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Theme

Numerical and Analytical Approaches to Free Surface and Potential Flow

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In the name of Allah, the Most Gracious, the Most Merciful

Dedication

To my parents,

who never hesitated to support me with their prayers and encouragement. They were a true source of strength throughout every stage of my life.

To my brothers and sisters, and their spouses,

thank you for your presence, your understanding during my busiest times, and for every kind word that made a difference when I needed it.

To their little children,

whose innocent smiles brought comfort during moments of exhaustion.

To my entire family,

each of you, in your own way, left a quiet but meaningful impact on me.

To my true friends,

who walked beside me with honesty and simplicity, offering silent support when it mattered most.

I dedicate this modest work to everyone who stood by me — even in silence.

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Finally, I would like to thank everyone who contributed in one way or another to the success of my academic journey.

• ملخص

يتناول هذا العمل دراسة جريان ذو سطح حرٍ لسائل مثالي ثنائي الأبعاد،منتظم و غير دوامي،داخل قناة أفقية يحتوي قاعها على عائق، يُفترض أن السائل غير قابل للإنضغاط مع إهمال تأثيرات التوتر السطحي وأخذ الجاذبية بعين الإعتبار،تعتمد الدراسة على تقريب الأمواج الطويلة،و يستخدم لهذا الغرض معاملا صغيرا ϵ بحيث $0 < \epsilon = (\frac{H}{l})^2 << 1$.

يوصف شكل القاع بالدالة $y = b(x)$،و السطح الحر ب $y = H + \eta(x,t)$ حيث تمثل η المجهول الرئيسي في هذه الدراسة.

الكلمات المفتاحية: مائع،الجريان الحر،سطح حر،سائل غير قابل للإنضغاط،حالة فوق حرجة،حالة تحت حرجة.

• Résumé

Ce travail porte sur l'étude de l'écoulement à surface libre d'un fluide idéal, bidimensionnel, stationnaire et irrotationnel, dans un canal horizontal comportant un obstacle au fond. Le fluide est supposé incompressible, les effets de la tension superficielle sont négligés, tandis que la gravité est prise en compte. L'étude repose sur l'approximation des longues ondes, avec un petit paramètre ε : $0 < \varepsilon = \left(\frac{H}{l}\right)^2 \ll 1$.

Le profil du fond est donné par la fonction $y = b(x)$ et la surface libre par $y = H + \eta(x,t)$, où η est l'inconnue principale de notre problème.

Mots clés : fluide, écoulement potentiel, surface libre, fluide incompressible, écoulement supercritique, écoulement subcritique.

• Abstract

This work deals with the study of a free surface flow of an ideal, two-dimensional, steady and irrotational fluid inside a horizontal channel with an obstacle on the bottom. The fluid is assumed to be incompressible, and surface tension effects are neglected, while gravity is taken into account.The study is based on the long-wave approximation, using a small parameter ε: $0 < \varepsilon = \left(\frac{H}{l}\right)^2 \ll 1$.

The bottom profile is described by the function $y = b(x)$ and the free surface by $y = H + \eta(x,t)$, where η is the main unknown of our problem.

Keywords: fluid, potential flow, free surface, incompressible fluid, supercritical flow, subcritical flow.

List of Symbols

In what follows, we will use the following notations.

m	the mass
V	the volume
$\frac{dV}{v}$	the fractional change
ρ	the density
P	the pressure
$\vec{v}$	the velocity vector
ϕ	the scalare function
Φ	the velocity potential of the fluid
σ	the surface tention coefficient
g	the gravity
Δ	Laplacian
$\dfrac{\partial}{\partial x}$	the partial derivative of x
η	the free surface elevation
F	the Froud number
H	the water depth
h	the height of the obstacle
U	the vilocity of uniform stream
$\dfrac{D}{Dt}$	The material derivative operator
ϵ	a small parameter

General Introduction

General Introduction

Fluid mechanics is a discipline concerned with the behavior of fluids in motion or at rest. It involves the study of forces acting on liquids and gases. There are many applications of fluid mechanics in the field of engineering. In fact, each branch of engineering focuses on specific aspects of fluid behavior.

Civil engineering deals with the design of piping and pumping systems for supplying drinking and irrigation water. It also includes water storage facilities and open channel flow systems.

Mechanical engineers are concerned with the design of pumps, gas compressors, and aircraft. These tasks require a deep understanding of fluid properties and flow characteristics.

In chemical engineering, the focus is on the flow of fluids in pipes and industrial systems. For instance, transporting a liquid between locations requires identifying the optimal pipe size, while handling solid particles involves evaluating pressure drops.

After this general introduction, this dissertation is structured into three chapters:

- **Chapter One** covers the fundamental concepts of fluid mechanics and essential governing equations, including the continuity equation and Bernoulli's principle, as well as boundary conditions for free surface flows.

- **Chapter Two** presents approximate analytical solutions for incompressible, irrotational flows disturbed by a bottom obstacle. A simplified model is derived in the form of the forced Korteweg–de Vries (fKdV) equation.

- **Chapter Three** focuses on the numerical resolution of the fKdV equation using two methods: the finite difference method for the supercritical case, and the fourth-order Runge–Kutta method for the subcritical case. Numerical results are illustrated with figures showing the influence of obstacle height and the Froude number on the free surface profile.

This study focuses particularly on modeling the deformation of the free surface caused by a localized bottom obstacle, using both analytical derivations and numerical simulations based on the forced Korteweg–de Vries equation.

Academic Note: The analytical and numerical parts of this study are inspired by the approach presented in [8], with reformulated derivations, adapted notations, and independently implemented numerical simulations to align with the objectives of the present work.

CHAPTER 1

BASIC CONCEPTS

1.1 Some Basic Concepts of Fluid Mecanics

In this chapter, we discuss some fundamental concepts of fluid mechanics, along with some essential equations and analytical solutions for simple potential flow problems

1.2 Definition of Fluids[[2][9]]

Fluids are materials that exist in nature two forms:liquids and gases, they are different from solids because they do not resist shear forces .

In fluids, the particles can move freely, which allows them to flow and change shape to fit the contrainer, this is defferent from solids, where the particles are strongly bounded and cannot move easily, another important characteristic of the fluid is the uniform disbution of the acting pressure to all parts and in all directions.

this research focuses on studying them, without considering shear resistance in flow analysis.

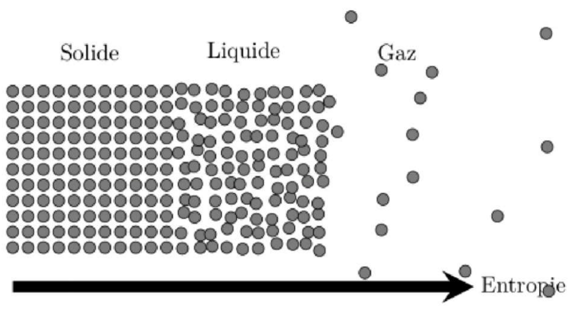

Figure 1.1: Microscopic interpretation of fluid types.

1.3 Properties of Fluids:

Fluids have a number of properties that determine their behavior in different conditions, some of these main properties are:

1.3.1 The Density [[2]]

The Density by definition is mass per unit volume, the case fluids, we can define the density(withe aide of fig 1.1) as the limit of this ratio when a measuring volume V shrinks to zero, this definition is important because density can change from one point to another, Also, in this picture, we can relate to a volume element in space that we can" control volume", wich moves with the fluid or can be stationary. therefore the definition of density at a point is

$$\rho = \lim_{v \to 0} \frac{m}{v} \tag{1.1}$$

,

typical units are kilograms per cubic meter (kg/m^3) or grams per cubic centimeter (g/cm^3)

Figure 1.2 Mass m in a control volume V. Density is thraction of m/v

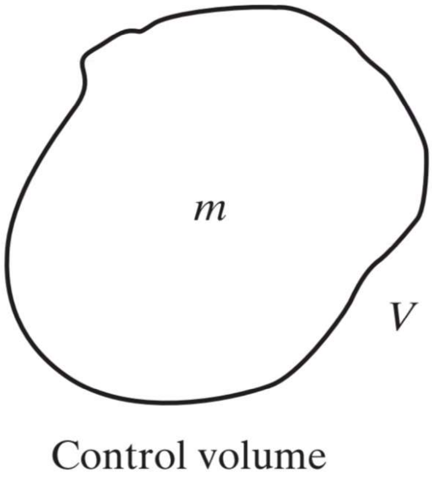

Figure 1.2: control volume.

1.3.2 The Viscosity [[2]]

The Viscosity is a quantitative of a fluid's resistance to flow, more specifically, it determines the fluid strain rate that is generated by a given applied shear stress, we can easily move through air, which has very low viscosity, mouvement is more difficult in water, which has 50 times higher viscosity still more resistance is found in SAE 30 oil, which is 300 times more viscous than water, try to slide your hand through glycerin, which is five times more viscous than SAE 30 oil, or blacks trap mol asses, another factor of five higher than glycerin fluid may have a vast range of viscosities these differences in viscosity directly affect flow behavior, especially infree surface and potential flow.

1.3.3 the pressure [[2]]

The pressure is the (compression)stress at a point in a static fluid next to velocity, the pressure P is the most dynamic variable in fluid mechanics, differences or gradients in pressure of ten drive a fluid flow, especially in ducts, in low-speed flows, the actual magnitude of the pressure is of ten not important , unless it drops so low as to cause vapore bubles to form in a liquid, for convenience, we set many such problem assignments at the level of $1atm = 2116Ibf/ft^2 = 101,300Pa.$ High-speed (compressible) gas flows, however, are indeed sensitive to the magnitude of pressure.

1.3.4 The compressibility [[5]]

The Compressibility is a measure of the change in volume of a fluid subjected to a change in pressure, it is quantified using the bulk modulus, defined as:

$$E = -\frac{dP}{\frac{dV}{V}}$$

Where:

- dP: The defferential change in pressure
- $\frac{dV}{V}$: The fractional change in volume.

The negative sing is included since an increase in pressure will cause a decrease in volume , the bulk modulus can be written as:

$$E = \frac{d\rho}{\frac{d\rho}{\rho}}$$

Fluids whith a high bulk modulus, such as liquids, are nearly incompressible, whereaas gases exhibit signficant compressibility due to their lower bulk modulus.

1.3.4.1 Incompressibility [[10]]

Incompressibility refers to the oroperty of a fluid whose density remains constant during flow, meaning that the density does not change with time. This assumption is mathematicaly expressed as:

$$\frac{\partial \rho}{\partial t} = 0.$$

1.3.5 The Velocity [[5]]

The Velocity is defined as the time rate of change of the position vector $r_A(x, y)$ of a particle since velocity is a vector quantity, it has both magnitude and direction, the manitude of velocity denoted as $|V|$, represents the speed of the fluid and it is given by the equation:

$$|V| = \sqrt{u^2 + v^2},$$

where u, v are the velocity components in the x, y derections, repectively, the velocity field can be expressed as:

$$\vec{V} = u(x, y)\hat{\imath} + v(x, y)\hat{\jmath}.$$

Figure 1.3 illustrates how the position vector of a particle changees over time, leading to the definition of velocity as the time derivative of position vector, given by:

$$V = \frac{dr_A}{dt}.$$

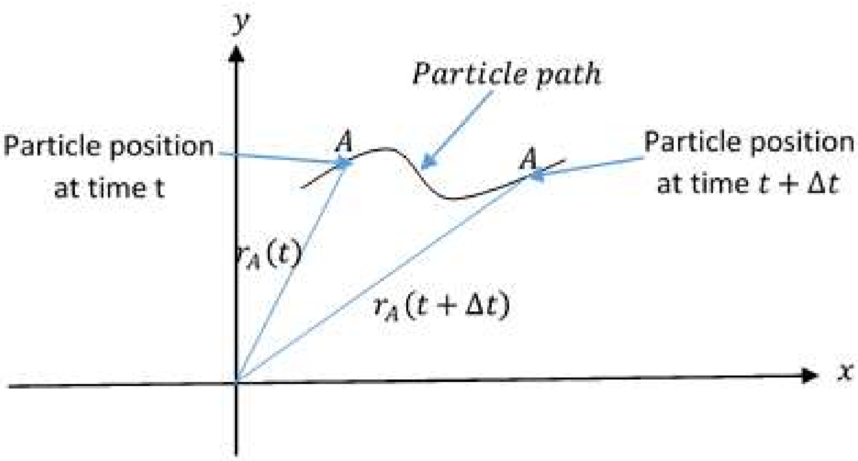

Figure 1.3: position vector of a moving particle in space and time

1.4 Some Equation in Fluid Mechanics:

In fluid mechanics study, it is necessary to known some governing equations and the associated fundamental concepts, such as:

1.4.1 The Continuity Equation (Mass conservation) [[4][1]]

The first necessary concept is the continuity equation, also known as the mass conservation equation is expressed as follows:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0. \tag{1.2}$$

For an incompressible fluid where density (ρ) remains constant with respect to time $(\frac{\partial \rho}{\partial t} = 0.)$, then the equation 1.2 simplifies to:

$$\nabla.\vec{v} = 0,$$

this equation ensures that mass is conserved in a fluid flow, meaning that the mass entering a control volume equals the mass leaving it.

Considerations,as shown in Fig 1.4, control volumes can alwas be defind whith in whith the system 's total mass can be stated as constant,if necessary these control volumes can comprise the whole earth $M = constant$

Figure 1.4: Urne with constant mass of water

1.4.2 The Potential Flow

In fluid mechanics, **potential flow** represents a special case where the vorticity is zero, Mathematically, this expressed as:

$$\nabla.\vec{v} = 0$$

In this type of flow, the governing equations cane be significantly simplified . Due to the irrotational nature of potential flow, we can define a **Velocity Potential** , typically denoted by ϕ

the velocity potential relates to the velocity vector by the following:

$$\vec{v} = \nabla\phi$$

This means that the velocity vector is the gradient of the scalare function ϕ, this property makes potential flow highly amenable to analysis using powerful mathematical tools.
When applying the continuity equation to incompressible fluid ($\nabla.\vec{v} = 0$) within the context of potential flow,and utilizing the definition of the velocity potential, we arrive at:

Laplace's Equation

$$\nabla.(\nabla\phi) = 0,$$

or,in another form

$$\Delta^2\phi = 0,$$

- In three-dimensional caratesian coordinate system,laplace's equation is written as:

$$\frac{\partial^2\phi}{\partial^2 x} + \frac{\partial^2\phi}{\partial^2 y}$$

1.4.3 The Euler equation [[11]]

The Euler equation of motion is one of the fundamental equations in fluid mechanics, describing the behavior of an ideal (inviscid) fluid under the influence of pressure and gravitational forces. For steady flow of an incompressible fluid along a streamline, the Euler equation can be expressed as:

$$\frac{dp}{\rho} + gdH + VdV = 0, \tag{1.3}$$

this equation represents the balance of forces acting on a fluid particle in motion and serves as the theoretical foundation from which Bernoulli's derived by integration along the streamline.

1.4.4 The Bernoulli equation [[1]]

The Bernoulli equation is one the most important equations in fluid mecanics, expressing the principle of energy conservation in systems where the flow is inviscid,and no external work is applied, it is derived from the energy equation, which, when simplified for friction flow no external work, takes the following from:

$$\frac{\Delta u^2}{2\alpha} + g\Delta y + \int V dp = 0.$$

For an incompressible fluid,such as liquid,we have:

$$\frac{\Delta \overline{u}^2}{2\alpha} + g\Delta y + V\Delta p = 0.$$

by substituting the relation between density ρ and specific volum $V = \frac{1}{\rho}$, the equation can be rewritten as:

$$\frac{\overline{u}_2^2}{2\alpha} + gy_2 + \frac{p_2}{\rho} = \frac{\overline{u}_1^2}{2\alpha} + gy_1 + \frac{p_1}{\rho} = 0,$$

this equation is known as the Bernoulli equation. The Bernoulii equation is applied to flow systems based on the following assumptions:

- The flow is **single-phase**
- The fluid is **incompressible**
- The flow is **one-dimensional**

- **No external work** is applied to the system .

Some times,by dividing all terms by g, another form of Bernoulli's equation,known as the head form, which is:

$$\frac{\overline{u}_2^2}{2\alpha g} + z_2 + \frac{p_2}{\rho g} = \frac{\overline{u}_1^2}{2\alpha g} + z_1 + \frac{p_1}{\rho g} = 0$$

The equation states that **the sum of kinetic energy,potential energy, and pressure energy remains constant** along a streamline. The terms represent:

- **Kitnetic Energy Head** : $\frac{\overline{u}_2^2}{2\alpha g}$, representing the height equivalent to kinetic energy
- **Potential Energy Head** : z, which is the elevation of the relative to a reference level
- **Pressure Head** :$\frac{p}{\rho g}$, representing the height equivalent to the pressure energy.

Bernoulli's equation for steady flow [[6]]

Bernoulli's equation for steady flow can be formulated as :

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gH = c$$

1.5 Boundary Conditions for The Free surface

In fluid mechanics,the free surface represents the boundary between two different fluids, such as water and air, where the surface can move freely in response to external forces like gravity,pressure, and surface tension, to describe its behavior, we define several boundary conditions that govern its motion and force balance, whivh are presented below:

1.5.1 Kinematic Boundary Condition [[7]]

this condition ensures that particles on the free surface remain onit at all times, meaning that the fluid cannot penetrate the surface, mathematically, it is expressed as:

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w$$

- η is the free surface elevation
- u, v are the horizontal and vertical velocity components
- w is the vertical velocity component at the free surface.

1.5.2 Dynamic Boundary condition [[7]]

This condition describes the pressure balance at the free surface,ensuring continuity of pressure across the interface according to bernoulli's equation:

$$p = p_{atm} + \rho g z$$

1.5.3 No-Penetration Condition [[7]]

This condition states that fluid cannot pentrate a submerged solid body,the fluid velocity normal to the surface must match the normal velocity of the body:

$$v.\eta.$$

1.5.4 surface tension Condition [[7]]

If surface tension is present,the pressure at the free surface must account for the curvature of surface, expressed as:

$$\Delta p = \sigma(\frac{1}{R_1} + \frac{1}{R_2})$$

- σ is the surface tension coefficient
- R_1, R_2 are the principal radii of curvature.

Chapter 2

Analytical approximate solutions of the flow over an obstacle

2.1 Introduction

The behavior of a free surface flow over obstacles is mainly governed by the Froude number $F = \frac{U}{\sqrt{gH}}$, which represents the ratio between the flow velocity and speed of gravity waves.It also depends on the obstacle height $h.$ Here U is the velocity of the uniform stream, H is the undisturbed water depth, and g is the gravity acceleration.As shown in Fig 2.1, the flow is considered subcritical if $F < 1$, and supercritical when $F > 1$. In subcritical regimes,complex wave patterns may develop over the obstacle, while in supercritical rigimes, the free surface is generally smoother.
Linear models provide reasonable predictions when the wave amplitude is small and F is far from 1. However, as the froude number approaches unity, the linear theory becomes inaccurate, requiring the application of weakly nonlinear approximations for a better description of the flow behavior.

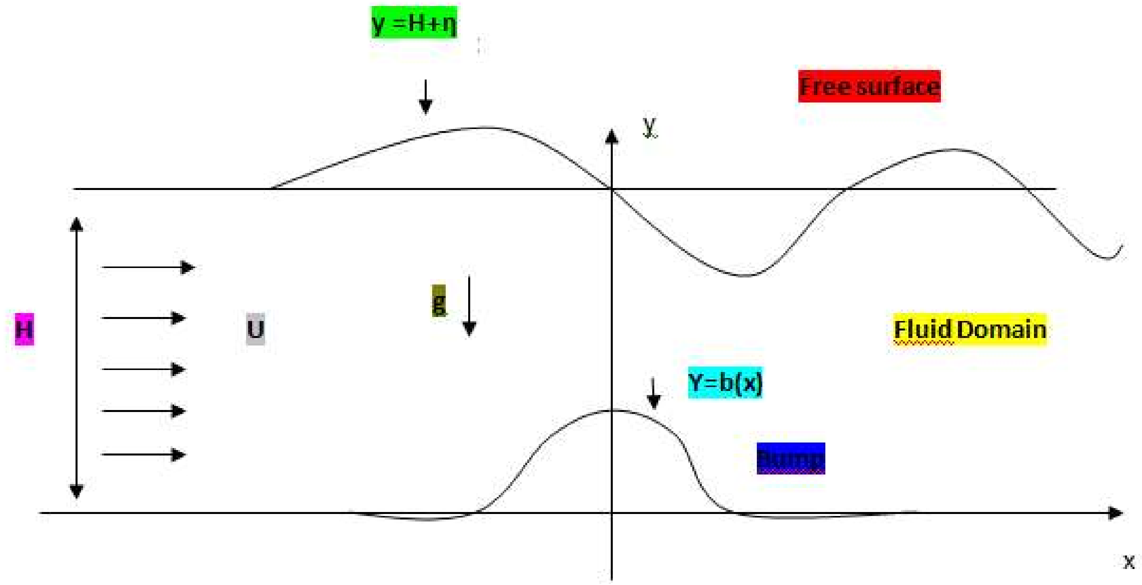

Figure 2.1: Sketch of the flow configuration

2.2 Fully Nonlinear Unsteady Governing Equations

We consider the free surface flow of an ideal fluid (inviscid and incompressible) in a horizontal channel with an obstacle placed on the bottom. The flow is assumed to be two-dimensional, irrotational, and of constant density. Surface tension effects are neglected, and only gravity is taken into account.
The motion of the fluid is governed by the Laplace equation for the velocity potential Φ

$$\Delta\Phi = 0, \qquad x \in \mathbb{R} \qquad , \qquad b(x) < y < H + \eta(x,t), \tag{2.1}$$

where:

- Φ is the velocity potential of the fluid
- $\Delta = \dfrac{\partial^2}{\partial^2 x} + \dfrac{\partial^2}{\partial^2 y} = 0$ is the two-dimensional Laplacian.

The governing system also includes the kinematic and dynamic boundary conditions.

Kinematic condition on the free surface:

$$\frac{D}{Dt}(y - H - \eta) = 0 \qquad \text{on} \qquad y = H + \eta(x,t) \tag{2.2}$$

Kinematic condition on the bottom:

$$\frac{D}{Dt}(y - b) = 0 \qquad \text{on} \qquad y = b(x), \tag{2.3}$$

Where $\frac{D}{Dt}$ denotes the material derivative.

Since the free surface is not known ($\eta(x,t)$ is an unknown of the problem),

Bernoulli's equation on the free surface:

$$\rho\frac{\partial \Phi}{\partial t} + \frac{\rho}{2}\left|(\nabla\Phi)^2\right| + g\rho(H+\eta) = \frac{\rho U^2}{2} + \rho g H \qquad \text{on} \qquad y = H + \eta(x,t), \tag{2.4}$$

where :

- ρ is the density of the fluid and U is the speed of the upstream flow.

After expressing the material derivatives, the boundary conditions become:

$$-\eta_t - \eta_x\Phi_x + \Phi_y = 0 \quad \text{on} \qquad y = H + \eta(x,t), \tag{2.5}$$

$$b_x + \Phi_x - \Phi_y = 0 \qquad \text{on} \qquad y = b(x), \tag{2.6}$$

where : η_x, Φ_x and Φ_y are the derivatives with respect to t,x and y of η and Φ respectively.

2.3 Perturbation of The Velocity Potential.

Since the base solution in the case of a flat channel is $(\Phi, \eta) = (U_x, 0)$, we introduce a perturbation of the velocity potential as follows:

$$\Phi(x,y,t) = U_x + \phi(x,y,t)$$

Substituting into the governing equations yields the perturbed system:

$$\Delta\phi = 0, \qquad x \in \mathbb{R} \qquad , \qquad b(x) < y < H + \eta(x,t) \tag{2.7}$$

$$-\frac{\partial \eta}{\partial t} - \eta_x(U + \phi_x) + \phi_y = 0, \qquad x \in \mathbb{R} \qquad , \qquad y = H + \eta(x,t), \tag{2.8}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\eta + U\phi_x = 0, \qquad x \in \mathbb{R} \qquad , \qquad y = H + \eta(x,t), \tag{2.9}$$

$$(U + \phi_x)b_x - \phi_y = 0, \qquad x \in \mathbb{R} \qquad , \qquad y = b(x). \tag{2.10}$$

2.4 Transformation of the Equations into Dimensionless Form

To simplify the analysis and highlight the key parameters, the governing equations are made dimensionless using the characteristic scales:

$$x^\star = \frac{x}{L} \qquad , \qquad y^\star = \frac{y}{H} \qquad , \qquad t^\star = \epsilon^{3/2}\sqrt{\frac{g}{H}}t \qquad , \qquad \eta^\star = \frac{\eta}{H}$$

$$\phi^\star = \frac{\phi}{L\sqrt{gH}} \qquad , b^\star = \epsilon^{-2}\frac{b}{H} \qquad , \qquad U^\star = \frac{U}{\sqrt{gH}} = F,$$

- ϵ is a small parameter given in the obstract.

  After applying the change of variables and omitting the stars $\star$ for convenience, the dimensionless system becomes:

$$\epsilon\phi_{xx} + \phi_{yy} = 0, \qquad x \in \mathbb{R} \qquad , \qquad \epsilon^2 b(x) < y < 1 + \eta(x,t), \tag{2.11}$$

$$\epsilon\eta_t + (F + \phi_x)\eta_x = \epsilon^{-1}\phi_y, \qquad \text{on}, \qquad y = 1 + \eta(x,t), \tag{2.12}$$

$$\epsilon\phi_t + \frac{1}{2}(\phi_x^2 + \epsilon^{-1}\phi_y^2) + F\phi_x + \eta = 0, \qquad \text{on}, \qquad y = 1 + \eta(x,t), \tag{2.13}$$

$$(F + \phi_x)b_x = \epsilon^{-3}\phi_y, \qquad \text{on}, \qquad y = \epsilon^2 b(x), \tag{2.14}$$

2.5 Fixed Domain and Taylor Expansions

2.5.1 Fixing the Flow Domain ($\mathbb{R} \times [0,1]$)

To simplify the analysis,the physical domain bounded by the free surface $y = 1 + \eta(x,t)$ and the bottom $\epsilon^2 b(x)$ is transformed into a fixed computation domain between $y = 0$ and $y = 1$. This domain transformation allows for systematic asymptotic expansions.

2.5.2 Taylor Expansions of the Boundary Conditions

Taylor series are applied to the boundary conditions at the free surface and bpttom around the reference levely $y = 1$ and $y = 0$, respectively, resulting in the following expressions: **Kinematic condition at the free surface:**

$$\epsilon\eta_t + (F + \phi_x + \phi_{xy}\eta)\eta_x = \epsilon^{-1}\phi_y + \epsilon^{-1}\phi_{yy}\eta, \qquad \text{on}, \qquad y = 1. \tag{2.15}$$

Dynamic condition at the free surface:

$$\epsilon\phi_t + \epsilon\phi_{ty}\eta + \frac{1}{2}(\phi_x^2 + \epsilon^{-1}\phi_y^2) + \phi_x\phi_{xy}\eta + \epsilon^{-1}\phi_y\phi_{yy}\eta + F\phi_x + F\phi_{xy}\eta + \eta = 0, \qquad \text{on}, \qquad y = 1. \tag{2.16}$$

Boundary condition at the bottom:

$$(F + \phi_x + \epsilon^2\phi_{xy}b)b_x = \epsilon^{-3}\phi_y + \epsilon^{-1}\phi_{yy}b, \qquad \text{on}, \qquad y = 0. \tag{2.17}$$

2.5.3 Expansion of the Froude Number

Since the flow is studied near the critical regime, the Froude number is expanded as follows:

$$F = F_0 + \epsilon F_1 + O(\epsilon^2), \tag{2.18}$$

where:

- F_0 is the critical speed of the upstream uniform flow which will be determined.
- F_1 is a measurement of the perturbation of the upstream uniform velocity F from the critical value F_0.

These expansions of nonlinearity and dispersion at different orders.

2.6 Asymptotic expansions

2.6.1 Formal Expansions of ϕ and η

To construct approximate solutions, the velocity potential $\phi(x, y, t, \epsilon)$ and the free surface elevation $\eta(x, t, \epsilon)$ are expanded in powers of ϵ:

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + O(\epsilon^4), \tag{2.19}$$

$$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + O(\epsilon^3). \tag{2.20}$$

These expansions are substituted into the governing system, and terms are grouped according to powrs of ϵ, resulting in successive boundary value problems.

2.7 Boundary Value Problems by Order

Lowest order system:

$$\begin{aligned} \phi_{1yy} &= 0 \qquad in \qquad 0 < y < 1, \\ \frac{1}{2}\phi_{1y}^2 + F_0\phi_{1x} + \eta_1 &= 0 \qquad on \qquad y = 1, \\ \phi_{1y} &= 0 \qquad on \qquad y = 1, \\ \phi_{1y} &= 0 \qquad on \qquad y = 0, \end{aligned} \tag{2.21}$$

First order system:

$$\begin{aligned} \phi_{1xx} + \phi_{2yy} = 0 \qquad in \qquad 0 < y < 1, \\ \phi_{1t} + \frac{1}{2}\phi_{1x}^2 + \phi_{1y}\phi_{2y} + \phi_{1y}\phi_{1xy}\eta_1 + F_0\phi_{2x} + F_1\phi_{1x} + F_0\phi_{1xy}\eta_1 + \eta_2 = 0 \qquad on \qquad y = 1, \end{aligned} \tag{2.22}$$

$$\begin{aligned} F_0\eta_{1x} - \phi_{2y} - \phi_{1yy}\eta_1 = 0 \qquad on \qquad y = 1, \\ \phi_{2y} = 0 \qquad on \qquad y = 0, \end{aligned} \tag{2.23}$$

Second order system:

$$\phi_{2xx} + \phi_{3yy} = 0 \qquad in \qquad 0 < y < 1, \tag{2.24}$$

$$\eta_{1t} + F_0\eta_{2x} + F_1\eta_{1x} + \phi_{1x}\phi_{1x} - \phi_{3y} - \phi_{1yy}\eta_2 - \eta_1\phi_{2yy} = 0 \qquad on \qquad y = 1, \tag{2.25}$$

$$F_0 b_x - \phi_{3y} - \phi_{1yy} b = 0 \qquad on \qquad y = 0. \tag{2.26}$$

2.7.1 Derivation of the Forced Korteweg-de Vries (fKdV) Equation

From integration of lowest order system

$$\phi_{1x} = -\frac{\eta(x,t)}{F_0}, \tag{2.27}$$

From the first-order system

$$\phi_2 = \frac{1}{2F_0}\eta_{1x}y^2 + G(x,t). \tag{2.28}$$

Using2.22,we obtain:

$$G_x = -\frac{\phi_{1t}}{F_0} - \frac{1}{2F_0^3}\eta_1^2 - \frac{1}{2F_0}\eta_{1xx} + \frac{F_1}{F_0^2}\eta_1 - \frac{\eta_2}{F_0}, \tag{2.29}$$

then

$$\phi_{2x} = \frac{1}{F_0}\eta_{1xx}y^2 - \frac{\phi_{1t}}{F_0} - \frac{1}{2F_0^3}\eta_1^2 - \frac{1}{2F_0}\eta_{1xx} + \frac{F_1}{F_0^2}\eta_1 - \frac{\eta_2}{F_0}. \tag{2.30}$$

From2.23 and**??**, it follows $(F_0^2-1)\eta_{1x}=0$. For a nontrivial solution η_{1x}, we have $\eta_{1x}\neq 0$,so

$$F_{0^2}=1. \tag{2.31}$$

Subtituting and simplifying leads to the following equation:

$$\eta_{1t}+F_0\eta_{2x}+F_1\eta_{1x}+\phi_{1x}\eta_{1x}-\eta_1\phi_{2yy}-F_0b_x=-[\frac{\eta_{1xxx}}{6F_0}-\frac{\phi_{1tx}}{F_0}-\frac{(\eta_1^2)_x}{2F_0^3}-\frac{\eta_{1xxx}}{2F_0}+\frac{F_1\eta_{1x}}{F_0^2}-\frac{\eta_{2x}}{F_0}]. \tag{2.32}$$

Finally, it results

$$\eta_{1t}+F_1\eta_{1x}-\frac{3}{2}\eta_1\eta_{1x}-\frac{1}{6}\eta_{1xxx}=\frac{1}{2}b_x. \tag{2.33}$$

Equation2.32 is called the forsed Korteweg-de Vries equation (fKdV). In the following of this work, we neglect η_{1x} in2.32,so we consider the steady equation:

$$F_1\eta_{1x}-\frac{3}{2}\eta_1\eta_{1x}-\frac{1}{6}\eta_{1xxx}=\frac{1}{2}b_x, \qquad x\in\mathbb{R} \qquad . \tag{2.34}$$

The unknown function $\eta_1(x)$ represents the first order elevation of the free surface of the fluid.Note that when the flow is supercritical, far the obstacle,upstream and downstream, η_1 tends to zero. Hwever, if the flow is subcritical, η_1 and η_{1x} tend to zero,upstream,far the obstacle.
Therefore the solution of equation2.34, is equivalent to solve the second order nonlinear ordinary differential equation:

$$F_1\eta_1-\frac{3}{4}\eta_1^2-\frac{1}{6}\eta_{1xx}=\frac{1}{2}b, \qquad x\in\mathbb{R} \qquad , \tag{2.35}$$

with

$$\lim_{x\to-\infty}\eta_1(x)=\lim_{x\to+\infty}\eta_1(x)=0 \qquad if \qquad F_1>0 \qquad (supercritical flow),$$
$$\lim_{x\to-\infty}\eta_1(x)=\lim_{x\to-\infty}\eta_{1x}(x)=0 \qquad if \qquad F_1<0 \qquad (subcritical flow).$$

For a given bump b and a given upstream near the critical flow speed $F=1+\epsilon F_1$,we can find an asymptotically approximate shape of the free surface $y=1\epsilon\eta_1(x)+O(\epsilon^2)$ .Since we solve numerically equation2.35, so we take x in $[-l, l], l \in \mathbb{R}$, then $\eta_1(-l) = \eta_1(l) = 0$ for $F > 1$ and $\eta_1(-l) = \eta_{1x}(-l) = 0$ for $F < 1$.

In the following section, we describe the numerical method which is used for solving equation2.35, and we present our results in different figures.

CHAPTER 3

NUMERICAL APPROXIMATE SOLUTIONS OF THE FLOW OVER AN OBSTACLE

Following the analytical derivation of the forced Korteweg-de Vries (fKdV) equation, a numerical approach is adopted to study the influence of a localized bottom obstacle on the free surface profile. The equation is treated under two flow regimes: first in the supercritical case($F_1 > 0 \quad or \quad F > 1$),with **the finite difference method** , second in the subcritical case ($F_1 < 0 \quad or \quad F < 1$), with **the fourth order Runge-Kutta method**.

3.1 Supercritical Flow: Finite Difference Method

In the supercritical case, where the flow speed exceeds the wave speed, the steady form of the fKdV equation is considered:

$$F_1\eta_1 - \frac{3}{4}\eta_1^2 - \frac{1}{6}\eta_{1xx} = \frac{1}{2}b, \qquad x \in \mathbb{R} \qquad .$$

3.1.1 Domain and Discretization

The computational domain is defined as a symmetric interval $[-l, l]$, it is discretized into $N+2$ uniformly spaced poins with spatial step P where: $x_j = -l + jp, j = 0, ..., N+1,$
$N \in \mathbb{N} \qquad and \qquad p = \dfrac{2l}{N+1}$

3.1.2 Nonlinear System and Numerical Solution

By approximating the second derivative in equation2.35 using a central finite difference scheme:

$$\eta_{1xx}(x_j) \approx \frac{\eta_{1j+1} - 2\eta_{1j} + \eta_{1j-1}}{p^2},$$

we obtain one discretized equation at each internal grid point j. This results in a nonlinear system of N equations to determine the N unknowns $\eta_{11}, \eta_{12}, ...\eta_{1N}$. As stated in [[8]]

The boundary conditions are defined for the supercritical regime as:

$$\eta_{10} = \eta_1(-l) = 0, \eta_{1N+1} = \eta_1(l) = 0$$

Which represent a uniform stream at the inlet and outlet , as described in the article [[8]]

Thus, for a given value of F_1 and a fixed obstacle $b(x)$, a nonlinear algebraic system is built and solve using **Newton's method**

once the solution is obtained, the free surface profile is reconstructed by plotting the values (x_j, η_{1j})

3.1.3 Obstacle shape

In the numerical scheme, the obstacle is modeled by a smooth, symmetric, localized function given by:

$$b(h,x) = \begin{cases} h\cos(\frac{\pi}{2}x) & if \quad |x| \leq 1 \\ 0 & else. \end{cases}$$

This type of obstacle is widely used in the literature to represent smooth bumps on the bottom surface due to its mathematical simplicity and symmetry. It was explicitly adopted in [[8]] to study the free surface response under varying flow conditions.

3.1.4 Numerical Results Analysis

After solving the nonlinear system numerically, the free surface elevation η_1 is plotted as a function of x. The results are illustrated through various plots that show the dependence of the free surface profile on both the **height of the obstacle** h and the **Froude number** F $(F = 1 + \epsilon F_1)$

The numerical solutions demonstrate that under supercritical conditions, the surface tends to follow the obstacle's shape locally, and the maximum elevation of the free surface decreases as the Froude number increases, increases the inverse relationship between flow speed and surface deformation caused by the obstacle.

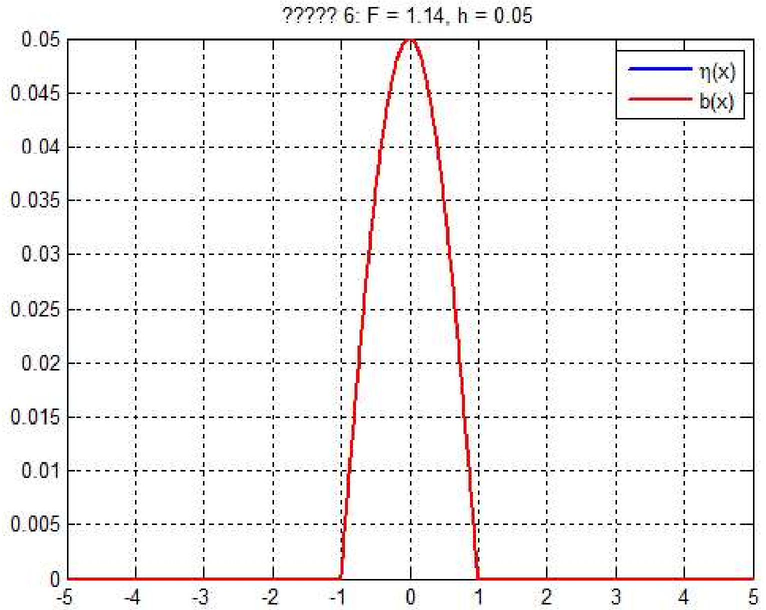

Figure 3.1: Free surface flow profile over a bump $b(h, x)$
for $F = 1.14$ and $h = 0.05$

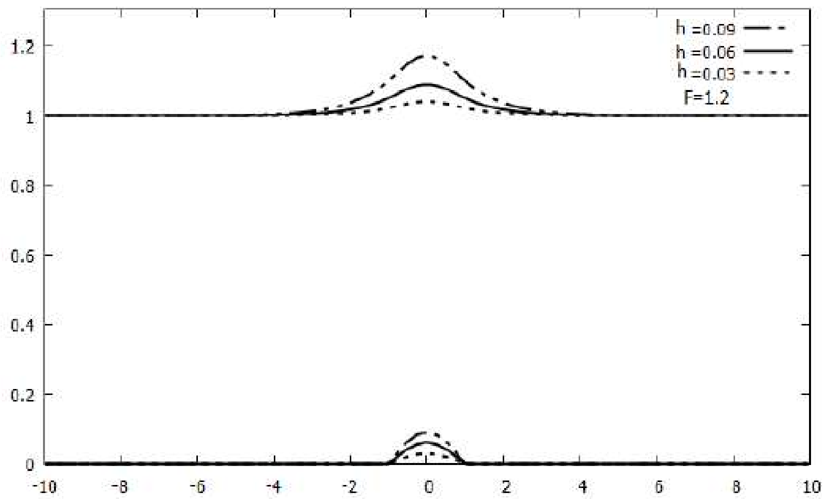

Figure 3.2: Plot of the free surface profile over a bump $b(h, x)$ for different heights h of the obstacle and the Froude number fixed .

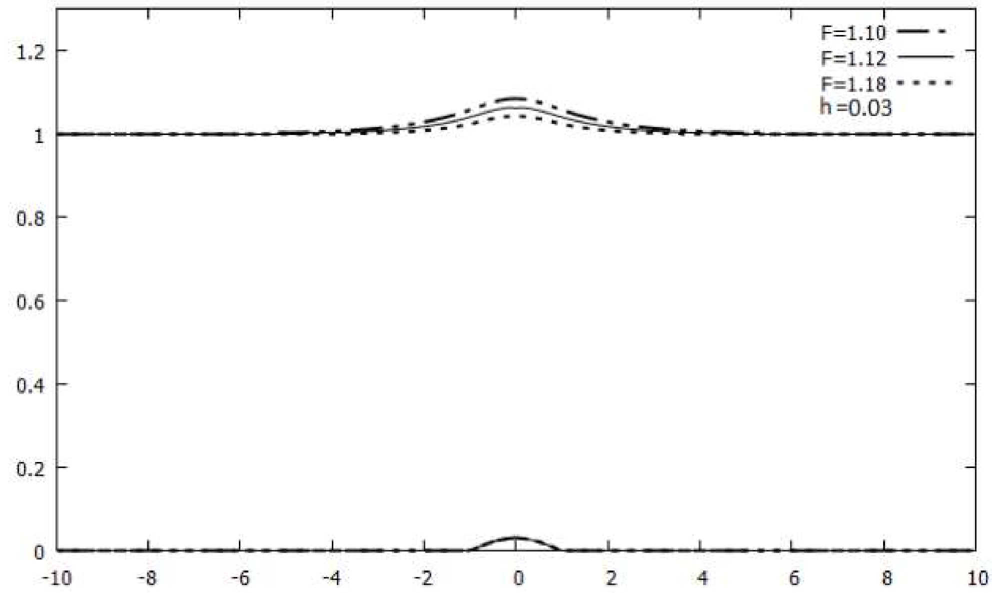

Figure 3.3: Plot of the free surface profile over a bump $b(h, x)$ for different values of the Froude number F and h fixed.

3.2 Subcritical case: Runge-Kutta Method

In the subcritical flow regime ($F < 1$), the following boundary conditins are imposed on the first-order elevation the free surface: $\eta_1(-l) = \eta_1'(-l) = 0$

To trasform the **(fKdV)** equation into a first-order system, we define the variables: $z_1 = \eta_1$ and $z_2 = \eta_1' = \eta_{1x}$, then $z_{1(-l)} = z_2(-l) = 0$

The equation 2.35 is then rewritten as the following system:

$$\begin{aligned} z_1\prime = \eta_1\prime = z_2 = f_1(x, z_1, z_2), \\ z_2\prime = \eta_1\prime\prime = \eta_{1xx} = 6F_1 z_1 - \frac{9}{2}z_1^2 - 3b(x) = f_2(x, z_1, z_2) \end{aligned} \tag{3.1}$$

Where the expression for $z_2\prime$ is derived directly from the original equation.

3.2.1 Domain descretization

As in the supercritical case, the spatial domain $[-l, l]$ is discretized into $N+2$ equally spaced poins using a step size: $p = \dfrac{2l}{N+1}$ With the grid points defined as:

$x_i = -l + ip, i = 0, ...N + 1.$ and the numerical approximations:

$z_{1,j} \approx z_1(x_j), \qquad z_{2,j} \approx z_2(x_j),$

3.2.2 Runge-Kutta implementation

To solve the system 3.1, the classical **fourth-order Runge-Kutta method** is used the update formulas are:

$$z_{1,i+1} = z_{1,i} + \frac{p}{6}(K_{11} + 2K_{21} + 2K_{31} + K_{41}),$$

$$i = 0, ...N$$

$$z_{2,i+1} = z_{2,i} + \frac{p}{6}(K_{12} + 2K_{22} + 2K_{32} + K_{42}).$$

where $z_{1,0} = z_1(-l)0$ and $z_{2,0} = z_2(-l)0$, with intermediate stages defined by:

$$K_{11} = f_1(x_i, z_{1,i}, z_{2,i}), \quad K_{12} = f_2(x_i, z_{1,i}, z_{2,i}),$$

$$K_{21} = f_1(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}K_{11}, z_{2,i} + \frac{p}{2}K_{12}),$$

$$K_{22} = f_2(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}K_{11}, z_{2,i} + \frac{p}{2}K_{12}),$$

$$K_{31} = f_1(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}K_{21}, z_{2,i} + \frac{h}{2}K_{22}),$$

$$K_{32} = f_2(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}K_{21}, z_{2,i} + \frac{p}{2}K_{22}),$$

$$K_{41} = f_1(x_i + p, z_{1,i} + pK_{31}, z_{2,i} + pK_{32}),$$

$$K_{42} = f_2(x_i + p, z_{1,i} + pK_{31}, z_{2,i} + pK_{32}),$$

The computation starts from the initial conditions :$z_1(-l) = 0$, $z_2(-l) = 0$

and the final numerical profile of the free surface $\eta_1(x)$ is obtained from the computed values of $z_{1,i}$.

3.2.3 Results Visualization

After performing the numerical integration using the Runge-Kutta method, the free surface profiles $\eta_1(x)$ are visualized through plots with respect to the spatial variable x. These profiles are displayed for various values of the obstacle height h and the Froude number F, in order to analyze thier effect on the wave shape. The results illustrate how the obstacle causes a depression in the free surface, followed by a train of oscillations whose amplitude and structure vary according to the flow parameters. This allows for precise comparison between different flow regimes and demonstrates the sensitivity of the model.

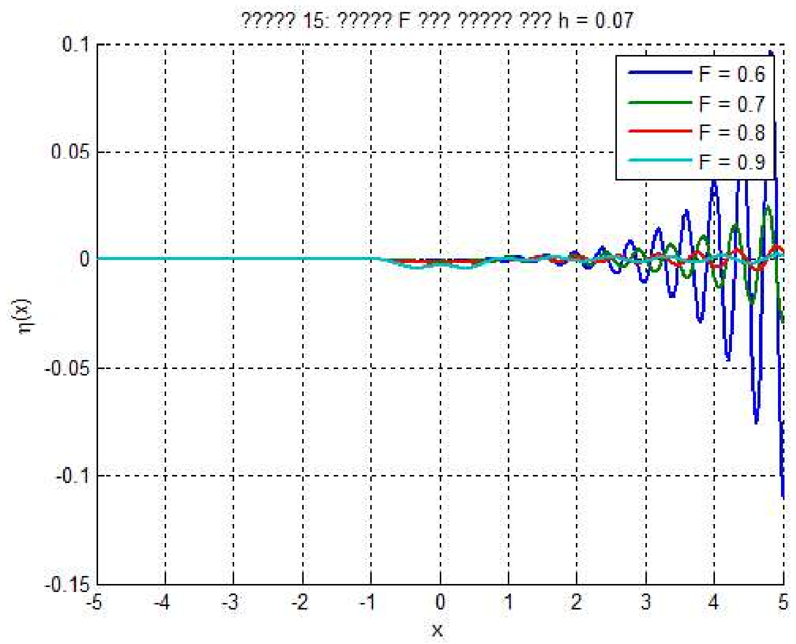

Figure 3.4: Plot of the free surface profile over a bump $b(h, x)$ for different values of the Froude number F and h fixed.

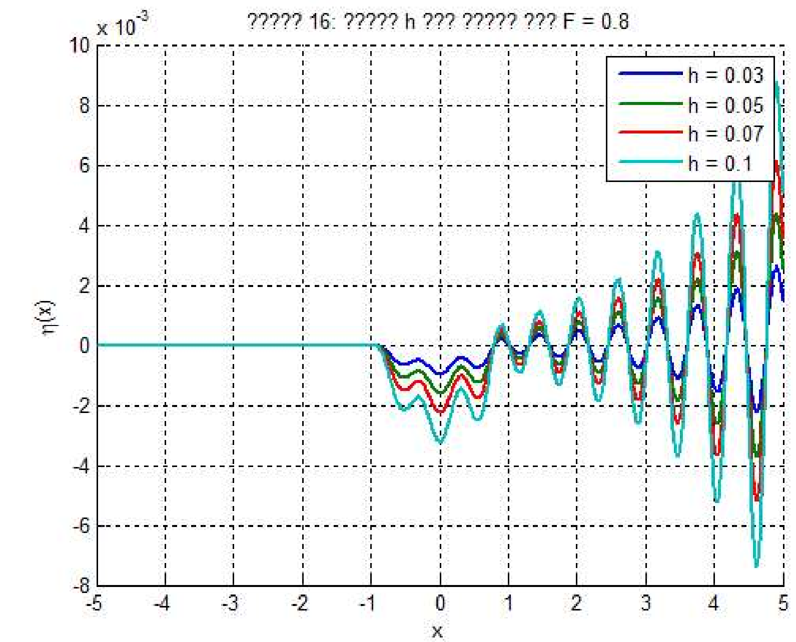

Figure 3.5: Plot of the free surface profile over a bump $b(h, x)$ for different heights h of the obstacle and the Froude number fixed.

Conclusion

In this dissertation, we studied a mathematical model representing the free surface flow of an ideal fluid inside a horizontal channel with an obstacle, starting from Laplace's equation for the velocity potential, along with boundary conditions on the free surface and the bottom.To facilitate the analysis, the ptential was reformulated as a pertubation around the refernce state, and a non-dimensional system was adopted using a small parameter ϵ.

The boundary conditions were simplified using Taylor expansions, and the domain was fixed between the bottom and the reference surface. This approach allowed the application of asymptotic expansions and led to the derivation of the forced Korteweg-de Vries (fKdV) equation, which describes the surface elevation as a function of the obstacle shape.

Two numerical methods were adopted to solve the resulting equation: The Runge-Kutta method and the finite difference method. This enabled numerical verification of the analytical solutions.

These findings provide a solid foundation for future work aiming at a deeper understanding of complex surface flows, particulary in systems of bottom topography on the structure of the free surface.

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