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Fundamentals of functional Analysis 1

كمرجع للدروس لطلبة **السنة الأولى ماستر تحليل دالي - رياضيات.**

وهذا بعد الاطلاع على التقارير الإيجابية للأستاذ الخبير المكلف بالمطبوعة.

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سلمت هذه الشهادة للمعني لاستعمالها في حدود ما يسمح به القانون.

University of Mohamed Boudiaf–M'sila

Faculty of Mathematics and Computer Science

Fundamentals of Functional Analysis 01

Prepared by

Dr Ahlem Alouani

Academic Year

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Abstract

This course introduces the main concepts and fundamental results of Functional Analysis for first-year Master's (M1) students at the University of M'sila. It covers normed and Banach spaces, Hilbert spaces, continuous linear operators, and dual spaces. Special emphasis is placed on the fundamental theorems of the theory, including the Hahn–Banach Theorem, the Uniform Boundedness Principle, the Open Mapping Theorem, and the Closed Graph Theorem. The course also presents weak topologies and weak convergence, highlighting their importance in operator theory and applications.

keywords: Normed spaces, Banach spaces, Hilbert spaces, inner product, orthogonality, linear operators, bounded operators Continuous linear functionals, dual spaces, Hahn–Banach theorem , open Mapping theorem, closed Graph theorem, uniform Boundedness Principle, weak convergenc, weak-* convergence, reflexive spaces.

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0.1 Introduction

This text presents a course in Functional Analysis intended for first-year Master's (M1) students at the University of M'sila (Fall 2025). Functional Analysis is a central area of modern mathematics that studies infinite-dimensional vector spaces and the linear operators acting on them. It provides a unifying structure for many branches of mathematics, including differential equations, harmonic analysis, probability, and mathematical physics.

The origins of Functional Analysis lie in the study of function spaces, where functions are viewed as elements of vector spaces. This perspective allows the use of algebraic and geometric methods to analyze fundamental notions such as convergence, continuity, and stability. Over time, this approach has developed into a rich and powerful theory with deep theoretical and practical implications.

At its foundation, Functional Analysis begins with normed vector spaces, which provide a natural setting to measure the size of elements and study convergence. The notion of completeness leads to Banach spaces, where every Cauchy sequence converges, ensuring a robust analytical setting. The study of dual spaces further enriches the theory by introducing continuous linear functionals, offering deeper insight into the structure of these spaces.

A particularly important class of spaces is given by Hilbert spaces, where the presence of an inner product allows for geometric methods, such as orthogonality and projections. Operators on Hilbert spaces play a fundamental role in both theory and applications, especially in areas such as quantum mechanics and spectral theory.

The development of Functional Analysis relies on several fundamental results that reveal the structure of infinite-dimensional spaces. Among these, Baire's Theorem occupies a central position, serving as a key tool in establishing many of the major results of the theory. Its consequences include the Uniform Boundedness Principle, which ensures that pointwise bounded families of operators are uniformly bounded, the Open Mapping Theorem, which guarantees that surjective continuous linear operators between Banach spaces are open, and the Closed Graph Theorem, which provides a useful criterion for the continuity of linear operators.

Another cornerstone of the theory is the Hahn–Banach Theorem, which allows the extension of linear functionals and plays a crucial role in the study of duality. Its consequences are far-reaching, particularly in separation theorems and the analysis of convex structures.

Finally, the introduction of weak topologies on Banach spaces provides a refined notion of convergence that is weaker than norm convergence but extremely powerful in applications, especially in the study of compactness and operator theory.

Chapter 1

Normed Vector Spaces

1.1 Metric Spaces

1.1.1 Fundamental Notions

Let X be an arbitrary nonempty set.

Definition

A *metric* (or distance) on X is a function

$$d : X \times X \to \mathbb{R}_+$$

such that for all $x, y, z \in X$, we have

(i) $d(x,y) = 0 \iff x = y$ (separation),

(ii) $d(x,y) = d(y,x)$ (symmetry),

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

A *metric space* (X,d) is a set X equipped with a metric d.

Example 1.1.1. *If we let $d(x,y) = |x-y|$, then $(\mathbb{R}, d)$ is a metric space. The first two conditions are obvious, and the third follows from the usual triangle inequality:*

$$d(x,y) = |x-y| = |(x-z)+(z-y)| \leq |x-z| + |z-y| = d(x,z) + d(z,y).$$

Example 1.1.2. *Let* $X = C([a,b];\mathbb{R})$ *be the set of all continuous functions* $f : [a,b] \to \mathbb{R}$. *There are many ways to measure the distance between two functions in* X, *for instance*

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx,$$

$$d_2(f,g) = \left(\int_a^b |f(x) - g(x)|^2\, dx \right)^{1/2},$$

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}.$$

Each of these defines a metric on X.

Definition (Convergent and Cauchy sequences in a metric space)

Let (X,d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X.

1. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ (or that x is the limit of $(x_n)_{n\in\mathbb{N}}$) if

   $$\lim_{n\to\infty} d(x_n, x) = 0,$$

   that is, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

   $$n \geq N \quad \Rightarrow \quad d(x_n, x) < \varepsilon.$$

   In this case we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

2. The sequence (x_n) is called a *Cauchy sequence* if

   $$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } m, n \geq N \quad \Rightarrow \quad d(x_n, x_m) < \varepsilon.$$

Definition (Complete metric space)

Let (X, d) be a metric space. We say that X is *complete* if every Cauchy sequence in X is convergent in X; that is,

$$\forall (x_n)_{n\in\mathbb{N}} \subset X, \quad \Big((x_n)_{n\in\mathbb{N}} \text{ is Cauchy } \Rightarrow \exists x \in X \text{ such that } x_n \to x \Big).$$

In this case, X is called a *complete metric space*.

1.1.2 Open and Closed Sets in Metric Spaces

Let (X, d) be a metric space.

Definition

The *open ball* of radius $r > 0$ centered at $x_0 \in X$ is the set

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

The *closed ball* of radius $r > 0$ centered at $x_0 \in X$ is the set

$$\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}.$$

Remark 1.1.1. *The closed ball $\overline{B}(x_0, r)$ is not always the closed $\overline{B(x_0, r)}$ of the open ball $B(x_0, r)$. For example, in $X = \mathbb{Z}$ with the usual distance, we have*

$$\overline{B(0,1)} = \{0\}, \quad \overline{B}(0,1) = \{-1, 0, 1\}.$$

However, we always have $\overline{B(x_0, r)} \subseteq \overline{B}(x_0, r)$, with equality if X is a normed space $\overline{B(x_0, r)} = \overline{B}(x_0, r)$.

Definition

Let $S \subseteq X$ (S be a subset in X).

1. A point $x_0 \in X$ belongs to the *interior* of S if $\exists r > 0$ such that $B(x_0, r) \subseteq S$. The set of all such points is denoted $\operatorname{Int}(S)$.
2. A point $x_0 \in X$ belongs to the *exterior* of S if $\exists r > 0$ such that $B(x_0, r) \cap S = \varnothing$. The set of all such points is denoted $\operatorname{Ext}(S)$.
3. $\overline{S} = \operatorname{Ext}(S)^c$, where $\overline{S}$ is the closure of S.
4. The *boundary* of S is $\operatorname{Fr}(S) = \overline{S} \setminus \operatorname{Int}(S)$.

Definition

Let (X, d) be a metric space.

1. A subset $S \subseteq X$ is *open* if for every $x_0 \in S$, there exists $r > 0$ such that $B(x_0, r) \subseteq S$.
2. A subset $S \subseteq X$ is *closed* if its complement $X \setminus S$ (or S^c) is open.

Proposition 1.1.1. *All open balls $B(x_0, r)$ are open sets, while all closed balls $\overline{B}(x_0, r)$ are closed sets.*

Proof. We prove the statement for open balls. Assume $x \in B(x_0, r)$. We must show that there exists $r_1 > 0$ such that $B(x, r_1) \subseteq B(x_0, r)$. Choose $r_1 = r - d(x_0, x) > 0$. If $y \in B(x, r_1)$, then by the triangle inequality,

$$d(x_0, y) \leq d(y, x) + d(x, x_0) < r_1 + d(x_0, x) = r.$$

Hence $y \in B(x_0, r)$.

The statement for closed balls follows similarly.
Recall that the closed ball centered at $x_0 \in X$ of radius $r > 0$ is

$$\overline{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}.$$

A subset $F \subseteq X$ is closed if its complement $X \setminus F$ is open.

Let $x \notin \overline{B}(x_0, r)$, i.e. $d(x, x_0) > r$. Define

$$\varepsilon = d(x, x_0) - r > 0.$$

If $y \in B(x, \varepsilon)$, then by the triangle inequality,

$$d(y, x_0) \;\geq\; d(x, x_0) - d(y, x) \;>\; (r + \varepsilon) - \varepsilon = r.$$

Thus $y \notin \overline{B}(x_0, r)$, which shows that

$$B(x, \varepsilon) \subseteq X \setminus \overline{B}(x_0, r).$$

Therefore $X \setminus \overline{B}(x_0, r)$ is open, and so $\overline{B}(x_0, r)$ is closed.

□

Theorem

Let (X, d) be a metric space.

1. The sets $\varnothing$ and X are open.
2. Any union of open sets is open.
3. Any finite intersection of open sets is open.

Proof. **By definition.** A set $U \subseteq X$ is **open** if for every $x \in U$, there exists $\varepsilon > 0$ such that the open ball

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

is contained in U.

(1)

- $\varnothing$ is open vacuously (no elements to check).
- X is open because for any $x \in X$, every ball $B(x, \varepsilon) \subseteq X$.

(2): Let $\{U_i\}_{i \in I}$ be a family of open sets and $U = \bigcup_{i \in I} U_i$. For any $x \in U$, there exists $i_0 \in I$ such that $x \in U_{i_0}$. Since U_{i_0} is open, there exists $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq U_{i_0} \subseteq U$. Hence, U is open.

(3): Let $U_1, \ldots, U_n$ be open sets and $V = \bigcap_{i=1}^{n} U_i$. For any $x \in V$, we have $x \in U_i$ for all i. For each i, there exists $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subseteq U_i$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$. Then $B(x, \varepsilon) \subseteq B(x, \varepsilon_i) \subseteq U_i$ for all i, so $B(x, \varepsilon) \subseteq V$. Hence, V is open.

□

Theorem (Complementary)

1. The sets $\varnothing$ and X are closed.
2. Any intersection of closed sets is closed.
3. Any finite union of closed sets is closed.

1.1.3 Continuous Functions

Let (X, d_X) and (Y, d_Y) be two metric spaces.

Definition (ε-δ continuous)

A function $f : X \to Y$ is *continuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \text{whenever } d_X(x, x_0) < \delta.$$

Equivalently, f is continuous at x_0 iff

$$\forall \varepsilon > 0,\ \exists \delta > 0 : B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon)).$$

Definition

A function $f : X \to Y$ is called *continuous* if it is continuous at every $x \in X$.

Definition (Sequential continuity)

A function $f : X \to Y$ is continuous at $x_0 \in X$ iff

$$\forall (x_n) \subset X, \quad x_n \to x_0 \implies f(x_n) \to f(x_0).$$

These two definitions are equivalent.

Proof. We prove both implications.

(1) ε-δ continuity $\Rightarrow$ Sequential continuity

Assume f is ε-δ continuous at x_0. Let (x_n) be a sequence in X with $x_n \to x_0$. We want to show $f(x_n) \to f(x_0)$.

Let $\varepsilon > 0$ be arbitrary. By ε-δ continuity, there exists $\delta > 0$ such that:

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon.$$

Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$d_X(x_n, x_0) < \delta.$$

Therefore, for all $n \geq N$:

$$d_Y(f(x_n), f(x_0)) < \varepsilon.$$

This shows $f(x_n) \to f(x_0)$.

(2) Sequential continuity ⇒ ε-δ continuity

We prove the contrapositive: if f is not ε-δ continuous at x_0, then f is not sequentially continuous at x_0.

Suppose f is not ε-δ continuous at x_0. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists some $x \in X$ with:

$$d_X(x, x_0) < \delta \quad \text{but} \quad d_Y(f(x), f(x_0)) \geq \varepsilon.$$

In particular, for each $n \in \mathbb{N}$, take $\delta = \frac{1}{n}$. Then there exists $x_n \in X$ such that:

$$d_X(x_n, x_0) < \tfrac{1}{n} \quad \text{and} \quad d_Y\big(f(x_n), f(x_0)\big) \geq \varepsilon,$$

The sequence (x_n) converges to x_0 since $d_X(x_n, x_0) < \frac{1}{n} \to 0$. However, $(f(x_n))$ does not converge to $f(x_0)$ because $d_Y(f(x_n), f(x_0)) \geq \varepsilon$ for all n.

Therefore, f is not sequentially continuous at x_0.

□

1.2 Normed and Banach Spaces

1.2.1 Definitions and Properties

A set X is a vector space (or linear space) over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ if

1. **Closure under Addition** for all $x, y \in X$ and $\lambda \in \mathbb{F}$, we have $x + y \in X$ and $\lambda x \in X$,

2. **Commutative of Addition** the set X is an Abelian group with respect to the addition $+$,

3. the following identities hold for all $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$:

   $\alpha(x+y) = \alpha x + \alpha y$, distributivity of scalar multiplication over vector addition

   $(\alpha+\beta)x = \alpha x + \beta x$, distributivity of scalar multiplication over field addition

   $(\alpha\beta)x = \alpha(\beta x)$, associative of scalar multiplication

   $1x = x$. identity

Definition

A *norm* $\|\cdot\|$ on a real or complex vector space X is a function

$$\|\cdot\| : X \to \mathbb{R}_+$$

such that for all $x, y \in X$ and $\lambda \in \mathbb{F}$,

(i) $\|x\| = 0 \iff x = 0$ (non-degeneracy),

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ (absolute homogeneity),

(iii) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality).

A *normed space* $(X, \|\cdot\|)$ is a vector space X equipped with a norm.

Exercise 1.2.1. *Every normed space is a metric space with associated metric*

$$d(x, y) = \|x - y\|.$$

The converse may not be true?

Conversly Using counterexample
Consider $\mathbb{R}$ with the discrete metric:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

This is a metric space. However, suppose for contradiction that there exists a norm $\|\cdot\|$ on $\mathbb{R}$ such that $d(x,y)=\|x-y\|$ for all $x,y\in\mathbb{R}$. Then for any $x\neq 0$, we have:

$$\|x\|=d(x,0)=1.$$

But by the homogeneity property of norms, $\|2x\|=|2|\cdot\|x\|=2$. On the other hand, since $2x\neq 0$, we have:

$$\|2x\|=d(2x,0)=1.$$

This leads to a contradiction: $2=1$. Hence, no such norm exists, and the discrete metric on $\mathbb{R}$ is not induced by any norm.

Example 2: $d(x,y)=\frac{|x-y|}{1+|x-y|}$ for all $x,y\in V$

By the triangle inequality for norms, we have:

Proposition 1.2.1 (Continuity of the Norm)**.** *For all $x,y\in X$, we have*

$$|\|x\|-\|y\||\leq\|x\mp y\|.$$

In particular, if $x_n\to x$, then $\|x_n\|\to\|x\|$.

indeed

$$|\|x\|-\|y\||\leq\|x-y\|.$$

Now, suppose $x_n\to x$ in X. Then by definition, $\|x_n-x\|\to 0$ as $n\to\infty$. Using the inequality above with $y=x_n$, we have:

$$|\|x\|-\|x_n\||\leq\|x-x_n\|=\|x_n-x\|\to 0.$$

Therefore, $\|x_n\|\to\|x\|$ as $n\to\infty$.

Definition

The open ball of radius $r>0$ centered at $x_0\in X$ is

$$B(x_0,r)=\{x\in X:\|x-x_0\|<r\}.$$

The closed ball of radius $r>0$ centered at $x_0\in X$ is

$$\overline{B}(x_0,r)=\{x\in X:\|x-x_0\|\leq r\}.$$

Remark 1.2.1. *In normed spaces, closed balls are exactly the closures of open balls.*

Proof. Let $(X, \|\cdot\|)$ be a normed space, $x_0 \in X$, and $r > 0$.

1. First, note that

$$B(x_0, r) = \{x \in X : \|x - x_0\| < r\}, \qquad \overline{B}(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}.$$

2. We show that every point of $\overline{B}(x_0, r)$ belongs to the closure of $B(x_0, r)$.
   - If $\|x - x_0\| < r$, then $x \in B(x_0, r)$, hence trivially $x \in \overline{B(x_0, r)}$.
   - If $\|x - x_0\| = r$, then x belongs to the closure of the open ball $B(x_0, r)$.

That is, we need to find a sequence of points in $B(x_0, r)$ that converges to x, consider the sequence

$$x_n = x_0 + \left(1 - \tfrac{1}{n}\right)(x - x_0), \quad n \geq 1.$$

Then $x_n \to x$ as $n \to \infty$, and

$$\|x_n - x_0\| = \left(1 - \tfrac{1}{n}\right)\|x - x_0\| = \left(1 - \tfrac{1}{n}\right) r < r.$$

Hence $x_n \in B(x_0, r)$ for all n, and thus x belongs to the closure of $B(x_0, r)$.

3. Therefore,

$$\overline{B}(x_0, r) \subseteq \overline{B(x_0, r)}.$$

4. Conversely, since $B(x_0, r) \subseteq \overline{B}(x_0, r)$ and $\overline{B}(x_0, r)$ is closed, it follows that

$$\overline{B(x_0, r)} \subseteq \overline{B}(x_0, r).$$

Combining both inclusions, we obtain

$$\overline{B}(x_0, r) = \overline{B(x_0, r)}.$$

Thus, in normed spaces, closed balls are precisely the closures of open balls. □

Remark 1.2.2. *If the normed vector space $(X, \|\cdot\|)$ is finite-dimensional, then there exists a basis $\{x_1, \ldots, x_n\}$ and any $x \in X$ can be written uniquely as*

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

where the family $\{x_1, \dots, x_n\}$ *are linearly independent.*

This identifies X *with* $\mathbb{R}^n$*. The map* $T : X \longrightarrow \mathbb{R}^n$ *defined by*

$$T(\alpha_1 x_1 + \cdots + \alpha_n x_n) = (\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n),$$

$$\begin{aligned} T : X &\longrightarrow \mathbb{R}^n \\ \sum_{i=1}^n \alpha_i x_i &\longmapsto \sum_{i=1}^n \alpha_i e_i \end{aligned}$$

where $(e_i)_{1 \le i \le n}$ *is the canonical basis.*

The operator T *is linear and bijictive. Therefore every finite dimentional normed space* X *is isomorphic to* $\mathbb{R}^n$*. Thus, the study of finite-dimensional normed spaces reduces to the study of norms on* $\mathbb{R}^n$*.*

Example 1.2.1 (The ℓ^p-norms)**.** *For* $p \ge 1$*, define*

$$\|x\| = \|(x_i)_{1 \le i \le n}\| = \|(x_1, \dots, x_n)\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

Special cases:

- $p = 2$*: Euclidean norm,*
- $p = 1$*: Manhattan (or taxicab) norm,*
- $p = \infty$*:* $\|(x_i)\|_\infty = \max_i |x_i|$.

If $0 < p < 1$ and the dimension $n > 1$, the triangle inequality fails. If $p \ge 1$, the triangle inequality is known as *Minkowski's inequality*.

Theorem (Young's Inequality)

Let $x, y \ge 0$ and $1 < p < \infty$. Then

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Equality holds iff $x^p = y^q$.

Method 1: Using the exponential function. The exponential function $\exp(t)$ is convex. By Jensen's inequality, for any $u, v \in \mathbb{R}$ and $\lambda \in [0,1]$,

$$\exp\big(\lambda u + (1-\lambda)v\big) \ \leq \ \lambda \exp(u) + (1-\lambda)\exp(v).$$

Choose $\lambda = \frac{1}{p}$ and $1-\lambda = \frac{1}{q}$, which is possible since $\frac{1}{p} + \frac{1}{q} = 1$. Now let

$$u = \ln(x^p), \qquad v = \ln(y^q).$$

Applying Jensen's inequality gives

$$\exp\Big(\tfrac{1}{p}\ln(x^p) + \tfrac{1}{q}\ln(y^q)\Big) \ \leq \ \tfrac{1}{p}\exp(\ln(x^p)) + \tfrac{1}{q}\exp(\ln(y^q)).$$

Simplifying each side:
- Left-hand side:

$$\exp\Big(\tfrac{1}{p}\ln(x^p) + \tfrac{1}{q}\ln(y^q)\Big) = \exp(\ln x + \ln y) = xy.$$

- Right-hand side:

$$\tfrac{1}{p}\exp(\ln(x^p)) + \tfrac{1}{q}\exp(\ln(y^q)) = \tfrac{1}{p}x^p + \tfrac{1}{q}y^q.$$

Therefore,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Equality condition: Equality holds in the convexity argument when $u = v$, i.e., $\ln(x^p) = \ln(y^q)$, which gives $x^p = y^q$.

The cases where $x = 0$ or $y = 0$ are trivial, as both sides equal 0.

Method 2:[variation of a function]

□

Proposition 1.2.2 (different forms of Young's inequality (and its multi-variable extension))**.** *For* $x, y > 0$

$$xy \leq \inf_{\varepsilon>0}\left\{\frac{(\varepsilon x)^p}{p} + \frac{(y/\varepsilon)^q}{q}\right\}.$$

Let $p, q, r > 1$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

and let $x, y, z \geq 0$. *Then*

$$xyz = \inf_{\varepsilon>0,\ \delta>0}\left\{\frac{(\varepsilon x)^p}{p} + \frac{(\delta y)^q}{q} + \frac{\left(\frac{z}{\varepsilon\delta}\right)^r}{r}\right\}. \tag{1.1}$$

Proposition 1.2.3. *Let* $x, y \geq 0$ *and let* $p > 0$. *Define*

$$\Phi(\varepsilon) := x^p \varepsilon^{1-p} + y^p (1-\varepsilon)^{1-p}, \qquad 0 < \varepsilon < 1.$$

1. *If* $p > 1$, *then*
$$(x+y)^p = \inf_{0<\varepsilon<1} \Phi(\varepsilon),$$
and the infimum is attained at $\varepsilon = \dfrac{x}{x+y}$ *(when* $x + y > 0$*).*

2. *If* $0 < p < 1$, *then*
$$(x+y)^p = \sup_{0<\varepsilon<1} \Phi(\varepsilon),$$
and the supremum is attained at $\varepsilon = \dfrac{x}{x+y}$ *(when* $x + y > 0$*).*

Theorem (Hölder's Inequality)

Suppose $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $x = (x_i), y = (y_i) \in \mathbb{F}^n$, we have

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$

Equality holds iff the ratios $|x_i|^p / |y_i|^q$ are constant.

If $p = 2$, it called **Cauchy-Schwarz inequality**.

Proof. Let $x = (x_i)$ and $y = (y_i)$ be elements of $\mathbb{F}^n$, and let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If either $x = 0$ or $y = 0$, the inequality is trivial. So assume both are nonzero, and define

$$a_i = \frac{|x_i|}{\left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}}, \qquad b_i = \frac{|y_i|}{\left(\sum_{j=1}^{n} |y_j|^q \right)^{1/q}}.$$

Then, by construction,

$$\sum_{i=1}^{n} a_i^p = 1 \quad \text{and} \quad \sum_{i=1}^{n} b_i^q = 1.$$

We will use **Young's inequality**, which states that for all nonnegative real numbers u, v,

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q},$$

with equality if and only if $u^p = v^q$.

Applying this to each pair (a_i, b_i) gives

$$a_i b_i \leq \frac{a_i^p}{p} + \frac{b_i^q}{q}.$$

Summing over i yields

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{p} \sum_{i=1}^n a_i^p + \frac{1}{q} \sum_{i=1}^n b_i^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Returning to x_i and y_i, we have

$$\sum_{i=1}^n |x_i y_i| = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \left(\sum_{i=1}^n |y_i|^q \right)^{1/q} \sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \left(\sum_{i=1}^n |y_i|^q \right)^{1/q}.$$

Equality condition: Equality in Hölder's inequality occurs precisely when equality holds in Young's inequality for every i, i.e.

$$a_i^p = b_i^q \quad \text{for all } i.$$

Equivalently,

$$\frac{|x_i|^p}{|y_i|^q} = \frac{\sum_{j=1}^n |x_j|^p}{\sum_{j=1}^n |y_j|^q} = \text{constant}.$$

Hence, equality holds if and only if the ratios $\frac{|x_i|^p}{|y_i|^q}$ are constant.

This completes the proof. ☐

Theorem (Minkowski's Inequality)

For $p \geq 1$ and $x = (x_i), y = (y_i) \in \mathbb{F}^n$, we have

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p \right)^{1/p}.$$

Proof of Minkowski's inequality. We give the proof for $p > 1$. One has

$$\sum_{i=1}^{n} (|x_i| + |y_i|)^p = \sum_{i=1}^{n} |x_i| \left(|x_i| + |y_i|\right)^{p-1} + \sum_{i=1}^{n} |y_i| \left(|x_i| + |y_i|\right)^{p-1}.$$

$$\begin{aligned}\sum_{i=1}^{n}(|x_i| + |y_i|)^p \leq & \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n}(|x_i| + |y_i|)^{q(p-1)}\right)^{1/q} \\ & + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n}(|x_i| + |y_i|)^{q(p-1)}\right)^{1/q},\end{aligned}$$

$$\sum_{i=1}^{n}(|x_i|+|y_i|)^p \leq \left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}\right) \left(\sum_{i=1}^{n}(|x_i| + |y_i|)^p\right)^{1-\frac{1}{p}},$$

where $q = \dfrac{p}{p-1}$ is the conjugate exponent of p.

If

$$\sum_{i=1}^{n}(|x_i| + |y_i|)^p = 0,$$

then all x_i and y_i are zero, and the inequality is trivial. If

$$\sum_{i=1}^{n}(|x_i| + |y_i|)^p > 0,$$

after simplification we obtain

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

□

The Lebesgue spaces, or L^p-spaces

Let $X \subset \mathbb{R}^N$ be an open (or measurable) set. For $1 \leq p < \infty$ and a measurable function $f : X \to \mathbb{R}$ (or $\mathbb{C}$), we define the L^p**-norm** as:

$$\|f\|_p := \left(\int_X |f(x)|^p dx\right)^{1/p}$$

The **Lebesgue space** $L^p(X)$ is defined as:

$$L^p(X) := \{f : X \to \mathbb{R} \text{ (or } \mathbb{C}) \mid f \text{ is measurable and } \|f\|_p < \infty\}$$

If $p = \infty$, we define the **essential supremum**:

$$\|f\|_\infty := \inf\{M \geq 0 : |f(x)| \leq M \text{ for almost every } x \in X\}$$

The space $L^\infty(X)$ is:

$$L^\infty(X) := \{f : X \to \mathbb{R} \text{ (or } \mathbb{C}) \mid f \text{ is measurable and } \|f\|_\infty < \infty\}$$

For $1 \leq p \leq \infty$, $L^p(X)$ is a vector space over $\mathbb{R}$ (or $\mathbb{C}$).

Remark 1.2.3. *The spaces ℓ^p and L^p are normed spaces for $1 \leq p \leq \infty$.*

1.2.2 Completeness and Banach Spaces

Definition

Let X be a normed space.

1. The sequence (x_n) is **convergent to** x if $\|x_n - x\| \to 0$. (x is the limit of (x_n)).

2. The sequence (x_n) is **convergent absolutely to** x if $\|x_n\| \to \|x\|$.

3. The sequence (x_n) is a **Cauchy sequence** if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N(\epsilon)$.

We know that every convergent sequence in a normed space is a Cauchy sequence.

Indeed. Let (x_n) be a sequence in a normed space X such that $x_n \to x$ for some $x \in X$. We show that (x_n) is a Cauchy sequence.

By the definition of convergence, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \implies \|x_n - x\| < \frac{\varepsilon}{2}.$$

Then, for all $m, n > N$, we have by the triangle inequality

$$\|x_m - x_n\| \leq \|x_m - x\| + \|x_n - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (x_n) is Cauchy.

However, the converse is not always true: a Cauchy sequence does not necessarily converge in every normed space. A normed space in which every Cauchy sequence converges is called a *complete space*. In other words, a normed space X is said to be *complete* if every Cauchy sequence in X converges to a limit in X.

In the study of normed spaces, one of the most fundamental notions is that of *completeness*. In simple terms, completeness means that the space contains all its limit points with respect to the norm topology. This property guarantees that Cauchy sequences—sequences whose elements become arbitrarily close to each other—actually converge within the space. A complete normed space is called a *Banach space*, a concept introduced and developed by the Polish mathematician Stefan Banach, one of the founders of modern functional analysis.

Definition

A *Banach space* is a complete normed space.

Example 1.2.2. *The spaces ℓ^p, L^p for $1 \leq p \leq \infty$, and $C(K)$ for K compact are Banach spaces.*

Proof. The space $\ell^\infty(\mathbb{R})$ is a Banach space. Let $(u_p)_{p\geq1}$ be a Cauchy sequence in $\ell^\infty(\mathbb{R})$ with $u_p = (u_p^n)_{n\geq1}$. For $n \in \mathbb{N}$, we have

$$|u_p^n - u_q^n| \leq \sup_{n\geq1} |u_p^n - u_q^n| = \|u_p - u_q\|_\infty.$$

This implies that, $(u_p^n)_{n\geq1}$ is a Cauchy sequence. As $\mathbb{R}$ is complete, this sequence converges and we can write

$$u_n = \lim_{p\to\infty} u_p^n.$$

We will show that this sequence $(u_n)_{n\geq1}$ belongs to the space $l^\infty(\mathbb{R})$ because the sequence $(u_p)_{p\geq1}$ converges to $(u_n)_{n\geq1}$.

As $(u_p)_{p\geq1}$ is a Cauchy sequence, hence it is bounded and there exists $M \in \mathbb{R}^+$ such that

$$\forall p \in N, \qquad \|u_p\|_\infty \leq M.$$

Let n be in $\mathbb{N}$. One has

$$|u_n| = \lim_{p\longrightarrow\infty} |u_p^n|.$$

For each fixed n and p, we have

$$|u_n^p| \leq \|u^p\|_\infty \leq M$$

Taking the limit as $p \to \infty$, and using the continuity of the absolute value function,

$$|u_n| = \lim_{p\to\infty} |u_p^n| \leq M.$$

Since this holds for all n,

$$\sup_{n\geq 1} |u_n| \leq M,$$

therefore, $(u_n)_{n\geq 1}$ is an element of the space $l^\infty(\mathbb{R})$.
Consider $\epsilon > 0$. Since (u_p) is Cauchy, there is $N \in \mathbb{N}$ such that

$$\forall p, q > N, \quad \|u_p - u_q\|_\infty \leq \epsilon.$$

This implies for all $n \geq 1$

$$|u_p^n - u_q^n| \leq \epsilon.$$

Now, fix $p > N$ and $n \in \mathbb{N}$. Taking the limit as $q \to +\infty$, we get

$$|u_n - u_p^n| \leq \epsilon$$

and this holds for all $n \in \mathbb{N}$. Thus

$$\|u - u_p\|_\infty = \sup_{n\geq 1} |u_p^n - u_n| \leq \epsilon.$$

Hence, $u_p \to u$ in $\ell^\infty(\mathbb{R})$. Therefore, $\ell^\infty(\mathbb{R})$ is complete and thus a Banach space. □

Let $K \subset \mathbb{R}$ be a compact set. The space $C(K)$ consists of the continuous functions $f : K \longrightarrow \mathbb{R}$. Addition and scalar multiplication of functions is defined pointwise in the usual way: if $f, g \in C(K)$ and $\alpha \in \mathbb{R}$, then

$$(f + g)(x) = f(x) + g(x), \qquad (\alpha f)(x) = \alpha f(x).$$

The sup-norm or ∞-norm of a function $f \in C(K)$ is defined by

$$\|f\|_\infty = \sup_{x \in K} |f(x)|.$$

Since a continuous function on a compact set attains its maximum and minimum value, for $f \in C(K)$ we can also write

$$\|f\|_\infty = \max_{x \in K} |f(x)|.$$

Example 1.2.3. *The space* $(\mathcal{C}([a,b]), \|\cdot\|_\infty)$ *is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}([a,b],\mathbb{R})$. This implies that

$$\forall \epsilon > 0 \ \exists n_0 > 0 : \forall n \geq n_0, \forall p \geq 0 \quad \|f_n - f_{n+p}\|_\infty < \epsilon.$$

Hence

$$\forall x \in [a,b], \forall n \geq n_0, \forall p \geq 0 \quad |f_n(x) - f_{n+p}(x)| < \|f_n - f_{n+p}\|_\infty < \epsilon.$$

For all $x \in [a,b]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. So there exists a limit denoted $f(x)$ of the sequence $\{f_n(x)\}$. We must show that f is continuous and $\|f(x) - f_n(x)\| \longrightarrow 0$ when $n \longrightarrow \infty$.

Since n_0 does not depend on x, so we can take the limit when $p \longrightarrow \infty$. We have

$$\forall x \in [a,b], \forall n \geq n_0, \quad |f(x) - f_n(x)| < \epsilon.$$

This implies that $f_n \xrightarrow{\text{UC}} f$.

Thus f is continuous.

Moreover, from the inequality $|f_n(x) - f(x)| \leq \epsilon$ for all x and $n \geq N$, we have

$$\|f_n - f\|_\infty < \epsilon.$$

Hence $f_n \to f$ in $C([a,b])$. Since every Cauchy sequence converges, $C([a,b])$ is complete and therefore a Banach space. $\square$

Remark 1.2.4. *The interval* $[a,b]$ *can be replaced by any compact subset* K *of any topological space. And* $\mathbb{R}$ *can be replaced by any complete metric space.*

Example 1.2.4. *Consider* $K = [a,b]$ *and define another norm on* $\mathcal{C}([a,b])$ *by*

$$\|f\|_1 = \int_a^b |f(x)|\,dx.$$

The space $C([a,b])$ *with the* 1*-norm* $\|\cdot\|_1$ *is a normed space but not a Banach space.*

Define the continuous functions $f_n : [0,1] \longrightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ n\left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} < x < \frac{1}{2} + \frac{1}{n}, \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} \le x \le 1. \end{cases}$$

If $n > m$, we have

$$\begin{aligned} \|f_n - f_m\|_1 &= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |f_n(x) - f_m(x)|\,dx \\ &= \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right) \quad \text{(area of a triangle)}. \end{aligned}$$

Hence (f_n) is a Cauchy sequence with respect to the 1-norm. We claim that if $\|f_n - f\|_1 \longrightarrow 0$ where $f \in C([0,1])$, then f would have to be

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

1.2.3 Riesz's characterization of finite-dimensional normed spaces

We study the properties of finite-dimensional spaces and subspaces. For any Banach or normed space X, we denote by B_X the closed unit ball in X. The following theorem, due to F. Riesz characterizes finite-dimensional spaces among normed linear spaces.

Theorem (F. Riesz's Theorem)

Let X be a normed linear space. The following statements are equivalent:

(a) The closed unit ball

$$B_X = \{x \in X : \|x\| \leq 1\}$$

is compact.

(b) Every closed and bounded subset of X is compact.

(c) The space X is finite-dimensional.

To prove this theorem, we will use a simple but important lemma, also due to F. Riesz, another leading contributor to functional analysis.

Riesz's lemma says that in any normed space, if you take a proper closed subspace M, you can find a unit vector that is almost at distance 1 from every point of M. This result highlights that closed subspaces of infinite-dimensional spaces are "far apart" in a geometric sense.

Lemma 1.2.1 (Riesz's Lemma)**.** *Let X be a normed linear space and let M be a proper closed linear subspace of X (i.e., $M \subsetneq X$) . Then for each $\epsilon > 0$ there exists a point $x \in X$ such that $\|x\| = 1$ and $d(x, M) \geq 1 - \epsilon$.*

Proof. Since M is a proper subspace, there exists $x_0 \in X \setminus M$. Let $d = d(x_0, M) = \inf_{y \in M} \|x_0 - y\| > 0$. Choose $y_0 \in M$ such that $\|x_0 - y_0\| < d + \varepsilon d$, and define

$$x = \frac{x_0 - y_0}{\|x_0 - y_0\|}.$$

Then $\|x\| = 1$. For any $y \in M$, we have

$$\|x - y\| = \frac{\|x_0 - (y_0 + \|x_0 - y_0\| y)\|}{\|x_0 - y_0\|} \geq \frac{d}{\|x_0 - y_0\|} > \frac{1}{1 + \varepsilon}.$$

Hence $d(x, M) \geq 1 - \varepsilon$. □

Proof of F. Riesz's Theorem

Proof of F. Riesz's Theorem. It is easy to see that conditions (a) and (b) are equivalent.

If X is finite-dimensional, then it is topologically isomorphic to $(\mathbb{R}^n, \|\cdot\|_\infty)$, where it is known that every closed and bounded set is compact; therefore, the same property holds for X.

Conversely, suppose that the closed unit ball

$$B(0,1) = \{x \in X : \|x\| \leq 1\}$$

is compact. For $\varepsilon > 0$, let $a_1, \dots, a_{n_\varepsilon} \in B(0,1)$ be such that

$$B(0,1) \subset \bigcup_{i=1}^{n_\varepsilon} B(a_i, \varepsilon),$$

that is, the family of balls $B(a_i, \varepsilon)$ covers $B(0,1)$.

Let M be the finite-dimensional subspace of X generated by $\{a_1, \dots, a_{n_\varepsilon}\}$.

Suppose that M is a proper subspace of X (i.e., $M \neq X$). Then, by F. Riesz's lemma, there exists $x \in X$ such that $\|x\| = 1$ and

$$d(x, M) \geq 1 - \varepsilon.$$

However, since $x \in B(0,1)$, there exists some a_i such that $\|x - a_i\| < \varepsilon$, which implies

$$d(x, M) \leq \varepsilon.$$

This is a contradiction because $1 - \varepsilon \leq d(x, M) \leq \varepsilon$.

Hence, our assumption that $M \neq X$ is false. Therefore, $X = M$, and thus X is finite-dimensional. □

Proposition 1.2.4. *A subspace E of a Banach space X is complete iff E is closed in X.*

Proof. ($\Rightarrow$) If E is a closed subspace of a complete space X and (x_n) is a Cauchy sequence in E, then (x_n) is also a Cauchy sequence in X. Since X is complete, (x_n) converges to some $x \in X$. Because E is closed, we have $x \in E$, which shows that E is complete.

($\Leftarrow$) Conversely, if E is not closed, then there exists a convergent sequence (x_n) in E whose limit x does not belong to E. Since the sequence converges, it is Cauchy, but it has no limit in E; therefore, E is not complete. □

1.2.4 Equivalent Norms

Two equivalent norms on the same vector space X are two norms for which the topologies induced on X are identical. This equivalence of norms on X translates into the equivalence of the associated distances.

Definition (Equivalent Norms)

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are said to be **equivalent** if there exist positive real numbers $a, b > 0$ such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1, \quad \text{for all } x \in X.$$

Proposition 1.2.5. *Let X be a vector space. Then, the following properties are equivalent:*

(i) The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on X;

(ii) The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ generate the same topology on X;

(iii) For every sequence (x_n) in X, one has

$$x_n \xrightarrow[\|\cdot\|_1]{} x \quad \textit{if and only if} \quad x_n \xrightarrow[\|\cdot\|_2]{} x.$$

Theorem

On finite-dimensional spaces, all norms are equivalent.

Proof. It suffices to consider $X = \mathbb{R}^n$ and show that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to the norm

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|, \qquad x = (x_1, \ldots, x_n).$$

Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$, where $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$. For any $x \in \mathbb{R}^n$ we have the decomposition

$$x = \sum_{i=1}^{n} x_i e_i,$$

hence by the triangle inequality and homogeneity of the norm,

$$\|x\| = \|\sum_{i=1}^{n} x_i e_i\| \leq \sum_{i=1}^{n} \|x_i e_i\| \leq \sum_{i=1}^{n} |x_i|\, \|e_i\| \leq \Big(\max_{1\leq i\leq n} |x_i|\Big) \sum_{i=1}^{n} \|e_i\|.$$

Put

$$C_1 := \sum_{i=1}^{n} \|e_i\| > 0.$$

Then

$$\|x\| \leq C_1 \|x\|_\infty \qquad \text{for all } x \in \mathbb{R}^n.$$

This gives the upper bound.

Define the function $f : \mathbb{R}^n \to \mathbb{R}_+$ by $f(x) = \|x\|$. We claim f is continuous with respect to the $\|\cdot\|_1$-norm. Indeed, for all $x, y \in \mathbb{R}^n$,

$$\big|\|x\| - \|y\|\big| \leq \|x - y\|$$

(by the reverse triangle inequality), and by the already proved upper bound applied to $x - y$,

$$\|x - y\| \leq C_1 \|x - y\|_\infty,$$

so

$$\big|\|x\| - \|y\|\big| \leq C_1 \|x - y\|_\infty.$$

Hence f is continuous on $(\mathbb{R}^n, \|\cdot\|_\infty)$.

Consider the unit sphere for $\|\cdot\|_\infty$,

$$S_n := \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}.$$

The set S is compact in $\mathbb{R}^n$ (Heine–Borel). Since f is continuous, f attains a minimum on S; let

$$C_2 := \min_{x\in S} f(x) = \min_{\|x\|_1 = 1} \|x\|.$$

Note that $C_2 > 0$ because $\|x\| = 0$ iff $x = 0$, and no point of S is the zero vector.

For arbitrary $x \in \mathbb{R}^n$, if $x = 0$ the desired inequality is trivial. If $x \neq 0$, then

$$\frac{x}{\|x\|_\infty} \in S,$$

so by definition of C_2,

$$\left\| \frac{x}{\|x\|_\infty} \right\| \geq C_2.$$

Multiplying both sides by $\|x\|_\infty$ gives

$$\|x\| \geq C_2 \|x\|_\infty.$$

Combining the two bounds we obtain positive constants C_2, C_1 such that, for all $x \in \mathbb{R}^n$,

$$C_2 \|x\|_\infty \leq \|x\| \leq C_1 \|x\|_\infty,$$

which means $\| \cdot \|$ and $\| \cdot \|_\infty$ are equivalent. Since every finite dimensional vector space is isomorphic to $\mathbb{R}^n$, the result follows. $\square$

Remark 1.2.5. *The norms $\| \cdot \|_p$ are equivalent for every n and for all $1 \leq p \leq \infty$. However, if n is infinite, the norms $\| \cdot \|_p$ are not equivalent.*

Indeed, consider the sequence

$$x_n = \left(\frac{1}{n} \right)$$

in ℓ^1 and in ℓ^2.

We have

$$\|x_n\|_1 = \sum_{k=1}^{n} \frac{1}{k} \quad \textit{and} \quad \|x_n\|_2 = \left(\sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2}.$$

Then $\|x_n\|_1 \to \infty$ while $\|x_n\|_2 \to \dfrac{\pi}{\sqrt{6}}$. Hence, the two norms are not equivalent.

If X is not finite-dimensional, the preceding theorem is false, as shown by this example.

Example 1.2.5. *Let $C([0,1])$ denote the set of continuous real-valued functions $f : [0,1] \to \mathbb{R}$. Then the norms*

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \quad \textit{(under which } C([0,1]) \textit{ is a Banach space)}$$

and

$$\|f\|_1 = \int_0^1 |f(x)|\, dx \quad \textit{(under which } C([0,1]) \textit{ is not a Banach space)}$$

are not equivalent on $C([0,1])$.

Corollary 1.2.1. *Every finite dimensional normed vector space is a Banach space.*

1.3 Series in Normed Spaces

Definition

Let $(X, \|\cdot\|)$ be a normed vector space. If $(x_n)_{n\geq 1}$ is a sequence in X, we say that the formal series

$$\sum_{n=1}^{\infty} x_n$$

is *convergent* if the sequence of partial sums

$$s_N = \sum_{n=1}^{N} x_n$$

converges to some point $x \in X$, that is,

$$\left\| x - \sum_{n=1}^{N} x_n \right\| \longrightarrow 0 \quad \text{as } N \to \infty.$$

In this case, we write

$$x = \sum_{n=1}^{\infty} x_n.$$

If the series

$$\sum_{n=1}^{\infty} \|x_n\|$$

converges in $\mathbb{R}$, we say that the series $\sum_{n=1}^{\infty} x_n$ is *absolutely convergent.*

Theorem

Let $(X, \|\cdot\|)$ be a Banach space. Then every absolutely convergent series in X is convergent in X. Moreover,

$$\left\|\sum_{n=1}^{\infty} x_n\right\| \leq \sum_{n=1}^{\infty} \|x_n\|.$$

Proof. Let (x_n) be a sequence in X such that the series

$$\sum_{n=1}^{\infty} \|x_n\|$$

is convergent. Then, by definition of convergence in $\mathbb{R}$, the sequence of partial sums

$$s_n = \sum_{k=1}^{n} \|x_k\|$$

is Cauchy.

We now show that the sequence of partial sums

$$S_n = \sum_{k=1}^{n} x_k$$

is Cauchy in X. For $p > q$, we have

$$\|S_p - S_q\| = \left\|\sum_{k=q+1}^{p} x_k\right\| \leq \sum_{k=q+1}^{p} \|x_k\|.$$

Since (s_n) is Cauchy in $\mathbb{R}$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|s_p - s_q| = \sum_{k=q+1}^{p} \|x_k\| < \varepsilon \quad \text{whenever } p, q \geq N.$$

Hence,

$$\|S_p - S_q\| < \varepsilon \quad \text{for all } p, q \geq N.$$

Therefore, (S_n) is a Cauchy sequence in X. Since X is complete (i.e., a Banach space), there exists $S \in X$ such that

$$S_n \to S.$$

Thus, the series $\sum_{n=1}^{\infty} x_n$ converges in X.

Finally, the inequality

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \|x_n\|$$

follows from the triangle inequality applied to the partial sums:

$$\left\| \sum_{k=1}^{n} x_k \right\| \leq \sum_{k=1}^{n} \|x_k\|,$$

and then passing to the limit as $n \to \infty$.

□

Recall that if a series

$$\sum_{n=1}^{\infty} a_n$$

of real (or complex) numbers converges absolutely, that is, if

$$\sum_{n=1}^{\infty} |a_n| \text{ converges,}$$

then the original series also converges.

Viewing $\mathbb{R}$ (or $\mathbb{C}$) as a normed space, this result is a special case of the following useful criterion for a normed space to be a Banach space.

Theorem

A normed space X is a Banach space if and only if the absolute convergence of a series in X implies its convergence.

1.4 Continuous Linear Operators on Normed Spaces

1.4.1 Definitions and Properties

We now turn our attention to a new class of normed linear spaces, namely, spaces of linear mappings.

Definition (Linear Operator)

Let X and Y be real (or complex) linear spaces. A *linear map* or *linear operator* T from X to Y is a function

$$T : X \to Y$$

such that, for all scalars $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$) and all $x, y \in X$, we have

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

We denote by $L(X, Y)$ the set of all linear maps $T : X \to Y$.

If X and Y are normed spaces, we can define the notion of a *bounded* or *continuous* linear operator. As we shall see, the boundedness of a linear operator is equivalent to its continuity.

Definition (Bounded Linear Operator)

Let X and Y be two normed linear spaces. We denote both ($X$and Y) norms by $\|\cdot\|$. A linear map $T : X \to Y$ is said to be *bounded* if there exists a constant $C > 0$ such that

$$\|T(x)\| \le C\|x\|, \quad \forall x \in X.$$

If $T : X \to Y$ is a bounded linear map, we define the *operator norm* of T by

$$\|T\| = \inf\big\{C > 0 : \|T(x)\| \le C\|x\| \text{ for all } x \in X\big\} = \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|T(x)\|}{\|x\|}.$$

We donete by $\mathcal{B}(X, Y)$ (or simply $\mathcal{L}(X, Y)$) the set of all bounded linear maps from X to Y.

If $Y = \mathbb{R}$ (or $\mathbb{C}$), then $\mathcal{B}(X, \mathbb{R}) = X^*$, which is called the *dual space* of X.

Remark 1.4.1. *(Equivalent expressions for the operator norm.) For a bounded linear operator $T : X \to Y$, we have the following equivalent formulas for its norm:*

$$\begin{aligned}\|T\| &= \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in X,\, x \neq 0\right\}\\ &= \sup\left\{\frac{\|T(x)\|}{\|x\|} : \|x\| \leq 1\right\}\\ &= \sup\left\{\|T(x)\| : \|x\| = 1\right\}.\end{aligned}$$

Proof. Indeed,

$$\|T\| = \sup_{x\in X,\, x\neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{x\in X,\, x\neq 0} \left\|T\left(\frac{x}{\|x\|}\right)\right\|.$$

Setting $x_0 = \dfrac{x}{\|x\|}$, we see that $x_0 \in S_X$,

where $S_X = \{x \in X : \|x\| = 1\}$ denotes the unit sphere of X. Hence,

$$\|T\| = \sup_{x_0 \in S_X} \|T(x_0)\|.$$

as $S_X \subset B_X = \{x \in X : \|x\| \leq 1\} \subset X$, the same supremum is obtained over B_X, so

$$\|T\| = \sup_{\|x\|\leq 1} \|T(x)\|.$$

□

Remark 1.4.2. *We can also write*

$$\|T\| = \sup\{\, \|T(x)\| : \|x\| < 1\,\},$$

since the supremum of a continuous function over the closed unit ball is the same as over the open unit ball; that is, $\overline{B_X} = B_X$.

Example 1.4.1 (Norm attained). *Let* $X = \ell^1$ *and define* $T : X \to \mathbb{C}$ *by*

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}, \quad \textit{for } x = (x_n) \in \ell^1.$$

Then T *is a linear operator and*

$$\|T\| = 1.$$

Indeed, for any $x \in \ell^1$,

$$|T(x)| = \left| \sum_{n=1}^{\infty} \frac{x_n}{n} \right| \leq \sum_{n=1}^{\infty} \frac{|x_n|}{n} \leq \|x\|_{\ell^1}.$$

Hence $\|T\| \leq 1$. *But for* $x = e_1 = (1, 0, 0, \ldots)$, *we have*

$$T(e_1) = 1,$$

so $\|T\| = 1$ *and the norm is attained at* e_1.

Example 1.4.2 (Norm not attained). *Let* $X = c_0$ *equipped with the norm* $\|\cdot\|_\infty$. *Define* $T : X \to \mathbb{C}$ *by*

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Then T *is linear, and since*

$$|T(x)| \leq \sum_{n=1}^{\infty} \frac{|x_n|}{2^n} \leq \|x\|_\infty,$$

we have $T \in \mathcal{B}(X, \mathbb{C})$ *and* $\|T\| \leq 1$.

On the other hand, for each $n \geq 1$, *note that*

$$\|e_1 + \cdots + e_n\|_\infty = 1 \quad \textit{and} \quad T(e_1 + \cdots + e_n) = \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}. \textit{ geomertic sequence}$$

Hence,

$$\|T\| \geq 1 - \frac{1}{2^n}.$$

Since n is arbitrary, it follows that $\|T\| \geq 1$, and therefore $\|T\| = 1$.

Now let $x \in c_0$ with $\|x\|_\infty \leq 1$. Then there exists $N \geq 1$ such that $|x_n| < \frac{1}{2}$ for all $n \geq N$. We have

$$|T(x)| \leq \sum_{i=1}^{N} \frac{|x_i|}{2^i} + \frac{1}{2} \sum_{i=N+1}^{\infty} \frac{1}{2^i} < 1 = \|T\|.$$

Split the series at N to get the estimate

$$|T(x)| \leq \sum_{i=1}^{N-1} \frac{|x_i|}{2^i} + \sum_{i=N}^{\infty} \frac{|x_i|}{2^i} \leq \sum_{i=1}^{N-1} \frac{|x_i|}{2^i} + \frac{1}{2} \sum_{i=N}^{\infty} \frac{1}{2^i}.$$

Using the trivial bound $|x_i| \leq 1$ for the first finite sum yields

$$|T(x)| \leq \sum_{i=1}^{N-1} \frac{1}{2^i} + \frac{1}{2} \sum_{i=N}^{\infty} \frac{1}{2^i} = \left(1 - \frac{1}{2^{N-1}}\right) + \frac{1}{2} \cdot \frac{1}{2^{N-1}} = 1 - \frac{1}{2^N} < 1.$$

Consequently, $|T(x)| < \|T\|$ for all x in the unit ball B_{c_0}, and hence the norm of T is not attained.

Remark 1.4.3. *In finite-dimensional normed spaces, every continuous linear functional T attains its norm, that is, there exists x_0 with $\|x_0\| = 1$ such that $|T(x_0)| = \|T\|$. However, in infinite-dimensional spaces like c_0, this property may fail. Intuitively, the sequence $\{e_1 + \cdots + e_n\}$ approaches a point where the supremum of $|T(x)|$ would be reached, but this limit vector does not belong to c_0 (it would be $(1, 1, 1, \dots)$, which is not in c_0). Thus, the functional T attains its supremum only "at infinity," not at any actual element of X.*

Proposition 1.4.1. *Let $T : X \to Y$ be a linear operator between the normed vector spaces X and Y. Then, the following assertions are equivalent:*

1. *T is continuous;*
2. *T is continuous at 0;*
3. *T is bounded (one may take $x_0 = \dfrac{\delta x}{2\|x\|}$).*

Theorem

Let X and Y be normed vector spaces. If Y is a Banach space, then $\mathcal{L}(X,Y)$ equipped with the operator norm

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|$$

is a Banach space.

Proof. Let (T_n) be an arbitrary Cauchy sequence in $\mathcal{L}(X,Y)$. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|T_n - T_m\| \leq \varepsilon, \quad \text{for all } m, n \geq N.$$

Therefore, for all $x \in X$ and all $m, n \geq N$, we have

$$\|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \, \|x\| \leq \varepsilon \|x\|. \tag{1.2}$$

Thus, for each fixed $x \in X$, the sequence $(T_n(x))$ in Y is Cauchy. Since Y is complete, there exists $y \in Y$ such that $T_n(x) \to y$.

We need to construct a limit of $(T_n(x))$ define $T : X \to Y$ by

$$T(x) = y = \lim_{n\to\infty} T_n(x).$$

By construction, T is linear:

$$\begin{aligned} T(\alpha x_1 + \beta x_2) &= \lim_{n\to\infty} T_n(\alpha x_1 + \beta x_2) \\ &= \lim_{n\to\infty} (\alpha T_n(x_1) + \beta T_n(x_2)) \\ &= \alpha \lim_{n\to\infty} T_n(x_1) + \beta \lim_{n\to\infty} T_n(x_2) \\ &= \alpha T(x_1) + \beta T(x_2). \end{aligned}$$

Next, for every $x \in X$, we have

$$\|T_n(x) - T(x)\| = \left\| T_n(x) - \lim_{m\to\infty} T_m(x) \right\| = \lim_{m\to\infty} \|T_n(x) - T_m(x)\| \overset{(1.2)}{\leq} \varepsilon \|x\|.$$

This implies that $\|T_n - T\| \leq \varepsilon$ (is bounded) .Since $T_n \in \mathcal{L}(X,Y) \Rightarrow T = T_n - (T_n - T)$,therefore $T$is bounded, hence (T_n) converges to T in the operator norm.

Therefore, $\mathcal{L}(X,Y)$ (or $\mathcal{B}(X,Y)$) is complete, i.e., a Banach space. $\square$

Remark 1.4.4. *The converse is true. Indeed, suppose that* $\mathcal{L}(X, Y)$ *is complete, and let* (y_n) *be a Cauchy sequence in* Y.

Select a vector $x_0 \in X$ *with* $\|x_0\| = 1$*; then there exists* $x^* \in X^*$ *such that* $\langle x_0, x^* \rangle = 1$.

Define, for each n*, the operator* $T_n \in \mathcal{L}(X, Y)$ *by*

$$T_n(x_0) = \langle x_{{}_0}, x^* \rangle \, y_n, \qquad x \in X.$$

We then have

$$\|T_n - T_m\| \leq \|x^*\| \, \|y_n - y_m\|.$$

Consequently, (T_n) *is a Cauchy sequence in* $\mathcal{L}(X, Y)$ *and therefore has a limit* $T \in \mathcal{L}(X, Y)$.

Finally,

$$\|y_n - T(x_0)\| = \|T_n(x_0) - T(x_0)\| \longrightarrow 0 \quad \text{as } n \to \infty,$$

whence $\lim\limits_{n \to \infty} y_n = T(x_0) \in Y$.

1.4.2 Normed Dual Space

Definition (Normed Dual Space)

Let X be a normed space. A *bounded linear functional* on X is a linear map

$$f : X \to \mathbb{K} \quad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})$$

such that there exists a constant $C > 0$ with

$$|f(x)| \le C\|x\| \quad \text{for all } x \in X.$$

The smallest such constant is called the *norm* of f and is given by

$$\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\|\le 1} |f(x)| = \sup_{\|x\|<1} |f(x)|.$$

The set of all bounded linear functionals on X is denoted by X^* and is called the *normed dual space* (or *topological dual*) of X. Equipped with the norm $\|\cdot\|$, the space X^* is always a Banach space.
For each $x^* \in X^*$ and $x \in X$, the value of x^* at x is written as

$$x^*(x) = \langle x^*, x\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the *duality pairing* between X^* and X.

Definition

Let $T \in \mathcal{B}(X, Y)$.

(1) We say that T is

- (a) an *isometry* if $\|T(x)\| = \|x\|$ for all $x \in X$;
- (b) an *isometric isomorphism* if T is a linear isometry which is also a surjection;
- (c) an *isomorphic isomorphism* if both T and T^{-1} are bounded.

(2) We say that X is *embeddable in* Y if it is isomorphic to a subspace of Y.

Theorem (Finite dimension)

Let X, Y be normed spaces and suppose that $\dim X = n < \infty$. Then every linear operator $T : X \to Y$ is bounded.

Proof. Let $(e_1, \ldots, e_n)$ be a basis for X, and let $x \in X$ be arbitrary. We can write

$$x = \sum_{i=1}^{n} \beta_i e_i.$$

Then

$$\|Tx\| = \left\| \sum_{i=1}^{n} \beta_i T(e_i) \right\| \le \sum_{i=1}^{n} |\beta_i| \, \|T(e_i)\| \le C \sum_{i=1}^{n} |\beta_i|,$$

where $C = \max\{\|T(e_i)\| : i = 1, \ldots, n\}$.

Since all norms are equivalent in finite-dimensional spaces, there exists a constant $\alpha > 0$ such that

$$\sum_{i=1}^{n} |\beta_i| \le \alpha \|x\|.$$

It follows that

$$\|Tx\| \le C\alpha \|x\|.$$

Hence T is bounded. □

1.4.3 Adjoint operator

The word adjoint has a number of related meanings. In linear analysis, it refers to the conjugate transpose and is most commonly denoted T^*.

Definition

Let X, Y be two Banach spaces and T be a bounded operator from X into Y. The adjoint (or dual or transpose) of T, noted T^*, is the bounded operator from Y^* to X^* such that

$$T^*(y^*)(x) = \langle T^*(y^*), x \rangle = \langle y^*, T(x) \rangle.$$

Theorem 1.4.1. *Let X, Y be two Banach spaces. The map from $\mathcal{B}(X, Y)$ into $\mathcal{B}(Y^*, X^*)$ which associates to T its adjoint T^* is an isometry (i.e., $\|T\| = \|T^*\|$ for all $T \in \mathcal{B}(X, Y)$).*

Proof. It is clear that the application T^* is linear. We have

$$\begin{aligned}\|T\| &= \sup_{\|x\|=1} \|T(x)\| \\ &= \sup_{\|x\|=1} \sup_{\|y^*\|=1} |\langle T(x), y^* \rangle| \\ &= \sup_{\|x\|=1} \sup_{\|y^*\|=1} |\langle x, T^*(y^*) \rangle| \\ &= \sup_{\|y^*\|=1} \sup_{\|x\|=1} |\langle x, T^*(y^*) \rangle| \\ &= \sup_{\|y^*\|=1} \|T^*(y^*)\| = \|T^*\|.\end{aligned}$$

This ends the proof. □

Proposition 1.4.2. *Let X, Y, and Z be three Banach spaces, and let $T, S \in \mathcal{B}(X, Y)$ and $R \in \mathcal{B}(Z, X)$. Then:*

(a) $(T + S)^* = T^* + S^*$ *and* $(T \circ R)^* = R^* \circ T^*$.

(b) If $T : X \to Y$ is a topological isomorphism, then $T^* : Y^* \to X^*$ is also a topological isomorphism, and in fact

$$(T^*)^{-1} = (T^{-1})^*.$$

(c) If $T : X \to Y$ is an isometric isomorphism, then $T^* : Y^* \to X^*$ is also an isometric isomorphism.

Quotient Space

This construction allows us to obtain new spaces from given ones. The *quotient topology* consists of all sets whose preimage under the canonical projection is open. The canonical projection maps each element to its equivalence class.

Proposition 1.4.3. *Let X be a Banach space and let M be a closed linear subspace of X. Then M is a Banach space.*

Proof. This is an immediate consequence of the elementary fact that a closed subspace of a complete metric space is complete. □

We also have the following easy proposition.

Proposition 1.4.4. *Let X be a normed linear space and let M be a linear subspace of X. Then M, equipped with the restriction of the norm of X, is also a normed linear space.*

Definition (Quotient Space)

Let X be a vector space and let M be a linear subspace of X. The *quotient space* X/M is defined as the linear space of cosets

$$x + M = \{\, x + m : m \in M \,\},$$

with the algebraic operations

$$(x + M) + (y + M) = (x + y) + M, \quad \text{for all } x, y \in X,$$

and

$$\lambda(x + M) = (\lambda x) + M, \quad \text{for all } x \in X,\ \lambda \in \mathbb{R} \text{ (or } \mathbb{C}).$$

The coset $0+M$ serves as the zero vector of X/M and is simply denoted by 0.
It is straightforward to verify that all the axioms of a vector space are satisfied.
Now, suppose that X is a normed linear space. We equip the quotient space X/M with the *quotient norm* defined by

$$\|x + M\|_{X/M} = d(0, x + M) = \inf_{y \in x+M} \|y\| = \inf_{m \in M} \|x + m\|.$$

We call $\|\cdot\|_{X/M}$ the *quotient norm* on the space X/M that is induced by the norm $\|\cdot\|$ on X. We now verify this property.

Proposition 1.4.5. *TD The space*

$$\big(X/M,\ \|\cdot\|_{X/M}\big)$$

is a normed space if and only if M is a closed linear subspace of X.

Proof. It is straightforward to verify the properties of homogeneity and the triangle inequality.

(1) For any $\lambda \in \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $x \in X$, we have

$$\|\lambda x\|_{X/M} = \inf_{m\in M} \|\lambda x+m\| = \inf_{m\in M} \|\lambda(x+\tfrac{1}{\lambda}m)\| = |\lambda| \inf_{m\in M} \|x+m\| = |\lambda| \, \|x\|_{X/M}.$$

(2) For all $x, y \in X$ and for any $m_1, m_2 \in M$, we have

$$\|(x+y)+(m_1+m_2)\| \leq \|x+m_1\| + \|y+m_2\|.$$

Taking the infimum over all $m_1, m_2 \in M$, we obtain

$$\|x+y\|_{X/M} \leq \|x\|_{X/M} + \|y\|_{X/M}.$$

(3) We now show that $\|x\|_{X/M} = 0$ if and only if $x \in M$.

($\Rightarrow$) Assume M is closed. Then the set $x+M$ is also closed, and

$$\|x\|_{X/M} = 0 \iff d(0, x+M) = 0 \iff 0 \in x+M \iff x \in M.$$

($\Leftarrow$) Conversely, suppose that M is not closed. Then there exists some $x \in \overline{M} \setminus M$. For such x, we have

$$\|x\|_{X/M} = \inf_{m\in M} \|x-m\| = 0 \quad \text{but} \quad x \notin M,$$

which means that $\|\cdot\|_{X/M}$ is not a true norm (it is only a seminorm).

Hence, the quotient space $\big(X/M, \|\cdot\|_{X/M}\big)$ is a normed space if and only if M is closed. ☐

Theorem

Let X be a Banach space and let M be a closed linear subspace of X. Then the quotient space X/M is a Banach space.

Proof. Let $\{x_n + M\}$ be a Cauchy sequence in X/M. Then for each positive integer k, there exists an integer n_k such that

$$\|(x_m+M)-(x_n+M)\|_{X/M} < \frac{1}{2^k} \quad \text{for all } m, n \geq n_k.$$

We may also assume that $n_1 < n_2 < n_3 < \cdots$.

Consider the subsequence $\{x_{n_k} + M\}$. By construction,

$$\|(x_{n_k} + M) - (x_{n_{k+1}} + M)\|_{X/M} < \frac{1}{2^k}.$$

By the definition of the quotient norm, for each k we can choose $y_k \in x_{n_k} + M$ such that

$$\|y_k - y_{k+1}\| < \frac{1}{2^k}.$$

It follows that $\{y_k\}$ is a Cauchy sequence in X. Since X is Banach, there exists $y \in X$ such that $y_k \to y$ in X.

Now, for each k, we have

$$\|(x_{n_k}+M)-(y+M)\|_{X/M} = \|(y_k+M)-(y+M)\|_{X/M} \le \|y_k-y\| \to 0 \quad \text{as } k \to \infty.$$

Hence, the subsequence $\{x_{n_k} + M\}$ converges to $y + M$ in X/M.

Since every Cauchy sequence in a metric space that has a convergent subsequence with the same limit is itself convergent, the whole sequence $\{x_n + M\}$ converges to $y + M$ in X/M.

Therefore, X/M is complete, and thus a Banach space. □

Theorem

The quotient map

$$q : X \longrightarrow X/M, \qquad x \longmapsto q(x) = x + M,$$

is a bounded linear operator with $\|q\| \le 1$. Moreover, q is surjective and open.

Proof. It is clear that q is linear and surjective by definition. Let us show that q is bounded. For any $x \in X$, we have

$$\|q(x)\|_{X/M} = \|x + M\|_{X/M} = \inf_{m\in M} \|x + m\| \le \|x\|,$$

hence $\|q\| \le 1$.

To prove that q is open, let $U \subset X$ be an open set, and let $\xi \in q(U)$. Then there exists $x \in U$ such that $\xi = q(x) = x + M$. Since U is open, there exists $\varepsilon > 0$ such that the open ball $B(x, \varepsilon) \subset U$. We claim that

$$B(\xi, \varepsilon) \subset q(U).$$

Indeed, if $\eta \in B(\xi, \varepsilon)$, then by definition of the quotient norm, there exists $y \in x + M$ such that

$$\|\eta - q(y)\|_{X/M} < \varepsilon.$$

This means there exists $m \in M$ with $\|(y-m)-x\| < \varepsilon$, so $y-m \in B(x,\varepsilon) \subset U$, and hence $\eta = q(y-m) \in q(U)$.

Therefore, $q(U)$ contains an open ball around each of its points, so it is open. Thus, q is an open mapping. □

1.5 Hilbert Spaces

1.5.1 Hilbert Spaces

Let H be a complex vector space.

Definition (Inner Product)

A *Hermitian inner product* on H is a function

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$$

such that for all $x, y, z \in H$ and all $a, b \in \mathbb{C}$, the following properties hold:

1. $$\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle. \text{ (linearity)}$$
2. $$\langle x, y \rangle = \overline{\langle y, x \rangle}. \text{ (hermitian symmetry)}$$
3. $$\langle x, x \rangle \geq 0, \quad \text{and } \langle x, x \rangle = 0 \text{ if and only if } x = 0. (positive definite)$$

Combining properties (1) and (2), we see that for fixed $z \in H$, the map

$$x \mapsto \langle z, x \rangle$$

is *anti-linear* (or *conjugate-linear*), for fixed $z \in H$ that is,

$$\langle z, ax + by \rangle = \overline{a}\, \langle z, x \rangle + \overline{b}\, \langle z, y \rangle.$$

If H is real, then $\langle \cdot, \cdot \rangle$ is bilinear. If H is complex, then $\langle \cdot, \cdot \rangle$ is said to be *sesquilinear*.

We often use the notation $\|x\|^2 = \langle x, x \rangle$. Then, for all $x, y \in H$, we have

$$\begin{aligned} \|x \pm y\|^2 &= \langle x - y, x - y \rangle \\ &= \|x\|^2 + \|y\|^2 \pm \langle x, y \rangle \pm \langle y, x \rangle \\ &= \|x\|^2 + \|y\|^2 \pm 2 \operatorname{Re} \langle x, y \rangle. \end{aligned}$$

Definition

A linear space equipped with a Hermitian inner product is called a *Hermitian inner product space* or a *pre-Hilbert space*.

Theorem (Schwarz–Bunyakovskii–Cauchy Inequality)

Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. Then, for all $x, y \in H$, we have

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|,$$

and equality holds if and only if x and y are linearly dependent.

Proof. If $y = 0$, the result holds trivially.

So assume that $y \neq 0$.

If $x = \lambda y$ for some $\lambda \in \mathbb{C}$, then

$$\langle x, y \rangle = \lambda \langle y, y \rangle = \lambda \|y\|^2,$$

and hence

$$|\langle x, y \rangle| = |\lambda| \, \|y\|^2 = \|x\| \, \|y\|.$$

Moreover, in this case we have

$$\lambda = \frac{\langle x, y \rangle}{\|y\|^2}.$$

Now suppose $x \in H$ is arbitrary. Define

$$z = x - \frac{\langle x, y \rangle}{\|y\|^2} y.$$

Then $\|z\|^2 \geq 0$, we compute:

$$\begin{aligned}
0 \leq \|z\|^2 &= \left\langle x - \frac{\langle x, y\rangle}{\|y\|^2} y,\, x - \frac{\langle x, y\rangle}{\|y\|^2} y \right\rangle \\
&= \|x\|^2 - 2\operatorname{Re}\left(\frac{\langle x, y\rangle}{\|y\|^2} \langle x, y\rangle \right) + \frac{|\langle x, y\rangle|^2}{\|y\|^4} \|y\|^2 \\
&= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}.
\end{aligned}$$

Hence,

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \geq 0,$$

which gives

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|.$$

Equality holds if and only if $z = 0$, that is,

$$x = \frac{\langle x, y\rangle}{\|y\|^2}\, y,$$

which means that x and y are linearly dependent. □

Corollary 1.5.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, and define*

$$\|x\| := \sqrt{\langle x, x\rangle}, \quad x \in H.$$

Then $\|\cdot\|$ is a norm on H.

Proof. We only need to verify the triangle inequality. For all $x, y \in H$, we have

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y\rangle \\
&\leq \|x\|^2 + \|y\|^2 + 2\,\|x\|\,\|y\| \quad \text{(by the Cauchy–Schwarz inequality)} \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

Taking square roots on both sides gives

$$\|x + y\| \leq \|x\| + \|y\|,$$

which is the triangle inequality. Hence $\|\cdot\|$ is a norm on H. □

Definition (Orthogonality)

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space.

1. Two elements $x, y \in H$ are said to be *orthogonal*, and we write $x \perp y$, if and only if
$$\langle x, y\rangle = 0.$$

2. More generally, if $A \subset H$ is a set, we say that $x \in H$ is *orthogonal to* A, and write $x \perp A$, if and only if
$$\langle x, y\rangle = 0 \quad \text{for all } y \in A.$$
The set of all vectors orthogonal to A is denoted by
$$A^{\perp} = \{\, x \in H : x \perp A \,\}.$$

3. A subset $S \subset H$ is said to be *orthogonal* if
$$x \perp y \quad \text{for all distinct } x, y \in S.$$

4. If, in addition, $\|x\| = 1$ for all $x \in S$, then S is called *orthonormal*.

Note also that if S is orthogonal, then
$$\left\{ \frac{x}{\|x\|} : x \in S \setminus \{0\} \right\}$$
is orthonormal.

Proposition 1.5.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. Then:*

1. ***(Parallelogram Law)*** *For all $x, y \in H$, we have*
$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big).$$

2. ***(Pythagorean Theorem)*** *If $S \subset H$ is a finite orthogonal set, then*
$$\left\| \sum_{x\in S} x \right\|^2 = \sum_{x\in S} \|x\|^2.$$

3. If $A \subset H$ is a set, then $A^\perp$ is a closed linear subspace of H.

Proof. **(1)** We compute:

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 2\|x\|^2 + 2\|y\|^2.$$

(2) Let S be a finite orthogonal set. Then

$$\left\|\sum_{x\in S} x\right\|^2 = \left\langle \sum_{x\in S} x, \sum_{y\in S} y \right\rangle = \sum_{x\in S}\sum_{y\in S}\langle x, y\rangle = \sum_{x\in S}\langle x, x\rangle = \sum_{x\in S}\|x\|^2,$$

since $\langle x, y\rangle = 0$ whenever $x \neq y$.

(3) The result follows from the continuity of the inner product and the fact that

$$A^\perp = \bigcap_{x\in A} x^\perp = \bigcap_{x\in A} \ker(\langle \cdot, x\rangle),$$

where each $\ker(\langle\cdot, x\rangle) = \{\, y \in H : \langle y, x\rangle = 0 \,\}$ is a closed subspace of H. Hence, $A^\perp$ is a closed linear subspace. □

Definition

A *Hilbert space* is a complete pre-Hilbert space $(H, \langle\cdot,\cdot\rangle)$.

Example 1.5.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $H := L^2(X, \mathcal{M}, \mu)$ endowed with the inner product*

$$\langle f, g\rangle = \int_X f(x)\,\overline{g(x)}\,d\mu(x)$$

is a Hilbert space.

We now state the most important theorem in Hilbert space theory.

Theorem (Projection Theorem)

Let H be a Hilbert space and K a closed subspace of H.

(i) For each $x \in H$, there exists a unique $y_0 \in K$ such that

$$\|x - y_0\| = d(x, K) = \min_{y \in K} \|x - y\|.$$

(ii) The element $y_0 \in K$ satisfies

$$\|x - y_0\| = \min_{y \in K} \|x - y\|$$

if and only if $y_0 \in K$ and $(x - y_0) \in K^\perp$, where $K^\perp$ denotes the orthogonal complement of K in H.

The element y_0 is called the *orthogonal projection* of x onto K.

Proof. **(i)** We have $d(x, y) \geq 0$ for every $y \in K$, thus $d(x, K)$ exists because it is the infimum of a nonempty subset of $\mathbb{R}$ bounded below by 0. Let $\varepsilon = d(x, K)$. By definition of the infimum, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in K such that for all $n \in \mathbb{N}$,

$$d(x, K) \leq \|x - x_n\| \leq d(x, K) + \frac{1}{n}.$$

We now show that (x_n) is a Cauchy sequence.

For $p, q \in \mathbb{N}$, we use the parallelogram identity:

$$\begin{aligned} 2\big(\|x - x_p\|^2 + \|x - x_q\|^2\big) &= \|(x - x_p) - (x - x_q)\|^2 + \|(x - x_p) + (x - x_q)\|^2 \\ &= \|x_p - x_q\|^2 + 4\left\|x - \frac{x_p + x_q}{2}\right\|^2. \end{aligned}$$

Hence,

$$\|x_p - x_q\|^2 = 2\big(\|x - x_p\|^2 + \|x - x_q\|^2\big) - 4\left\|x - \frac{x_p + x_q}{2}\right\|^2.$$

Since $\frac{x_p + x_q}{2} \in K$ (because K is a subspace), we have

$$\left\|x - \frac{x_p + x_q}{2}\right\| \geq d(x, K) = \varepsilon.$$

Therefore,

$$\|x_p - x_q\|^2 \leq 2\big(\|x - x_p\|^2 + \|x - x_q\|^2\big) - 4\varepsilon^2 = 2\big((\|x - x_p\|^2 - \varepsilon^2) + (\|x - x_q\|^2 - \varepsilon^2)\big).$$

As $\|x - x_n\|^2 \to \varepsilon^2$, for every $\delta > 0$ there exists n_0 such that for all $p, q \geq n_0$,

$$\big|\|x - x_p\|^2 - \varepsilon^2\big| < \frac{\delta^2}{4}, \quad \big|\|x - x_q\|^2 - \varepsilon^2\big| < \frac{\delta^2}{4}.$$

Thus,

$$\|x_p - x_q\|^2 < \delta^2, \quad \text{hence} \quad \|x_p - x_q\| < \delta.$$

So (x_n) is a Cauchy sequence in K. Since H is complete and K is closed, there exists $y_0 \in K$ such that $x_n \to y_0$. By continuity of the norm,

$$\|x - y_0\| = \lim_{n \to \infty} \|x - x_n\| = d(x, K).$$

Uniqueness. Let $y_0, y_1 \in K$ be such that

$$\|x - y_0\| = \|x - y_1\| = d(x, K).$$

Consider the sequence

$$x_n = \begin{cases} y_0, & \text{if } n \text{ is even,} \\ y_1, & \text{if } n \text{ is odd.} \end{cases}$$

The same argument as above shows that (x_n) must converge in K to some limit $y \in K$. But since the sequence alternates between two points, we must have $y_0 = y_1$. Hence, the projection is unique.

(ii) We first show that if y_0 is a minimizing vector, then $x - y_0$ must be orthogonal to K.

Suppose, to the contrary, that there exists an element $m \in K$ which is not orthogonal to the error $x - y_0$. Without loss of generality, we may assume that $\|m\| = 1$ and that

$$\langle x - y_0, m \rangle = \beta \neq 0.$$

Define the vector $m_1 \in K$ by

$$m_1 = y_0 + \beta m.$$

Then

$$\begin{aligned}\|x-m_1\|^2 &= \|x-y_0-\beta m\|^2\\ &= \|x-y_0\|^2 - \langle x-y_0, \beta m\rangle - \langle \beta m, x-y_0\rangle + |\beta|^2\|m\|^2\\ &= \|x-y_0\|^2 - 2\Re\big(\beta\langle x-y_0, m\rangle\big) + |\beta|^2.\end{aligned}$$

Since $\langle x-y_0, m\rangle = \beta$ and $\|m\| = 1$, we have

$$\|x-m_1\|^2 = \|x-y_0\|^2 - |\beta|^2 < \|x-y_0\|^2.$$

This contradicts the minimality of y_0. Hence, $x-y_0$ must be orthogonal to K.

Conversely, assume that $x-y_0 \perp K$. For any $m \in K$, we have

$$\begin{aligned}\|x-m\|^2 &= \|x-y_0+y_0-m\|^2\\ &= \|x-y_0\|^2 + \|y_0-m\|^2 + 2\Re\langle x-y_0, y_0-m\rangle.\end{aligned}$$

Since $x-y_0 \perp K$ and $y_0-m \in K$, we have $\langle x-y_0, y_0-m\rangle = 0$. Thus,

$$\|x-m\|^2 = \|x-y_0\|^2 + \|y_0-m\|^2 \geq \|x-y_0\|^2,$$

with equality if and only if $m = y_0$.

Therefore, y_0 is the unique element in K minimizing $\|x-y\|$.

□

Corollary 1.5.2. *Let H be a Hilbert space and K a closed subspace of H. For each $x \in H$ let $p_K(x) \in K$ be the (unique) element characterized by*

$$\|x-p_K(x)\| = \min\{\|x-y\| : y \in K\}.$$

Then the map $p_K : H \to K$ (the orthogonal projection onto K) satisfies:

1. *p_K is linear, continuous, $\|p_K\| = 1$, and $p_K|_K = \mathrm{Id}_K$.*
2. *$x \in K$ if and only if $p_K(x) = x$.*
3. *$K \oplus K^\perp = H$ (every $x \in H$ decomposes uniquely as an element of K plus an element of $K^\perp$).*
4. *If $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$, then $p_K(x) = x_1$.*
5. *$x \in K^\perp$ if and only if $p_K(x) = 0$.*

1.5.2 Characterization of Hilbert Spaces

Dual of Hilbert Spaces

Let H be a Hilbert space. For each $x \in H$, we define a linear functional

$$\varphi_x : H \to \mathbb{K}$$

by

$$\varphi_x(y) = \langle y, x \rangle, \quad \text{for all } y \in H.$$

Proposition 1.5.2. *Consider φ_x as defined above.*

(i) The linear form φ_x is continuous and satisfies $\|\varphi_x\| = \|x\|$.

(ii) The mapping

$$\Phi : H \longrightarrow H^*, \qquad x \longmapsto \varphi_x$$

is an antilinear isometry (linear isometry if $\mathbb{K} = \mathbb{R}$).

Proof. **(i)** For all $y \in H$, by the Cauchy–Schwarz inequality, we have

$$|\varphi_x(y)| = |\langle y, x \rangle| \leq \|x\| \, \|y\|.$$

This shows that $\varphi_x \in H^*$ and $\|\varphi_x\| \leq \|x\|$. On the other hand,

$$|\varphi_x(x)| = |\langle x, x \rangle| = \|x\|^2,$$

which implies $\|\varphi_x\| \geq \|x\|$. Therefore, $\|\varphi_x\| = \|x\|$.

(ii) We verify that for all $x_1, x_2, x \in H$ and $\lambda \in \mathbb{K}$,

$$\varphi_{x_1+x_2} = \varphi_{x_1} + \varphi_{x_2}, \qquad \varphi_{\lambda x} = \overline{\lambda} \, \varphi_x.$$

Hence, Φ is antilinear. By (i), it is an isometry. □

The Hilbert Space Representation Theorem

At least two theorems are usually referred to as the *Riesz representation theorem*:

- The **Riesz–Markov representation theorem** relates linear functionals on spaces of continuous functions on a locally compact space to measures in measure theory.

- The **Fréchet–Riesz representation theorem**, which concerns continuous linear forms on a Hilbert space.

This latter theorem establishes an important connection between a Hilbert space H and its continuous dual space H^* (the space of all continuous linear functionals from H into the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). If the field is $\mathbb{R}$, the two spaces are isometrically isomorphic; if the field is $\mathbb{C}$, they are isometrically anti-isomorphic.

Theorem (Riesz's Theorem)

Let H be a Hilbert space. The antilinear isometry $\Phi : H \to H^*$ defined by $\Phi(x) = \varphi_x$ is bijective.

Proof. It suffices to prove that Φ is surjective.

If $\dim(H)$ is finite, then Φ is surjective because

$$\dim(H) = \dim(H^*).$$

If $\dim(H)$ is infinite, let $f \in H^*$ and set

$$M = \ker(f) = \{y \in H : f(y) = 0\}.$$

- If $\ker(f) = \{0\}$, then $f = 0$, hence $f = \varphi_0$.
- If $\ker(f) \neq \{0\}$, then by the orthogonal decomposition theorem Corollary1.5.2, we have

$$H = M \oplus M^{\perp}, \qquad \text{and} \qquad \dim(M^{\perp}) = 1.$$

Let $z \in M^{\perp} \setminus \{0\}$. Then $M^{\perp} = \mathrm{Span}\{z\}$.
For any $x \in M^{\perp}$, define

$$w = x - \frac{f(x)}{f(z)}\, z.$$

We have

$$f(w) = f(x) - \frac{f(x)}{f(z)} f(z) = 0,$$

so $w \in M$. But $w \in M^{\perp}$ as well, hence $w = 0$, and consequently

$$x = \frac{f(x)}{f(z)}\, z.$$

Now, let

$$x = \frac{f(z)}{\|z\|^2}\, z \in M^{\perp} \setminus \{0\}.$$

We will show that $\varphi_x = f$.

First, observe that

$$f(x) = \frac{f(z)}{\|z\|^2} f(z) = \frac{|f(z)|^2}{\|z\|^2} = \langle x, x \rangle.$$

Now, for any $y \in H$, set

$$w = y - \frac{f(y)}{\|x\|^2}\, x.$$

Then $f(w) = 0$, so $w \in M$, and therefore $w \perp x$. It follows that

$$\langle y, x \rangle = \langle y - w, x \rangle = \frac{f(y)}{\|x\|^2} \langle x, x \rangle = f(y).$$

Hence $\varphi_x(y) = f(y)$ for all $y \in H$, i.e.

$$f = \varphi_x.$$

Thus, every $f \in H^*$ is of the form φ_x for some $x \in H$, so Φ is surjective.

□

Hilbertian Bases, Bessel's Inequality, and Parseval's Equality

Friedrich Wilhelm Bessel developed the special functions that bear his name. The following inequality is one of the fundamental results in the theory of Hilbert spaces.

Theorem (Bessel's Inequality)

Let H be a Hilbert space, and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in H. Then, for every $x \in H$, the series with general term $|\langle x, e_n \rangle|^2$ is convergent, and we have

$$\sum_{n\in\mathbb{N}} |\langle x, e_n \rangle|^2 \le \|x\|^2.$$

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in H, and let $x \in H$. For each $n \in \mathbb{N}$, consider the vector

$$x - \sum_{k=0}^{n} \langle x, e_k\rangle e_k.$$

Since the norm is nonnegative, we have

$$0 \leq \left\| x - \sum_{k=0}^{n} \langle x, e_k\rangle e_k \right\|^2.$$

Expanding this expression using the inner product, we get

$$\begin{aligned}\left\| x - \sum_{k=0}^{n} \langle x, e_k\rangle e_k \right\|^2 &= \left\langle x - \sum_{k=0}^{n} \langle x, e_k\rangle e_k,\ x - \sum_{j=0}^{n} \langle x, e_j\rangle e_j \right\rangle \\ &= \langle x, x\rangle - \sum_{j=0}^{n} \langle x, e_j\rangle\langle e_j, x\rangle - \sum_{k=0}^{n} \langle x, e_k\rangle\langle x, e_k\rangle + \sum_{k=0}^{n} \langle x, e_k\rangle\langle e_k, e_k\rangle\langle e_k, x\rangle \\ &= \|x\|^2 - \sum_{k=0}^{n} |\langle x, e_k\rangle|^2.\end{aligned}$$

$$0 \leq \left\| x - \sum_{k=0}^{n} \langle x, e_k\rangle e_k \right\|^2 = \|x\|^2 - \sum_{k=0}^{n} |\langle x, e_k\rangle|^2.$$

Since the left-hand side is nonnegative, it follows that

$$\sum_{k=0}^{n} |\langle x, e_k\rangle|^2 \leq \|x\|^2 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the sequence of partial sums $S_n = \sum_{k=0}^{n} |\langle x, e_k\rangle|^2$ is increasing and bounded above by $\|x\|^2$, hence convergent.

Thus,

$$\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 \leq \|x\|^2,$$

which proves the result. □

Definition (Basis)

Let H be a Hilbert space. A family $(e_i)_{i\in I}$ is called a **basis** of H if it satisfies the following properties:

(a) **Orthogonality:** for all $i \neq j$ in I, we have $\langle e_i, e_j\rangle = 0$;

(b) **Normalization:** for all $i \in I$, we have $\|e_i\| = 1$;

(c) **Totality:** the closed linear span of $\{e_i : i \in I\}$ is the whole space, that is,

$$H = \overline{\operatorname{span}}\{e_i : i \in I\}.$$

A family $(e_i)_{i\in I}$ satisfying these conditions is called an **orthonormal basis** (or **Hilbertian basis**) of H.

Proposition 1.5.3. *Every Hilbert space admits an orthonormal basis.*

This means that there exists an orthonormal family $(e_i)_{i\in I}$ in H such that every vector $x \in H$ can be written uniquely as

$$x = \sum_{i\in I} \langle x, e_i\rangle e_i,$$

where the series converges in the norm of H.

Proof. Let $B = (e_i)_{i\in I}$ be an orthonormal set in H. If B is not a basis, let V be the closure of the linear span of B, that is, the closure (with respect to the norm of H) of the set of finite linear combinations of elements of B.

Choose $x \in V^{\perp}$ with $x \neq 0$, and set

$$B_0 = B \cup \left\{ \frac{x}{\|x\|} \right\}.$$

Then B_0 is an orthonormal set that strictly contains B.

It is easy to verify that the union of an increasing family of orthonormal sets is itself orthonormal. Hence, by Zorn's lemma, there exists a maximal orthonormal set $B_{\max}$ in H.

If $B_{\max}$ were not a basis, then there would exist a nonzero vector $x \in (\overline{\operatorname{span}}\, B_{\max})^{\perp}$. But then

$$B_{\max} \cup \left\{ \frac{x}{\|x\|} \right\}$$

would be a strictly larger orthonormal set, contradicting maximality. Therefore, $B_{\max}$ is an orthonormal basis of H. $\square$

Remark 1.5.1. *A Hilbert space that possesses a countable Hilbertian basis is clearly separable, since the linear span of the basis vectors is dense and generated by countably many elements.*

For the converse we have the following

Theorem

If a Hilbert space is separable, then it possesses a countable Hilbertian basis.

Proof. Let H be a separable Hilbert space, and let $(f_n)_{n\in\mathbb{N}}$ be a total sequence in H. We shall construct from this sequence a Hilbert basis.

First, remove inductively any f_i that is a linear combination of its predecessors. We may therefore assume that the sequence $(f_n)_{n\in\mathbb{N}}$ is linearly independent.

To orthogonalize this sequence, we proceed recursively using the Gram–Schmidt process. Define a new sequence $(e_n)_{n\in\mathbb{N}}$ by

$$e_1 = f_1, \qquad e_2 = f_2 - P_{F_1}(f_2), \qquad \ldots, \qquad e_n = f_n - P_{F_{n-1}}(f_n),$$

where F_n denotes the subspace of H generated by the first n vectors $f_1, f_2, \ldots, f_n$, and $P_{F_{n-1}}$ is the orthogonal projection onto F_{n-1}.

By construction, the sequence $(e_n)_{n\in\mathbb{N}}$ consists of pairwise orthogonal vectors, and for each integer n, the subspace generated by $\{e_1, e_2, \ldots, e_n\}$ coincides with F_n. Hence, $(e_n)_{n\in\mathbb{N}}$ is total in H.

Finally, by normalizing each vector, that is, setting

$$u_n = \frac{e_n}{\|e_n\|},$$

we obtain an orthonormal sequence $(u_n)_{n\in\mathbb{N}}$ which is total in H. Therefore, $(u_n)_{n\in\mathbb{N}}$ is a Hilbertian (orthonormal) basis of H. $\square$

Theorem (Characterization of Hilbert Bases)

Let H be a Hilbert space, and let $(e_i)_{i\in I}$ be an orthonormal family in H. Then the following assertions are equivalent:

(1) If $\langle x, e_i\rangle = 0$ for all $i \in I$, then $x = 0$.

(2) For all $x \in H$, we have Parseval's equality:

$$\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2.$$

(3) Every element $x \in H$ can be written as

$$x = \sum_{i\in I} \langle x, e_i\rangle e_i,$$

where the series is Cauchy and hence convergent in H.

Proof. When (1) holds, we say that the orthonormal set is *complete*. Assertion (2) is known as *Parseval's identity*. In (3), the convergence is with respect to the norm of H, and implies that only countably many terms on the right-hand side are nonzero.

First, we show that (1) ⇒ (3). Let $x \in H$. By Bessel's inequality, there can be at most countably many indices i such that $|\langle x, e_i\rangle|^2 \neq 0$. Let $(e_n)_{n\in\mathbb{N}}$ be an enumeration of these. Then, by Bessel's inequality, the series

$$\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2$$

converges. Since (e_i) is orthonormal, we have

$$\left\| \sum_{i=m}^{n} \langle x, e_i\rangle e_i \right\|^2 = \sum_{i=m}^{n} |\langle x, e_i\rangle|^2 \to 0 \quad \text{as } m, n \to \infty.$$

Thus, the sequence of partial sums is Cauchy, and hence convergent. Let

$$z = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i.$$

Then

$$\langle z-x, e_i\rangle = 0 \quad \text{for each } i.$$

By (1), this implies $z-x=0$, that is, $x=z$.

Next, we show that (3) $\Rightarrow$ (2). Indeed, for every $x \in H$,

$$\|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 = \|x\|^2 - \left\| \sum_{i=1}^{n} \langle x, e_i\rangle e_i \right\|^2 \longrightarrow 0,$$

as $n \to \infty$, proving Parseval's equality.

Finally, (2) $\Rightarrow$ (1) is clear, since if $\langle x, e_i\rangle = 0$ for all i, then $\|x\|^2 = 0$, hence $x = 0$. ☐

Corollary 1.5.3. *Any separable Hilbert space H of infinite dimension is isometrically isomorphic to $\ell^2(\mathbb{N}, \mathbb{F})$.*

Proposition 1.5.4 (Pythagoras' Theorem)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be an orthogonal sequence in a Hilbert space H. Then the following properties are equivalent:*

(1) The numerical series $\sum_{n\in\mathbb{N}} \|x_n\|^2$ converges in $\mathbb{R}$.

(2) The series $\sum_{n\in\mathbb{N}} x_n$ converges in H.

In this case, we have

$$\left\| \sum_{n\in\mathbb{N}} x_n \right\|^2 = \sum_{n\in\mathbb{N}} \|x_n\|^2.$$

Proof. For all $m < n$ in $\mathbb{N}$, we have

$$\left\| \sum_{i=m+1}^{n} x_i \right\|^2 = \sum_{i=m+1}^{n} \|x_i\|^2,$$

since the sequence (x_n) is orthogonal.

This shows that $(\sum_{i=0}^{n} x_i)_{n\in\mathbb{N}}$ is a Cauchy sequence in H if and only if $(\sum_{i=0}^{n} \|x_i\|^2)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

As both H and $\mathbb{R}$ are complete, the conditions (1) and (2) are equivalent.

Moreover, under either hypothesis, letting $n \to \infty$ yields the equality

$$\left\| \sum_{n\in\mathbb{N}} x_n \right\|^2 = \sum_{n\in\mathbb{N}} \|x_n\|^2.$$

☐

1.5.3 Operators on Hilbert Spaces

Let H and K be Hilbert spaces.

Theorem

Let H be a Hilbert space, and let $T : H \to H$ be a bounded linear operator. Recall that

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}.$$

Then,

$$\|T\| = \sup\{|\langle T(x), y\rangle| : \|x\| = \|y\| = 1\}.$$

Proof. By definition,

$$\|T\| = \sup_{\|x\|=1} \|T(x)\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle T(x), y\rangle|.$$

Hence,

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle T(x), y\rangle|,$$

which proves the result. □

Theorem

There exists a unique linear operator $T^* : H \to H$ such that the following properties hold:

(a) $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in H$;

(b) $\|T\| = \|T^*\|$;

(c) $(T^*)^* = T$.

The continuous linear operator T^* is called the *adjoint* of T.

Proof. **(a)** For each fixed $y \in H$, the map

$$x \longmapsto \langle T(x), y\rangle$$

is a continuous linear functional on H. By the Riesz representation theorem, there exists a unique element $T^*(y) \in H$ such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \quad \text{for all } x \in H.$$

(b) To prove $\|T\| = \|T^*\|$, observe that

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle T(x), y\rangle| = \sup_{\|x\|=\|y\|=1} |\langle x, T^*(y)\rangle| = \|T^*\|.$$

(c) Finally, we verify that $(T^*)^* = T$. Indeed, for all $x, y \in H$,

$$\langle x, (T^*)^*(y)\rangle = \langle T^*(y), x\rangle = \langle y, T(x)\rangle = \langle x, T(y)\rangle,$$

hence $(T^*)^* = T$. □

Proposition 1.5.5. *Let H, K, and L be Hilbert spaces, and let $T \in \mathcal{B}(H, K)$ and $S \in \mathcal{B}(K, L)$. Then:*

(a) $(S \circ T)^* = T^* \circ S^*$,

(b) $\|T^*T\| = \|T\|^2$.

Proof. **(a)** For all $x \in L$ and $y \in H$, we have

$$\langle (S \circ T)(y), x\rangle = \langle T(y), S^*(x)\rangle = \langle y, T^*(S^*(x))\rangle = \langle y, (T^* \circ S^*)(x)\rangle.$$

Hence, by uniqueness of the adjoint,

$$(S \circ T)^* = T^* \circ S^*.$$

(b) Let $x \in H$ with $\|x\| = 1$. By the Cauchy–Schwarz inequality,

$$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle T^*T(x), x\rangle \leq \|T^*T\| \, \|x\|^2 = \|T^*T\|.$$

Taking the supremum over all unit vectors $x \in H$, we obtain

$$\|T\|^2 \leq \|T^*T\|.$$

Conversely, since $\|T^*\| = \|T\|$, we have

$$\|T^*T\| \leq \|T^*\| \, \|T\| = \|T\|^2.$$

Therefore,

$$\|T^*T\| = \|T\|^2.$$

□

1.6 Compact Operators

In this section, we suppose that H is a separable Hilbert space. Let

$$B = \overline{B}(0,1) = B_H = \{x \in H : \|x\| \leq 1\}$$

be the closed unit ball of H. We start by recalling some properties concerning compact spaces.

Definition (Borel–Lebesgue)

A subspace A of a Hausdorff topological space X is said to be *compact* if and only if every open cover of A by open sets in X has a finite subcover.

This is a topological property of finiteness.

Theorem (Bolzano–Weierstrass)

A subset A of a metric space is compact if and only if every sequence in A has a convergent subsequence.

Theorem

The following properties are verified:

1. Every closed subset A of a compact space X is compact.
2. Every compact subset A of a Hausdorff space is closed.
3. Any finite union of compact subsets of a space X is also compact.
4. **Tychonoff's theorem:** The product of any collection of compact topological spaces is compact with respect to the product topology.

After this reminder, we now give the definition of a compact operator between Hilbert spaces.

Definition

An operator $T : H \to H$ is said to be *compact* if $\overline{T(B)}$ is compact in H
We denote by $\mathcal{K}(H)$ the set of all compact operators $T : H \to H$.

Proposition 1.6.1. *The following properties are equivalent for a linear operator $T : H \to H$:*

(a) The operator T is compact.

(b) Every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in H has a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $(T(x_{n_k}))_{k \in \mathbb{N}}$ converges in H.

(c) Every sequence $(x_n)_{n \in \mathbb{N}}$ in the closed unit ball B_H has a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $(T(x_{n_k}))_{k \in \mathbb{N}}$ converges in H.

Proof. **(a) ⇒ (b).** Suppose that T is compact and let $(x_n)_n$ be a bounded sequence in H. Then $(x_n)_n$ is contained in a (closed) ball $B(0, r)$. We have $B(0, r) = rB$, and hence

$$T(B(0, r)) = r\,T(B).$$

Since the map $x \mapsto rx$ is a homeomorphism on H, it follows that $T(B(0, r))$ is compact. Thus, the sequence $(T(x_n))_n$ has a convergent subsequence. This proves that (a) implies (b).

(b) ⇒ (c). This is obvious.

(c) ⇒ (a). Let $(y_n)_n$ be a sequence in $T(B)$. For each n, there exists $x_n \in B$ such that

$$\|y_n - T(x_n)\| \leq \frac{1}{n} \qquad \text{(because } y_n \in T(B)\text{)}.$$

By hypothesis, the sequence $(x_n)_n$ has a subsequence $(x_{n_k})_k$ such that $(T(x_{n_k}))_k$ is convergent to some $y \in H$, and since the limit of $T(B)$ is contained in $T(B)$, we have $y \in T(B)$.

Moreover,

$$\|y_{n_k} - T(x_{n_k})\| \leq \frac{1}{n_k} \longrightarrow 0,$$

so the subsequence $(y_{n_k})_k$ also converges to y. Thus every sequence in $T(B)$ has a convergent subsequence, which shows that $T(B)$ is compact. Hence T is a compact operator.

□

Remark 1.6.1. *Let H be a Hilbert space.*

1. *Let $T \in \mathcal{K}(H)$. Then T is continuous (i.e., $T \in \mathcal{B}(H)$). Indeed, $T(B_H)$ being compact and bounded implies that $T \in \mathcal{B}(H)$.*
2. *Suppose that $T \in \mathcal{B}(H)$ is of finite rank, i.e., the image $T(H)$ of T is finite-dimensional. Then $T \in \mathcal{K}(H)$. Indeed, since $T(H)$ is finite-dimensional, it is closed in H. Therefore, $T(B_H)$ is bounded and closed in a finite-dimensional space, hence compact.*
3. *The operator Id_H is compact if and only if H is finite-dimensional (de Riesz's theorem).*
4. *$\mathcal{K}(H)$ is a subspace of $\mathcal{B}(H)$.*
5. *For all $T \in \mathcal{K}(H)$ and $S \in \mathcal{B}(H)$, we have $T \circ S \in \mathcal{K}(H)$ and $S \circ T \in \mathcal{K}(H)$.*
6. *If H is infinite-dimensional and $T \in \mathcal{K}(H)$, then T is not invertible. Suppose T is invertible. Then, the identity operator $\mathrm{Id}_H = T^{-1}T$ would be compact. Since H is infinite-dimensional, this contradicts Riesz's theorem.*
7. *If T is compact, then $\mathrm{Im}(T)$ is separable. Indeed, since T is compact, the sets $R_n = T(B(0,n))$ are relatively compact and hence separable. As $\mathrm{Im}(T) = \bigcup_{n=1}^{\infty} R_n$, it is a countable union of separable sets, and thus separable.*

Example 1.6.1. *Consider the linear operator $T_N : \ell^2 \to \ell^2$ defined by*

$$T_N(x) = (x_1, x_2, \ldots, x_N, 0, 0, \ldots).$$

Then T_N is compact. Indeed, T_N is of finite rank (its image is at most N-dimensional), and any finite-rank operator on a Hilbert space is compact.

Chapter 2

Baire's Theorem and their consequences

2.1 Baire's Theorem

Exercise 2.1.1. *Give an example of a Banach space X and of two (non-closed) sets A and B such that* $\mathrm{Int}(A \cup B) \neq \varnothing$*, but* $\mathrm{Int}(A) = \mathrm{Int}(B) = \varnothing$*.*

Consider the Banach space $X = \mathbb{R}$ with the usual norm. Define

$$A = \mathbb{Q} \cap [0,1], \qquad B = (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1].$$

- Both A and B are *not closed*, since $\mathbb{Q}$ is dense in $\mathbb{R}$ but does not contain all its limit points.
- The interior of each set is empty:

$$\mathrm{Int}(A) = \mathrm{Int}(B) = \varnothing,$$

$$A \cup B = (\mathbb{Q} \cap [0,1]) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]) = [0,1].$$
$$\mathrm{Int}(A \cup B) = \mathrm{Int}([0,1]) = (0,1) \neq \varnothing.$$

Hence, this provides the requested example.

Exercise 2.1.2. *Let (X,d) be a complete metric space and $A \subset X$. Show that the following properties are equivalent:*

A is dense in X;

$\operatorname{Int}(X \setminus A) = \varnothing$.

Indeed,

- A is dense in X iff $A \cap U \neq \varnothing$ for every nonempty open set U;
- this is equivalent to $U \not\subset X \setminus A$ for every nonempty open U;
- equivalently $\operatorname{Int}(X \setminus A) = \varnothing$.

So (i) and (ii) are equivalent.

Lemma 2.1.1 (Cantor's lemma)**.** *Let (X, d) be a complete metric space and $(F_n)_{n\geq 1}$ a decreasing sequence of nonempty closed subsets of X such that*

$$\lim_{n\to\infty} \operatorname{diam}(F_n) = 0.$$

Then the intersection $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

Proof. For each n, choose a point $x_n \in F_n$. The sequence $(x_n)_{n\geq 1}$ so constructed is Cauchy, because

$$\operatorname{diam}(F_n) \longrightarrow 0.$$

Since X is complete, the sequence (x_n) converges to some point $a \in X$.

For every n, all points x_m with $m \geq n$ lie in F_n. As each F_n is closed, it contains the limit of the sequence $(x_m)_{m\geq n}$; hence $a \in F_n$ for all n. Therefore,

$$a \in \bigcap_{n\geq 1} F_n.$$

The hypothesis on the diameters of the sets F_n implies that the intersection $\bigcap_{n\geq 1} F_n$ contains at most one point. Thus,

$$\bigcap_{n\geq 1} F_n = \{a\},$$

which proves the lemma. □

2.1.1 First Version

Theorem (Baire's Lemma)

Let X be a Banach space. Then the intersection of every countable collection of dense open subsets of X is dense in X.

Proof. Let $(V_n)_{n\geq 1}$ be a sequence of dense open subsets of X, and set

$$V := \bigcap_{n\geq 1} V_n.$$

Let U be any nonempty open subset of X. We must show that

$$U \cap V \neq \varnothing,$$

which will imply that V is dense in X.

Since V_1 is dense, we have $U \cap V_1 \neq \varnothing$; choose $x_1 \in U \cap V_1$. Because $U \cap V_1$ is open, there exists $0 < r_1 < \frac{1}{2}$ such that

$$\overline{B}(x_1, r_1) \subset U \cap V_1.$$

Since V_2 is dense in X, we have

$$V_2 \cap B(x_1, r_1) \neq \varnothing.$$

Choose

$$x_2 \in V_2 \cap B(x_1, r_1).$$

Because $V_2 \cap B(x_1, r_1)$ is open, there exists

$$0 < r_2 < \frac{1}{2^2}$$

such that

$$\overline{B}(x_2, r_2) \subset V_2 \cap B(x_1, r_1).$$

Assume $n \geq 3$. Since V_n is dense and $\overline{B}(x_{n-1}, r_{n-1})$ is nonempty and open,

$$V_n \cap \overline{B}(x_{n-1}, r_{n-1}) \neq \varnothing.$$

Choose

$$x_n \in V_n \cap \overline{B}(x_{n-1}, r_{n-1}),$$

and because this set is open, choose

$$0 < r_n < 2^{-n} \quad \text{such that} \quad \overline{B}(x_n, r_n) \subset V_n \cap \overline{B}(x_{n-1}, r_{n-1}).$$

Thus we obtain a sequence of closed balls

$$\overline{B}(x_1, r_1) \supset \overline{B}(x_2, r_2) \supset \cdots \supset \overline{B}(x_n, r_n) \supset \cdots$$

which is nested and nonempty.

Application of Cantor's lemma
The diameters satisfy

$$\operatorname{diam}\big(\overline{B}(x_n, r_n)\big) \leq 2r_n \longrightarrow 0,$$

hence by Cantor's lemma (Lemma 3), the intersection

$$\bigcap_{n\geq 1} \overline{B}(x_n, r_n)$$

is a singleton, say $\{x\}$.

Since

$$\overline{B}(x_n, r_n) \subset V_n \quad \text{for all } n,$$

we have

$$x \in \bigcap_{n\geq 1} V_n = V.$$

Also,

$$\overline{B}(x_1, r_1) \subset U,$$

so $x \in U$.

Therefore $x \in U \cap V \neq \varnothing$. As U was arbitrary, this shows that V is dense in X. □

Remark 2.1.1. *We can take X to be any complete metric space.*

Remark 2.1.2. *Baire's Theorem asserts that the intersection of the sets* V_n *is dense. However, this intersection is* not necessarily open.

For example, consider the complete metric space $(\mathbb{R}, |\cdot|)$*. Since* $\mathbb{Q}$ *is countable, we can write* $\mathbb{Q} = \{r_n : n \geq 0\}$*. Define*

$$V_n = \mathbb{R} \setminus \{r_n\}.$$

Then each V_n *is open and dense in* $\mathbb{R}$*.*

By Baire's Theorem, the intersection

$$\bigcap_{n\geq 0} V_n = \mathbb{R} \setminus \mathbb{Q}$$

is dense in $\mathbb{R}$*, neither open nor closed.*

2.1.2 Second Version

An equivalent statement, obtained by passing to complements $V_n = X \setminus A_n$, is the following.

Theorem (Baire's Theorem, second version)

Let X be a complete metric space. Let $(A_n)_{n\geq 1} \subset X$ be a sequence of **closed subsets** of X such that

$$\mathrm{Int}(A_n) = \varnothing \quad \text{for all } n \geq 1.$$

Then

$$\mathrm{Int}\left(\bigcup_{n\geq 1} A_n\right) = \varnothing.$$

In other words, the **countable union of closed sets with empty interior** also has empty interior.

Remark 2.1.3. *Let* X *be a complete metric space (or a Banach space). Let* $(A_n)_{n\geq 1} \subset X$ *be a sequence of closed subsets of* X*.*

If

$$X = \bigcup_{n\geq 1} A_n,$$

then there exists some $N \in \mathbb{N}$ *such that*

$$\mathrm{Int}(A_N) \neq \varnothing.$$

Explanation. *Suppose, for contradiction, that*

$$\operatorname{Int}(A_n) = \varnothing \quad \textit{for all } n \geq 1.$$

Define

$$U_n := X \setminus A_n.$$

Then each U_n is open and dense in X (by Exercise 2). Consider

$$U := \bigcap_{n \geq 1} U_n.$$

By Baire's theorem, U is dense in X.

But then

$$X \setminus U = \bigcup_{n \geq 1} A_n$$

would have empty interior (by Exercise 2), which contradicts the assumption that

$$X = \bigcup_{n \geq 1} A_n$$

has nonempty interior.

Hence, there must exist some N such that $\operatorname{Int}(A_N) \neq \varnothing$.

Remark 2.1.4. *This theorem is often applied when*

$$X = \bigcup_{n \geq 1} A_n.$$

Remark 2.1.5. *Without the assumption of countability, Baire's theorem may fail. Indeed, consider the complete metric space $(\mathbb{R}, |\cdot|)$. Then*

$$X = \bigcup_{x \in \mathbb{R}} \{x\},$$

where each singleton $\{x\}$ is closed with empty interior. However, $\mathbb{R}$ itself does not have empty interior, giving a contradiction.

Example 2.1.1. *The real line $\mathbb{R}$ is uncountable. Suppose, for contradiction, that*

$$\mathbb{R} = \bigcup_{n=0}^{\infty} \{x_n\}.$$

Since singletons are closed, there exists at least one $n_0 \in \mathbb{N}$ *such that*

$$\operatorname{Int}\{x_{n_0}\} \neq \varnothing,$$

but this implies $\mathbb{R}$ *is countable, which is false. Thus, we obtain a contradiction.*

2.2 Consequences

Applications to basic theorems of functional analysis.

2.2.1 Uniform Boundedness Principle

Theorem (Banach–Steinhaus)

Let X be a Banach space and Y a normed space. Assume that T_n is a sequence of bounded linear operators in $\mathcal{L}(X, Y)$ which is *pointwise bounded* bounded at each point of X, i.e.,

$$\forall x \in X \ \exists C_x > 0 \text{ such that } \|T_n(x)\| \leq C_x, \quad \forall n \in \mathbb{N}. \tag{2.1}$$

Then the sequence T_n is uniformly bounded: there exists a constant $C > 0$, independent of x, such that

$$\|T_n\| \leq C, \qquad \forall n \in \mathbb{N}.$$

Proof. For each $k \in \mathbb{N}^*$, define the set

$$A_k = \{x \in X : \|T_n(x)\| \leq k, \ \forall n \in \mathbb{N}\} = \bigcap_{n \in \mathbb{N}} \left(\|T_n(\cdot)\|^{-1}([0, k])\right).$$

Since each map $x \mapsto \|T_n(x)\|$ is continuous, every A_k is closed.

From hypothesis (2.1), we have

$$X = \bigcup_{k=1}^{\infty} A_k, \qquad \text{i.e., } \forall x \in X \ \exists k \in \mathbb{N}^* : x \in A_k.$$

Suppose, by contradiction, that this is false; i.e.

$$\exists x \in X \text{ such that } x \notin A_k, \ \forall k \in \mathbb{N}^*.$$

Then

$$\|T_n(x)\| > k, \qquad \forall k \in \mathbb{N}^*,$$

which contradicts (2.1).

By Baire's theorem, at least one of the sets A_k contains a non-empty open ball. Thus, for some $k_0 \in \mathbb{N}^*$, there exist $x_0 \in X$ and $\varepsilon > 0$ such that

$$B(x_0, \varepsilon) \subset A_{k_0}.$$

Now take an arbitrary $x \in X$ and define

$$y = x_0 + \frac{\varepsilon}{2\|x\|}\, x.$$

Then $y \in B(x_0, \varepsilon) \subset A_{k_0}$, so

$$\|T_n(y)\| \leq k_0, \qquad \|T_n(x_0)\| \leq k_0, \qquad \forall n \in \mathbb{N}.$$

For each n,

$$\begin{aligned} \|T_n(x)\| &= \frac{2}{\varepsilon}\|x\|\,\|T_n(y - x_0)\| \\ &\leq \frac{2}{\varepsilon}\|x\|\,\big(\|T_n(y)\| + \|T_n(x_0)\|\big) \\ &\leq \frac{2}{\varepsilon}\|x\|\,(k_0 + k_0) = \frac{4k_0}{\varepsilon}\,\|x\|. \end{aligned}$$

Thus,

$$\|T_n\| \leq \frac{4k_0}{\varepsilon}, \qquad \forall n \in \mathbb{N}.$$

Setting $C = \dfrac{4k_0}{\varepsilon}$ yields the conclusion. □

Remark 2.2.1. *If X is not a Banach space, the Banach–Steinhaus theorem is not valid in general. Indeed, let*

$$X = c_{00} = \left\{(x_n)_n \in \mathbb{C} : \exists N \in \mathbb{N},\ x_n = 0 \text{ for all } n \geq N\right\},$$

equipped with the norm $\|\cdot\|_\infty$. Define $T_n : X \to \mathbb{C}$ by

$$T_n(x) = \sum_{k=1}^{n} k x_k, \qquad x = (x_k)_k \in X.$$

Then

$$|T_n(x)| \leq \sum_{k=1}^{n} k\,|x_k| \leq \left(\sum_{k=1}^{n} k\right) \|x\|_\infty,$$

so each T_n *belongs to* $\mathcal{B}(X, \mathbb{C})$.

On the other hand, for every $x \in X$,

$$\lim_{n\to\infty} T_n(x) = \sum_{k=1}^{\infty} kx_k,$$

and this sum contains only finitely many nonzero terms since $x \in c_{00}$. *Thus the sequence* $(T_n(x))_n$ *is bounded for every* $x \in X$. *Consequently, the hypothesis of the Banach–Steinhaus theorem (pointwise boundedness) is satisfied.*

However, the conclusion fails. Indeed, fix $x = (x_k)_k \in c_{00}$ *given by*

$$x_1 = x_2 = \cdots = x_n = 1, \qquad x_k = 0 \text{ for } k > n.$$

Then $\|x\|_\infty = 1$, *and*

$$|T_n(x)| = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \geq n.$$

Hence

$$\|T_n\| \geq |T_n(x)| \geq n \xrightarrow[n\to\infty]{} \infty,$$

so the sequence (T_n) *is not uniformly bounded.*

Conclusion. *The space* $(c_{00}, \|\cdot\|_\infty)$ *is not complete; this shows that completeness of* X *is essential in the Banach–Steinhaus theorem.*

Corollary 2.2.1. *Let* X *be a Banach space and* Y *a normed space. Let* (T_n) *be a sequence of bounded operators from* X *into* Y *such that*

$$T_n(x) \longrightarrow T(x), \qquad \forall x \in X.$$

Then T *is a bounded linear operator from* X *into* Y.

Proof. The linearity of T follows immediately from the pointwise limit of linear maps.

Since (T_n) converges pointwise, the sequence $(T_n(x))_n$ is bounded for every $x \in X$. By the Banach–Steinhaus theorem, the sequence (T_n) is uniformly bounded; that is,

$$M := \sup_{n\in\mathbb{N}} \|T_n\|_{\mathcal{B}(X,Y)} < \infty.$$

For all n and all $x \in X$, we have

$$\|T_n(x)\|_Y \leq \|T_n\|_{\mathcal{B}(X,Y)} \, \|x\|_X \leq M\|x\|_X.$$

Passing to the limit as $n \to \infty$ gives

$$\|T(x)\|_Y = \lim_{n\to\infty} \|T_n(x)\|_Y \leq M\|x\|_X.$$

Thus T is bounded and $\|T\| \leq M$, which completes the proof. ☐

Remark 2.2.2. *In general, the sequence (T_n) does not converge to T in the operator norm. For example, let $X = H$ be a Hilbert space and let $(e_n)_n$ be an orthonormal basis. Define*

$$T_n(x) = \langle x, e_n \rangle, \qquad x \in H.$$

Then $T_n(x) \to 0$ for every $x \in H$, but

$$\|T_n\| = 1 \qquad \textit{for all } n.$$

Thus (T_n) converges pointwise to 0 but does not converge in operator norm.

2.2.2 The Open Mapping Theorem

The Open Mapping Theorem, due to Schauder (1930) and also known as the Banach–Schauder Theorem, is a fundamental result in functional analysis.

Theorem (Open Mapping Theorem)

Let X and Y be Banach spaces, and let

$$u : X \longrightarrow Y$$

be a surjective continuous linear operator. Then u is an open mapping, that is, if U is an open subset of X, then $u(U)$ is an open subset of Y.

Proof. Since u is surjective, we have

$$Y = \bigcup_{n=1}^{\infty} u\big(B_X(0,n)\big) = \bigcup_{n=1}^{\infty} \overline{u\big(B_X(0,n)\big)}.$$

By Baire's theorem, there exists $n_0 \in \mathbb{N}$ such that

$$\operatorname{Int} \overline{u\big(B_X(0,n_0)\big)} \neq \varnothing.$$

Hence, there exists $\varepsilon > 0$ such that

$$B_Y(0,\varepsilon) \subset \overline{u(B_X(0,n_0))}.$$

Let $y \in \overline{u(B_X(0,n_0))}$. Then there exists $x_0 \in B_X(0,n_0)$ such that

$$\|y - u(x_0)\| < \frac{\varepsilon}{2}.$$

Thus

$$y - u(x_0) \in B_Y\Big(0, \frac{\varepsilon}{2}\Big) \subset \overline{u\Big(B_X\Big(0, \frac{n_0}{2}\Big)\Big)}.$$

Therefore, there exists $x_1 \in B_X\big(0, \frac{n_0}{2}\big)$ such that

$$\|y - u(x_0) - u(x_1)\| < \frac{\varepsilon}{4}.$$

Repeating this argument, we construct a sequence (x_n) with

$$x_n \in B_X\Big(0, \frac{n_0}{2^n}\Big)$$

and such that

$$\left\| y - u\left(\sum_{i=0}^{n} x_i\right) \right\| < \frac{\varepsilon}{2^{\,n+1}}.$$

Let

$$s_n := \sum_{i=0}^{n} x_i.$$

Then

$$\|s_n\| \le \sum_{i=0}^{n} \|x_i\| \le \sum_{i=0}^{n} \frac{n_0}{2^i} \le 2n_0,$$

so (s_n) is a Cauchy sequence in X. Since X is Banach, it converges. Let

$$x = \lim_{n\to\infty} s_n.$$

By continuity of u, we obtain

$$u(x) = \lim_{n\to\infty} u(s_n) = \lim_{n\to\infty} y = y.$$

Thus, for every $y \in B_Y(0,\varepsilon)$ there exists $x \in B_X(0, 2n_0)$ such that

$$u(x) = y.$$

Hence,

$$B_Y(0,\varepsilon) \subset u\big(B_X(0, 2n_0)\big).$$

Therefore, u maps a neighborhood of 0 in X onto a neighborhood of 0 in Y, which proves that u is an open mapping. □

Remark 2.2.3. ***Another proof (to be understood later).*** *Consider the factorization of the operator u:*

$$X \xrightarrow{q} X/\ker(u) \xrightarrow{v} Y,$$

where q is the canonical quotient map and v is defined by

$$v\big(x + \ker(u)\big) = u(x).$$

Then v is a well-defined linear bijection. Using the bounded inverse theorem, one concludes that v is an isomorphism of Banach spaces, and hence u is an open mapping.

Remark 2.2.4. *The surjectivity of the operator u is essential. Indeed, if u is an open mapping, then its range $u(X)$ is an open vector subspace of Y. Since the only open vector subspace of a normed space is the whole space, it follows that*

$$u(X) = Y.$$

Hence u must be surjective.

Remark 2.2.5. *The converse of the Open Mapping Theorem is also true: if a linear operator $u : X \to Y$ is continuous and open, then it must be surjective.*

2.2.3 The Bounded Inverse Theorem

The Bounded Inverse Theorem, also called the Inverse Mapping Theorem (*Théorème des isomorphismes de Banach*), is a direct corollary of the Open Mapping Theorem.

Theorem (Bounded Inverse Theorem)

Let X and Y be Banach spaces and let

$$u : X \longrightarrow Y$$

be a continuous linear operator which is one-to-one and onto (i.e., bijective). Then the inverse mapping

$$u^{-1} : Y \longrightarrow X$$

is also continuous (and hence bounded).

Proof. Since u is continuous and surjective, the Open Mapping Theorem implies that u is an open mapping. Therefore, its inverse

$$(u^{-1})^{-1} = u$$

being open is equivalent to u^{-1} being continuous. □

Remark 2.2.6. *In particular, if X is a vector space equipped with two complete norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that*

$$\|x\|_1 \leq C_1 \|x\|_2 \quad \textit{for all } x \in X$$

for some positive constant C_1, then there exists another positive constant C_2 such that

$$\|x\|_2 \leq C_2 \|x\|_1 \quad \textit{for all } x \in X.$$

That is, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. The identity map $I : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is linear, bijective and continuous by hypothesis; since both spaces are Banach, the Bounded Inverse Theorem implies I^{-1} is continuous. Hence $\|x\|_2 \leq C_2 \|x\|_1$ for some $C_2 > 0$, and the norms are equivalent. □

Corollary 2.2.2. *Let* $u : X \to Y$ *be a surjective continuous linear operator between two Banach spaces. Then* u *induces, by passing to the quotient, an isomorphism*

$$\widetilde{u} : X/\ker u \longrightarrow Y$$

between the Banach spaces $X/\ker u$ *and* Y*.*

Proof. Define the map

$$\widetilde{u} : X/\ker u \longrightarrow Y, \qquad \widetilde{u}\big(x + \ker u\big) = u(x).$$

Well-defined and linear

If $x + \ker u = x' + \ker u$, then $x - x' \in \ker u$, hence

$$u(x - x') = 0 \quad \Rightarrow \quad u(x) = u(x').$$

Thus $\widetilde{u}$ is well-defined. Linearity follows immediately from the linearity of u.

Bijective

Surjectivity of u implies surjectivity of $\widetilde{u}$. If $\widetilde{u}(x + \ker u) = 0$, then $u(x) = 0$, so $x \in \ker u$, and therefore

$$x + \ker u = 0 + \ker u.$$

Thus, $\widetilde{u}$ is injective.

Continuity

Since u is continuous and $\ker u$ is a closed subspace of the Banach space X, the quotient space $X/\ker u$ is again a Banach space. The canonical projection

$$\pi : X \to X/\ker u$$

is continuous, and we have

$$\widetilde{u} \circ \pi = u.$$

Therefore, $\widetilde{u}$ is continuous.

Continuity of the inverse

Since $\widetilde{u}$ is a continuous bijection between Banach spaces, the Open Mapping Theorem implies that $\widetilde{u}^{-1}$ is also continuous.

Hence $\widetilde{u}$ is a Banach space isomorphism. □

Remark 2.2.7. *Let* $X = c_{00}$ *be the space of finitely supported sequences, equipped with the norm* $\|\cdot\|_1$*. Consider the linear operator*

$$T : X \longrightarrow X, \qquad T\big((x_n)_n\big) = \big(x_n/n\big)_n.$$

Then $T \in \mathcal{B}(X;X)$*,* T *is bijective, but its inverse*

$$T^{-1}\big((x_n)_n\big) = (nx_n)_n$$

is not continuous. Hence, the Inverse Mapping Theorem fails if X *is not complete.*

2.2.4 The Closed Graph Theorem

The Closed Graph Theorem is a fundamental result in functional analysis which characterizes continuous linear operators between Banach spaces in terms of their graph. It was established by Banach in 1932.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Define a norm on $X \times Y$ by

$$\|(x,y)\|_1 := \|x\|_X + \|y\|_Y.$$

Let $X \oplus_1 Y$ denote the product space $X \times Y$ equipped with this norm $\|\cdot\|_1$. Note that the norm topology on $X \oplus_1 Y$ coincides with the product topology on $X \times Y$.

Exercise 2.2.1. *Show that if* X *and* Y *are Banach spaces, then* $X \oplus_1 Y$ *is also a Banach space.*

Exercise 2.2.2. *Let* $p_X : X \oplus_1 Y \longrightarrow X$ *be defined by* $p_X(x,y) = x$*, and similarly* $p_Y : X \oplus_1 Y \longrightarrow Y$*. Verify that* $\|p_X\| = \|p_Y\| = 1$*.*

Theorem (Closed Graph Theorem)

Let X and Y be Banach spaces, and let

$$u : X \longrightarrow Y$$

be a linear operator. Then u is continuous if and only if its graph

$$\mathcal{G}(u) := \{(x, u(x)) : x \in X\} \subset X \times Y$$

is closed.

Proof. Suppose u is continuous. If $(x_n, u(x_n)) \to (x, y) \in X \times Y$, then $u(x_n) \to u(x) = y$, so $(x, y) \in \mathcal{G}(u)$. Hence, $\mathcal{G}(u)$ is closed.

Conversely, assume $\mathcal{G}(u)$ is closed. Then $\mathcal{G}(u)$ is a closed linear subspace of $X \times Y$, so it is a Banach space. Consider the projections

$$p_1 : \mathcal{G}(u) \longrightarrow X, \quad (x, u(x)) \mapsto x, \qquad p_2 : X \times Y \longrightarrow Y, \quad (x, y) \mapsto y.$$

Both projections are continuous, and p_1 is bijective. By the Banach Isomorphism Theorem, p_1^{-1} is continuous. Hence,

$$u = p_2 \circ p_1^{-1}$$

is continuous, completing the proof. □

Remark 2.2.8. *The requirement that X is complete in the Closed Graph Theorem is essential.*

Let $X = C^1([0,1])$ be the space of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$, with $f' \in C([0,1])$, equipped with the sup-norm

$$\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|.$$

Let $Y = C([0,1])$ with the sup-norm. Note that X is not a Banach space.

Define $u : X \to Y$ by $u(f) = f'$. Clearly, u is linear. To show $\mathcal{G}(u)$ is closed, let $(f_n, f_n') \to (f, g)$ in $X \oplus_1 Y$. Then $f_n' \to g$ uniformly, so

$$f_n(x) - f_n(0) = \int_0^x f_n'(t)\,dt \longrightarrow \int_0^x g(t)\,dt.$$

But $f_n(x) - f_n(0) \to f(x) - f(0)$, hence

$$f(x) = f(0) + \int_0^x g(t)\,dt \qquad \Longrightarrow \qquad f' = g.$$

Thus, $(f, g) \in \mathcal{G}(u)$ and $G(u)$ is closed in $X \oplus_1 Y$.

However, u is not bounded. For example, consider $f_n(x) = \frac{1}{n}\sin(nx)$. Then $f_n \to 0$ in X, but $u(f_n) = f_n' = \cos(nx)$ does not converge to 0 in Y. Hence u is unbounded.

Example 2.2.1. *Let E be the space $C_0([0,1])$ endowed with the L^1-norm, and let F be the same space endowed with the sup-norm $\|\cdot\|_\infty$. Consider $T : E \to F$ the identity mapping. This application is clearly* not continuous, *but its graph is* closed.

Remark 2.2.9. *We notice that the Open Mapping Theorem* $\Longrightarrow$ *the Inverse Mapping Theorem* $\Longrightarrow$ *the Closed Graph Theorem. In fact, the three theorems are equivalent.*

These theorems are equivalent because in Banach spaces, surjective continuous maps are open (Open Mapping), bijective continuous maps have continuous inverses (Inverse Mapping), and linear maps with closed graphs are continuous (Closed Graph). Thus, each theorem provides a different perspective on when a linear operator between complete normed spaces is continuous.

Chapter 3

Hahn–Banach Theorems and its Consequences

The Hahn–Banach theorem is one of the two most important and fundamental theorems in basic Functional Analysis. The other one is *Baire's theorem* and their applications (the Uniform Boundedness Principle, the Closed Graph Theorem, etc.).

Recall. Let X be a Banach space and let X^* be its topological dual. Let $u \in X^*$. By definition, we have

$$\|u\| = \sup_{\|x\|=1} |u(x)|.$$

3.1 Hahn–Banach Theorems

3.1.1 Analytic Form

Theorem (Hahn–Banach)

Let X be a Banach space and let $X_0 \subset X$ be a subspace. Let $u_0 : X_0 \to \mathbb{R}$ be a bounded linear functional. Then there exists an extension $u : X \to \mathbb{R}$ of u_0 (i.e. $u|_{X_0} = u_0$) such that

$$\|u\| = \|u_0\|.$$

$u_0 = u \circ i$ (i is the canonical embedding)

$$\begin{tikzcd}
X \arrow[r, "u"] & \mathbb{R} \\
& X_0 \arrow[ul, "i"'] \arrow[u, "u_0"']
\end{tikzcd}$$

Proof. Let $z \in X \setminus X_0$. We want to extend u_0 from the subspace X_0 to the larger subspace

$$X_1 = \operatorname{span}(X_0 \cup \{z\}).$$

Every element of X_1 can be written in the form

$$x + \lambda z, \qquad x \in X_0,\ \lambda \in \mathbb{R}.$$

So, to define a linear extension u, it is enough to choose a value for $u(z)$. Then we define

$$u(x + \lambda z) = u_0(x) + \lambda u(z).$$

Fix $z \in X \setminus X_0$. We must find a value for $u(z)$ such that for all $x \in X_0$,

$$|u(z) - u_0(x)| \le \|u_0\|\,\|x - z\|, \qquad \forall x \in X_0.$$

Equivalently,

$$u_0(y) - \|u_0\|\,\|y - z\| \le u(z) \le u_0(x) + \|u_0\|\,\|x - z\|, \qquad \forall x, y \in X_0.$$

Hence,

$$\sup_{y \in X_0} \big(u_0(y) - \|u_0\|\,\|y - z\|\big) \le u(z) \le \inf_{x \in E} \big(u_0(x) + \|u_0\|\,\|x - z\|\big).$$

This is possible because for all $x, y \in X_0$, we have

$$u_0(x) - u_0(y) \le \|u_0\|\,\|x - y\| \le \|u_0\|\big(\|x - z\| + \|y - z\|\big).$$

We define

$$u(z) = \inf_{x \in X_0} \big(u_0(x) + \|u_0\|\,\|x - z\|\big).$$

We finish by Zorn's lemma .

Let $(\mathcal{P}, \le)$ be a partially ordered set. If every chain in $\mathcal{P}$ has an upper bound in $\mathcal{P}$, then $\mathcal{P}$ has a maximal element. Define $\mathcal{P}$ as the set of all pairs (Y, v) such that:

- $X_0 \subset Y \subset X$,

- Y is a subspace of X,
- $v : Y \to \mathbb{R}$ is linear,
- $v|_{X_0} = u_0$,
- $\|v\| = \|u_0\|$.

We define an order relation on $\mathcal{P}$ by

$$(Y_1, v_1) \leq (Y_2, v_2) \quad \Longleftrightarrow \quad Y_1 \subset Y_2 \text{ and } v_2|_{Y_1} = v_1.$$

This means that (Y_2, v_2) is an *extension* of (Y_1, v_1).

Zorn's Lemma guarantees the existence of a maximal element $(Y_{\max}, v_{\max}) \in \mathcal{P}$. Moreover, we have

$$Y_{\max} = X \quad \text{and} \quad \|v_{\max}\| = \|u_0\|.$$

□

Corollary 3.1.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $x \in X$. Then there exists $u \in X^*$ such that $\|u\| = \|x\|$ and*

$$u(x) = \langle u, x \rangle = \|x\|^2, \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

Proof. Let $x \in X$ and define the one-dimensional subspace

$$X_0 := \mathbb{R}x.$$

Define a linear functional u_0 on X_0 by

$$u_0(\lambda x) := \lambda \|x\|^2, \quad \text{for all } \lambda \in \mathbb{R}.$$

We compute its norm:

$$\begin{aligned} \|u_0\| &:= \sup_{\|y\|=1} |u_0(y)| = \sup_{\|\lambda x\|=1} |u_0(\lambda x)| \\ &= \sup_{\|\lambda x\|=1} |\lambda| \, \|x\|^2 = \sup_{|\lambda| \, \|x\|=1} |\lambda| \, \|x\|^2 \\ &= \sup_{|\lambda|=\frac{1}{\|x\|}} \frac{1}{\|x\|} \|x\|^2 = \|x\|. \end{aligned}$$

By the Hahn–Banach Theorem, there exists an extension $u \in X^*$ of u_0 such that

$$\|u\| = \|u_0\| = \|x\|.$$

Moreover,

$$\langle u, x\rangle = \langle u|_{X_0}, x\rangle = \langle u_0, x\rangle = \|x\|^2.$$

This completes the proof. □

Corollary 3.1.2. *For every $x \in X$, we have*

$$\|x\| = \sup_{\|u\|_{X^*}=1} |\langle u, x\rangle| = \max_{\|u\|_{X^*}=1} |\langle u, x\rangle|,$$

that is, the supremum is attained.

Proof. For every $u \in X^*$ with $\|u\| = 1$, we have

$$|\langle u, x\rangle| \leq \|u\|\,\|x\| = \|x\|.$$

Hence,

$$\sup_{\|u\|_{X^*}=1} |\langle u, x\rangle| \leq \|x\|.$$

Let u_0 be as in Corollary 3.1.1, so that

$$\|u_0\| = \|x\| \quad \text{and} \quad \langle u_0, x\rangle = \|x\|^2.$$

Define

$$u_1 := \frac{u_0}{\|x\|} \in X^*.$$

Then

$$\|u_1\| = 1 \quad \text{and} \quad \langle u_1, x\rangle = \|x\|.$$

So u_1 attains the supremum Therefore,

$$\|x\| = \sup_{\|u\|_{X^*}=1} |\langle u, x\rangle| = \max_{\|u\|_{X^*}=1} |\langle u, x\rangle|.$$

□

Remark 3.1.1. *Let X be a Banach space.*

1. *Recall that*
$$\|u\| \overset{def}{=} \sup_{\|x\|=1} |u(x)|$$
for $u \in X^*$. *In general, this supremum is* not *attained. The supremum is attained for every* $u \in X^*$ *if and only if* X *is* reflexive *(see R. C. James, 1951).*

2. *In contrast, by Corollary3.1.2, for every* $x \in X$, *the supremum*
$$\sup_{\|u\|_{X^*}=1} |\langle u, x\rangle|$$
is always attained.

3.1.2 Hahn–Banach Theorem: Geometric Form

In this subsection we consider only real Banach spaces.

Definition (31)

A *hyperplane* in X is a subset $H \subset X$ of the form
$$H = u^{-1}(\alpha) = \{x \in X : u(x) = \alpha\}, \qquad \alpha \in \mathbb{R},$$
where $u : X \to \mathbb{R}$ is a linear functional.
We say that H is the hyperplane of equation $u = \alpha$.

Proposition 3.1.1. *The hyperplane* H *is closed if and only if* u *is bounded (i.e. continuous).*

Proof. ($\Rightarrow$) Suppose that H is closed. Consider the kernel
$$\ker u = u^{-1}(0).$$
Since
$$\ker u = H - \alpha = \{x - y : x \in H,\ u(y) = \alpha\},$$
it is a translate of a closed set, hence closed. A linear functional is continuous if and only if its kernel is closed. Therefore, u is bounded.

($\Leftarrow$) Suppose that u is bounded. Then u is continuous, and since $\{\alpha\}$ is a closed subset of $\mathbb{R}$, its inverse image
$$H = u^{-1}(\{\alpha\})$$
is closed in X. □

Definition

Let $A, B \subset X$. We say that a hyperplane

$$H = \{x \in X : u(x) = \alpha\}$$

separates at large the sets A and B if

$$u(x) \leq \alpha \quad \text{for all } x \in A \quad \text{and} \quad u(y) \geq \alpha \quad \text{for all } y \in B.$$

It *strictly separates* A and B if there exists $\varepsilon > 0$ such that

$$u(x) \leq \alpha - \varepsilon \quad \text{for all } x \in A \quad \text{and} \quad u(y) \geq \alpha + \varepsilon \quad \text{for all } y \in B.$$

Theorem (32 – Hahn–Banach, first geometric form)

Let $A, B \subset X$ be two nonempty, disjoint, convex subsets of X. Suppose that A is open. Then there exists a closed hyperplane

$$H = \{x \in X : u(x) = \alpha\}$$

which separates A and B at large, that is,

$$u(x) \leq \alpha \quad \text{for all } x \in A, \qquad u(y) \geq \alpha \quad \text{for all } y \in B.$$

Theorem (33 – Hahn–Banach, second geometric form)

Let $A, B \subset X$ be two nonempty, disjoint, convex subsets of X. Suppose that A is closed and that B is compact. Then there exists a closed hyperplane

$$H = \{x \in X : u(x) = \alpha\}$$

which *strictly* separates A and B, that is, there exists $\varepsilon > 0$ such that

$$u(x) \leq \alpha - \varepsilon \quad \text{for all } x \in A, \qquad u(y) \geq \alpha + \varepsilon \quad \text{for all } y \in B.$$

Given the current situation, we admit the proofs. We can consult them in the literature.

Corollary 3.1.3. *Let X be a Banach space and $X_0 \subset X$ be a subspace such* *that* $\overline{X}_0 \neq X$. *Then there exists* $u \in X^* \setminus \{0\}$ *such that*

$$\langle u, x \rangle = 0, \quad \forall x \in X_0.$$

Proof. Consider $x_0 \in X \setminus \overline{X}_0$. We use the Hahn-Banach second geometric form with

$$A = \overline{X}_0, \quad B = \{x_0\}.$$

Hence, there exists a hyperplane $u \neq 0$ and $\varepsilon > 0$ such that

$$\langle u, x \rangle < \beta - \varepsilon < \beta + \varepsilon < \langle u, x_0 \rangle, \quad \forall x \in \overline{X}_0.$$

This implies

$$\langle u, x \rangle < \beta < \langle u, x_0 \rangle, \quad \forall x \in \overline{X}_0.$$

Consequently, for all $\lambda \in \mathbb{R}$,

$$\langle u, \lambda x \rangle < \beta.$$

Therefore, for all $\lambda \in \mathbb{R}^+$,

$$\langle u, \lambda x \rangle < \beta,$$

and for all $\lambda \in \mathbb{R}^-$,

$$\langle u, \lambda x \rangle > \beta.$$

Taking the limits as $|\lambda| \to \infty$, we obtain

$$\langle u, x \rangle = 0, \quad \forall x \in X_0.$$

□

3.2 Consequences

3.2.1 bidual of normed space

Definition

The norm dual of X^* is called the **second dual** (or **bidual**, or **double dual**) of X and is denoted by X^{**}. The normed space X can be embedded isometrically in X^{**} in a natural way. Define the canonical map

$$i : X \longrightarrow X^{**}, \qquad x \longmapsto i(x),$$

by the formula

$$i(x)(x^*) = x^*(x) = \langle x^*, x\rangle, \quad \text{for each } x^* \in X^*.$$

Lemma 3.2.1. *Let X be a Banach space. For each $x \in X$, we have*

$$\|i(x)\| = \max_{\|x^*\|=1} |x^*(x)| = \|x\|.$$

Proof. By definition,

$$\|i(x)\| = \sup_{\|x^*\|=1} |i(x)(x^*)| = \sup_{\|x^*\|=1} |x^*(x)|.$$

Since $|x^*(x)| \leq \|x^*\| \, \|x\|$, we obtain $\|i(x)\| \leq \|x\|$.

Now, set $V = \{\lambda x : \lambda \in \mathbb{R}\}$ and define $f : V \to \mathbb{R}$ by

$$f(\lambda x) = \lambda \|x\|.$$

If we define $p(y) = \|y\|$, then clearly $|f(\lambda x)| = |\lambda| \|x\| = \|\lambda x\| = p(\lambda x)$. By the *Hahn–Banach Theorem* (to be seen in Chapter 3), the functional f can be extended to all of X so that

$$|f(y)| \leq p(y) = \|y\| \quad \text{for every } y \in X.$$

Thus, $f \in X^*$, $\|f\| \leq 1$, and $f(x) = \|x\|$.

It follows that

$$\|i(x)\| = \sup_{\|x^*\|=1} |x^*(x)| \geq |f(x)| = \|x\|.$$

Combining both inequalities, we conclude

$$\|i(x)\| = \sup_{\|x^*\|=1} |x^*(x)| = \max_{\|x^*\|=1} |x^*(x)| = \|x\|.$$

□

Remark 3.2.1. *Let X be a Banach space. The mapping*

$$x \longmapsto i(x)$$

from the Banach space X into its bidual X^{**} is a linear isometry. *Hence, X can be identified with the subspace $i(X) \subset X^{**}$.*

When the linear isometry

$$x \longmapsto i(x)$$

from X into X^{**} is ***surjective****, the Banach space X is said to be* ***reflexive****. That is, we have the following definition.*

Definition

A Banach space X is called **reflexive** if

$$i(X) = X^{**},$$

that is, if the canonical embedding

$$i : X \longrightarrow X^{**}$$

is *surjective*.

3.2.2 Weak Topologies on Banach Spaces

The Weak Topology $\sigma(X, X^*)$

For $\lambda \in \mathbb{F}$, the notation $|\lambda|$ denotes the absolute value of λ if $\lambda \in \mathbb{R}$, and the modulus of λ if $\lambda \in \mathbb{C}$.

Definition

Let E be a vector space over the field $\mathbb{F}$. A *semi-norm* on E is a function

$$p : E \longrightarrow \mathbb{R}$$

satisfying, for all $(\lambda, x, y) \in \mathbb{F} \times E \times E$, the following conditions:

1. $p(\lambda x) = |\lambda| \, p(x)$;
2. $p(x + y) \leq p(x) + p(y)$.

Let $\mathcal{P}$ be a family of semi-norms on E. It is said to be *separating* if

$$\big(p(x) = 0 \text{ for all } p \in \mathcal{P}\big) \quad \Longrightarrow \quad x = 0.$$

The Weak Topology $\sigma(X, X^*)$ For $x^* \in X^*$, define

$$p_{x^*}(x) := |x^*(x)|, \qquad x \in X.$$

p_{x^*} it is a semi-norm on X.

We set

$$\mathcal{P} := \{\, p_{x^*} \mid x^* \in X^* \,\}.$$

Let $p \in \mathcal{P}$. For $x_0 \in X$ and $r > 0$, we call the *p-open ball* of center x_0 and radius r the set

$$B_p(x_0, r) := \{\, x \in X : p(x - x_0) < r \,\} = p^{-1}\big(\,]\, p(x_0) - r,\ p(x_0) + r\, [\, \big).$$

Finally, we call a *$\mathcal{P}$-open ball* of center x_0 any finite intersection of p_i-open balls of center x_0, that is,

$$\bigcap_{i=1}^{n} B_{p_i}(x_0, r_i), \qquad p_i \in \mathcal{P},\ r_i > 0.$$

Definition

A subset $U \subset X$ is said to be *open* if, for every $x_0 \in U$, there exists a $\mathcal{P}$-open ball centered at x_0 that is contained in U.
One easily verifies that this definition indeed defines a topology on X.

In other words, the *weak topology* on X is the weakest topology such that every linear functional $x^* \in X^*$ is continuous.

We denote by $(X, \mathcal{P})$ the space X equipped with this topology. This topology is called the *weak topology* on X (in contrast with the topology induced by the norm, which is called the *strong topology*). It is often denoted by

$$\sigma(X, X^*).$$

By definition of the topology associated with a family of semi-norms, the sets

$$\bigcap_{i=1}^{n} B_{p_i}(x_0, r_i), \qquad p_i \in \mathcal{P},\ r_i > 0.$$

This is exactly

$$\begin{aligned} V_\varepsilon(x_0; x_1^*, \ldots, x_n^*) &:= \Big\{ x \in X : |x_i^*(x_0 - x)| < \varepsilon,\ i = 1, \ldots, n \Big\} \\ &= \bigcap_{i=1}^{n} (x_i^*)^{-1}\big( ]x_i^*(x_0) - \varepsilon,\ x_i^*(x_0) + \varepsilon[\big), \end{aligned}$$

where $n \in \mathbb{N}$ and $\varepsilon > 0$,
form a basis of open neighborhoods of x_0 in X.

This topology (weak topology) is *linear* , that is, the maps

$$\begin{aligned} f &: X \times X \longrightarrow X, \quad (x, y) \longmapsto x + y, \\ g &: X \longrightarrow X, \qquad\quad x \longmapsto \lambda x, \end{aligned}$$

are continuous.

It is also *locally convex.*

Indeed, the projection maps

$$\begin{aligned} p_1 &: X \times X \longrightarrow X, \quad (x, y) \longmapsto x, \\ p_2 &: X \times X \longrightarrow X, \quad (x, y) \longmapsto y, \end{aligned}$$

are continuous, and hence $f = p_1 + p_2$ is continuous.

Moreover,

$$g^{-1}\big(V_\varepsilon(x_0; x_1^*, \ldots, x_n^*)\big) = \frac{1}{\lambda} V_\varepsilon(x_0; x_1^*, \ldots, x_n^*).$$

Consider now, $x_1, x_2 \in V_\varepsilon(x_0; x_1^*, \dots, x_n^*)$ and $t \in [0,1]$. We show that

$$tx_1 + (1-t)x_2 \in V_\varepsilon(x_0; x_1^*, \dots, x_n^*).$$

Let $1 \le i \le n$. Then

$$\begin{aligned} \left|x_i^*(x_0 - (tx_1 + (1-t)x_2))\right| &= \left|x_i^*\big(t(x_0 - x_1) + (1-t)(x_0 - x_2)\big)\right| \\ &\le t\left|x_i^*(x_0 - x_1)\right| + (1-t)\left|x_i^*(x_0 - x_2)\right| \\ &< t\varepsilon + (1-t)\varepsilon = \varepsilon. \end{aligned}$$

Thus, $V_\varepsilon(x_0; x_1^*, \dots, x_n^*)$ is convex.

Remark 3.2.2. *The open sets of the weak topology are rather large.*

If U is a weak neighborhood of 0 in an infinite-dimensional Banach space X. By definition, there exist $\varepsilon > 0$ and a finite family $\{x_1^*, \dots, x_n^*\} \subset X^*$ such that

$$\Big\{x \in X : |x_i^*(x)| < \varepsilon,\ i = 1, \dots, n\Big\} \subset U.$$

Hence, U contains the closed subspace

$$\ker(x_1^*) \cap \cdots \cap \ker(x_n^*),$$

which is infinite-dimensional.

Proposition 3.2.1. *The topology $\sigma(X, X^*)$ is separating (Hausdorff).*

Proof. Let $x, y \in X$ with $x \neq y$. We shall find two disjoint open subsets U_x and U_y of $(X, \sigma(X, X^*))$ such that

$$x \in U_x, \qquad y \in U_y, \qquad U_x \cap U_y = \varnothing.$$

Consider the sets $A = \{x\}$ and $B = \{y\}$. They are non-empty, convex, and disjoint. By the strict geometric form of the Hahn–Banach theorem, there exist $x^* \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$x^*(x) < \alpha < x^*(y).$$

Define

$$U_x := (x^*)^{-1}(]-\infty, \alpha[), \qquad U_y := (x^*)^{-1}(]\alpha, +\infty[).$$

Then U_x and U_y are weakly open subsets of X, satisfying $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \varnothing$. $\square$

Proposition 3.2.2. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *and let* $x \in X$*. Then* (x_n) *converges to* x *in the weak topology* $\sigma(X, X^*)$ *if and only if*

$$\lim_{n\to\infty} x^*(x_n) = x^*(x), \qquad \text{for all } x^* \in X^*.$$

Moreover, a subset $E \subset X$ *is bounded in the topology* $\sigma(X, X^*)$ *if and only if* $x^*(E)$ *is a bounded subset of* $\mathbb{R}$ *(for* $\mathbb{C}$ *if* X *is complex) for every* $x^* \in X^*$*.*

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$.

1. The sequence (x_n) converges *strongly* (that is, with respect to the norm topology $\tau_{\|\cdot\|}$) to x, and we write

$$x_n \xrightarrow{\|\cdot\|} x,$$

if

$$\|x_n - x\| \longrightarrow 0.$$

2. The sequence (x_n) converges *weakly* to x, and we write

$$x_n \xrightarrow{\sigma(X,X^*)} x,$$

if x_n converges to x in the weak topology $\sigma(X, X^*)$.

A simple consequence of the fact that

$$\sigma(X, X^*) \subset \tau_{\|\cdot\|}$$

is that

$$x_n \xrightarrow{\|\cdot\|} x \quad \Longrightarrow \quad x_n \xrightarrow{\sigma(X,X^*)} x.$$

That is, every strongly convergent sequence is weakly convergent.

Proposition 3.2.3. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *and* $x \in X$*. If*

$$x_n \xrightarrow{\sigma(X,X^*)} x,$$

then the sequence $(\|x_n\|)_{n\in\mathbb{N}}$ *is bounded. Moreover,*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Proof. Assume that $x_n \to x$ weakly in X. Using the canonical embedding of X into its bidual X^{**}, we may view each x_n as a continuous linear functional on X^*.

For every $x^* \in X^*$, the sequence of real (or complex) numbers

$$\langle x_n, x^* \rangle = x^*(x_n)$$

converges to $\langle x, x^* \rangle = x^*(x)$, and hence is bounded. Thus, for each $x^* \in X^*$, there exists a constant $M_{x^*} > 0$ such that

$$|\langle x_n, x^* \rangle| \leq M_{x^*}, \qquad \text{for all } n \in \mathbb{N}.$$

By the Uniform Boundedness Principle (Banach–Steinhaus theorem), it follows that the sequence (x_n) is bounded in norm, that is, there exists a constant $M > 0$ such that

$$\|x_n\| \leq M, \qquad \text{for all } n \in \mathbb{N}.$$

Recall the standard formula:

$$\|y\| = \sup_{\|x^*\| \leq 1} |x^*(y)|, \qquad \text{for all } y \in X.$$

Now choose $x^* \in X^*$ with $\|x^*\| = 1$ such that

$$|x^*(x)| = \|x\|$$

(which is possible by Corollary 3.1.2). Then

$$\|x\| = |x^*(x)| = \lim_{n\to\infty} |x^*(x_n)| = \liminf_{n\to\infty} |x^*(x_n)| \leq \|x^*\| \liminf_{n\to\infty} \|x_n\| = \liminf_{n\to\infty} \|x_n\|.$$

This concludes the proof. $\square$

Proposition 3.2.4. *Let X be a Banach space such that $\dim X < \infty$. Then*

$$(X, \sigma(X, X^*)) = (X, \|\cdot\|).$$

Proof. By definition of the weak topology, we always have

$$(X, \sigma(X, X^*)) \subset (X, \|\cdot\|).$$

For the converse inclusion, let $x_0 \in X$ and let V_{x_0} be a neighborhood of x_0 in the norm topology. Then there exists $r > 0$ such that

$$B(x_0, r) \subset V_{x_0}.$$

So it is enough to find a weak neighborhood inside $B(x_0, r)$ Since $\dim X = n < \infty$, let $(e_1, \dots, e_n)$ be a basis of X. Every $x \in X$ can be written uniquely as

$$x = \sum_{i=1}^{n} \alpha_i e_i.$$

Define linear functionals $u_i : X \to \mathbb{R}$ by

$$u_i(x) := \alpha_i, \qquad i = 1, \dots, n.$$

Then $u_i \in X^*$ for all i, and there exists a constant (which we may take equal to 1 after renorming) such that

(In a finite-dimensional vector space, all norms are equivalent.

that means there exists a constant $C > 0$ such that, for any X,$\|x\| \leq C \sum_{i=1}^{n} |u_i(x)|$.)

$$\|x\| \leq \sum_{i=1}^{n} |u_i(x)|.$$

Define

$$U_{x_0} := \left\{ x \in X : |u_i(x - x_0)| < \frac{r}{n},\ i = 1, \dots, n \right\} = \bigcap_{i=1}^{n} u_i^{-1}\Big( ]u_i(x_0) - \tfrac{r}{n},\ u_i(x_0) + \tfrac{r}{n}[\Big).$$

Then U_{x_0} is a weakly open neighborhood of x_0.

For any $x \in U_{x_0}$, we have

$$\|x - x_0\| \leq \sum_{i=1}^{n} |u_i(x - x_0)| \leq \sum_{i=1}^{n} \frac{r}{n} = r.$$

Thus,

$$U_{x_0} \subset B(x_0, r) \subset V_{x_0}.$$

Hence every norm neighborhood contains a weak neighborhood, and therefore

$$(X, \|\cdot\|) \subset (X, \sigma(X, X^*)).$$

The two topologies coincide. $\square$

Let $F \subset X$ be a closed subset in $(X, \sigma(X, X^*))$. Then F is also closed in $(X, \|\cdot\|)$. In general, the converse is not true. However, it is true if F is convex, as shown by the following theorem.

Theorem (Mazur)

Let $C \subset X$ be a convex set. Then

$$\overline{C}^{\|\cdot\|} = \overline{C}^{\sigma(X,X^*)},$$

i.e., C is norm-closed if and only if it is weakly closed.

Proof. Suppose that C is closed in the norm topology. Equivalently, the complement $\{C := X \setminus C$ is open in $(X, \|\cdot\|)\}$.

We show that $\{C$ is also open in $(X, \sigma(X, X^*))\}$. Let $x_0 \in \{C\}$. We must find a weakly open neighborhood $V_{x_0} \subset \{C$ of x_0.

Set $A = \{x_0\}$ and $B = C$. By the strict geometric form of the Hahn–Banach theorem, since $x_0 \notin C$, there exists $u \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$u(x_0) < \alpha < u(x), \qquad \text{for all } x \in C.$$

Define

$$V_{x_0} := u^{-1}\big(]-\infty, \alpha[\big).$$

Then V_{x_0} is weakly open and satisfies

$$V_{x_0} \cap C = \varnothing \quad \Longrightarrow \quad V_{x_0} \subset \{C.$$

Hence, every point of $\{C$ has a weak neighborhood contained in $\{C$, and so C is weakly closed. □

Remark 3.2.3. *Let X be a Banach space.*

1. *If* $\dim X = +\infty$, *then*

$$(X, \sigma(X, X^*)) \subsetneq (X, \|\cdot\|),$$

i.e., the weak topology is strictly weaker than the norm topology. Indeed, the unit ball

$$B_X := \{x \in X : \|x\| < 1\}$$

is not *open in the weak topology (exercise).*

2. *Moreover, the weak topology is in general* not metrizable, *except when X is separable.*

The Weak*-Topology $\sigma(X^*, X)$

Let X be a Banach space, X^* its dual equipped with the dual norm, and X^{**} its bidual equipped with the bidual norm

$$\|x^* * \|_{X^{**}} := \sup_{\|u\|_{X^*}=1} |\langle x^* *, u\rangle|, \qquad x^* * \in X^{**}.$$

Let

$$i : X \longrightarrow X^{**}$$

be the canonical injection defined by

$$\langle i(x), x^*\rangle_{X^{**},X^*} = \langle x^*, x\rangle_{X^*,X}, \quad \text{for all } x^* \in X.$$

That is, $i(x) = x$ in the sense of the natural embedding.

The map i is an injective linear isometry. Indeed, for any $x \in X$,

$$\|i(x)\| = \sup_{\|x^*\|_{X^*}=1} |\langle i(x), x^*\rangle| = \sup_{\|x^*\|_{X^*}=1} |\langle x^*, x\rangle| = \|x\|.$$

In general, i is not surjective, so X is a subspace of X^{**}.

Definition

The *weak**-*topology* $\sigma(X^*, X)$ on X^* is the weakest topology such that for each $x \in X$, the map

$$x : X^* \longrightarrow \mathbb{R}, \quad x^* \mapsto \langle x, x^*\rangle$$

is continuous.

A base of neighborhoods of 0 in X^* is given by

$$V_\varepsilon(0; x_1, \dots, x_n) := \{\, x^* \in X^* : |\langle x_i, x^*\rangle| < \varepsilon,\ i = 1, \dots, n \,\} = \bigcap_{i=1}^{n} x_i^{-1}(]-\varepsilon, \varepsilon[),$$

where $n \in \mathbb{N}$ and $\varepsilon > 0$.

On X^*, we have three topologies, ordered by strength:

$$(X^*, \sigma(X^*, X)) \ \subset\ (X^*, \sigma(X^*, X^{**})) \ \subset\ (X^*, \|\cdot\|).$$

Like the weak topology, the weak* topology is locally convex.

In the sequel, we denote elements of X^* by x^* and elements of $X^* *$ by $x^* *$. As in the preceding proposition, we have the following results.

Proposition 3.2.5. *The topology $\sigma(X^*, X)$ is separating; that is, the space*

$$(X^*, \sigma(X^*, X))$$

is Hausdorff.

Proposition 3.2.6. *Let $(x_n^*)_{n\geq 1} \subset X^*$ be a sequence. Then x_n^* converges to x^* in $\sigma(X^*, X)$ if and only if*

$$\lim_{n\to\infty} x_n^*(x) = x^*(x), \qquad \text{for all } x \in X.$$

Proposition 3.2.7. *If*

$$x_n^* \xrightarrow{\sigma(X^*,X)} x^*,$$

then the sequence $(\|x_n^*\|)_{n\geq 1}$ is bounded. Moreover,

$$\|x^*\| \leq \liminf_{n\to\infty} \|x_n^*\|.$$

Remark 3.2.4. *An inconvenience of the weak* topology, compared with the weak topology, is that in general, convex closed sets in X^* (closed either in the norm topology or in the weak topology) are* not *weak*-closed.*

However, an important compensating property is given by the Banach–Alaoglu Theorem*: the unit ball of X^* is compact in the weak* topology. This result was first discovered by S. Banach in 1932 for separable Banach spaces and extended to the general case by Leonidas Alaoglu in 1940.*

Theorem (Banach–Alaoglu)

Let X be a Banach space. Then the closed unit ball

$$B_{X^*} := \{x^* \in X^* : \|x^*\| \leq 1\}$$

is compact in the weak* topology $\sigma(X^*, X)$ (and hence in particular, weak*-closed).

Proof. For $x \in X$, define the interval

$$I_x := \{t \in \mathbb{R} : |t| \leq \|x\|\},$$

and consider the Cartesian product

$$\mathcal{P} := \prod_{x \in X} I_x.$$

Recall that $\mathcal{P}$ is the set of all functions f on X such that $f(x) \in I_x$ for all $x \in X$, endowed with the product topology. This is the weakest topology making all evaluation maps

$$T_x : \mathcal{P} \longrightarrow I_x, \quad f \mapsto f(x)$$

continuous.Each I_x is compact in $\mathbb{R}$ By Tychonoff's theorem (An arbitrary product of compact spaces is compact (for the product topology).), $\mathcal{P}$ is compact.

Each $x^* \in B_{X^*}$ defines a function

$$f_{x^*} : X \longrightarrow \mathbb{R}, \qquad f_{x^*}(x) := x^*(x),$$

which belongs to $\mathcal{P}$.

Thus,

$$B_{X^*} \subset \mathcal{P}.$$

The topology induced on B_{X^*} by $\mathcal{P}$ coincides with the weak* topology $\sigma(X^*, X)$.

For each $x, y \in X$ and each $c \in \mathbb{R}$, define

$$F_{x,y}(f) := f(x)+f(y)-f(x+y) = 0, \mathit{linearity} \quad G_{x,c}(f) := f(cx)-cf(x) = 0.\mathit{homogenity}$$

$f \in \bigcap_{x,y \in X} F_{x,y}^{-1}(\{0\})$ and $f \in \bigcap_{x \in X, c \in \mathbb{R}} G_{x,c}^{-1}(\{0\})$ These functions are continuous on $\mathcal{P}$ (The product topology is the coarsest topology making all coordinate projections continuous.) , and

Since $f \in \mathcal{P}$, we have

$$|f(x)| \leq \|x\| \qquad \text{for all } x \in X.$$

This implies that

$$\|f\| \leq 1.$$

Indeed, by definition of the operator norm,

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| \leq 1.$$

so $f \in B_{X^*}$

Conversely, if $x^* \in B_{X^*}$, then:

- x^* is linear, so for all $x, y \in X$ and $c \in \mathbb{R}$,

  $$F_{x,y}(x^*) = x^*(x) + x^*(y) - x^*(x+y) = 0, \qquad G_{x,c}(x^*) = x^*(cx) - c\,x^*(x) = 0;$$

- $|x^*(x)| \leq \|x\|$, so $x^* \in \mathcal{P}$.

Hence,

$$x^* \in \bigcap_{x,y \in X} F_{x,y}^{-1}(\{0\}) \;\cap \bigcap_{x \in X,\, c \in \mathbb{R}} G_{x,c}^{-1}(\{0\}).$$

Thus

$$B_{X^*} = \bigcap_{x,y \in X} F_{x,y}^{-1}(\{0\}) \;\cap \bigcap_{x \in X, c \in \mathbb{R}} G_{x,c}^{-1}(\{0\}).$$

$f \in \bigcap_{x,y \in X} F_{x,y}^{-1}(\{0\})$ and $f \in \bigcap_{x \in X, c \in \mathbb{R}} G_{x,c}^{-1}(\{0\})$ These functions are continuous on $\mathcal{P}$ and $\{0\}$ is closed in $\mathbb{R}$ Hence B_{X^*} is a closed subset of the compact set $\mathcal{P}$ and therefore compact itself. $\square$

Proposition 3.2.8. *Let* $(x_n^*)_{n \in \mathbb{N}} \subset X^*$. *If*

$$x_n^* \xrightarrow{\sigma(X^*,X)} x^*,$$

then the sequence $(\|x_n^*\|)_{n \in \mathbb{N}}$ *is bounded. Moreover,*

$$\|x^*\| \leq \liminf_{n \to \infty} \|x_n^*\|.$$

Proof. Set $C := \liminf_{n \to \infty} \|x_n^*\|$ and let $\varepsilon > 0$ be arbitrary. There exists a subsequence $(x_{n_k}^*)$ such that

$$\|x_{n_k}^*\| \leq C + \varepsilon, \quad \text{for all } k \in \mathbb{N}.$$

The closed ball of radius $C + \varepsilon$ in X^*,

$$B_{C+\varepsilon} := \{x^* \in X^* : \|x^*\| \leq C + \varepsilon\},$$

is weak*-compact (by Banach–Alaoglu) and hence weak*-closed. Since $x^* \in \overline{\{x_{n_k}^*\}}^{\sigma(X^*,X)}$, it follows that

$$\|x^*\| \leq C + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$\|x^*\| \leq \liminf_{n\to\infty} \|x_n^*\|.$$

This also implies that the sequence $(\|x_n^*\|)$ is bounded. ☐

Theorem (H.-H. Goldstine, 1938)

Let X be a Banach space. Then the closed unit ball

$$B_X := \{x \in X : \|x\| \leq 1\}$$

is $\sigma(X^{**}, X^*)$-dense in the closed unit ball of the bidual

$$B_{X^{**}} := \{x^{**} \in X^{**} : \|x^{**}\| \leq 1\}.$$

Corollary 3.2.1. *Let X be a Banach space. Then the closed unit ball B_X is weakly compact if and only if X is reflexive.*

Proof. Suppose B_X is weakly compact. Then B_X is also weak* compact when viewed as a subset of X^{**}. By Goldstine's theorem, the image of B_X under the canonical embedding $i : X \to X^{**}$ is dense in $B_{X^{**}}$. Since B_X is weak* closed, it follows that $i(X) = X^{**}$, so X is reflexive.

Conversely, if X is reflexive, then $X = X^{**}$, and by the Banach–Alaoglu theorem, B_X is weakly compact. ☐

Remark 3.2.5. *The closed unit ball B_X is closed in $B_{X^{**}}$ for the strong topology because X is complete and the canonical embedding i is an isometry. However, in general, B_X is* not dense *in $B_{X^{**}}$ for the strong topology, except when X is reflexive (in which case $B_X = B_{X^{**}}$).*

Proposition 3.2.9. *Let $(x_n^*)_n$ be a sequence in X^* such that*

$$x_n^* \xrightarrow[n\to\infty]{} x^* \quad \text{in } \sigma(X^*, X).$$

Then the sequence $(\|x_n^*\|)$ is bounded. Moreover,

$$\|x^*\| \leq \liminf_{n\to\infty} \|x_n^*\|.$$

Proof. Let

$$C = \liminf_{n\to\infty} \|x_n^*\|$$

and let $\varepsilon > 0$ be arbitrary. Then there exists a subsequence (still denoted (x_n^*)) such that

$$\|x_n^*\| \le C + \varepsilon \quad \text{for all } n.$$

The closed ball of radius $C+\varepsilon$ in X^* is weak*-compact, hence weak*-closed. Since $x_n^* \to x^*$ in the weak* topology, we must have

$$\|x^*\| \le C + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the result follows. ☐

Theorem (Hahn–Banach–Goldstine, 1938)

Let X be a Banach space. Then the closed unit ball

$$B_X = \{x \in X : \|x\| \le 1\}$$

is $\sigma(X^{**}, X^*)$-dense in the closed unit ball $B_{X^{**}}$ of X^{**}.

Corollary 3.2.2. *Let X be a Banach space. The closed unit ball B_X is weakly compact if and only if X is reflexive.*

Proof. Assume first that B_X is weakly compact. Viewing B_X as a subset of X^{**} via the canonical embedding $i : X \hookrightarrow X^{**}$, it is weak*-compact and hence weak*-closed. By Theorem 3.2.2, B_X is weak*-dense in $B_{X^{**}}$, which implies

$$i(B_X) = B_{X^{**}}.$$

Therefore $i(X) = X^{**}$ and X is reflexive.

Conversely, if X is reflexive, then $B_X = B_{X^{**}}$, which is weak*-compact by the Banach–Alaoglu theorem. Hence B_X is weakly compact. ☐

Remark 3.2.6. *The closed unit ball B_X is closed in $B_{X^{**}}$ for the strong topology since X is complete and the canonical embedding $i : X \to X^{**}$ is an isometry. Therefore, in general, B_X is not dense in $B_{X^{**}}$ for the strong topology, except in the reflexive case where*

$$B_X = B_{X^{**}}.$$

This case will be discussed in the next section.

3.2.3 Reflexive Spaces

Let X be a normed space and let X^{**} denote its bidual. For every $x \in X$, we define a linear functional φ_x on X^* by

$$\varphi_x(f) = f(x), \qquad f \in X^*.$$

Since

$$|\varphi_x(f)| = |f(x)| \leq \|x\| \, \|f\|,$$

it follows that φ_x is continuous and hence belongs to X^{**}.

Theorem

Let X be a normed space. Then X is isometrically isomorphic to a subspace $\widetilde{X}$ of its bidual X^{**}. More precisely, the map

$$J : X \longrightarrow X^{**}, \qquad J(x) = \varphi_x,$$

is a linear isometry.

Proof. It is immediate that J is linear. Moreover, for all $x \in X$,

$$\|J(x)\| = \|\varphi_x\| = \sup_{\|f\|=1} |\varphi_x(f)| = \sup_{\|f\|=1} |f(x)| \leq \|x\|,$$

so J is continuous.

It remains to show that J preserves the norm. By Corollary 3.1.1, for every $x \in X$ there exists $f \in X^*$ such that

$$f(x) = \|x\| \quad \text{and} \quad \|f\| = 1.$$

Therefore,

$$\|J(x)\| = \sup_{\|f\|=1} |\varphi_x(f)| \geq |\varphi_x(f)| = |f(x)| = \|x\|.$$

Combining this with the previous inequality yields

$$\|J(x)\| = \|x\|.$$

Thus J is an isometry, and the proof is complete. □

The isometry $J : X \longrightarrow X^{**}$ is called the *canonical isometry* (or *canonical embedding*) of X into X^{**}.

Definition

A normed vector space X is said to be *reflexive* if

$$J(X) = X^{**}.$$

Remark 3.2.7. *If X is reflexive, then X is isometrically isomorphic to X^{**}. The converse is false: R. C. James (1951) constructed a Banach space X and a bijective isometry*

$$j : X \longrightarrow X^{**}$$

such that X is not reflexive.

Example 3.2.1. *1. Every finite-dimensional normed space is reflexive.*

2. The spaces $L^p(\Omega)$ are reflexive for $1 < p < \infty$.

Theorem (Kakutani, 1939)

Let X be a Banach space. Then the closed unit ball

$$B_X = \{x \in X : \|x\| \leq 1\}$$

is $\sigma(X, X^*)$-compact if and only if X is reflexive.

Corollary 3.2.3. *Let X and Y be Banach spaces, and assume that X is reflexive.*

(a) If $A \subset X$ is bounded, then A is relatively $\sigma(X, X^*)$-compact.

(b) If $A \subset X$ is convex, bounded, and closed in the norm topology, then A is $\sigma(X, X^*)$-compact.

(c) If $T : X \to Y$ is a bounded linear operator, then $T(B_X)$ is $\sigma(Y, Y^*)$-compact.

Proof. (a) Since A is bounded, there exists $r > 0$ such that $A \subset rB_X$. As B_X is $\sigma(X, X^*)$-compact by Theorem 3.2.3, so is rB_X. Hence A is relatively $\sigma(X, X^*)$-compact.

(b) By part (a), A is relatively $\sigma(X, X^*)$-compact. Since A is convex and norm closed, it is also $\sigma(X, X^*)$-closed. Therefore A is $\sigma(X, X^*)$-compact.

(c) The operator T is weak-to-weak continuous. Since B_X is $\sigma(X, X^*)$-compact, its image $T(B_X)$ is $\sigma(Y, Y^*)$-compact.

□

Corollary 3.2.4. *For every Banach space X, the space X is reflexive if and only if X^* is reflexive.*

Proof. Assume that X is reflexive. Then the closed unit ball B_{X^*} is $\sigma(X^*, X)$-compact. Since

$$\sigma(X^*, X) = \sigma(X^*, X^{**}),$$

it follows from Theorem 3.2.3 that X^* is reflexive.

Conversely, suppose that X^* is reflexive. Then the closed unit ball $B_{X^{**}}$ is $\sigma(X^{**}, X^*)$-compact. By Goldstine's theorem, $J(B_X)$ is $\sigma(X^{**}, X^*)$-dense in $B_{X^{**}}$. Since $J(B_X)$ is also $\sigma(X^{**}, X^*)$-closed, we obtain

$$J(B_X) = B_{X^{**}},$$

and hence X is reflexive. □

Corollary 3.2.5. *Let $T : X \to Y$ be a bounded linear operator between Banach spaces. If either X or Y is reflexive, then T is weakly compact; that is,*

$$T(B_X) \quad \textit{is } \sigma(Y, Y^*)\textit{-compact.}$$

Proposition 3.2.10. *Let Z be a reflexive Banach space. Then every closed subspace Y of Z is reflexive.*

Proof. By the Hahn–Banach theorem, every continuous linear functional on Y can be extended to a continuous linear functional on Z. Consequently, the weak topology $\sigma(Y, Y^*)$ on Y coincides with the trace on Y of the weak topology $\sigma(Z, Z^*)$ on Z.

Since Z is reflexive, its closed unit ball B_Z is weakly compact. The closed unit ball B_Y of Y satisfies

$$B_Y = B_Z \cap Y,$$

hence B_Y is weakly closed in Z because Y is closed and convex. Therefore, B_Y is weakly compact as a weakly closed subset of the weakly compact set B_Z.

It follows that Y is reflexive. □

3.2.4 Separable Spaces

The notion of separability was introduced by Fréchet in his thesis.

Definition

A Banach space X is called *separable* if it contains a countable dense subset. That is, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$\overline{\{x_n : n \in \mathbb{N}\}} = X.$$

Example 3.2.2. *1. If K is a compact* metrizable *Hausdorff space, then the Banach space $C(K)$ is separable. In particular, $C([0,1])$ is separable.*

2. The Lebesgue spaces $L^p(\Omega)$ are separable for any $1 \le p < \infty$, provided that $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space.

3. The spaces ℓ^∞ and $L^\infty([0,1])$ are not separable.

Proposition 3.2.11. *The Banach space ℓ^∞ is not separable.*

Proof. For every subset $A \subset \mathbb{N}$, define the sequence $\mathbf{1}_A \in \ell^\infty$ by

$$\mathbf{1}_A(n) = \begin{cases} 1, & \text{if } n \in A, \\ 0, & \text{if } n \notin A. \end{cases}$$

Clearly, $\|\mathbf{1}_A\|_\infty = 1$.

Moreover, for any two distinct subsets $A, B \subset \mathbb{N}$, we have

$$\|\mathbf{1}_A - \mathbf{1}_B\|_\infty = 1.$$

Assume by contradiction that ℓ^∞ is separable. Then there exists a countable dense subset $D \subset \ell^\infty$.

For each $A \subset \mathbb{N}$, choose an element $x_A \in D$ such that

$$\|\mathbf{1}_A - x_A\|_\infty < \frac{1}{2}.$$

This defines a mapping

$$q : \mathcal{P}(\mathbb{N}) \longrightarrow D, \qquad q(A) = x_A.$$

We claim that q is injective. Indeed, if $A \neq B$, then by the triangle inequality,

$$\begin{aligned} 1 &= \|\mathbf{1}_A - \mathbf{1}_B\|_\infty \\ &\leq \|\mathbf{1}_A - x_A\|_\infty + \|x_A - x_B\|_\infty + \|x_B - \mathbf{1}_B\|_\infty \\ &< \frac{1}{2} + \|x_A - x_B\|_\infty + \frac{1}{2} = 1 + \|x_A - x_B\|_\infty. \end{aligned}$$

Hence $\|x_A - x_B\|_\infty > 0$, which implies $x_A \neq x_B$.

Therefore,

$$\mathrm{Card}(D) \geq \mathrm{Card}(\mathcal{P}(\mathbb{N})).$$

Since $\mathcal{P}(\mathbb{N})$ has the cardinality of the continuum, this contradicts the assumption that D is countable.

Thus, ℓ^∞ is not separable. □

Proposition 3.2.12. *The Banach space $C([1, +\infty))$, endowed with the supremum norm, is not separable.*

Proof. We argue by contradiction. Suppose that $C([1, +\infty))$ is separable. Then there exists a countable dense subset

$$\{f_1, f_2, \dots, f_n, \dots\} \subset C([1, +\infty)).$$

For each $n \in \mathbb{N}$, define a sequence

$$x_n = \big(f_n(1), f_n(2), f_n(3), \dots\big) \in \ell^\infty.$$

Let

$$D := \{x_n : n \in \mathbb{N}\} \subset \ell^\infty.$$

Now let $y = (y_m)_{m\in\mathbb{N}} \in \ell^\infty$. Define a function $f \in C([1, +\infty))$ such that

$$f(m) = y_m \quad \text{for all } m \in \mathbb{N}.$$

(Such a function exists, for example, by piecewise linear interpolation.)

Since $\{f_n\}$ is dense in $C([1, +\infty))$, for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\|f - f_n\|_\infty < \varepsilon.$$

Then

$$\begin{aligned}\|y - x_n\|_\infty &= \sup_{m\in\mathbb{N}} |f(m) - f_n(m)| \\ &\le \sup_{x\in[1,+\infty)} |f(x) - f_n(x)| \\ &= \|f - f_n\|_\infty < \varepsilon.\end{aligned}$$

This shows that D is dense in ℓ^∞, contradicting the fact that ℓ^∞ is not separable.

Therefore, $C([1,+\infty))$ is not separable. $\square$

Proposition 3.2.13. *Let X and Y be Banach spaces such that X is separable, and let $T : X \to Y$ be a bounded linear operator. Then $T(X)$ is separable.*

Proof. Let $(x_n)_{n\ge1}$ be a countable dense subset of X. Since T is continuous, the set $\{T(x_n) : n \ge 1\}$ is countable and dense in $T(X)$. Hence $T(X)$ is separable. $\square$

Lemma 3.2.2. *Let X be a normed space and $(x_n)_{n\ge1}$ a sequence in X. Let E be the vector subspace spanned by $(x_n)_{n\ge1}$. If E is dense in X, then X is separable.*

Proof. Let E_0 be the vector space over $\mathbb{Q}$ spanned by $(x_n)_{n\ge1}$. Then E_0 is countable and dense in E. Since E is dense in X, it follows that E_0 is dense in X. Hence X is separable. $\square$

Proposition 3.2.14. *Let X be a Banach space and let $X_0 \subset X$ be a closed subspace. Then:*

1. *If X^* is separable, then X is separable.*
2. *If X is separable, then X_0 is separable.*

Proof of (1). Let $(x_n^*)_{n\ge1}$ be a countable dense sequence in X^*. By the definition of the norm in the dual space, for each $n \ge 1$ there exists $x_n \in X$ with $\|x_n\| = 1$ such that

$$|\langle x_n^*, x_n\rangle| \ge \frac{1}{2}\|x_n^*\|.$$

Let E be the vector subspace of X spanned by $(x_n)_{n\ge1}$. We claim that E is dense in X.

Let $x^* \in X^*$ such that $x^*_{|E} = 0$. We show that $x^* = 0$. Given $\varepsilon > 0$, there exists $n \geq 1$ such that

$$\|x^* - x_n^*\| < \varepsilon.$$

Then

$$\frac{1}{2}\|x_n^*\| \leq |\langle x_n^*, x_n \rangle| = |\langle x_n^* - x^*, x_n \rangle| \leq \|x_n^* - x^*\|.$$

Hence $\|x_n^*\| \leq 2\varepsilon$, and therefore

$$\|x^*\| \leq \|x^* - x_n^*\| + \|x_n^*\| < 3\varepsilon.$$

Since this holds for all $\varepsilon > 0$, we conclude that $x^* = 0$.

Thus E is dense in X. By Lemma 3.2.2, X is separable. $\square$

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University of Mohamed Boudiaf – M'Sila
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Exercise Sheet – Fundamentals of Functional Analysis 01

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1 Normed Vector Spaces

Exercise 1.1. *Let* $(X, \|\cdot\|)$ *be a normed space,* $a \in X$ *and* $r > 0$. *The open ball*

$$B(a,r) = \{x \in X : \|x - a\| < r\}$$

and the closed ball

$$\overline{B}(a,r) = \{x \in X : \|x - a\| \leq r\}$$

are convex sets.

Exercise 1.2. *Let* X *be a normed space. Find all linear subspaces* $M \subset X$ *which are contained in some (closed or open) ball* $B(a,r) = \{x \in X : \|x - a\| < r\}$ *(or* $\leq r$*).*

Exercise 1.3. *If* $0 < p < 1$, *then* ℓ^p *is a vector space, but*

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$

is not a norm on ℓ^p.

Exercise 1.4. *The spaces* ℓ^p, L^p *for* $1 \leq p \leq \infty$, *and* $C(K)$ *for* K *compact are Banach spaces.*

Exercise 1.5. *(a) Show that* ℓ^∞ *equipped with the norm* $\|\cdot\|_\infty$ *is a Banach space.*

(b) Let c_0 *be the space of real (or complex) sequences converging to* 0. *Show that* c_0 *is a closed subspace of* ℓ^∞.

(c) Let c_{00} *denote the space of all real (or complex) sequences* $x = (x_n)_{n\geq 1}$ *for which there exists* n_0 *with* $x_n = 0$ *for all* $n \geq n_0$*. Equip* c_{00} *with the sup-norm* $\|x\|_\infty = \sup_{n\geq 1} |x_n|$*. We show that* $(c_{00}, \|\cdot\|_\infty)$ *is* not *a Banach space.*

Exercise 1.6. *Consider the space*

$$C[0,1) = \{f : [0,1) \to \mathbb{R} \mid f \text{ is continuous on } [0,1)\},$$

equipped with the supremum norm

$$\|f\|_\infty = \sup_{x\in[0,1)} |f(x)|.$$

This space is not complete.

Exercise 1.7. *Show that he application* $f \mapsto \|f\|_2 = \left(\int_{-1}^{1} |f(x)|^2 \, dx\right)^{\frac{1}{2}}$ *is a norm on* $C([-1,1])$*, but the vector space* $(C([-1,1]), \|\cdot\|_2)$ *is not a Banach space.*

Indication *we can choose the sequence of functions:*

$$f_n(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq -\frac{1}{n}, \\ nx+1 & \text{if } -\frac{1}{n} \leq x \leq 0, \\ 1 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Exercise 1.8. *Show that the following norms on* $\mathbb{R}^n$ *are* equivalent*:*

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \qquad \|x\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*.*

Exercise 1.9. *Let*

$$E := \{f \in C^1([0,1];\mathbb{R}) : f(0) = 0\}.$$

Define two norms on E *by*

$$\|f\|_1 := \sup_{x\in[0,1]} |\, f(x) + f'(x)\,| \qquad \text{and} \qquad \|f\|_2 := \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|.$$

Show that the norms $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent on* E*.*

Exercise 1.10. ***

We have for all $(x_i)_{1\leq i\leq n}$ *in* $\mathbb{R}^n$ *and for all* $1 \leq p \leq q \leq \infty$*,*

$$\left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \leq n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q}.$$

Exercise 1.11. *Let $(X, \|\cdot\|)$ be a Banach space. Show that every absolutely convergent series in X is convergent in X. Moreover,*

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \|x_n\|.$$

Exercise 1.12. ***

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, and define

$$\|x\| := \sqrt{\langle x, x\rangle}, \quad x \in H.$$

Then $\|\cdot\|$ is a norm on H.

Exercise 1.13. *Let M be a closed subspace of a Hilbert space H. Prove that*

$$\left(M^{\perp}\right)^{\perp} = M.$$

Exercise 1.14. *(a) Show that*

$$M := \{x = (x_n) \in \ell^2 : x_{2n} = 0 \text{ for all } n \in \mathbb{N}\}$$

is a closed subspace of the Hilbert space ℓ^2.

(b) Find $M^{\perp}$.

Exercise 1.15. ** Let $T : X \to Y$ be a linear operator between the normed vector spaces X and Y. Then, the following assertions are equivalent:*

1. *T is continuous;*
2. *T is continuous at 0;*
3. *T is bounded (one may take $x_0 = \dfrac{\delta x}{2\|x\|}$).*

Exercise 1.16 (Norm of a Linear Functional)**.**

(a) Let $X = \ell^1$ and define $T : X \to \mathbb{C}$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}, \quad \text{for } x = (x_n) \in \ell^1.$$

- *(i) Show that T is a bounded linear operator on ℓ^1.*
- *(ii) Compute $\|T\|$ and show that $\|T\| = 1$.*
- *(iii) Determine whether the norm of T is attained.*

(b) Let $X = c_0$, the space of all real (or complex) sequences converging to 0, equipped with the norm $\|\cdot\|_\infty$. Define $T : X \to \mathbb{C}$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

(i) Show that T is a bounded linear operator on c_0.

(ii) Compute $\|T\|$.

(iii) Determine whether the norm of T is attained.

Exercise 1.17. *Let T be the unilateral shift operator on ℓ^2 defined by*

$$T : \ell^2 \to \ell^2, \qquad T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

Prove that T is a bounded linear operator and $\|T\| = 1$.

Exercise 1.18. *Let $a < b$ be real numbers. Consider the Hilbert space $L^2[a, b]$ over $\mathbb{R}$ and the operator $T : L^2[a, b] \to \mathbb{R}$ defined by*

$$Tf = \int_a^b f(x)\,dx, \qquad f \in L^2[a, b].$$

(a) Show that T is bounded. Compute $\|T\|$.

(b) By Riesz's theorem there exists $g \in L^2[a, b]$ such that $Tf = \langle f, g\rangle$ for all $f \in L^2[a, b]$. Find such a g and verify $\|g\|_{L^2} = \|T\|$.

Exercise 1.19. *Let (e_n) be an orthonormal basis of a Hilbert space H and let u be the operator defined by*

$$u(e_n) = e_{n+1}.$$

Show that $\|u + \mathrm{Id}\| = 2$.

Exercise 1.20. *Consider $T \in \mathcal{B}(H)$ a normal operator. Show that*

$$\ker(T) = \ker(T^*).$$

Exercise 1.21. *Let H be a Hilbert space and $A \in \mathcal{B}(H)$ a normal operator. Show that if A is invertible, then there exists a constant $c > 0$ such that*

$$\|A(x)\| \geq c\,\|x\| \quad \textit{for all } x \in H,$$

and conversely.

Correction

Ex 01 Let $x, y \in B(a, r)$ and $\lambda \in [0, 1]$. Then

$$\|\lambda x + (1 - \lambda)y - a\| = \|\lambda x + \lambda a - \lambda a + (1 - \lambda)y - a\|$$

$$\|\lambda x + (1 - \lambda)y - a\| \leq \lambda\|x - a\| + (1 - \lambda)\|y - a\| < \lambda r + (1 - \lambda)r = r.$$

Thus $\lambda x + (1 - \lambda)y \in B(a, r)$. The same reasoning works for $\overline{B}(a, r)$ using $\leq$ instead of $<$. Hence both sets are convex.

Ex 0.2. If $M \subset B(a,r)$ for some $a \in X$ and $r > 0$, then for any $x \in M$, since M is linear, $tx \in M$ for all $t \in \mathbb{R}$. Thus $\|tx - a\| < r$ for all t, which is impossible unless $x = 0$. et us look at what happens as $|t|$ becomes very large. We can write:

$$\|tx - a\| \geq \|tx\| - \|a\| = |t|\,\|x\| - \|a\|.$$

If $x \neq 0$, then $\|x\| > 0$. As $|t| \to \infty$, the term $|t|\,\|x\| - \|a\| \to \infty$. Hence, for large enough t,

$$\|tx - a\| > r.$$

That means $tx \notin B(a,r)$. Hence $M = \{0\}$ is the only linear subspace contained in a ball (open or closed).

Ex 0.3. The function $\|\cdot\|_p$ satisfies positivity, definiteness, and homogeneity, but not the triangle inequality. Indeed, for $x, y \in \ell^p$ we have

$$\|x+y\|_p^p = \sum |x_n + y_n|^p > \sum (|x_n|^p + |y_n|^p)$$

may fail since for $0 < p < 1$, the function $t \mapsto t^p$ is concave. Thus

$$|x_n + y_n|^p \leq |x_n|^p + |y_n|^p$$

does not hold in general, and so the triangle inequality fails. Hence $\|\cdot\|_p$ is not a norm, though ℓ^p remains a vector space.

textbfCounterexample (showing the triangle inequality fails for $0 < p < 1$). Let $0 < p < 1$ and consider the sequences $e_1 = (1, 0, 0, \dots)$ and $e_2 = (0, 1, 0, \dots)$ in ℓ^p. Then

$$\|e_1\|_p = \big(|1|^p\big)^{1/p} = 1, \qquad \|e_2\|_p = 1,$$

while

$$\|e_1 + e_2\|_p = \Big(|1|^p + |1|^p\Big)^{1/p} = 2^{1/p}.$$

Since $0 < p < 1$ we have $1/p > 1$, hence $2^{1/p} > 2$. Therefore

$$\|e_1 + e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p + \|e_2\|_p,$$

which is a direct violation of the triangle inequality. Thus $\|\cdot\|_p$ is not a norm on ℓ

Ex 0.4.

(a) The space $\ell^\infty(\mathbb{R})$ is a Banach space. Let $(u_p)_{p\geq 1}$ be a Cauchy sequence in $\ell^\infty(\mathbb{R})$ with $u_p = (u_p^n)_{n\geq 1}$. For $n \in \mathbb{N}$, we have

$$|u_p^n - u_q^n| \leq \sup_{n\geq 1} |u_p^n - u_q^n| = \|u_p - u_q\|_\infty.$$

This implies that, $(u_p^n)_{n\geq 1}$ is a Cauchy sequence. As $\mathbb{R}$ is complete, this sequence converges and we can write

$$u_n = \lim_{p\to\infty} u_p^n.$$

We will show that this sequence $(u_n)_{n\geq 1}$ belongs to the space $l^\infty(\mathbb{R})$ because the sequence $(u_p)_{p\geq 1}$ converges to $(u_n)_{n\geq 1}$.

As $(u_p)_{p\geq 1}$ is a Cauchy sequence, hence it is bounded and there exists $M \in \mathbb{R}^+$ such that

$$\forall p \in N, \qquad \|u_p\|_\infty \leq M.$$

Let n be in $\mathbb{N}$. One has

$$|u_n| = \lim_{p \longrightarrow \infty} |u_p^n|.$$

For each fixed n and p, we have

$$|u_n^p| \leq \|u^p\|_\infty \leq M$$

Taking the limit as $p \to \infty$, and using the continuity of the absolute value function,

$$|u_n| = \lim_{p\to\infty} |u_p^n| \leq M.$$

Since this holds for all n,

$$\sup_{n\geq 1} |u_n| \leq M,$$

therefore, $(u_n)_{n\geq 1}$ is an element of the space $l^\infty(\mathbb{R})$.
Consider $\epsilon > 0$. Since (u_p) is Cauchy, there is $N \in \mathbb{N}$ such that

$$\forall p, q > N, \quad \|u_p - u_q\|_\infty \leq \epsilon.$$

This implies for all $n \geq 1$

$$|u_p^n - u_q^n| \leq \epsilon.$$

Now, fix $p > N$ and $n \in \mathbb{N}$. Taking the limit as $q \to +\infty$, we get

$$|u_n - u_p^n| \leq \epsilon$$

and this holds for all $n \in \mathbb{N}$. Thus

$$\|u - u_p\|_\infty = \sup_{n\geq 1} |u_p^n - u_n| \leq \epsilon.$$

Hence, $u_p \to u$ in $\ell^\infty(\mathbb{R})$. Therefore, $\ell^\infty(\mathbb{R})$ is complete and thus a Banach space.

(b) If $(x^{(n)}) \subset c_0$ and $x^{(n)} \to x$ in $\|\cdot\|_\infty$, then $\|x^{(n)} - x\|_\infty \to 0$. Thus $x_n^{(n)} \to x_n$ uniformly in n, and since $x^{(n)} \in c_0$, we have $\lim_{k\to\infty} x_k = 0$. Hence $x \in c_0$, and c_0 is closed.

(c) We observe that $c_{00} \subset c_0 \subset \ell^\infty$.

(a) Consider the sequence $x^{(n)}$ defined by

$$x^{(n)} = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, 0, 0, \ldots) \in c_{00}.$$

Then, for $n, m \geq N$,

$$\|x^{(n)} - x^{(m)}\|_\infty = \begin{cases} \frac{1}{m+1}, & \text{if } n \geq m, \\ \frac{1}{n+1}, & \text{if } n \leq m. \end{cases}$$

In both cases, for any $N > 0$, we have

$$\|x^{(n)} - x^{(m)}\|_\infty \leq \frac{1}{N+1} \quad \text{for all } n, m \geq N.$$

So the sequence $(x^{(n)})$ is a Cauchy sequence in c_{00}. Evidently, it is also a Cauchy sequence in ℓ^∞.

Now consider the sequence

$$x = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \ldots) \in \ell^\infty.$$

We have $x^{(n)} \to x$ in ℓ^∞, but $x \notin c_{00}$ (since $x_n \neq 0$ for all n).

Now consider the sequence

$$x = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right) \in \ell^\infty.$$

For each fixed k and each n,

$$x_k^{(n)} = \begin{cases} \frac{1}{k}, & k \leq n, \\ 0, & k > n. \end{cases}$$

Thus for every fixed k we have $x_k^{(n)} \to \frac{1}{k} = x_k$ as $n \to \infty$. Hence the pointwise limit of $x^{(n)}$ is x.

$$\|x^{(n)} - x\|_\infty = \sup_{k \geq 1} \left| x_k^{(n)} - x_k \right|.$$

If $k \leq n$ then $x_k^{(n)} - x_k = 0$. If $k > n$ then $|x_k^{(n)} - x_k| = \left|0 - \frac{1}{k}\right| = \frac{1}{k}$. The maximum of $\frac{1}{k}$ for $k > n$ is attained at $k = n + 1$, hence

$$\|x^{(n)} - x\|_\infty = \frac{1}{n+1} \xrightarrow{n \to \infty} 0.$$

Therefore $x^{(n)} \to x$ in the norm of ℓ^∞.

By definition c_{00} is the set of finitely supported sequences (sequences that are zero from some index onward). The sequence

$$x = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\right)$$

has infinitely many nonzero terms (it never becomes identically zero after some index), so $x \notin c_{00}$. Since $x^{(n)} \in c_{00}$ for all n, we have produced a Cauchy sequence in c_{00} (with respect to $\|\cdot\|_\infty$) that converges in ℓ^∞ to an element not in c_{00}. Hence c_{00} is not complete (equivalently, c_{00} is not closed in ℓ^∞).

We have $x^{(n)} \to x$ in ℓ^∞. However, we claim that $x \notin c_{00}$.

(b) The closure of c_{00} in ℓ^∞ is c_0, the space of all sequences converging to zero.

Ex 0.6 The mapping $f \mapsto \|f\|_2 = \left(\int_{-1}^{1} |f(x)|^2 dx\right)^{1/2}$ is a norm on $C([-1,1])$, but $(C([-1,1]), \|\cdot\|_2)$ is not a Banach space.

(1) Norm properties. For $f, g \in C([-1,1])$, $\|f\|_2 = 0$ implies $f(x) = 0$ for all x. Homogeneity is clear, and the triangle inequality follows from Minkowski's inequality. Hence $\|\cdot\|_2$ is a norm.

(2) Non-completeness. Define

$$f_n(x) = \begin{cases} 0, & -1 \le x \le -\frac{1}{n}, \\ nx+1, & -\frac{1}{n} \le x \le 0, \\ 1, & 0 \le x \le 1. \end{cases}$$

Each f_n is continuous, and (f_n) is Cauchy in $\|\cdot\|_2$, since

$$\|f_n - f_m\|_2^2 = \int_{-1}^{0} |f_n(x) - f_m(x)|^2 dx \to 0.$$

However, the pointwise limit

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0, \end{cases}$$

is not continuous. Thus $f \notin C([-1,1])$, so $C([-1,1])$ is not complete under $\|\cdot\|_2$.

Ex07 First note the easy inequality

$$|f(x) + f'(x)| \le |f(x)| + |f'(x)| \le \|f\|_2,$$

so taking suprema over $x \in [0,1]$ gives

$$\|f\|_1 \le \|f\|_2.$$

Thus one side of equivalence holds with constant 1.

For the other direction, we have

$$\|f\|_2 = \|f\|_1 + \|f'\|_1.$$

Calculation of $\|f'\|_\infty$:

$$\|f'\|_\infty = \sup_{x\in[0,1]} |f'(x)| = \sup_{x\in[0,1]} |f'(x) + f(x) - f(x)| \le \sup_{x\in[0,1]} |f'(x) + f(x)| + \sup_{x\in[0,1]} |f(x)| \le \|f\|_\infty + \|f\|_1$$

Calculation of $\|f\|_\infty$:

The function f is continuous on $[0,1]$, hence it is bounded and attains its supremum. Let $x_0 \in [0,1]$ be such that $\|f\|_1 = |f(x_0)|$.

- If $0 < x_0 < 1$, then $f'(x_0) = 0$, and therefore

$$\|f\|_\infty = |f(x_0)| = |f(x_0) + f'(x_0)| \le \|f\|_1.$$

- If $x_0 = 1$, then f and f' have the same sign on some interval $[1-\varepsilon, 1]$. On this interval we have $|f(x)| \le |f(x) + f'(x)|$, hence

$$\|f\|_\infty = |f(1)| \le \|f\|_1.$$

- If $x_0 = 0$, then $f(0) = 0$, which implies $f \equiv 0$ in a neighborhood of 0, and the inequality is obvious.

Consequently,

$$\|f\|_2 = \|f\|_1 + \|f'\|_1 \le \|f\|_1 + 2\|f\|_1 = 3\|f\|_1.$$

Hence the two norms are equivalent:

$$\|f\|_1 \le \|f\|_2 \le 3\|f\|_1, \qquad \forall f \in E.$$

Ex 0.8

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We recall the three norms:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \qquad \|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

Step 1. Show that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.

Indeed, since $\|x\|_\infty = \max_i |x_i|$, we have for each i:

$$|x_i|^2 \le \|x\|_\infty^2.$$

Summing over i gives

$$\sum_{i=1}^{n} |x_i|^2 \le n\,\|x\|_\infty^2,$$

and hence

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \le \sqrt{n}\,\|x\|_\infty.$$

But also, since at least one coordinate satisfies $|x_i| = \|x\|_\infty$, we get

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \ge |x_i| = \|x\|_\infty.$$

Thus

$$\boxed{\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty.}$$

Step 2. Show that $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$.

First, since $|x_i|^2 \le |x_i|\,|x_i|$, it follows that

$$\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \le \sum_{i=1}^{n} |x_i|,$$

so $\|x\|_2 \le \|x\|_1$.

For the other direction, by the Cauchy–Schwarz inequality,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \le \sqrt{n} \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{n}\, \|x\|_2.$$

Thus,

$$\boxed{\|x\|_2 \le \|x\|_1 \le \sqrt{n}\, \|x\|_2.}$$

Step 3. Combine the estimates.

From the above we have:

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\, \|x\|_\infty.$$

Hence, all three norms are equivalent on $\mathbb{R}^n$.

Ex08

Let $1 \le p \le q < \infty$. Then

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r} \quad \Longrightarrow \quad 1 = \frac{p}{q} + \frac{p}{r} \quad \Longrightarrow \quad 1 = \frac{1}{q/p} + \frac{1}{r/p}.$$

By Hölder's inequality, we have

$$\sum_{i=1}^{n} 1 \cdot |x_i|^p \le \left(\sum_{i=1}^{n} 1^{\,r/p} \right)^{p/r} \left(\sum_{i=1}^{n} \left(|x_i|^p\right)^{q/p} \right)^{p/q}.$$

That is,

$$\sum_{i=1}^{n} |x_i|^p \le n^{p/r} \left(\sum_{i=1}^{n} |x_i|^q \right)^{p/q}.$$

This implies

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \le n^{1/r} \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} = n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q}.$$

For the converse, consider $x \in S_{\ell_p^n}$ (i.e., $\|x\|_p = 1$). We show that $\|x\|_q \le 1$.

Indeed, for every i, we have $|x_i| \le 1$, and hence

$$\sum_{i=1}^{n} |x_i|^q \le \sum_{i=1}^{n} |x_i|^p.$$

Therefore,

$$\left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} \le 1.$$

Now, let $x_0 = \dfrac{x}{\|x\|_p}$. By homogeneity of the norm,

$$\|x_0\|_p = \left\| \frac{x}{\|x\|_p} \right\|_p = \frac{\|x\|_p}{\|x\|_p} = 1.$$

This implies that $\|x_0\|_q \leq 1$, and consequently

$$\frac{\|x\|_q}{\|x\|_p} \leq 1.$$

Ex09 Let (x_n) be a sequence in X such that the series

$$\sum_{n=1}^{\infty} \|x_n\|$$

is convergent. Then, by definition of convergence in $\mathbb{R}$, the sequence of partial sums

$$s_n = \sum_{k=1}^{n} \|x_k\|$$

is Cauchy.

We now show that the sequence of partial sums

$$S_n = \sum_{k=1}^{n} x_k$$

is Cauchy in X. For $p > q$, we have

$$\|S_p - S_q\| = \left\| \sum_{k=q+1}^{p} x_k \right\| \leq \sum_{k=q+1}^{p} \|x_k\|.$$

Since (s_n) is Cauchy in $\mathbb{R}$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|s_p - s_q| = \sum_{k=q+1}^{p} \|x_k\| < \varepsilon \quad \text{whenever } p, q \geq N.$$

Hence,

$$\|S_p - S_q\| < \varepsilon \quad \text{for all } p, q \geq N.$$

Therefore, (S_n) is a Cauchy sequence in X. Since X is complete (i.e., a Banach space), there exists $S \in X$ such that

$$S_n \to S.$$

Thus, the series $\sum_{n=1}^{\infty} x_n$ converges in X.

Finally, the inequality

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \|x_n\|$$

follows from the triangle inequality applied to the partial sums:

$$\left\| \sum_{k=1}^{n} x_k \right\| \leq \sum_{k=1}^{n} \|x_k\|,$$

and then passing to the limit as $n \to \infty$.

Ex10 We only need to verify the triangle inequality. For all $x, y \in H$, we have

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y\rangle \\
&\le \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| \quad \text{(by the Cauchy–Schwarz inequality)} \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

Taking square roots on both sides gives

$$\|x+y\| \le \|x\| + \|y\|,$$

which is the triangle inequality. Hence $\|\cdot\|$ is a norm on H.

Ex11 In general, if M is a subset of H, then $M \subset (M^\perp)^\perp$. Indeed,

$$M^\perp := \{x \in H : x \perp M\}.$$

So we have

$$x \in M \Rightarrow x \perp M^\perp \Rightarrow x \in (M^\perp)^\perp.$$

Now suppose M is a closed subspace of H and $x \in (M^\perp)^\perp$. Since $x \in H = M \oplus M^\perp$, we can write

$$x = u + v, \quad u \in M,\ v \in M^\perp.$$

Since $M \subset (M^\perp)^\perp$, we have

$$x = u + v, \quad u \in (M^\perp)^\perp,\ v \in M^\perp.$$

Because $x - u \in (M^\perp)^\perp$ and $v \in M^\perp$ with $v = x - u$, we obtain

$$v \in M^\perp \cap (M^\perp)^\perp.$$

This implies $v = 0$. Hence, $x = u \in M$.

Ex12 (a) *Linear subspace.* If $x = (x_n), y = (y_n) \in M$ and $\alpha, \beta \in \mathbb{C}$, then for every n,

$$(\alpha x + \beta y)_{2n} = \alpha x_{2n} + \beta y_{2n} = 0 + 0 = 0,$$

so $\alpha x + \beta y \in M$. Thus M is a linear subspace of ℓ^2.

Closedness. Let $(x^{(k)})_{k\ge 1}$ be a sequence in M with $x^{(k)} \to x$ in the ℓ^2-norm. For each fixed coordinate index m the map

$$\ell^2 \to \mathbb{C}, \qquad z = (z_n) \mapsto z_m$$

is a continuous linear functional (indeed $|z_m| \le \|z\|_{\ell^2}$). Hence $x_m^{(k)} \to x_m$ for every m. In particular, for every $n \in \mathbb{N}$ we have $x_{2n}^{(k)} = 0$ for all k, so the limit satisfies $x_{2n} = 0$. Therefore $x \in M$ and M is closed.

(b) Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on ℓ^2,

$$\langle x, y \rangle = \sum_{m=1}^{\infty} x_m \overline{y_m}.$$

We compute the orthogonal complement $M^\perp = \{y \in \ell^2 : \langle x, y \rangle = 0 \text{ for all } x \in M\}$.

If $y \in M^\perp$, then for each fixed odd index $2n-1$ choose the vector $x \in M$ defined by $x_{2n-1} = 1$ and $x_m = 0$ for $m \neq 2n-1$ (this x belongs to M since all even coordinates are 0). Then

$$0 = \langle x, y \rangle = 1 \cdot \overline{y_{2n-1}},$$

so $y_{2n-1} = 0$ for every n. Thus every $y \in M^\perp$ has all odd entries zero; equivalently

$$M^\perp \subseteq \{y = (y_m) \in \ell^2 : y_{2n-1} = 0 \text{ for all } n\}.$$

Conversely, any $y \in \ell^2$ with $y_{2n-1} = 0$ for all n is supported only on even indices. For such a y and any $x \in M$ (which is supported only on odd indices), the inner product satisfies

$$\langle x, y \rangle = \sum_{m=1}^{\infty} x_m \overline{y_m} = 0$$

because at every coordinate at least one factor is zero. Hence every such y belongs to $M^\perp$. Therefore

$$M^\perp = \{y = (y_m) \in \ell^2 : y_{2n-1} = 0 \text{ for all } n\},$$

i.e. the subspace of ℓ^2 consisting of sequences supported on even indices.

Let $M = \{x = (x_n) \in \ell^2 : x_{2n} = 0 \ \forall n \geq 0\}$. Then

$$z \in M^\perp \iff \langle z, x \rangle = 0 \quad \text{for all } x \in M.$$

Writing the inner product and using that every $x \in M$ has only odd-index coordinates possibly nonzero,

$$0 = \langle z, x \rangle = \sum_{n=0}^{\infty} z_{2n+1} \overline{x_{2n+1}}$$

for all choices of scalars $(x_{2n+1})_{n \geq 0}$ with $\sum_{n=0}^{\infty} |x_{2n+1}|^2 < \infty$. This implies $z_{2n+1} = 0$ for every $n \geq 0$ (otherwise one could choose $x_{2n+1} \neq 0$ supported at that index). Hence

$$M^\perp = \{z = (z_n) \in \ell^2 : z_{2n+1} = 0 \text{ for all } n \geq 0\},$$

i.e. the subspace of ℓ^2 supported on even indices.

Ex14 [Norm attained] Let $X = \ell^1$ and define $T : X \to \mathbb{C}$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}, \quad \text{for } x = (x_n) \in \ell^1.$$

Then T is a linear operator and

$$\|T\| = 1.$$

Indeed, for any $x \in \ell^1$,

$$|T(x)| = \left|\sum_{n=1}^{\infty} \frac{x_n}{n}\right| \leq \sum_{n=1}^{\infty} \frac{|x_n|}{n} \leq \|x\|_{\ell^1}.$$

Hence $\|T\| \leq 1$. But for $x = e_1 = (1, 0, 0, \ldots)$, we have

$$T(e_1) = 1,$$

so $\|T\| = 1$ and the norm is attained at e_1.

[Norm not attained] Let $X = c_0$ equipped with the norm $\|\cdot\|_\infty$. Define $T : X \to \mathbb{C}$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Then T is linear, and since

$$|T(x)| \leq \sum_{n=1}^{\infty} \frac{|x_n|}{2^n} \leq \|x\|_\infty,$$

we have $T \in \mathcal{B}(X, \mathbb{C})$ and $\|T\| \leq 1$.

On the other hand, for each $n \geq 1$, note that

$$\|e_1 + \cdots + e_n\|_\infty = 1 \quad \text{and} \quad T(e_1 + \cdots + e_n) = \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}. \text{ geomertic sequence}$$

Hence,

$$\|T\| \geq 1 - \frac{1}{2^n}.$$

Since n is arbitrary, it follows that $\|T\| \geq 1$, and therefore $\|T\| = 1$.

Now let $x \in c_0$ with $\|x\|_\infty \leq 1$. Then there exists $N \geq 1$ such that $|x_n| < \frac{1}{2}$ for all $n \geq N$. We have

$$|T(x)| \leq \sum_{i=1}^{N} \frac{|x_i|}{2^i} + \frac{1}{2} \sum_{i=N+1}^{\infty} \frac{1}{2^i} < 1 = \|T\|.$$

Split the series at N to get the estimate

$$|T(x)| \leq \sum_{i=1}^{N-1} \frac{|x_i|}{2^i} + \sum_{i=N}^{\infty} \frac{|x_i|}{2^i} \leq \sum_{i=1}^{N-1} \frac{|x_i|}{2^i} + \frac{1}{2} \sum_{i=N}^{\infty} \frac{1}{2^i}.$$

Using the trivial bound $|x_i| \leq 1$ for the first finite sum yields

$$|T(x)| \leq \sum_{i=1}^{N-1} \frac{1}{2^i} + \frac{1}{2} \sum_{i=N}^{\infty} \frac{1}{2^i} = \left(1 - \frac{1}{2^{N-1}}\right) + \frac{1}{2} \cdot \frac{1}{2^{N-1}} = 1 - \frac{1}{2^N} < 1.$$

Consequently, $|T(x)| < \|T\|$ for all x in the unit ball B_{c_0}, and hence the norm of T is *not attained.*
Ex15 Linearity of T is immediate from its coordinatewise definition.

For $x = (x_1, x_2, x_3, \dots) \in \ell^2$ we compute

$$\|Tx\|_{\ell^2}^2 = |0|^2 + |x_1|^2 + |x_2|^2 + \cdots = \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_{\ell^2}^2.$$

Hence $\|Tx\|_{\ell^2} = \|x\|_{\ell^2}$ for every $x \in \ell^2$, so T is bounded and in fact an isometry. In particular,

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \|x\| = 1.$$

(Alternatively, note that for the standard orthonormal basis vectors e_n we have $\|e_n\| = 1$ and $Te_n = e_{n+1}$ so $\|Te_n\| = 1$, which shows $\|T\| \geq 1$, combined with the equality $\|Tx\| = \|x\|$ gives $\|T\| = 1$.)

Ex16 (a) Linearity of T is immediate. To show boundedness and compute the operator norm, apply the Cauchy–Schwarz inequality: for any $f \in L^2[a, b]$,

$$|Tf| = \left| \int_a^b f(x)\, dx \right| = \left| \int_a^b f(x) \cdot 1\, dx \right| \leq \|f\|_{L^2[a,b]} \, \|1\|_{L^2[a,b]} = \|f\|_{L^2[a,b]} \sqrt{b-a}.$$

Thus T is bounded and $\|T\| \leq \sqrt{b-a}$.

To see equality, take the function $f_0 \equiv \dfrac{1}{\sqrt{b-a}}$ (which has $\|f_0\|_{L^2} = 1$). Then

$$Tf_0 = \int_a^b \frac{1}{\sqrt{b-a}}\, dx = \frac{b-a}{\sqrt{b-a}} = \sqrt{b-a},$$

so $\|T\| \geq \sqrt{b-a}$. Hence $\|T\| = \sqrt{b-a}$.

(b) Consider the constant function $g \equiv 1$ on $[a, b]$. For every $f \in L^2[a, b]$,

$$\langle f, g \rangle_{L^2} = \int_a^b f(x)\, g(x)\, dx = \int_a^b f(x)\, dx = Tf,$$

so $g = 1$ represents T via the Riesz representation. Its L^2 norm is

$$\|g\|_{L^2[a,b]} = \left(\int_a^b 1^2\, dx \right)^{1/2} = \sqrt{b-a}.$$

Therefore $\|g\|_{L^2} = \sqrt{b-a} = \|T\|$, as required.

Problem sup. Let X, Y be normed spaces and let $T \in B(X, Y)$. Consider the following statement:

$$T \text{ is an isometry } \iff \|T\| = 1.$$

Do you agree with it?

[Solution] The statement is *false in general.*

($\Rightarrow$) Suppose T is an isometry. Then, by definition,

$$\|Tx\| = \|x\| \quad \text{for all } x \in X.$$

Hence

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \|x\| = 1.$$

Thus, every isometry has operator norm 1.

($\Leftarrow$) The converse is *not true.* The condition $\|T\| = 1$ only means that

$$\|Tx\| \leq \|x\| \quad \text{for all } x \in X,$$

and equality holds for at least one x with $\|x\| = 1$, but not necessarily for all x.

Counterexample. Let $X = Y = \mathbb{R}^2$ with the Euclidean norm and define

$$T(x, y) = (x, 0).$$

Then

$$\|T(x, y)\| = |x| \leq \sqrt{x^2 + y^2} = \|(x, y)\|,$$

so $\|T\| = 1$. However, T is not an isometry because, for example,

$$\|T(0, 1)\| = 0 \neq 1 = \|(0, 1)\|.$$

Conclusion. We always have

$$T \text{ is an isometry } \Rightarrow \|T\| = 1, \quad \text{but the converse does not hold.}$$

Ex17 It is clear that u is an isometry. Hence,

$$\|u + \mathrm{Id}\| \leq \|u\| + \|\mathrm{Id}\| = 1 + 1 = 2.$$

To show equality, we construct a sequence of unit vectors (x_n) such that $\|(u + \mathrm{Id})x_n\| \to 2$ as $n \to \infty$.

Let x_n be the vector whose coordinates in the basis (e_k) are given by

$$x_n = \frac{1}{\sqrt{n}}(e_1 + e_2 + \cdots + e_n).$$

Then $\|x_n\| = 1$.

We have

$$u(x_n) = \frac{1}{\sqrt{n}}(e_2 + e_3 + \cdots + e_{n+1}),$$

and therefore

$$(u + \mathrm{Id})(x_n) = \frac{1}{\sqrt{n}}(2e_2 + 2e_3 + \cdots + 2e_n + e_1 + e_{n+1}).$$

Hence,

$$\|(u + \mathrm{Id})(x_n)\|^2 = \frac{1}{n}\big(4(n-1) + 2\big) = \frac{1}{n}(4n - 2),$$

and thus

$$\|(u + \mathrm{Id})(x_n)\| = \sqrt{\frac{4n-2}{n}} = 2\sqrt{1 - \frac{1}{2n}}.$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|(u + \mathrm{Id})(x_n)\| = 2.$$

Therefore,

$$\|u + \mathrm{Id}\| = 2.$$

Ex18 Let $x \in \ker(T)$. Then $T(x) = 0$, and hence

$$\|T^*(x)\|^2 = \langle T^*(x), T^*(x)\rangle = \langle TT^*(x), x\rangle \quad \text{(since } T \text{ is normal)} = \langle T^*T(x), x\rangle = \langle T^*(0), x\rangle = 0.$$

Therefore, $T^*(x) = 0$, and so $x \in \ker(T^*)$. Thus,

$$\ker(T) \subseteq \ker(T^*).$$

Now, since T is normal, T^* is also normal. Applying the same reasoning to T^*, we obtain

$$\ker(T^*) \subseteq \ker(T^{**}) = \ker(T).$$

Hence,

$$\ker(T) = \ker(T^*).$$

Ex19 (⇒) Suppose that A is invertible. For any $x \in H$, we can write

$$\|x\| = \|A^{-1}A(x)\| \leq \|A^{-1}\|\,\|A(x)\|.$$

Hence,

$$\|A(x)\| \geq \frac{1}{\|A^{-1}\|}\,\|x\|.$$

Therefore, the inequality holds with $c = \dfrac{1}{\|A^{-1}\|} > 0$.

(⇐) Conversely, assume there exists $c > 0$ such that

$$\|A(x)\| \geq c\,\|x\| \quad \forall\, x \in H.$$

Then $\ker(A) = \{0\}$, because if $A(x) = 0$, this inequality gives $\|x\| = 0$, so $x = 0$.

Since A is normal, we have

$$\ker(A) = \ker(A^*) = (\operatorname{Im} A)^{\perp}.$$

Thus, $\operatorname{Im} A$ is dense in H.

To show that $\operatorname{Im} A$ is also closed, let $(A(x_n))$ be a Cauchy sequence in $\operatorname{Im} A$. Then for m, n large,

$$\|A(x_n - x_m)\| \to 0.$$

By the assumed inequality, we have

$$c\,\|x_n - x_m\| \leq \|A(x_n - x_m)\| \to 0,$$

so (x_n) is a Cauchy sequence in H and hence converges to some $x \in H$. By continuity of A, $A(x_n) \to A(x)$, which shows that $\operatorname{Im} A$ is closed.

Since $\operatorname{Im} A$ is both dense and closed in H, it follows that $\operatorname{Im} A = H$; hence A is surjective.

Finally, A is bijective and bounded, so by the inverse mapping theorem, A is invertible.

Exercise Sheet 2– Fundamentals of Functional Analysis 01

2 Baire's Theorem and their consequences

Exercise 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{R}$ and let $(x_n)_{n\in\mathbb{N}} \subset X$.*

1. *Assume that for every $\varphi \in X^*$ there exists a constant $M_\varphi > 0$ (depending on φ) such that*

$$|\varphi(x_n)| \leq M_\varphi \qquad \text{for all } n \in \mathbb{N}.$$

   Using the uniform boundedness principle, prove that there exists $C > 0$ with $\|x_n\| \leq C$ for all n.

2. *Deduce from (1) that any weakly convergent sequence in a normed space X is norm-bounded.*

Exercise 2.2. *Let $\mathbb{R}[X]$ be the vector space of real polynomials, viewed as a subspace of $C(\mathbb{R})$. Show that* no *norm on $\mathbb{R}[X]$ makes it a Banach space; equivalently, every normed topology on $\mathbb{R}[X]$ is incomplete.*

Exercise 2.3. *Let X be a Banach space and let $T : X \to X$ be a bounded (continuous) linear operator such that*

$$\forall x \in X \ \exists n \in \mathbb{N} : \ T^n(x) = 0.$$

Prove that T is nilpotent; that is, show that there exists $N \in \mathbb{N}$ with $T^N = 0$.

Exercise 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

1. *Prove that the graph $G(f) = \{(x, f(x)) : x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$.*
2. *Explain why f is not continuous.*

Exercise 2.5 (A generalization of the Hellinger–Toeplitz theorem)**.** *Let T be a linear operator on a Hilbert space H. Suppose there exists a map $\widetilde{T} : H \to H$ such that*

$$\forall x, y \in H, \qquad \langle \widetilde{T}(x), y\rangle = \langle x, \widetilde{T}(y)\rangle.$$

Prove that T is continuous.

Correction

Ex2.1

(1) For each $n \in \mathbb{N}$ define the linear functional

$$T_n : X^* \to \mathbb{R}, \qquad T_n(\varphi) = \varphi(x_n).$$

Each T_n is linear. Moreover, for every $\varphi \in X^*$ we have

$$|T_n(\varphi)| = |\varphi(x_n)| \leq \|\varphi\| \, \|x_n\|,$$

so T_n is bounded and $\|T_n\| \leq \|x_n\|$. Conversely, by the dual norm representation

$$\|x_n\| = \sup_{\|\varphi\|=1} |\varphi(x_n)| = \sup_{\|\varphi\|=1} |T_n(\varphi)| = \|T_n\|.$$

Thus $\|T_n\| = \|x_n\|$ for every n.

By hypothesis, for each fixed $\varphi \in X^*$ the set $\{T_n(\varphi) : n \in \mathbb{N}\}$ is bounded (indeed $|T_n(\varphi)| \leq M_\varphi$). The Banach–Steinhaus (uniform boundedness) theorem applies because the domain X^* is a Banach space. Hence the family $\{T_n\}$ is uniformly bounded: there exists $C > 0$ such that $\|T_n\| \leq C$ for all n. Using $\|T_n\| = \|x_n\|$ we obtain $\|x_n\| \leq C$ for all n, as required.

(2) If (x_n) converges weakly to some $x \in X$, then for every $\varphi \in X^*$ the scalar sequence $\varphi(x_n)$ converges to $\varphi(x)$ and in particular is bounded. Thus the hypothesis of part (1) is satisfied, so (x_n) is norm-bounded.

Ex 2.2

Write the usual countable algebraic basis of $\mathbb{R}[X]$ as

$$\{1, x, x^2, \ldots, x^n, \ldots\},$$

and set for each $n \geq 0$

$$X_n := \operatorname{span}\{1, x, \ldots, x^n\},$$

so each X_n is finite-dimensional and

$$\mathbb{R}[X] = \bigcup_{n=0}^{\infty} X_n.$$

Fix any norm $\| \cdot \|$ on $\mathbb{R}[X]$ and regard $(\mathbb{R}[X], \| \cdot \|)$ as a normed linear space. We will show it cannot be complete.

Facts we will use:

- Every finite-dimensional subspace of a normed space is closed.
- A finite-dimensional proper subspace of an infinite-dimensional normed space has empty interior.

- Baire's category theorem: a complete metric space (equivalently a Banach space) is not a countable union of closed sets with empty interior.

Each X_n is finite-dimensional, hence closed in $(\mathbb{R}[X], \|\cdot\|)$. Also $X_n \neq \mathbb{R}[X]$, so $\mathrm{Int}(X_n) = \varnothing$. Because

$$\mathbb{R}[X] = \bigcup_{n=0}^{\infty} X_n$$

is a countable union of closed sets with empty interior, Baire's theorem shows that $(\mathbb{R}[X], \|\cdot\|)$ cannot be complete. Therefore no norm on $\mathbb{R}[X]$ turns it into a Banach space.

Ex 2.3

For each $n \in \mathbb{N}$ set

$$\ker T^n = \{x \in X : \ T^n(x) = 0\}.$$

By hypothesis every $x \in X$ is killed by some power of T, hence

$$X = \bigcup_{n=1}^{\infty} \ker T^n.$$

Each $\ker T^n$ is a closed linear subspace of X because T^n is continuous. Since X is a (complete) Banach space, Baire's category theorem implies that at least one of the closed sets $\ker T^N$ has nonempty interior for some $N \in \mathbb{N}$.

Let $M := \ker T^N$ and suppose $\mathrm{Int}(M) \neq \varnothing$. Because M is a linear subspace, an interior point forces $M = X$: indeed, pick $u \in \mathrm{Int}(M)$. Then $u + U \subset M$ for some neighbourhood U of 0, so U itself (after translation) contains a neighbourhood of 0 contained in M. Hence there exists $\varepsilon > 0$ such that the ball $B(0, \varepsilon) \subset M$. For any $x \in X$ choose $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $0 < |\lambda| < \varepsilon/\|x\|$; then $\lambda x \in B(0, \varepsilon) \subset M$, and since M is a subspace it follows that $x = (1/\lambda)(\lambda x) \in M$. Thus $M = X$.

Therefore $\ker T^N = X$, so $T^N = 0$ on X. This shows T is nilpotent.

Ex 2.4

(1) $G(f)$ is closed.

We show that $G(f)$ contains the limit of every convergent sequence of its points. Let $((x_n, f(x_n)))_{n\geq 1}$ be a sequence in $G(f)$ and suppose

$$(x_n, f(x_n)) \longrightarrow (x, y) \quad \text{in } \mathbb{R}^2.$$

Then in particular $x_n \to x$ and $f(x_n) \to y$ as $n \to \infty$.

If $x \neq 0$, then for n large enough $x_n \neq 0$ and the function $x \mapsto 1/x$ is continuous at x, hence

$$y = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x} = f(x),$$

so $(x, y) = (x, f(x)) \in G(f)$.

If $x = 0$, then $x_n \to 0$. If infinitely many x_n are nonzero, then $|f(x_n)| = |1/x_n| \to \infty$, which contradicts the assumption that $f(x_n) \to y$ is finite. Therefore eventually $x_n = 0$, so for all sufficiently large n we have $f(x_n) = 0$, and thus $y = \lim f(x_n) = 0 = f(0)$. Hence again $(x, y) = (0, f(0)) \in G(f)$.

In all cases the limit point (x, y) belongs to $G(f)$, so $G(f)$ is closed.

(2) f is not continuous.

The function is clearly continuous at every $x \neq 0$ because it coincides there with the continuous map $x \mapsto 1/x$. At $x = 0$ it is not continuous: if f were continuous at 0 then for every sequence $x_n \to 0$ we would have $f(x_n) \to f(0) = 0$. But take $x_n = \dfrac{1}{n}$; then $x_n \to 0$ while

$$f(x_n) = \frac{1}{x_n} = n \longrightarrow \infty \neq 0,$$

a contradiction. Hence f is not continuous at 0 (and therefore not continuous on $\mathbb{R}$).

Ex 2.5

We shall show that the graph of T is closed and then apply the closed graph theorem.

Let $(x_n)_{n \geq 1}$ be a sequence in H with $x_n \to x$ and $T(x_n) \to y$ in H. We must prove $y = T(x)$.

Compute

$$\|y - T(x)\|^2 = \langle y - T(x),\, y - T(x) \rangle.$$

Since $T(x_n) \to y$ and $T(x_n) - T(x) \to y - T(x)$, we may write

$$\|y - T(x)\|^2 = \lim_{n \to \infty} \langle T(x_n) - T(x),\, y - T(x) \rangle.$$

Using the hypothesis and linearity of the inner product,

$$\langle T(x_n) - T(x),\, y - T(x) \rangle = \langle x_n - x,\, \widetilde{T}(y - T(x)) \rangle.$$

Because $x_n \to x$, the right-hand side tends to 0 as $n \to \infty$. Hence

$$\|y - T(x)\|^2 = 0,$$

so $y = T(x)$. This shows the graph of T is closed.

Finally, since T is a linear operator between Banach spaces (Hilbert spaces are complete), the closed graph theorem implies that T is bounded (continuous).

3 Hahn–Banach Theorems and its Consequences

Exercise 3.1 (Strict separation in $\mathbb{R}^2$). *Let*

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, \qquad B = \{(2, 0)\}.$$

Find explicitly a linear functional that strictly separates A and B.

definition Let E be a vector space over the field $\mathbb{F}$. A *semi-norm* on E is a function

$$p : E \longrightarrow \mathbb{R}$$

satisfying, for all $x, y \in E$ and $\lambda \in \mathbb{F}$, the following conditions:

1. $p(\lambda x) = |\lambda| \, p(x)$,
2. $p(x + y) \leq p(x) + p(y)$.

Let $\mathcal{P}$ be a family of semi-norms on E. We say that $\mathcal{P}$ is *separating* if

$$p(x) = 0 \quad \forall p \in \mathcal{P} \implies x = 0.$$

Exercise 3.2. *Let p be a semi-norm on E. Prove that:*

1. $p(0) = 0$*;*
2. $p(x) \geq 0$ *for all* $x \in E$*;*
3. *the set* $p^{-1}(\{0\}) = \{x \in E : p(x) = 0\}$ *is a subspace of* E*;*
4. *for all* $x, y \in E$*,*
$$|p(x) - p(y)| \leq p(x - y).$$

Exercise 3.3. ** Let u be a non-trivial linear form on E. Show that*

$$p(x) = |u(x)|$$

defines a semi-norm on E.

Exercise 3.4. *Let X, Y be two Banach spaces and $T : X \to Y$ be a linear operator.*

1. *Prove the equivalence of the following properties:*
    - *(a)* $T : (X, \| \cdot \|) \to (Y, \| \cdot \|)$ *is continuous.*
    - *(b)* $T : (X, \sigma(X, X^*)) \to (Y, \sigma(Y, Y^*))$ *is continuous.*

   (c) $T : (X, \|\cdot\|) \to (Y, \sigma(Y, Y^*))$ *is continuous.*

2. *Show that if*

$$T : (X, \sigma(X, X^*)) \to (Y, \|\cdot\|)$$

   is continuous, then

$$\dim \operatorname{Im}(T) < \infty.$$

Exercise 3.5. *Let* E *be a normed space.*

1. *Justify that weakly open sets are strongly open, and the same for closed sets.*
2. *Check that if* E *is finite-dimensional, the weak and strong topologies coincide.*
3. *Show that if* E *is infinite-dimensional, then its open unit ball*

$$B_E = \{x \in E : \|x\| < 1\}$$

   is not *open for the weak topology. We have therefore verified the converse of the previous question.*

Correction

Ex 3.1 *Closedness.* Define the function

$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x,y) = x^2 + y^2.$$

The function g is continuous and the interval $(-\infty, 1]$ is closed in $\mathbb{R}$. Hence,

$$A = g^{-1}((-\infty, 1])$$

is closed as the inverse image of a closed set under a continuous map.

Convexity. Let $(x_1, y_1), (x_2, y_2) \in A$ and let $\lambda \in [0,1]$. Set

$$(x,y) = \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2).$$

Using the convexity of the norm in $\mathbb{R}^2$, we have

$$\|(x,y)\| \leq \lambda \|(x_1, y_1)\| + (1-\lambda)\|(x_2, y_2)\|.$$

Since

$$\|(x_i, y_i)\|^2 = x_i^2 + y_i^2 \leq 1 \quad (i = 1, 2),$$

it follows that

$$\|(x,y)\| \leq \lambda + (1-\lambda) = 1,$$

and therefore

$$x^2 + y^2 \leq 1.$$

Thus $(x,y) \in A$, which proves that A is convex.

Since $B = \{(2,0)\}$ is a singleton in $\mathbb{R}^2$, it is compact. The set $B = \{(2,0)\}$ is closed and bounded in $\mathbb{R}^2$, hence compact. The set B is bounded, since it is contained in the closed ball of radius 1 centered at $(2,0)$. Hence, by the Heine–Borel theorem, B is compact.

Define the linear functional

$$u : \mathbb{R}^2 \to \mathbb{R}, \qquad u(x,y) = x.$$

For every $(x,y) \in A$, we have $x^2 + y^2 \leq 1$, hence $|x| \leq 1$, and therefore

$$u(x,y) = x \leq 1.$$

Moreover,

$$u(2,0) = 2.$$

Choose

$$\alpha = \frac{3}{2}, \qquad \varepsilon = \frac{1}{2}.$$

Then

$$u(x, y) \leq \alpha - \varepsilon = 1 \quad \text{for all } (x, y) \in A,$$

and

$$u(2, 0) = 2 \geq \alpha + \varepsilon = 2.$$

Hence, the hyperplane

$$H = \{(x, y) \in \mathbb{R}^2 : u(x, y) = \alpha\}$$

strictly separates the sets A and B.

Ex 3.2 definition Let E be a vector space over the field $\mathbb{F}$. A *semi-norm* on E is a function

$$p : E \longrightarrow \mathbb{R}$$

satisfying, for all $x, y \in E$ and $\lambda \in \mathbb{F}$, the following conditions:

1. $p(\lambda x) = |\lambda|\, p(x)$,
2. $p(x + y) \leq p(x) + p(y)$.

Let $\mathcal{P}$ be a family of semi-norms on E. We say that $\mathcal{P}$ is *separating* if

$$p(x) = 0 \quad \forall p \in \mathcal{P} \implies x = 0.$$

1. Since p is a semi-norm, for any $x \in E$ and scalar λ,

   $$p(\lambda x) = |\lambda|\, p(x).$$

   Take $x = 0$ and $\lambda = 0$. Then

   $$p(0) = p(0 \cdot x) = |0|\, p(x) = 0.$$

2. Again, using the semi-norm property with $\lambda = -1$:

   $$p(x) = p(x - 0) \leq p(x) + p(0) = p(x) + 0 = p(x),$$

   which shows $p(x) \geq 0$. More directly, a semi-norm is always non-negative by definition.

3. Let

   $$N := p^{-1}(\{0\}) = \{x \in E : p(x) = 0\}.$$

   Take $x, y \in N$ and $\alpha \in \mathbb{F}$. Then

   $$p(x + y) \leq p(x) + p(y) = 0 + 0 = 0 \implies x + y \in N,$$

   and

   $$p(\alpha x) = |\alpha| p(x) = |\alpha| \cdot 0 = 0 \implies \alpha x \in N.$$

   Hence N is a subspace of E.

4. For $x, y \in E$, using the triangle inequality for the semi-norm:

$$p(x) = p((x-y)+y) \leq p(x-y) + p(y) \implies p(x) - p(y) \leq p(x-y),$$

and similarly,

$$p(y) = p((y-x)+x) \leq p(y-x) + p(x) = p(x-y) + p(x) \implies p(y) - p(x) \leq p(x-y).$$

Therefore,

$$|p(x) - p(y)| \leq p(x-y).$$

Ex3.3

We need to verify the two properties of a semi-norm for $p(x) = |u(x)|$.
For any scalar $\lambda \in \mathbb{F}$ and $x \in E$,

$$p(\lambda x) = |u(\lambda x)| = |\lambda u(x)| = |\lambda|\,|u(x)| = |\lambda|\,p(x).$$

For any $x, y \in E$,

$$p(x+y) = |u(x+y)| = |u(x) + u(y)| \leq |u(x)| + |u(y)| = p(x) + p(y).$$

Since both properties are satisfied, $p(x) = |u(x)|$ is indeed a semi-norm on E.

Ex 3.4

Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator.

1. Equivalence of the three continuity properties.

(a) ⇒ (b).

Assume that $T : (X, \|\cdot\|) \to (Y, \|\cdot\|)$ is continuous. Since T is linear, this is equivalent to T being bounded.

Let $y^* \in Y^*$. Then the composition $y^* \circ T$ is a bounded linear functional on X, hence $y^* \circ T \in X^*$.

By definition, the weak topology $\sigma(X, X^*)$ is the weakest topology on X making all elements of X^* continuous. Therefore, for every $y^* \in Y^*$, the map

$$x \longmapsto y^*(Tx)$$

is $\sigma(X, X^*)$-continuous.

This is exactly the definition of continuity of T from $(X, \sigma(X, X^*))$ into $(Y, \sigma(Y, Y^*))$. Hence $(a) \Rightarrow (b)$.

(b) ⇒ (c).

The weak topology $\sigma(Y, Y^*)$ is weaker than the norm topology on Y. Moreover, the norm topology on X is stronger than $\sigma(X, X^*)$.

Therefore, if

$$T : (X, \sigma(X, X^*)) \to (Y, \sigma(Y, Y^*))$$

is continuous, then T remains continuous when the topology on X is strengthened to the norm topology. Hence

$$T : (X, \|\cdot\|) \to (Y, \sigma(Y, Y^*))$$

is continuous, and $(b) \Rightarrow (c)$ follows.

$\underline{(c) \Rightarrow (a).}$

Assume that $T : (X, \|\cdot\|) \to (Y, \sigma(Y, Y^*))$ is continuous. Then for every $y^* \in Y^*$, the mapping

$$x \longmapsto y^*(Tx)$$

is continuous on $(X, \|\cdot\|)$, hence $y^* \circ T \in X^*$.

Consider the family

$$\mathcal{F} = \{y^* \circ T : y^* \in Y^*,\ \|y^*\| \leq 1\} \subset X^*.$$

For each $x \in X$, we have

$$\sup_{f \in \mathcal{F}} |f(x)| = \sup_{\|y^*\| \leq 1} |y^*(Tx)| = \|Tx\|.$$

Thus, the family $\mathcal{F}$ is pointwise bounded. By the Uniform Boundedness Principle, $\mathcal{F}$ is bounded in X^*. Hence there exists $C > 0$ such that

$$\|Tx\| \leq C\|x\| \quad \text{for all } x \in X.$$

This shows that T is bounded, and therefore continuous from $(X, \|\cdot\|)$ to $(Y, \|\cdot\|)$. Thus $(c) \Rightarrow (a)$.

We conclude that (a), (b), and (c) are equivalent.

2. Weak-to-norm continuity implies finite-dimensional range.

Assume that

$$T : (X, \sigma(X, X^*)) \to (Y, \|\cdot\|)$$

is continuous.

The closed unit ball

$$B_X = \{x \in X : \|x\| \leq 1\}$$

is compact in the weak topology $\sigma(X, X^*)$ by the Banach–Alaoglu theorem.

Since T is continuous, the image $T(B_X)$ is compact in $(Y, \|\cdot\|)$. In particular, $T(B_X)$ is a compact neighborhood of 0 in $\operatorname{Im}(T)$.

If $\operatorname{Im}(T)$ were infinite-dimensional, then its closed unit ball could not be compact in the norm topology (by Riesz's theorem). This contradicts the compactness of $T(B_X)$.

Therefore,

$$\dim \operatorname{Im}(T) < \infty.$$