(t)]\right)$. Hence

$$v(t) = \varpi(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}[f_v(t) + g(t)].$$

Then, for all $t \in \Omega$, we get

$$\begin{aligned} \left|v(t) - \varpi(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f_v(t)\right| &= \left|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}[f_v(t) + g(t)] - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f_v(t)\right| = \left|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} g(t)\right| \\ &\leq \frac{[\rho]^{1-\alpha}_{q}}{\Gamma_{q^\rho}(\alpha)} \int_a^t s^{\rho-1}\left(t^\rho - (qs)^\rho\right)^{(\alpha-1)}_{q^\rho} |g(s)|\, d_q s. \\ &\leq \ell\varepsilon. \end{aligned}$$

The proof is complete. □

Theorem 5.3. *Let $u, v \in \mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$, where v is a solution of the inequality (5.14) and u is the unique solution of equation (5.1) with the conditions*

$${}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+} u(a) = {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+} v(a), \textit{ for each } k = 1, 2, \ldots, n,$$

such that

$$\left\| {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}(v-u)\right\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{c_0 - c_1}{c_2} \|v-u\|_{\mathcal{L}^1_{q,\rho}}, \textit{ for some } c_0 > c_1.$$

If the assumption $(\mathcal{H}_1)$ holds and $c_0\lambda < 1$, then equation (5.1) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$, we define $v \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ as a solution of the inequality (5.14) and $u \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ as the unique solution of equation (5.1) with the conditions

$$ {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+} u(a) = {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+} v(a) \text{ , for } k = 1,\ 2, \ldots,\ n. $$

Thus

$$ u(t) = \varpi(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f_u(t). $$

And

$$ \begin{aligned} |v(t) - u(t)| &\leq \left| v(t) - \varpi(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f_v(t) \right| \\ &\quad + \left| {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f_v(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} f_u(t) \right|. \end{aligned} $$

Using Lemma 5.4 and $(\mathcal{H}_1)$ makes us obtain

$$ \begin{aligned} \|v - u\|_{\mathcal{L}^1_{q,\rho}} &\leq \frac{(b^\rho - a^\rho)\,\ell}{[\rho]_q}\varepsilon + \left\| {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}(f_v - f_u) \right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \frac{(b^\rho - a^\rho)\,\ell}{[\rho]_q}\varepsilon + \lambda \|f_v - f_u\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \frac{(b^\rho - a^\rho)\,\ell}{[\rho]_q}\varepsilon + \lambda \left(c_1 \|v - u\|_{\mathcal{L}^1_{q,\rho}} + c_2 \left\| {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}(v - u) \right\|_{\mathcal{L}^1_{q,\rho}} \right) \\ &\leq \frac{(b^\rho - a^\rho)\,\ell}{[\rho]_q}\varepsilon + c_0\lambda \|v - u\|_{\mathcal{L}^1_{q,\rho}}. \end{aligned} $$

Then

$$ \|v - u\|_{\mathcal{L}^1_{q,\rho}} \leq \mu\varepsilon. $$

where $\mu = \frac{(b^\rho - a^\rho)\ell}{[\rho]_q(1 - c_0\lambda)}$. Definition 5.1 helps us infer that equation (5.1) is Ulam-Hyers stable on Ω. This completes the proof.

If we select $\theta(\varepsilon) = \mu\varepsilon$, it follows that $\theta(0) = 0$. Subsequently, according to Definition 5.2, it can be inferred that equation (5.1) manifests generalized Ulam-Hyers' stability.

□

5.4 Illustrative Examples

In this section, we give some examples to show the validity of the main results.

Example 5.1. Consider the following problem

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\left(u-h\right)(t)=\frac{(t^{\rho}-a^{\rho})^{(\alpha)}_{q^{\rho}}|\arctan(t)|\left(2+|u(t)|+\left|{}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}(u-h)(t)\right|\right)}{20q\,\Gamma_{q^{\rho}}(\alpha+1)\left(1+|u(t)|+\left|{}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}(u-h)(t)\right|\right)}, & t\in\left[\frac{1}{2},\frac{3}{2}\right], \\ \left({}^{\rho}\mathcal{D}^{\gamma-1}_{q,a^+}u\right)(a)=\omega\in\mathbb{C}, \text{ and } \left({}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+}u\right)(a)=0, & \text{for } k=\ 2,\ 3,\ldots,\ n. \end{cases} \tag{5.15}$$

We set

$$f\left(t,u,v\right)=\frac{(t^{\rho}-a^{\rho})^{(\alpha)}_{q^{\rho}}\left|\arctan\left(t\right)\right|\left(2+|u|+|v|\right)}{20q\,\Gamma_{q^{\rho}}\left(\alpha+1\right)\left(1+|u|+|v|\right)},$$

where $\alpha=\frac{5}{3}$, $\eta=\frac{3}{4}$, $q=\frac{1}{4}$, $\beta=\frac{1}{3}$, and $\rho=1$. Then $\gamma=\frac{23}{12}$, $\theta=\frac{5}{6}$, and $n=2$.
The function f is continuous. For any $u,v,\tilde{u},\tilde{v}\in\mathbb{C}$, and $\forall t\in\left[\frac{1}{2},\frac{3}{2}\right]$, we get

$$\left|f\left(t,u,v\right)-f\left(t,\tilde{u},\tilde{v}\right)\right|\leq\frac{(b^{\rho}-a^{\rho})^{(\alpha)}_{q^{\rho}}\,\pi}{40q\,\Gamma_{q^{\rho}}\left(\alpha+1\right)}\left(|u-\tilde{u}|+|v-\tilde{v}|\right).$$

Hence, the hypothesis $(\mathcal{H}_1)$ is satisfied with $c_1=c_2=\frac{(b^{\rho}-a^{\rho})^{(\alpha)}_{q^{\rho}}\pi}{40q\,\Gamma_{q^{\rho}}(\alpha+1)}\simeq 0,33368179$.
We also have

$$\left|f\left(t,u,v\right)\right|\leq\frac{(t^{\rho}-a^{\rho})^{(\alpha)}_{q^{\rho}}\left|\arctan\left(t\right)\right|\left(2+|u|+|v|\right)}{20q\,\Gamma_{q^{\rho}}\left(\alpha+1\right)}.$$

Thus, the hypothesis $(\mathcal{H}_2)$ is satisfied with

$$y_1\left(t\right)=\frac{(t^{\rho}-a^{\rho})^{(\alpha)}_{q^{\rho}}\left|\arctan\left(t\right)\right|}{10q\,\Gamma_{q^{\rho}}\left(\alpha+1\right)},\quad y_2\left(t\right)=y_3\left(t\right)=\frac{(t^{\rho}-a^{\rho})^{(\alpha)}_{q^{\rho}}\left|\arctan\left(t\right)\right|}{20q\,\Gamma_{q^{\rho}}\left(\alpha+1\right)},$$

and

$$y_1^{*}=\left\|y_1\right\|_{\mathcal{L}^1_{q,\rho}}\leq\frac{(b^{\rho}-a^{\rho})^{(\alpha+1)}_{q^{\rho}}\,\pi}{20q\ \left[\rho\right]_q\Gamma_{q^{\rho}}\left(\alpha+2\right)}=\frac{\pi}{5\Gamma_{\frac{1}{4}}\left(\frac{11}{3}\right)},$$

also, we have

$$\begin{aligned} y_2^* &= y_3^* = \frac{\left(b^\rho - a^\rho\right)_{q^\rho}^{(\alpha)} \pi}{40q\,\Gamma_{q^\rho}(\alpha+1)} \simeq 0,33368179 \\ \lambda &= \frac{\left([1]_q\right)^{-\frac{5}{3}} \left(\frac{3}{2}-\frac{1}{2}\right)_q^{\left(\frac{5}{3}\right)}}{\Gamma_q\left(\frac{5}{3}+1\right)} \simeq 1,06214213 \\ \kappa &= \frac{\left([1]_q\right)^{-\frac{5}{3}+\frac{1}{3}} \left(\frac{3}{2}-\frac{1}{2}\right)_q^{\left(\frac{5}{3}-\frac{1}{3}\right)}}{\Gamma_q\left(\frac{5}{3}-\frac{1}{3}+1\right)} \simeq 1,02830547. \end{aligned}$$

Condition (5.9) satisfies

$$\frac{\lambda y_2^*}{(1-\kappa y_3^*)} = 0,53955237 < 1.$$

It follows from Theorem 5.1 that problem (5.15) has at least one solution.

Example 5.2. Consider the following problem

$$\begin{cases} {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}(u-h)(t) = \frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)\left(1+|u(t)|+\left|{}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}(u-h)(t)\right|\right)}, & t\in\left[\frac{1}{2},1\right], \\ \left({}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1}u\right)(a) = \omega\in\mathbb{C}, \text{ and } \left({}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}u\right)(a) = 0, & \text{for } k = 2,\ 3,\ldots,\ n. \end{cases} \tag{5.16}$$

We set

$$f(t,u,v) = \frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)(1+|u|+|v|)}, \text{ for } u,v\in\mathbb{C},$$

where $\alpha=\frac{7}{3}$, $q=\eta=\frac{1}{2}$, $\beta=\frac{4}{3}$, and $\rho=1$. Then $\gamma=\frac{8}{3}$, $\theta=\frac{5}{3}$, and $n=3$.
The function f is continuous and for any $u,v,\tilde{u},\tilde{v}\in\mathbb{C}$, and $t\in\left[\frac{1}{2},1\right]$, we have

$$\begin{aligned} |f(t,u,v)-f(t,\tilde{u},\tilde{v})| &= \left|\frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)(1+|u|+|v|)} - \frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)(1+|\tilde{u}|+|\tilde{v}|)}\right| \\ &= \frac{1}{\sqrt{q}+1}\left|\frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{(1+|u|+|v|)} - \frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{(1+|\tilde{u}|+|\tilde{v}|)}\right| \\ &\leq \frac{1}{\sqrt{q}+1}\left(|u-\tilde{u}|+|v-\tilde{v}|\right). \end{aligned}$$

Hence, the hypothesis $(\mathcal{H}_1)$ is satisfied with

$$c_1 = c_2 = \frac{1}{\sqrt{q}+1} = \frac{1}{\sqrt{\frac{1}{2}}+1} \simeq 0,58578648,$$

and

$$\begin{aligned}\lambda &= \frac{\left([1]_q\right)^{-\frac{7}{3}}\left(1-\frac{1}{2}\right)_q^{\left(\frac{7}{3}\right)}}{\Gamma_q\left(\frac{7}{3}+1\right)} \simeq 0,19842513.\\ \kappa &= \frac{\left([1]_q\right)^{-\frac{7}{3}+\frac{4}{3}}\left(1-\frac{1}{2}\right)_q^{\left(\frac{7}{3}-\frac{4}{3}\right)}}{\Gamma_q\left(\frac{7}{3}-\frac{4}{3}+1\right)} = 0,5.\end{aligned}$$

Now, show that condition (5.13)

$$\frac{c_1\lambda}{1-\kappa c_2} \simeq 0,16438076 < 1.$$

Is satisfied. It follows from Theorem 5.2 that problem (5.16) has a unique solution.

Conclusion

In this thesis, the existence, uniqueness, and stability of solutions for certain classes of fractional q-differential equations were studied. These results were obtained by using the properties of the Banach contraction principle, as well as Schauder's and the nonlinear alternative Leray-Schauder's fixed point theorems. The theoretical results were also supported by illustrative examples for each studied problem.

The thesis introduced some fundamental concepts and definitions within the framework of classical and fractional q-calculus theory. Investigated n-nonlinear fractional q-differential systems in the sense of CAPUTO fractional q-derivatives with the initial conditions. Analyzed the stability of nonlinear and time-varing q-fractional order systems using a q-fractional extension of Lyapunov's direct method. Studied and analyzed the boundary value problem for nonlinear fractional q-differential equation in the sense of the KATUGAMPOLA fractional q-derivative, and addressed the initial value problem for nonlinear fractional q-differential equations in the sense of HILFER-KATUGAMPOLA q-derivatives of moving orders within Banach space $\mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$.

This work opens broad prospects for many future studies in the field of quantum fractional differential equations, which are represented as follows:

- Generalising the obtained results to include quantum fractional partial differential equations within the framework of quantum calculus.
- Studying similar problems using other fractional operators such as RIEMANN–LIOUVILLE, HADAMARD, and ERDELYI–KOBER.
- Developing numerical methods for solving nonlinear quantum fractional differential equations, with a focus on methods that naturally converge to classical models when $q \to 1$.

- Studying applications to real-world models such as biological systems, control systems, and quantum and discrete dynamical systems.

Bibliography

[1] T.A. ERNST, The history of q-calculus and new method (Licentiate Thesis), U.U.D.M. Report 2000: http://math.uu.se/thomas/Lics.pdf.

[2] F.H. JACKSON, On q-definite integrals. Quart. J. Pure Appl. Math. 41, 193–203 (1910).

[3] F.M. ATICI, P.W. ELOE, Fractional q-calculus on a time scale, J. Math. Phys. 14 (2007) 341–352.

[4] F.H. JACKSON, On q-functions and a certain difference operator. Trans. Roy. Soc. Edinb. 46, 64–72 (1908).

[5] T.A. ERNST, Comprehensive Treatment of q-Calculus; Springer: Basel, Switzerland, 2012.

[6] V. KAC, P. CHEUNG, Quantum Calculus (Springer, New York, 2002).

[7] W. A. AL-SALAM: q-Analogues of Cauchy's Formulas. Proc. Amer. Math. Soc., 17, No. 3 (1966), 616–621.

[8] W. A. AL-SALAM, A. VERMA, A fractional Leibniz q-formula, Pacific Journal of Mathematics, vol 60, (1975), 1–9.

[9] W. A. AL-SALAM, Some fractional q-integrals and q-derivatives, Proceedings of the Edinburgh Mathematical Society, 15 (1966), 135–140.

[10] R. P. AGARWAL, Certain fractional q-integrals and q-derivatives, Proc. Cambridge Philos. Soc. 66 (1969), 365–370.

[11] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, Fractional Integral and Derivatives (Theory and Applications), Gordon and Breach, Switzerland, 1993.

[12] I. PODLUBNY, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, New York, 1999.

[13] A. A. KILBAS, H. M. SRIVASTAVA AND J. TRUJILLO, Theory and Applications of Fractional Differential Equations, Elsevier Science, USA, 2006.

[14] K. DIETHELM, The Analysis of Fractional Differential Equations, Springer Berlin, 2010.

[15] M. PREDRAG RAJKOVIÂ´C, D. SLAD-ANA MARINKOVIÂ´C, S. MIOMIR STANKOVIÂ´C, Fractional integrals and derivatives in q-calculus, Applicable Analysis and Discrete Mathematics, 1 (2007), 311–323. http://pefmath.etf.bg.ac.yu.

[16] X. SU, S. ZHANG, Solution to boundary value problems for nonlinear differential equations of fractional order. Electron. J. Differ. Equ. 2009, 26 (2009).

[17] M.M. MATAR, Boundary value problem for fractional integro-differential equations with nonlocal conditions. Int. J. Open Probl. Comput. Sci. Math. 3(4), 481–489 (2010).

[18] TY. IAN, Y. ZHOU, Positive solutions for multipoint boundary value problem of functional differential equations. J. Appl. Math. Comput. 38, 417–427 (2012).

[19] V. LAKSHMIKANTHAM, S. LEELA, M. SANBANDHAN, Lyapunov theory for fractional differential equations. Commun Appl Anal 2008; 12: 365–76.

[20] V. LAKSHMIKANTHAM, S. LEELA, SJ. SASUNDARA DEVI, Theory of fractional dynamic systems. Cambridge scientific publishers; 2009.

[21] KS. NARENDRA, AM. ANNASWAMY, Atable adaptive systems. Dover publications Inc.; 2005.

[22] M. E. SAMI, Existence of solutions for a system of singular sum fractional q-diffrential equations via quantum calculus, Adv. Diffr. Equ. 2020 (2020), 23.

[23] S. SHAIMARDANA, L. E. PERSSON, AND N. S. TOKMAGAMBETOV, Existence and uniqueness of some Cauchy type problems in fractional q-difference calculus, Filomat, 34(13) (2020), 4429–4444.

[24] R. FERREIRA, Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. 2010, 2010, 1–10.

[25] B. AHMAD, S. K. NTOUYAS, AND I. K. PURNARAS, Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations, Adv. Differ. Equ., 2012 (2012), 1–15.

[26] W. X. ZHOU AND H. Z. LIU, Existence solutions for boundary value problem of nonlinear fractional q-difference equations, Adv. Differ. Equ., 2013 (2013), 1–12.

[27] C. ZHAI, AND J. REN, The unique solution for a fractional q-difference equation with three-point boundary conditions. Indag. Math. 2018 (29), 948–961.

[28] N. S. TOKMAGAMBETOV, B. K. TOLGEN, q-Analogues of Lyapunov-type inequalities involving Riemann-Liouville fractional derivatives, Mathematics Series, No. 3 (119), 2025, pp. 214–230.

[29] A. GRANAS AND J. DUGUNDJI, Fixed Point Theory, Springer-Verlag, New York, 2003.

[30] S.M. ULAM, A Collection of Mathematical Problems. Interscience, New York (1968).

[31] D.H. HYERS, On the stability of the linear functional equations. Proc. Natl. Acad. Sci. USA 27, 222–224 (1941).

[32] G. GASPER, M. RAHMAN, Basic Hypergeometric Series, 2nd edn. (Cambridge university Press, Cambridge, 2004).

[33] J. THOMAE, Beitrage zur theorie der durch die heinesche reihe : : :. J. Reine Angew. Math. 70, 258–281 (1869).

[34] F.H. JACKSON, A generalization of the function .n/ and xn. Proc. Roy. Soc. Lond. 74, 64–72 (1904).

[35] R. ASKEY, The q-gamma and q-beta functions. Appl. Anal. 8(2), 125–141 (1978/1979).

[36] ATICI F.M. AND ELOE P. W., Fractional q-calculus on a time scale , Journal of Nonlinear Mathematical Physics 14, 3, (2007), 333–344.

[37] P. M. RAJKOVIÂ´C, S. D. MARINKOVIÂ´C AND M. S. STANKOVIÂ´C, Fractional integrals and derivatives in q-calculus, Appl. Anal. Discrete Math. 1 (2007), no. 1, 311–323.

[38] P. M. RAJKOVIC, S. MARINKOVIC, AND M. STANKOVIC, On q–analogues of Caputo derivative and Mittag-Leffl function, Fract. Cal. Appl. Anal. 10 (4) (2007), 359–373.

[39] M.H. ANNABY, Z.S. MANSOUR, q-fractional calculus and equations Springer, Heidelberg. (2012).

[40] U.N. KATUGAMPOLA. (2011), New approach to a generalized fractional integral. Applied Mathematics and Computation, 218(3), 860–865.

[41] M. MOMENZADEH AND N. MAHMODOV, Study of new class of q-fractional integral operator, Filomat 33 (17) (2019), 5713–5721.

[42] M. MOMENZADEH AND S. NOROZPOUR, Study of new class of q-fractional derivative operator and its properties, Int. J. Adv. Sci. Techno., 29 (8) (2020), 2871–2878.

[43] L. CHANCHLANI, S. ALHA, AND I. MALLAH, On Katugampola fractional q-integral and q-derivative, JJnanabha, 52(1) (2022), 101–112.

[44] I. A. MALLAH, L. CHANACHANI, AND S. ALHA, q-analogue of Hilfer-Katugampola fractional derivative and applications, South East Asian J. Math. Math. Sci., 19(2) (2023), p77.

[45] Z. BAI AND H. LUÂ¨ , Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005), 495–505.

[46] N.K. MAHDI, AND A.R. KHUDAIR, On Caputo Delta q-fractional Dynamical systems: Lyapunov stability, Malaysian Journal of Mathematical Sciences 18 (4) (2024), 775–783.

[47] I. AOUINA, B. BASTI, Y. ARIOUA AND A. SALIM, On n-nonlinear Caputo fractional q-diffrential systems, Filomat 39:21 (2025), 7383–7395. https://doi.org/10.2298/FIL2521383A.

[48] A. SALIM, S. ABBAS, M. BENCHOHRA AND J. E. LAZREG, Caputo fractional q-diffrence equations in Banach spaces. J. Innov. Appl. Math.Comput. Sci. 3 (1) (2023), 1–14. https://doi.org/10.58205/jiamcs.v3i1.67.

[49] N. LALEDJ, A. SALIM, J. E. LAZREG, S. ABBAS, B. AHMAD AND M. BENCHOHRA, On implicit fractional q-diffrence equations: Analysis and stability. Math. Methods Appl. Sci. 45 (17) (2022), 10775–10797. https://doi.org/10.1002/mma.8417

[50] B. BASTI AND Y. ARIOUA, Existence study of solutions for a system of n-nonlinear fractional diffrential equations, J. Math. Phys. Anal. Geom. 18 (2) (2022), 350–367.

[51] B. BASTI, I. AOUINA AND Y. ARIOUA, Nonlinear fractional Q-differential equations involving HILFER-KATUGAMPOLA derivatives of moving orders, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 32 (2025) 283–303. http://www.watam.org.

Abstract: This thesis aims to study and analysis of several results concerning the existence, uniqueness, and stability of solutions for nonlinear fractional q-differential equations with initial and boundary conditions, using fractional q-derivatives of Caputo, Katugampola and Hilfer-Katugampola. In addition, the stability of fractional-order systems is studied using the direct Lyapunov method. The results have been proved using some fixed-point theorems of Schauder, Banach's contraction principle, and the nonlinear alternative technique of Leray-Schauder.

Key-words: Fractional q-differential equations, initial value problem, boundary value problem, fixed point theorems, existence, uniqueness, stability.

Résumé: Ce travail aborde l'étude et l'analyse de plusieurs résultats concernant l'existence, L'unicité et la stabilité des solutions des équations différentielles fractionnaires quantiques non linéaires avec des conditions initiales et aux limites, en utilisant les dérivées fractionnaires quantiques de Caputo, katugampola et Hilfer-Katugampola. En outre, la stabilité des systèmes d'ordre fractionnaire est étudiée l'aide de la méthode directe de Lyapunov. Les résultats ont été démontrés en utilisant certains théorèmes du point fixe de Schauder, le principe de contraction de Banach et la technique de l'alternative non linéaire de type Leray-Schauder.

Mots-clés: équations différentielles q-fractionnaires, problème de valeur initiale, problème aux limites, théorèmes du point fixe, existence, unicité, stabilité.

ملخص: تهدف هذه الأطروحة إلى دراسة عدة مسائل رياضية في إطار حساب التفاضل والتكامل الكسري من نوع كمومي. يتناول هذا العمل دراسة وتحليل عدة نتائج تتعلق بوجود ووحدانية واستقرار حلول المعادلات التفاضلية الكسرية غير الخطية من نوع كمومي مع شروط ابتدائية وحدودية، وذلك باستخدام المشتقات الكسرية من نوع كمومي, كابتو, كاتوغامبولا و هيلفر-كاتوغامبولا. بالإضافة إلى ذلك، تتم دراسة استقرار الأنظمة ذات الرتبة الكسرية باستخدام الطريقة المباشرة لدالة ليابونوف. وقد تم إثبات النتائج باستخدام بعض مبرهنات النقطة الثابتة مثل مبرهنة شاودر، ومبدأ الإ نكماش لباناخ، وتقنية البديل غير الخطي لليراي-شاودر.

كلمات مفتاحية: معادلات تفاضلية كسرية من نوع كمومي, مسائل بقيم إبتدائية, مسألة بقيم حدودية, نظريات النقطة الثابتة, الوجود, الوحدانية, الاستقرار.