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On Spectral Theory of Ordinary Differential Operator Of Fractional Order

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Table of contents

Notations

$\Gamma(.)$	The Gamma function
$\beta(.,.)$	The Beta function
$E_\alpha(.)$	The function of Mittag-Leffler
$D^\alpha f$	The fractional derivative of the function f
${}^{RL}D^\alpha f$	The fractional derivative in the sense of Riemann-Liouville of the function f of order α
${}^{C}D^\alpha f$	The fractional derivative in the Caputo sense of the function f of order α
${}_aD_t^\alpha f(t)$	TheLeft-sided Riemann-Liouville derivatives
${}_a^CD_t^\alpha f(t)$	The left-sided Caputon fractional derivative
${}_tD_b^\alpha f(t)$	The right-sided Riemann-Liouville derivatives
${}_t^CD_b^\alpha f(t)$	The right-sided Caputon fractional derivative
${}_aR_b^\alpha f$	The fractional Riesz derivative of the function f
${}^{RL}D^\alpha f$	Fractional derivative in Riemann-Liouville type of order α
$\mathcal{C}(X,Y)$	The set of all closed linear operators from X into Y
$\ker(T)$	The nul space of the operator T
$\mathrm{Im}(T)$	The range of the operator T
A^*	Adjoint operator of the operator A
I	Operator of identity
$\sigma(T)$	The spectrum of the operator T
$\rho(T)$	The resolvent set of the operator T
$S_p(A_\rho)$	The Trace of operator A_ρ

Introduction

Operator theory is important in mathematics (mathematics, physics and chemistry). It has proven to be a necessity for the progress of several theories in physics and especially quantum mechanics. The theory ensures a certain number of essential operations and it has provided mathematical tools that physicistsso needed. The theory of operators is based on linear operators. Differential equations describe the evolution of numerous phenomena in various fields. A differential equation is an equation involving one or more derivatives of an unknown function. In mathematics, and more particularly in analysis, a spectral theory is a theory extending to operators defined on general functional spaces the elementary theory of eigenvalues and eigenfunctors , Although these ideas originally come from the development of linear algebra they are also linked to the study of analytic functions, because the spectral properties of an operator are linked to those of analytique functions on the values of its spectrum. This spectral theory is an outgrowth of fundamental work by David Hilbert between 1900 and 1910 on the analysis of integral operators on infinite-dimensional. Fractional calculus is the branch of mathematical analysis which studies the generalization of the notions of derivation and integration to orders that are not necessarily integers (real or complex). The origins of fractional calculation date back to the end of the 17th century, starting from some speculations by G. W. Leibniz concerning the Hospital question, posed on September 30, 1695, on the meaning of $\dfrac{d^n f}{dt^n}$ if $n = \dfrac{1}{2}$. Since then, many mathematicians have contributed to the development of this theory, we cite among others P. S. Laplace, J. B. J. Fourier, N. H. Abel, J. Liouville, A. K. Grunwald, A. V. Letnikov, O. Heaviside, H. Weyl and M. Riesz etc. However, fractional calculus has long been considered a simple mathematical theory without any real or practical explanation. Indeed, the interest in this concept in fundamental sciences and engineering only became apparent in the second half of the 20th century. Since then, many contributions, both theoretical and practical, have shown the importance of fractional order systems and their interest in different disciplines such as mechanics, electricity, biology, chemistry, automation, etc. iii. Fractional derivatives have a non-local character which makes them a powerful tool for the description of the hereditary and mnemonic effects of various substances, as well as for the modeling of certain dynamic processes. The concept of fractional order operators was defined in the 19 th century by Riemann and Liouville, their goal was to extend derivation or fractional order integration by employing not only an entire order but also non-integer orders. My work is devided in three chapters. The First including the basic properties that we need to present the notion of derivative of fractional order. The Second Chapter is devoted to spectrum theory of closed linear operators collecting some principal notions and properties such that adjoint of linear operator, spectrum, and essential spectrum. the Third Chapter concerns with application of spectrum decomposition in the differential operator of fractional order to solve a boundary value problem.

Chaptre 1

Basic Properties

1.1 Special functions

1.1.1 Gamma function

The Gamma function is one of the basic functions in fractional calculus, and it is also called Euler's Gamma function

Definition 1.1.1 *The Gamma function* $\Gamma(z)$ *is defined by the integral* $\Gamma(z) = \int\limits_{0}^{\infty} e^{-t}t^{z-1}dt$

Example 1.1.1 *Let's calculate* $\Gamma(2)$

$$\begin{aligned} \Gamma(2) &= \int\limits_{0}^{\infty} e^{-t}t\,dt \\ &= \left[-te^{-t} - e^{-t}\right]_{0}^{\infty} \\ &= 1 \end{aligned}$$

Proposition 1.1.1 *[43]*

1) $\Gamma(z+1) = z\Gamma(z)$

2) For every natural number n : $\Gamma(n+1) = n!$

3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof. *1)Representations* $\Gamma(z+1)$ *by Euler's integral and let's integrate in parts*

$$\begin{aligned}\Gamma(z+1) &= \int_0^{\infty} t^z e^{-t} dt \\ &= [-t^z e^{-t}]_0^{\infty} + z\int_0^{\infty} t^z e^{z-1} dt \\ &= z\Gamma(z)\end{aligned}$$

2)It's sufficient to apply property 1 for $z = n$

$$\Gamma(n+1) = \int_0^{+\infty} t^n e^{-t} dt$$

Let's integrate by part n *times, we get*

$$\begin{aligned}\Gamma(n+1) &= n(n-1)(n-2).....1 \\ \Gamma(n+1) &= n!\end{aligned}$$

3)

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt$$

We put $t = x^2, dt = 2xdx$, *So*

$$\begin{aligned}\Gamma(\frac{1}{2}) &= 2\int_0^{+\infty} \frac{1}{x} e^{-x^2} x dx \\ &= 2\int_0^{+\infty} e^{-x^2} dx \\ [\Gamma(\frac{1}{2})]^2 &= 4\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)} dx dy \\ &= 4\int_0^{\frac{\pi}{2}}\int_0^{+\infty} e^{-r^2} dr d\theta = \pi \\ \Gamma(\frac{1}{2}) &= \sqrt{\pi}\end{aligned}$$

■

Remark 1.1.1 *The Gamma function does not have simple poles for point* $z = 0; -1; -2; -3; ...$
$\Gamma(0_+) = +\infty, \ \ \Gamma(1) = 1$
$\Gamma(n)$ *Decreasing function for* $0 < n \leq 1$

1.1.2 Beta function

Definition 1.1.2 *For all* $(u, \upsilon) \in \mathbb{C} \times \mathbb{C},$ *Where* $\text{Re}(u) > 0; \text{Re}(\upsilon) > 0,$ *We defined the Beta function as follows[27]*

$$\beta(u, \upsilon) = \int_0^1 t^{u-1}(1-t)^{\upsilon-1} dt$$

$$\begin{aligned} \beta(1,2) &= \int_0^1 (1-t)dt \\ &= t]_0^1 - \frac{1}{2}t^2]_0^1 \\ &= 1 - \frac{1}{2} \\ \beta(1,2) &= \frac{1}{2} \end{aligned}$$

Proposition 1.1.2 *For each* $(u, \upsilon) \in \mathbb{C} \times \mathbb{C},$ *Where* $\text{Re}(u) > 0; \text{Re}(\upsilon) > 0$

a) $\beta(u, \upsilon) = \beta(\upsilon, u)$

b) $\beta(u, \upsilon + 1) = \dfrac{\upsilon}{u}\beta(u + 1, \upsilon)$

c) $\beta(u + 1, \upsilon) = \dfrac{u}{u + \upsilon}\beta(u, \upsilon)$

d) $\beta(u, n + 1) = \dfrac{n}{u(u+1)(u+2)...(u+n)}, \ \ n \in \mathbb{N}^*$

e) Integral can be found using the Beta and Gamma function

$$\int_0^{\frac{\pi}{2}} \sin^{2u-1}\theta\cos^{2\upsilon-1}\theta d\theta = \frac{1}{2}\beta(u,\upsilon)$$

$$\int_0^{\frac{\pi}{2}} \frac{t^{p-1}}{1+t}dt = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad 0<p<1$$

1.1.3 Relation between Gamma and Beta

Theorem 1.1.1 *The relationship between the Gamma and Beta function is given by*

Proof.

$$\beta(u,\upsilon) = \frac{\Gamma(u)\Gamma(\upsilon)}{\Gamma(u+\upsilon)}, \quad \forall u,\upsilon>0 \tag{1.1.1}$$

Example 1.1.2 *Let* $H = [0,+\infty[\times[0,+\infty[$, *We have*

$$\Gamma(u)\Gamma(\upsilon) = \int_0^{\infty} e^{-x}x^{u-1}dx\int_0^{\infty} e^{-y}y^{\upsilon-1}dy,$$
$$= \int_0^{\infty}\int_0^{\infty} e^{-(x+y)}x^{u-1}y^{\upsilon-1}dxdy;$$

Let's put; $y = p-x; dy = dp$

$$\Gamma(u)\Gamma(\upsilon) = \int_0^{\infty}\int_0^{p} e^{-p}x^{u-1}(p-x)^{\upsilon-1}dxdp;$$
$$\Gamma(u)\Gamma(\upsilon) = \int_0^{\infty} e^{-p}\int_0^{p} x^{u-1}(p-x)^{\upsilon-1}dxdp;$$

We put; $x = tp; dx = pdt$,

$$\Gamma(u)\Gamma(\upsilon) = \int_0^{\infty} e^{-p}\int_0^{1} t^{u-1}p^{u-1}(1-t)^{\upsilon-1}p^{\upsilon}dtdp;$$
$$= \int_0^{\infty} e^{-p}p^{u+\upsilon-1}dp\int_0^{\infty} t^{u-1}(1-t)^{\upsilon-1}dt,$$
$$\Gamma(u)\Gamma(\upsilon) = \Gamma(u+\upsilon)\beta(u,\upsilon)$$

Therefore

$$\beta(u,\upsilon)=\frac{\Gamma(u)\Gamma(\upsilon)}{\Gamma(u+\upsilon)}$$

■

Example 1.1.3 *We calculate* $\beta(2,1)$

$$\beta(2,1)=\frac{\Gamma(2)\Gamma(1)}{\Gamma(2+1)}=\frac{0!1!}{2!}=\frac{1}{2}$$

1.1.4 The function of Mittag-Leffler

The Mittag-Leffler function plays a very important role in the theory of integer differential equations. It is also widely used in the search for solutions for fractional order differenatial equations, this functions was introduced by G.M. Mittag-Leffler[36]

Definition 1.1.3 *[39]for* $z\in\mathbb{C}$, *the function of Mittag-Leffler* $E_\alpha(z)$ *is defined by*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad \alpha>0$$

The two-parameter Mittag-Leffler function also plays a very important role in fractional computing theory. The latter was introduced by Wiman [45] and is defined by the following serial developpement

$$E_\alpha(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta>0$$

$E_0(z)=\frac{1}{1-z},\quad |z|<1.$

$E_1(z)=E_{1,1}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{+\infty}\frac{z^k}{k!}=e^z.$

$E_2(z)=E_{2,1}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(2k+1)}=\sum_{k=0}^{+\infty}\frac{(\sqrt{z})^{2k}}{(2k)!}=\cosh\sqrt{z}.$

$$E_{2,2}(z)=\sum_{k=0}^{+\infty}\frac{z^{2k}}{\Gamma(2k+2)}=\frac{1}{z}\sum_{k=0}^{+\infty}\frac{z^{2k}}{(2k+1)!}=\frac{1}{z}\sinh z.$$

1.2 derivative of Fractional order

1.2.1 Fractional derivative within the meaning of Rieman-Liouville

Definition 1.2.1 *On the interval* $[a,b]$, $T>0$, *The function* $f \in L^1([0,1]), T>0$, *the fractional derivative in the sense of Riemann-Liouville of the function* f *of order* $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha)>0)$, *denoted* ${}^{RL}D^{\alpha}f$ *is defined by*

$$\begin{aligned} {}^{RL}D^{\alpha}f(x) &= D^{n}I^{n-\alpha}f(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{a}^{x}(x-t)^{n-\alpha-1}f(t)dt \end{aligned}$$

Proposition 1.2.1 *Let* $\alpha,\beta>0$ *such that* $n-1<\alpha\leq n$ $,m-1<\beta\leq m$ *such that* $n,m\in\mathbb{N}$

If $\alpha>\beta>0$, *Then for* $f\in L^1([0,T])T\rangle 0$, *The relation*

$$D^{\beta}(I^{\alpha}f(x)=I^{\alpha-\beta}f(x)$$

is true almost everywhere on $t\in[0,T]$

If $\beta>\alpha>0$ *and the fractional derivative* $D^{\beta-\alpha}f$ *exists, Then we have*

$$D^{\beta}(I^{\alpha}f(x))=D^{\beta-\alpha}f(x)$$

For $\alpha>0, k\in\mathbb{N}^{*}$, *If the fractional derivatives* $D^{\alpha}f$ *and* $D^{k+\alpha}f$ *exist,Then*

$$D^{k}(D^{\alpha}f(x))=D^{k+\alpha}f(x)$$

Example 1.2.1 *Let* $f(x)=x^2$, *We calculate the derivative of order* $\frac{3}{2}$ *of* f

We have the definition

$$D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{0}^{x}(x-t)^{n-\alpha-1}f(t)dt$$

For $\alpha = \dfrac{3}{2}$ *therefore* $n = 2$ *we find*

$$D^{\frac{3}{2}} f(x) = \frac{1}{\Gamma(\frac{1}{2})} \frac{d^2}{dx^2} \int\limits_0^x (x-t)^{-\dfrac{1}{2}} t^2 dt$$

We set $I = \int\limits_0^x (x-t)^{-\dfrac{1}{2}} t^2 dt$*, and according to the integral by parts twice, We obtain*

$$\begin{aligned}
I &= \left[(-2) t^2 (x-t)^{\dfrac{1}{2}} \right]_0^x + \int\limits_0^x 4t(x-t)^{\dfrac{1}{2}} dt \\
&= 4 \int\limits_0^x t(x-t)^{\dfrac{1}{2}} dt \\
&= \left[4.(-\frac{2}{3}).t(x-t)^{\dfrac{3}{2}} \right]_0^x + \frac{8}{3} \int\limits_0^x (x-t)^{\dfrac{3}{2}} dt \\
&= \frac{8}{3} \int\limits_0^x (x-t)^{\dfrac{3}{2}} dt \\
&= \left[\frac{8}{3}.(-\frac{2}{5})\, (x-t)^{\dfrac{5}{2}} \right]_0^x \\
&= \frac{16}{15} x^{\dfrac{5}{2}}
\end{aligned}$$

Therefore

$$\begin{aligned}
D^{\dfrac{3}{2}} f(x) &= \frac{1}{\Gamma(\frac{1}{2})} \frac{d^2}{dx^2} (\frac{16}{15} x^{\dfrac{5}{2}}) \\
&= 4\sqrt{\frac{x}{\pi}}
\end{aligned}$$

Notation 1.2.1 *The main advantage of Caputo 's approach is that the initial conditions of thz fractional derivative in the Caputo sense of fractional differential equations take the same form as in the case of integer order differential equations*

Lemma 1.2.1 *The fractional derivative of Caputo for the constant function is equal to zero i.e*

$^cD^\alpha c = 0,$ *such as* $c = const$

Proof. *Let* $\alpha > 0$ *and* $\alpha \in]n-1; n[\,, n \in \mathbb{N}, i.e\ n \geq 1$ *apply the deffinition of the Caputo derivative*

$$^cD^\alpha f(x) = I^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, s > 0$$

And since the n^{th} *derivative* $c^{(n)}$ *is equal to* 0 *with* $n \in \mathbb{N}$ *and* $n \geq 1$, *it follows*

$$^cD_c^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{c^{(n)}(t)}{(t-s)^{\alpha+1-n}} ds = 0$$

■

Example 1.2.2 *We calculate the derivative of order* $\frac{3}{2}$ *of* g *in the Caput sense, such that* $g(x) = x^2$ *we have the definition*

$$^cD_t^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt$$

For $\alpha = \frac{3}{2}$ *So* $n = 2$ *we find*

$$^cD^{\frac{3}{2}} f(x) = \frac{1}{\Gamma(\frac{1}{2})} \int_a^x \frac{2}{(x-t)^{\frac{1}{2}}} dt$$

According to the integral per part, we get

$$\begin{aligned} ^cD^{\frac{3}{2}}(x^2) &= \frac{1}{\Gamma(\frac{1}{2})} 2 \int_a^x (x-t)^{-\frac{1}{2}} dt \\ &= \frac{1}{\Gamma(\frac{1}{2})} \left[-4\left[(x-t)^{\frac{1}{2}}\right]_0^x\right] \\ &= 4\sqrt{\frac{x}{\pi}} \end{aligned}$$

Proposition 1.2.2 *Let $f\colon [0;b] \mapsto \mathbb{R}$ and $\alpha > 0$, we have*

The relationship between the Caputo derivative and that of Rieman-Liouville is given by

$$D^{\alpha} f(x) =^{RL} D^{\alpha} f(x) - \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \tag{1.2.1}$$

$f(x) = (x-a)^{p},$ *With* $p > n-1,$ *Then*

$$D^{\alpha} f(x) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-a)^{p-\alpha}, 0 \leq n-1 \leq \alpha \leq n, \tag{1.2.2}$$

1.2.2 Some other Types of fractional derivatives

Left-sided Riemann-Liouville derivatives are defined as follows [39]

$$_{a}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{t} \frac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, (n-1 < \alpha < n) \tag{1.2.3}$$

Another definition of the left-sided Caputon fractional derivative was introduced by [20]

$$^{C}_{a}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{n}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, (n-1 < \alpha < n) \tag{1.2.4}$$

Relationship between the left-sided Riemann-Liouville and the left-sided Caputo fractional derivatives has discussed by [39].

The Right-sided Riemann-Liouville derivative is defined as follows [39]

$$_{t}D_{b}^{\alpha} f(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{t}^{b} \frac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, (n-1 < \alpha < n) \tag{1.2.5}$$

Inspite of the remark made by [39] , the definition of the right-sided Caputo derivative did not appear in the literature until very recent works in [3, 4]

The right-sided Caputo derivative is defined by analogy with the right-sided Riemann-Liouville

$$^{C}_{t}D_{b}^{\alpha} f(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{t}^{b} \frac{f^{n}(\tau)d\tau}{(\tau-t)^{\alpha+1-n}}, (n-1 < \alpha < n) \tag{1.2.6}$$

The Riesz potential [39, 41](sometimes it is written with a factor of $[2\cos(\pi\alpha/2)]^{-1}$ in the right hand side

$$ {}_aR_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^b f(\tau)\,|\tau - t|^{\alpha-1}\, d\tau $$

Is the sum of the left-sided and the right-sided Riemann-Liouville fractional integrals

$$ {}_aR_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)\,|t - \tau|^{\alpha-1}\, d\tau + \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)\,|\tau - t|^{\alpha-1}\, d\tau $$

Based on this, we can consider the fractional Riesz derivative

$$ {}_aR_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^b \frac{f(\tau)d\tau}{|t-\tau|^{\alpha+1-n}}, (n-1<\alpha<n) $$

And we can also introduce the fractional Riesz-Caputo derivative

$$ {}_a^CR_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^b \frac{f^{(n)}(\tau)d\tau}{|t-\tau|^{\alpha+1-n}}, (n-1<\alpha<n) $$

Chaptre 2

Spectrum Theory of Closed Linear Operator

2.1 Closed Linear Operator

2.1.1 Basic Properties of Closed Linear Operators

Unbounded Linear Operator

Definition 2.1.1 *[29] A Linear operator of a sub space D_A of a Banach space X in itself is called an unbounded operator of domaine D_A. We say that A is bounded if there exists a constant $C > 0$, Such that*

$$\|AU\|_X \leq C \|U\|_X \quad , \forall U \in D_A$$

Closed Linear Operator

Let X be a complex Banach space and $A : D_A \subset X \to X$ a Linear Operator

Definition 2.1.2 *We say That A is a closed linear operator if for any sequence*

$$(U_n)_{n \in \mathbb{N}} \subset D_A$$

Such that

$$U_n \to U, AU_n \to f \ and U \in D_A then AU = f$$

This is equivalent to saying that the graphe of A

$$G(A) = \{(x, Ax)\} \in X \times Y; x \in D_A$$

is closed in $X \times Y$

- The set of all closed linear operators from X into Y in denoted by $\mathcal{C}(X, Y)$, and by $\mathcal{C}(X)$ if $X = Y$

Theorem 2.1.1 *The operator A is closed if and only if* $(D_A, \|.\|_{D_A})$ *is a Banach space, with* $\|f\|_{D_A} = \|f\|_X + \|Af\|_X$ *(graphe standard norm)*

Example 2.1.1 *Let* $A\colon D_A = C^1([a,b]) \subset C([a,b]); \quad f \to Af = f'$.
D_A *equipped with the graphe standard norm* $\|f\|_{D_A} = \|f\|_\infty + \|f'\|_\infty$ *is complete. So A is closed operator*

Definition 2.1.3 ***(Closable Linear operator)*** A_0 *is said closable if and only if* A_0 *admits a closed extension A. We note* $A_0 \subset A$ *if* $D_{A_0} \subset D_A$ *and* $A_0 x = Ax, \forall x \in D_A$*. The smallest closed extension of A is then denoted* $\overline{A}$ *and is called the closure of A.*

Proposition 2.1.1 *The linear operator A is closable if and only if* $\forall (U_n)_{n \geq 0} \in D_A$.

$$\left\{ \begin{array}{c} U_n \to 0 \\ AU_n \to f \end{array} \right\} \Longrightarrow f = 0$$

Lemma 2.1.1 *Let* X, Y *be two Banach space,* $A : X \to Y$ *a bounded linear operator. A is closable and its closure* $\overline{A}$ *is a bounded linear operator with*

$$\|\overline{A}\|_{L(X,Y)} = \|A\|_{L(X,Y)} \tag{2.1.1}$$

Example 2.1.2 *We define the operator* $A_0 : C^\infty([a,b]) \to C([a,b])$, $A_0 f = f'$

$C^\infty([a,b])$ represents the space of the infinitely differentiable functions on $([a,b])$, then A_0 is closable

Lemma 2.1.2 *If A is closable (resp, closed) operator and B a bounded operator with* $D_A \subset D_B$ *then the operator* $A + B$ *is a closable (resp,closed) operator.*

Proof. [21] Let $(U_n)_{n\in\mathbb{N}}$ be a sequence in D_A such that: $\lim_n U_n = 0$ and $\lim(A+B)U_n = y$ show that $y = 0$, We have

$$\left\{\begin{array}{c}\lim_n AU_n = Z \\ \lim_n U_n = 0\end{array}\right\} \Longrightarrow Z = 0$$

and $\lim_n U_n = 0 \Longrightarrow \lim_n BU_n = 0$. Because $\|BU_n\| \leq M\,\|U_n\|$, So $\lim_n(A+B)U_n = \lim_n(AU_n + BU_n) = \lim_n AU_n = 0 = y$ ■

2.1.2 Adjoint Operators

Definition 2.1.4 *Let H a Hilbert space and $T \in \mathcal{C}(H)$ and D_A is dense in X, then there exists a unique operators $T^* \in \mathcal{C}(H)$ such that $\forall x \in H, \forall y \in H, \langle Tx, y\rangle_{H'} = \langle x, T^*y\rangle_H$ the operator T^* is called the adjoint of T*

Example 2.1.3 **I** *The identity operator, let $x \in H$ it is clear that $\langle Ix, y\rangle = \langle x, y\rangle = \langle x, Iy\rangle$ from where $I^* = I$*

Proposition 2.1.2 *Let H be a Hilbert space and $T \in L(H)$, Then T admits a unique adjoint T^* Which satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ and $\|T^*\| = \|T\|$*

Proof. *Let $y \in H$, $f : x \to \langle Tx, y\rangle$ is an element of H^*, So according to Riez 's Theorem, there exists a unique $T^*y \in H$ Satisfies*

$$\begin{aligned}
\|T^*y\|^2 &= \langle T^*y, T^*y\rangle \Longrightarrow \|T^*y\|^2 = \langle y, T(T^*y)\rangle \\
&= \|T^*y\|^2 \leq \|y\|\,\|T\|\,\|T^*y\| \\
\|T^*y\| &\leq \|T\|\,\|y\| \\
&\Longrightarrow \|T^*\| \leq \|T\|
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|Tx\|^2 &= \langle Tx, Tx\rangle \Longrightarrow \|Tx\|^2 = \langle x, T^*(Tx)\rangle \\
&= \|Tx\|^2 \leq \|x\|\,\|T^*\|\,\|Tx\| \\
&= \|Tx\| \leq \|T^*\|\,\|x\| \\
&\Longrightarrow \|T\| \leq \|T^*\|
\end{aligned}$$

So

$$T^* = T$$

■

Example 2.1.4 *Let $H = L^2(\mathbb{R})$ and the operator $S \in L^2(\mathbb{R})$ defined by*

$$S\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

$$x = (x_n)_n \to S(x) = (0, x_1, x_2, ..., x_n, ...)$$

The debuty operator of S is

$$S^* L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

$$y = (y_n)_n \to S(y) = (0, y_1, y_2, ..., y_n, ...)$$

Lemma 2.1.3 *Let H be a space of Hilbert and $T \in L(H)$, we have*

1. $\ker T = (\mathrm{Im}(T^*))^\perp$
2. $\overline{\mathrm{Im}(T)} = (\ker T^*)^\perp$

Proof.

1. x Belongs to $\ker T$ if and only if $Tx = 0$, Then

$$\begin{aligned} x &\in \ker T \Longleftrightarrow \forall y \in \mathrm{Im}(T), \langle Tx, y\rangle = 0 \\ &\Longleftrightarrow \langle x, T^* y\rangle = 0 \\ x &\in (\mathrm{Im}(T^*))^\perp \end{aligned}$$

2. By (1), if we take T^* instead of T,we obtain

$$\begin{aligned} \ker(T^*) &= (\mathrm{Im}(T))^\perp \\ &\Longleftrightarrow (\ker T^*)^\perp = ((\mathrm{Im}(T))^\perp)^\perp \end{aligned}$$

■

Theorem 2.1.2 *An operator $A \in L(E,F)$ is invertible if and only if A^* is invertible and we have $(A^*)^{-1} = (A^{-1})^*$*

Proof. According to the previous proposition we have

$$\begin{aligned} (A^{-1})^* A^* &= (AA^{-1})^* = (Id_E)^* = Id_E \\ A^*(A^{-1})^* &= (A^{-1}A)^* = (Id_E)^* = Id_E \end{aligned}$$

So $A^* \in L(E,F)$ is invertile and $(A^*)^{-1} = (A^{-1})^*$ Now if $A^* \in L(E,F)$ is invertile, then the previous steep shows that $A \in L(E,F)$ is invertile ■

2.1.3 Symmetric and Self-Adjoint Operators

Definition 2.1.5 *An operator $(A, D(A)$ is said to be symmetric if $A \subset A^*$ That's to say: $D(A) \subset D(A^*)$ and $AU = A^*U$ for $U \in D(A)$ In other words*

$$\forall x, y \in D(A) \quad \langle Ax, y\rangle = \langle x, Ay\rangle$$

Definition 2.1.6 *We say that an operator T is self-adjoint if $T^* = T$ i.e $D(T) = D(T^*)$ and $Tx = T^*x, \forall x \in D(T)$*

Example 2.1.5 ***(Multiplication operator)*** *Let (E, μ) be a measured space with a positive measure on E and a measurable function of E in $\mathbb{C}$, we define the operator T on $L^2(E)$ by*

$$\begin{aligned} T\varphi &= \alpha.\varphi \\ D(T) &= \left\{\varphi \in L^2(E);\ \alpha.\varphi \in L^2(E)\right\} \end{aligned}$$

T is symmetric if and only if is real allmost every where in this case T is self-adjoint.

Remark 2.1.1 *Let T a symmetric operator, we have*

1. If $\operatorname{Im}(T)$ is dense, So T is injective.

2. If T is Self-adjoint, and T is injective, So $\operatorname{Im}(T)$ is dense and T^{-1} is Self-adjoint

2.1.4 Essentially Self-adjoint operator

Definition 2.1.7 *Let $(T, D(T))$be a Symmetric operator in a Hilbert space H . T is said to be essentially self-adjoint if $\overline{T}$ is self-adjoint or $(\overline{T})^* = \overline{T} = T^*$*

Proposition 2.1.3 *If T is essentially self-adjoint, Then T has a unique Self-adjoint extension*

Proof. S is a self-adjoint extension of T, as S is closed then $T \subset S$ Therefore $S^* \subset (\overline{T})^*$ and $S = \overline{T}, S^* = S$ is $(\overline{T})^* = \overline{T}$ and we have$\overline{T} \subset S$, Let's show that $S \subset \overline{T}$.

S^* is self-adjoint

$$\Longrightarrow S^* = S \tag{2.1.2}$$

S is closed $\Longrightarrow S^* = S = \overline{S}$, On the other hand

$$S^* \subset (\overline{T})^* \tag{2.1.3}$$

and since T is essentially self-adjoint then

$$\overline{S} = (\overline{T})^* \tag{2.1.4}$$

and then (2.1.2) and (2.1.3) give $S^* \subset \overline{T}$and of (2.1.2) and (2.1.4) $\Longrightarrow S = S^* \Longrightarrow S \subset \overline{T}$ ■

2.2 Spectrum of a Closed linear operator

Definition 2.2.1 *Let $T \in \mathcal{C}(H)$, we say that $\lambda \in \mathbb{C}$ belongs to the solving set of T if $T - \lambda I$ is a bijection from H into H, the solving set of T is denoted*

$$\rho(T) = \{\lambda \in \mathbb{C}, T - \lambda I \ \textit{invertible}\}$$

The spectrum of T is

$$\sigma(T) = \mathbb{C} \diagdown \rho(T)$$

The resolvent of T is

$$\begin{aligned} R_\lambda(T)\colon \rho(T) &\rightarrow L(H) \\ \lambda &\rightarrow (T - \lambda I)^{-1} \end{aligned}$$

Where $L(H)$ is the space of continuous linear operators acting in H

Definition 2.2.2 ***(Eigenvalues and Eigenvectors of an operator)*** *Let $T \in L(H)$, The complex number A, is called the eigenvalue of T if there exists a vector x in $H - \{O_H\}$ (is called the eigenvector associated with λ, such that*

$$(T - \lambda I)x = 0 \text{ i.e } Tx = \lambda x$$

Example 2.2.1 *Let I_H be the identity on H and $\mu \in \mathbb{C}$, So*

$$\begin{aligned}\sigma(\mu I) &= \{\lambda \in \mathbb{C} \colon (\mu I - \lambda I), I \textit{ is not invertible}\} \\ &= \{\lambda \in \mathbb{C} \colon (\mu - \lambda)I, I \textit{ is not invertible}\} \\ &= \mu\end{aligned}$$

Definition 2.2.3 *The Spectrum $\sigma(T)$ is decomposed into three disjoint part*

Point spectrum

$$\sigma_P(T) = \{\lambda \in \mathbb{C}, (T - \lambda I) \textit{ is not injective}\}$$

Continious spectrum

$$\sigma_C(T) = \left\{\lambda \in \mathbb{C}, (T - \lambda I) \textit{ is injective and } \overline{T - \lambda I} = H\right\}$$

Residual spectrum

$$\sigma_R(T) = \left\{\lambda \in \mathbb{C}, (T - \lambda I) \text{ is injective and } \overline{T - \lambda I} \neq H\right\}$$

Proposition 2.2.1 *Let T be a closed operator on a Hilbert H of dense domain then*

- $\rho(T)$ is an opening in $\mathbb{C}$.
- The map $\lambda \to R_\lambda(T)$ of $\rho(T)$ in $B(H)$ is holomorphic on any connected component of $\rho(T)$.

Theorem 2.2.1 *Let $T \in L(H)$, if $\|T\| \leq |\lambda|$, So $\lambda \in \rho(T)$*

Proof. We have

$$\begin{aligned} T_\lambda &= (T-\lambda I) = -\lambda(I-\frac{1}{\lambda}T) \\ R_\lambda &= (T-\lambda I)^{-1} = -\frac{1}{\lambda}(I-\frac{T}{\lambda})^{-1} = \frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{T^k}{\lambda^k} \end{aligned}$$

The series is convergent if and only if $\|T\| \leq |\lambda|$, In this case T_λ exists and $\lambda \in \rho(T)$ ■

Theorem 2.2.2 *Let H be a Hilbert space and $T \in L(H)$, The spectrum of T is a closed set*

2.3 Adjoint Of Fractional Differential Expressions and Operators

2.3.1 Basic Integrations by Parts

In order to determine the adjoint of differential operator, we will need the two basic formulas for integration by parts. The first one is the formula known for fractional integrals [22]

$$\int_a^b f(t)({}_aD_t^{-\alpha}g(t))dt = \int_a^b g(t)({}_tD_b^{-\alpha}f(t))dt, (\alpha > 0) \tag{2.3.1}$$

Due to an analogy with the classical formula for integration by parts, the formula (2.3.1) is called fractional integration by parts.

Besides this, we will need also the classical formula for repeated integration by parts, which, under the assumptionthat the functions $w(t), y(t)$, and $z(t)$ are k times continuously differentiable, is

$$\int_a^b wzy^k dt = \left[wzy^{(k-1)} - (wz)'y^{(k-2)} + ... + (-1)^{k-1}(wz)^{(k-1)}y\right]_{t=a}^{t=b} + (-1)^k\int_a^b y(wz)^{(k)}dt \tag{2.3.2}$$

Case of the left-sided Riemann-Liouville Derivatives

Putting $w(t) = 1, y(t) ={}_a D_t^{-\alpha}g(t), z(t) = f(t)$, the equation (2.3.2) takes the form

$$\int_a^b f(t)_a D_t^{k-\alpha} g(t)dt$$
$$= [f(t)_a D_t^{k-\alpha} g(t)dt - f'(t)_a D_t^{k-2-\alpha} g(t)dt + ... + (-1)^{k-1} f^{(k-1)}(t)_a D_t^{-\alpha} g(t)]_{t=a}^{t=b} \quad (2.3.3)$$
$$+(-1)^k \int_a^b {}_a D_t^{-\alpha} g(t) f^{(k)}(t)dt$$

Applying the formula (2.3.1) to the integral in the right-hand sided gives

$$\int_a^b {}_a D_t^{-\alpha} g(t) f^{(k)}(t)dt = \int_a^b g(t)({}_t D_b^{-\alpha} f^{(k)})(t)dt, \quad (2.3.4)$$

where we recognize the right-sided Caputo derivative of order $k-\alpha$. If $k=0$, We obtain the known fact that left-sided and right-sided Riemann-Liouville fractional integrales are adjoint [25]

Denoting $k-\alpha = v(k-1\langle v\langle k)$,we can write relationship (2.3.3) as

$$\int_a^b f(t)_a D_t^v g(t)dt = \sum_{r=0}^{k-1} (-1)^r [f_a^{(r)} D_t^{v-1-r} g(t)dt]_{t=a}^{t=b} + (-1)^k \int_a^b g(t)({}_t^C D_b^v f(t))dt, \quad (2.3.5)$$

Note that in the left-hand side we see the left-sided Riemann-Liouville derivative, and in the right-hand side there is the right-sided Caputo derivative.

Example 2.3.1 *Let us start with a simple example or two-term fractional defferential expression, which (for various λ and $w(t)$ frequently appears in applications and in sample studies, and which providess the quick preview of the notions introduced in the subsequent sections.*

$$l(f) =_a D_t^\alpha f(t) + (\lambda + w(t))f(t)$$

Assuming that $0 < \alpha < 1$ and that functions f and g are good enough, and using the formula (2.3.5) we obtain

$$\int_a^b l(f)g dt = [g(t)_a D_t^{\alpha-1}]_{t=a}^{t=b} - \int_a^b f(t)_t^C D_b^v g(t) + \int_a^b (\lambda + w(t))f(t)g(t)dt \quad (2.3.6)$$

$$= g(b)[_a D_t^{\alpha-1} f(t)]_{t=b} - g(a)[_a D_t^{\alpha-1} f(t)]_{t=a} + \int_a^b f l^*(g) dt \quad (2.3.7)$$

Where $l^*(g) = -({}_t^C D_b^\alpha g(t)) + (\lambda + w(t))g(t)$. The fractional differential expression $l^*(g)$ is adjoint to the fractional differential expression $l(f)$.

In this example $0 < \alpha < 1$, so we can consider one condition on the function $f(t)$ at the boundary of the interval $[a, b]$, for instance, of the type

$$[_a D_t^\alpha f(t)]_{t=a} = 0 \quad (2.3.8)$$

Then the boundary condition on the function $g(t)$, which is adjoint to the boundary condition (2.3.8), is

$$g(b) = 0. \quad (2.3.9)$$

The initial value problem in terms of left-sided Riemann-Liouville derivatives

$$_a D_t^\alpha f(t) + (\lambda + w(t))f(t) = h(t), \quad (2.3.10)$$

$$(a\langle t\langle b), [_a D_t^\alpha f(t)]_{t=a} = 0, \quad (2.3.11)$$

And the initial value problem in terms of the left-sided Caputo derivatives

$$-({}_t^C D_b^\alpha g(t)) + (\lambda + w(t))g(t) = h(t), \quad (2.3.12)$$

$$(a\langle t\langle b), g(b) = 0 \quad (2.3.13)$$

Are adjoint initial value problems.

Fractional differential operator L^*, defined by the fractional differential expression $l^*(g)$ and the boundary condition (2.3.9), is an adjoint fractional differential operator to the fractional differential operator L defined by the fractional differential expression $l(f)$ and the boundary condition (2.3.9).

From (2.3.7), which is an analog of the classical Langrange identity, and from boundary conditions (2.3.8) and (2.3.9) it follows that for fractional differential operators L and L^* we have

$$\int_a^b L(f)g dt = \int_a^b f L^*(g) dt, \tag{2.3.14}$$

Or, using standard and well-known notation,

$$(Lf, g) = (f, L^* g). \tag{2.3.15}$$

Using the relationship (2.3.21), we see that, under the conditions (2.3.8) and (2.3.9), the operator L is adjoint to L^*.

$$L^{**} = L. \tag{2.3.16}$$

2.3.2 Adjoint Fractional Differential Expressions constant coefficients

Now let us consider a general $(N+1)$-term linear fractional differential expression with left-sided Riemann-Liouville derivatives and with constant coefficients

$$l_N(f) = \sum_{j=0}^{N} p_j \ {}_a D_l^{\alpha_j} f(t), (0 < \alpha_1, \alpha_2, ..., < \alpha_N) \tag{2.3.17}$$

Assuming that $k_j - 1 < \alpha_j < k_j$ and that functions f and g are good enough, and using the formula (2.3.5) we can obtain

$$\int_a^b l_N(f) g dt = P(\eta, \xi) + \int_a^b f l_N^*(g) dt, \tag{2.3.18}$$

Where the adjoint differential expressions appears

$$l_N^*(g) = \sum_{j=0}^{N} p_j (-1)^{k_j} \ {}_t^C D_b^{\alpha_j} g(t), \tag{2.3.19}$$

And $P(\eta, \xi)$ is a bilinear form with respect to the variables

$$\eta = (\widetilde{f}^{(0)}(a), \widetilde{f}^{(1)}(a), ..., \widetilde{f}^{(k_N-1)}(a), \widetilde{f}^{(0)}(b), \widetilde{f}^{(1)}(b), ..., \widetilde{f}^{(k_N-1)}(b)), \tag{2.3.20}$$

$$\xi = (g(a), g'(a), ..., g^{(k_N-1)}(a), ..., g(b), g'(b), ..., g^{(k_N-1)}(b)), \tag{2.3.21}$$

$$\widetilde{f}^{(r)}(a) = [\sum_{j=0}^{r} p_j \ {}_a D_t^{\alpha_j-1-r} f(t)]_{t=a}, \tag{2.3.22}$$

$$\widetilde{f}^{(r)}(b) = [\sum_{j=0}^{r} p_j \ {}_a D_t^{\alpha_j-1-r} f(t)]_{t=b} \tag{2.3.23}$$

2.3.3 Adjoint Boundary Conditions

The generalized Lagrange identity (2.3.18) can be written as:

$$\int_a^b l_N(f) g dt = U_1 V_{2k_N} + U_2 V_{2k_N} + ... + U_{2k_N} V1 + \int_a^b f l_N^*(g) dt \tag{2.3.24}$$

where $V_{2k_N}, V_{2k_N-1}, ..., V_1$ are linear forms of the variables

$$g(a), g'(a), ..., g^{(k_N-1)}(a), ..., g(b), g'(b), ..., g^{(k_N-1)}(b).$$

It can proved that the linear forms $V_1, V_2, ..., V_{2k_N}$ are linearly independant.

By analogy with the classical integer order case [38], we say that the boundary conditions (or any other equivalent boundary conditions)

$$V_1 = 0, V_2 = 0, ..., V_{2k_N-m} = 0 \tag{2.3.25}$$

Are adjoint to the boundary conditions

$$U_1 = 0, U_2 = 0, ..., U_m = 0, \tag{2.3.26}$$

Fractional differential operator L^*, defined by the fractional differential expression $l^*(g)$, equation (2.3.19) and the boundary condition (2.3.25), is an adjoint fractional diffrential operator to the fractional operator L defined by the fractional differential expression $l(f)$, equation (2.3.17), and the boundary conditions (2.3.26).

2.4 The completeness of the system of eigenfunctions and associated functions

In this section we will study the completenss of the eigenfunction and associated function of the operator in the space $L^2(0,1)$ defined by

$$-A_\rho(u) = \int_0^1 G(x,t)u(t)dt = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt\right]$$

where $0 < \rho < 2$.

Livshits Theorem

Theorem 2.4.1 ***(Livshits).*** *If $K(x,y)(a \le x, y \le b)$ - is a limited kernel and " real part" $\frac{1}{2}(K+K^*)$ of it is non-negative kernel , then the inequality is hold*

$$\sum_{j=1}^{\infty} \mathrm{Re}(\frac{1}{\lambda_j}) \le \int_a^b \mathrm{Re}\,K(t,t)dt \tag{2.4.1}$$

Where λ_j - is the characteristic numbers of kernel K. The system of main functions of the kernel K is complete in domain of values of the integral operator Kf if and only if, When there is an equal sign in inequality above.

In his paper [23] M. M Dzhrbashian wrote, that " the question about the completeness of the systems of eigenfunctions of the operator A_ρ or a finer question about wether these systems compose a basis in $L_2(0,1)$, has a certain interest but its solving is apparently associated with significant analytic difficulties" . The questions of the completeness of the systems of eigenfunctions and associated functions for similar problemes were studies by A. V. Agibalove in [1, 2] Undoubtedly, we shall note the fundamental results of M. M. Malamud and L. L. Oridoroga [32, 33] obtained in this direction.

In [7, 16] (see also [5, 17]), using the theorem of Matsaevand Palant, it was established that the system of eigenfunctions of the operator A_ρ is complete in $L_2(0,1)$. And this fact used by M. Ali, S. Aziz and S.A Malik in their paper [18].

As noted above, a similar result was obtained using the well-known Livishchits theorem.

Definition 2.4.1 *If a series of s-numbers [26]of the completely continuous operator is convergent, that is, $\sum\limits_{k=1}^{\infty} s_k(A) < \infty$ then such operator called as trace-class operator.*

Lemma 2.4.1 *Let $1 < \rho < 2$, then the operator A_ρ is trace-class and*

$$sp(A_\rho) = \frac{\Gamma(\rho^{-1})}{\Gamma(2\rho^{-1})}.$$

Proof. To find the trace spA_ρ of the operator $A_{\rho'}$let's rewrite A_ρ as $A_\rho u = A_1 u - A_0 u$ where

$$A_0 u = \frac{1}{\Gamma(\rho^{-1})} \int\limits_0^x (x-t)^{\frac{1}{\rho}-1} u(t)dt \tag{2.4.2}$$

$$A_1 u = \frac{1}{\Gamma(\rho^{-1})} \int\limits_0^x x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t)dt$$

Clearly, for $1 < \rho < 2$, the operators A_0 ans A_1 are trace class. Hence

$$spA_\rho = sp(A_1 - A_0) = sp(A_1) - sp(A_0)$$

Moreover, it is clear that $sp(A_0) = 0$. Thus

$$spA_\rho = sp(A_1)$$

Since operator A_1 is one-dimensional, it is easy to find a trace. consider the equation

$$u(x) - \frac{\lambda}{\Gamma(\rho^{-1})} \int\limits_0^1 x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t)dt = 0$$

The Fredholdm determinant

$$d(\lambda) = |1 - \lambda K_{11}|\,,$$

Where

$$K_{11} = \frac{1}{\Gamma(\rho^{-1})}\int_0^1 t^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}dt = \frac{\Gamma(2-\nu)}{\Gamma(4-2\nu)}(\nu = 2-\rho^{-1}). \tag{2.4.3}$$

From above follow that

$$sp(A_1) = \frac{\Gamma(2-\nu)}{\Gamma(4-2\nu)} \tag{2.4.4}$$

Which proves the lemma 2.4.1 ■

Remark 2.4.1 *Of course $\rho > \frac{1}{2}$, nuclearity of the operator A_ρ follows from Well-known Dzhrbaschian lemma ([24],p.142).*

Lemma 2.4.2 *(Dzhrbaschian-Nersisian).*
All zeros of fuctions $E_\rho(z;\mu) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu+\dfrac{n}{\rho})}$ (Where $\rho > \frac{1}{2}, \rho \neq 1; \operatorname{Im}\mu = 0$) With largest absolute values, are prime.
The following asymptotic formulas are valid
$\gamma_k^{\pm} = e^{\pm i\dfrac{\pi}{2\rho}}(2\pi k)^{\frac{1}{\rho}}(1+O(\frac{\log k}{k})), k \to \infty$, and the fact that the value λ_j is an eigenvalue of the operator A_ρ if and only if

$$E_\rho(\lambda_j;\frac{1}{\rho}) = 0$$

. *The completness of the eigenfunctions system is given by the followinf theorem*

Theorem 2.4.2 *The system of eigenfunctions and associated functions of the operator A_ρ, Where $0 < \rho < 1$, is complete in $L_2(0,1)$.*

Proof. We denote the kernel of A_ρ as $K(x,y)$. In [7]the authors have proved that this kernel is non-negative by by the following way: Let us rewrite A_ρ as

$$A_\rho = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (x-xt)^{\dfrac{1}{\rho}-1}u(t)dt - \int_0^x (x-t)^{\dfrac{1}{\rho}-1}u(t)dt\right]$$

Clearly, for $\rho < 1$, the kernel of A_ρ is non-negative. By the same way, we may show that the kernel $K^*(x,y)$ for adjoint operator

$$A_\rho^* u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (t-xt)^{\dfrac{1}{\rho}-1}u(x)dx - \int_0^x (t-x)^{\dfrac{1}{\rho}-1}u(x)dx\right]$$

Is non-negative too. Thus $\frac{1}{2}(K + K^*)$ is non-negative. Let us show that the following expression holds

$$\sum_{j=1}^{\infty} \mathrm{Re}(\frac{1}{\lambda_j}) = \int_0^1 \mathrm{Re}\, K(t,t)dt.$$

If $\lambda_j = \alpha_j + i\beta_j$ is eigenvalue of the operator A_ρ, then complex conjugate $\overline{\lambda_j} = \alpha_j - i\beta_j$ is eigenvalue of the operator A_ρ too. Thus

$$spA_\rho = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \sum_{j=1}^{\infty} \mathrm{Re}(\frac{1}{\lambda_j})$$

So, taking to acccount lemma2.4.1 , we obtain that the system of eigenfunction and associated functions of the operators A_ρ for $0 < \rho < 1$, is complete in $L_2(0,1)$ ■

Remark 2.4.2 *For $(\frac{1}{\rho} - 1) > 0$ the kernel of the operator A_ρ is continuous. Therefore, as it was showed by Lalesko[31], the fredholm determinant of this kernel is whole functions of zero kind. In this case [31]*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_j} = \int_0^1 K(t,t)dt,$$

That is, the equation

$$\sum_{j=1}^{\infty} \mathrm{Re}(\frac{1}{\lambda_j}) = \int_0^1 \mathrm{Re} K(t,t)dt$$

We can get by the behavious way.

Theorem 2.4.3 *The system of eigenfunctions and associated functions of the operator A_ρ, where $1 < \rho < 2$, is complete in $L_2(0,1)$.*

Proof. *For $1 < \rho < 2$ the kernel of the operator A_ρ is not fixed-sign, thus we cannot use the Livishits theorem2.4.1, used above. To prove the formulated theorem, let us consider the value of the form $(A_\rho u, \overline{u})$ [6]. Let us introduce the following designation*

$$A_\rho u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (x - xt)^{\frac{1}{\rho}-1} u(t)dt - \int_0^x (x-t)^{\frac{1}{\rho}-1} u(t)dt\right] = \upsilon(x)$$

So

$$(A_\rho u, \overline{u}) = (\upsilon, D_{0x}^{\frac{1}{\rho}}\upsilon) = \int_0^1 \upsilon(x)\left[D_{0x}^{\frac{1}{\rho}}\upsilon\right]dx$$
$$= \int_0^\varepsilon \upsilon(x)\left[D_{0x}^{\frac{1}{\rho}}\upsilon\right]dx + \int_\varepsilon^1 \upsilon(x)\left[D_{0x}^{\frac{1}{\rho}}\upsilon\right]dx$$

Where

$$D_{0x}^{\alpha} f = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}},$$

$n = \{[\alpha]+1, [\alpha]\}$ *is the integer part of* α*, called the operator of fractional differential in the strum-Liouville sense of order* α*. As was mentioned in Reference[6] (see also the reference therein), the study of forms*

$$\int_\varepsilon^1 \upsilon(x)\left[D_{0x}^{\frac{1}{\rho}}\upsilon\right]dx \tag{2.4.5}$$

Was provided in the paper and there, in particular, were established the values of those forms lying in $|\arg\lambda| < \frac{\pi\rho}{2}$*. Clearly, for small values* ε*, the operator* A_ρ *is sectorial. Since the operator* A_ρ *is sectorial and a trace-class operator, by Lidskii's theorem[30] the system of eigenfunctions and associated functions of* A_ρ *are complete in* $L_2(0,1)$ ■

Corollary 2.4.1 *Since the operator* A_ρ *does not generate any associated functions [8], we prove the completeness of system*

$$\chi_n(x) = x^{\frac{1}{\rho}-1} E_\rho(\lambda_n x^{\frac{1}{\rho}}; \frac{1}{\rho})$$

In $L_2(0,1)$*. But the system of these eigenfunctions, unfortunately, is not orthogonal, therefore, for solving inverse problems, and in Reference [18] the corresponding biorthogonal system was used.*

By the same method, we can provide spectral analysis of the operator

$$A_\rho^{[\alpha^{-1},\rho]} u = \frac{1}{\Gamma(\rho^{-1})}\int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\alpha-1}u(t)dt - \frac{1}{\Gamma(\rho^{-1})}\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt$$

Considered in Reference [7]

Theorem 2.4.4 *Let* $0 < \rho < 2, \alpha < \dfrac{1}{\rho}$. *Then, the system of eigenfunctions and associated functions of the operator* $A_\rho^{[\alpha^{-1},\rho]}$ *is complete in* $L_2(0,1)$.

Proof. We carry out the proof of theorem 2.4.4 in the same way as the proof of theorem 2.4.3. It can easily be shown that the kernel $M(x,t)$ of the operator $A_\rho^{[\alpha^{-1},\rho]}$ is non-negative. Elementary calculations show that the kernel $M^*(x,t)$ of the operator adjoint to the operator $A_\rho^{[\alpha^{-1},\rho]}$ will be non-negative too.Thus $\dfrac{1}{2}(M + M^*)$ will be non-negative too. The fact that

$$\sum_{j=1}^{\infty} \mathrm{Re}(\frac{1}{\mu_j}) = \int_0^1 \mathrm{Re}\, M(t,t)dt$$

where μ_j are eigenvalue of the operator $A_\rho^{[\alpha^{-1},\rho]}$, shown in the same way as in theorem 2.4.2

■

Chaptre 3

Application on boundary value problem

3.1 Problem statement

Let's in $D = \{0 < x < 1, 0 < t < 1\}$ consider the first boundary value problem for equations of vibration of a string with a fractional derivative of order α with respect to patial variable

$$\frac{\partial^2 u}{\partial t^2} = m\frac{\partial^2 u}{\partial x^2} + C_1 D_{0x}^{\alpha} u, \quad 0 < \alpha < 2 \tag{3.1.1}$$

$$u(0,t) = u(1,t) = 0 \tag{3.1.2}$$

$$u(x,0) = \varphi(x) \tag{3.1.3}$$

$$u_t'(x,0) = \Psi(x) \tag{3.1.4}$$

Here, $0 < \alpha < 1$, c-is an arbitrary constant, $D_{0x}^{\alpha} u$ - is fractional derivative in Riemann-Liouville type of order α.

In order to solve the problem (3.1.1)-(3.1.2)-(3.1.3)-(3.1.14) we use the Fourier method

$$u(x,t) = X(x)T(t) \tag{3.1.5}$$

Substituting the expression (3.1.5) into (3.1.1) we obtain the two-point boundary value problem for $X(x)$

$$\begin{aligned} XT" &= mX"T + TC_1 D^{\alpha}_{0x} X(x) \\ XT" &= (mX" + C_1 D^{\alpha}_{0x} X(x))T \\ \frac{T"}{T} &= \frac{mX" + C_1 D^{\alpha}_{0x} X(x)}{X(x)} = \gamma \\ mX" + C_1 D^{\alpha}_{0x} X(x) &= \gamma X \end{aligned}$$

Thus

$$X"(x) + C D^{\alpha}_{0x} X = \lambda X(x) \tag{3.1.6}$$

$$X(0) = X(1) = 0 \tag{3.1.7}$$

The solution of the problem (3.1.6)-(3.1.7) was written out in [5, 7]. In particular, there was shown that the value λ is the an eigenvalue of the problem (3.1.6)-(3.1.7) iff λ is the zero of the function.

$$\omega(\lambda) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda^{n-k} (-C_1)^k}{\Gamma(2n - k\alpha + 2)}$$

And corresponding eigen functions $X_j(x)$ are

$$X_j(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k} (-C_1)^k}{\Gamma(2n - k\alpha + 2)} x^{2n+1-k\alpha} \qquad j = 1, 2, 3, ... \tag{3.1.8}$$

(here λ_j -is the j -th eigenvalue of the problem (3.1.6)-(3.1.7). The system of eigenfunctions (3.1.8) is complete [31, 1, 2, 15, 13, 18] but non-biorthogonal, in this way we construct the system

$$\widetilde{X}_j(x) = (1-x) - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \lambda_j^{n-k} (-C_1)^k}{\Gamma(2n - k\alpha + 2)} (1-x)^{2n+1-k\alpha}, \quad j = 1, 2, 3, ... \tag{3.1.9}$$

Wich will be biorthogonal to the system of eigenfunctions

$$X_j = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{C_n^k\lambda_j^{n-k}(-C_1)^k}{\Gamma(2n-k\alpha+2)}x^{2n+1-k\alpha}$$

In order to construct this biorthogonal system, together with problem (3.1.6)-(3.1.7), we consider a problem conjugate to problem (3.1.6)-(3.1.7).

To pose the problem conjugate to problem (3.1.6)-(3.1.7), in the class $C^2(0,1)\bigcap C([0,1])$ we consider the following Cauchy problem: find a solution to the equation

$$u" + \frac{d^\alpha}{d(1-x)^\alpha}u + \lambda u = 0 \tag{3.1.10}$$

$$u(1) = 0, u'(1) = -1, \tag{3.1.11}$$

Where $\frac{d^\alpha}{d(1-x)^\alpha}$- is the operator adjoint to the fractional differential operator D_{0x}^α of order α [23].

3.2 Integral Equation of iterated kernels

Theorem 3.2.1 *[12]The problem (3.1.10), (3.1.11) is equivalent to the equation*

$$u(x) = \int_x^1 K(x,t)u(t)dt + (1-x),$$

Where

$$K(x,t) = \begin{Bmatrix} 0, 0 < t < x < 1, \\ \frac{(t-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda(t-x), 0 < x < t < 1 \end{Bmatrix}$$

Theorem 3.2.2 *[12]If we put the sequence of iterated kernels through recurrence relations* $K_{n+1}(x,t) = \int\limits_x^1 K_n(x,s)K_1(s,t)ds$ *Then*

$$K_n(x,t,\lambda) = \sum_{m=0}^{n} \frac{C_n^m \lambda^{n-m}}{\Gamma(2n-m\alpha)}(t-x)^{2n-1-m\alpha}$$

Then

$$K_2(x,t) = \int\limits_x^1 K_1(x,s)K_1(s,t)ds$$

$$K_1(x,s) = \begin{Bmatrix} 0 & ,0 < s < x < 1 \\ \frac{(s-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda(s-x), 0 < x < s < 1 \end{Bmatrix}$$

$$K_1(s,t) = \begin{Bmatrix} 0 & ,0 < t < s < 1, \\ \dfrac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda(t-s), 0 < s < t < 1 \end{Bmatrix}$$

$$K_2(x,t) = \int\limits_x^t K_1(x,s)K_1(s,t)ds$$

$$= \int\limits_x^t \left(\frac{(s-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda(s-x)\right)\left(\frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda(t-s)\right)$$

Put $s = (1-\theta)x + \theta t, \;\; 0 < \theta < 1$

$s - x = -\theta x + \theta t = (t-x)\theta$

$t - s = (1-\theta)(t-x)$

$ds = (-x+t)d\theta = (t-x)d\theta$

$$
\begin{aligned}
K_2(x,t) &= \int_0^1 \left(\frac{\theta^{1-\alpha}(t-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda\theta(t-x)\right)\left(\frac{(1-\theta)^{1-\alpha}(t-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda(1-\theta)(t-x)\right)(t-x)d\theta \\
&= \int_0^1 \frac{(t-x)^{3-2\alpha}\theta^{1-\alpha}(1-\theta)^{1-\alpha}}{\Gamma^2(2-\alpha)}d\theta + \int_0^1 \frac{\lambda(t-x)^{3-\alpha}\theta^{1-\alpha}(1-\theta)}{\Gamma(2-\alpha)}d\theta \\
&\quad + \int_0^1 \frac{\lambda(t-x)^{3-\alpha}\theta(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)}d\theta + \int_0^1 \lambda^2\theta(1-\theta)(t-x)^3 d\theta \\
&= \frac{(t-x)^{3-2\alpha}}{\Gamma^2(2-\alpha)}\beta(2-\alpha,2-\alpha) + \lambda(t-x)^{3-\alpha}\beta(2,2-\alpha) \\
&\quad + \lambda(t-x)^{3-\alpha}\beta(2-\alpha,2) + \lambda^2(t-x)^3\beta(2,2) \\
&= \frac{(t-x)^{3-2\alpha}}{\Gamma^2(2-\alpha)}\frac{\Gamma(2-\alpha)\Gamma(2-\alpha)}{\Gamma(4-2\alpha)} + \frac{\lambda(t-x)^{3-\alpha}}{\Gamma(2-\alpha)}\frac{\Gamma(2)\Gamma(2-\alpha)}{\Gamma(4-\alpha)} \\
&\quad + \frac{\lambda(t-x)^{3-\alpha}}{\Gamma(2-\alpha)}\frac{\Gamma(2-\alpha)\Gamma(2)}{\Gamma(4-\alpha)} + \lambda^2(t-x)^3\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \\
K_2(x,t) &= \frac{\lambda^2}{\Gamma(4)}(t-x)^3 + \frac{2\lambda}{\Gamma(4-\alpha)}(t-x)^{3-\alpha} + \frac{1}{\Gamma(4-2\alpha)}(t-x)^{3-2\alpha}
\end{aligned}
$$

$$
\begin{aligned}
K_3(x,t) &= \int_x^t K_2(x,s)K_1(s,t)ds \\
&= \int_x^t \left(\tfrac{\lambda^2}{\Gamma(4)}(s-x)^3 + \tfrac{2\lambda}{\Gamma(4-\alpha)}(s-x)^{3-\alpha} + \tfrac{1}{\Gamma(4-2\alpha)}(s-x)^{3-2\alpha}\right) \\
&\qquad \left(\tfrac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda(t-s)\right) ds \\
&= \int_x^t \left(\tfrac{\lambda^2}{\Gamma(4)}(t-x)^3\theta^3 + \tfrac{2\lambda}{\Gamma(4-\alpha)}(t-x)^{3-\alpha}\theta^{3-\alpha} + \tfrac{1}{\Gamma(4-2\alpha)}(t-x)^{3-2\alpha}\theta^{3-2\alpha}\right) \\
&\qquad \left(\tfrac{(t-x)^{1-\alpha}}{\Gamma(2-\alpha)}(1-\theta)^{1-\alpha} + \lambda(t-x)(1-\theta)\right) \\
&\qquad (t-x)d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda^2(t-x)^{5-\alpha}}{\Gamma(4)\Gamma(2-\alpha)}\int_0^1 \theta^3(1-\theta)^{1-\alpha}d\theta + \frac{2\lambda(t-x)^{5-2\alpha}}{\Gamma(4-\alpha)\Gamma(2-\alpha)}\int_0^1 \theta^{3-\alpha}(1-\theta)^{1-\alpha}d\theta \\
&\quad + \frac{(t-x)^{5-3\alpha}}{\Gamma(4-2\alpha)\Gamma(2-\alpha)}\int_0^1 \theta^{3-2\alpha}(1-\theta)^{1-\alpha}d\theta + \frac{\lambda^3(t-x)^5}{\Gamma(4)}\int_0^1 \theta^3(1-\theta)d\theta \\
&\quad + \frac{2\lambda^2(t-x)^{5-\alpha}}{\Gamma(4-\alpha)}\int_0^1 \theta^{3-\alpha}(1-\theta)d\theta + \frac{\lambda(t-x)^{5-2\alpha}}{\Gamma(4-2\alpha)}\int_0^1 \theta^{3-2\alpha}(1-\theta)d\theta \\
&= \frac{\lambda^2(t-x)^{5-\alpha}}{\Gamma(4)\Gamma(2-\alpha)}\frac{\Gamma(4)\Gamma(2-\alpha)}{\Gamma(6-\alpha)} + \frac{2\lambda(t-x)^{5-2\alpha}}{\Gamma(4-\alpha)\Gamma(2-\alpha)}\frac{\Gamma(4-\alpha)\Gamma(2-\alpha)}{\Gamma(6-2\alpha)} \\
&\quad + \frac{(t-x)^{5-3\alpha}}{\Gamma(4-2\alpha)\Gamma(2-\alpha)}\frac{\Gamma(4-2\alpha)\Gamma(2-\alpha)}{\Gamma(6-3\alpha)} + \frac{\lambda^3(t-x)^5}{\Gamma(4)}\frac{\Gamma(4)\Gamma(2)}{\Gamma(6)} \\
&\quad + \frac{2\lambda^2(t-x)^{5-\alpha}}{\Gamma(4-\alpha)}\frac{\Gamma(4-\alpha)\Gamma(2)}{\Gamma(6-\alpha)} + \frac{\lambda(t-x)^{5-2\alpha}}{\Gamma(4-2\alpha)}\frac{\Gamma(4-2\alpha)\Gamma(2)}{\Gamma(6-\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
K_3(x,t) &= \frac{\lambda^3(t-x)^5}{\Gamma(6)} + \frac{3\lambda^2(t-x)^{5-\alpha}}{\Gamma(6-\alpha)} \\
&\quad + \frac{3\lambda(t-x)^{5-2\alpha}}{\Gamma(6-2\alpha)} + \frac{1}{\Gamma(6-3\alpha)}(t-x)^{5-3\alpha}
\end{aligned}
$$

$$
\begin{aligned}
K_4(x,t) &= \frac{\lambda^4(t-x)^7}{\Gamma(8)} + \frac{4\lambda^3(t-x)^{7-\alpha}}{\Gamma(8-\alpha)} + \frac{6\lambda^2(t-x)^{7-2\alpha}}{\Gamma(8-2\alpha)} \\
&\quad + \frac{4\lambda(t-x)^{7-3\alpha}}{\Gamma(8-3\alpha)} + \frac{1}{\Gamma(8-4\alpha)}(t-x)^{7-4\alpha}
\end{aligned}
$$

$$\vdots$$

$$K_n(x,t) = \frac{\lambda^n (t-x)^{2n-1}}{\Gamma(2n)} + \frac{\binom{n}{1}\lambda^{n-1}(t-x)^{2n-1-\alpha}}{\Gamma(2n-\alpha)} + \frac{\binom{n}{2}\lambda^{n-2}(t-x)^{2n-1-2\alpha}}{\Gamma(2n-2\alpha)} \tag{3.2.1}$$

$$+\frac{\binom{n}{n-1}(t-x)^{2n-1-(n-1)\alpha}}{\Gamma(2n-(n-1)\alpha)} + \frac{\binom{n}{n}}{\Gamma(2n-n\alpha)}(t-x)^{2n-1-n\alpha} \tag{3.2.2}$$

$$K_n(x,t,\lambda) = \sum_{m=0}^{n} \frac{C_n^m \lambda^{n-m}}{\Gamma(2n-m\alpha)}(t-x)^{2n-1-m\alpha} \tag{3.2.3}$$

3.3 The Resolvent

Theorem 3.3.1 *[12] The resolvent will be*

$$R(x,t,\lambda) = \sum_{n=1}^{\infty}\sum_{m=0}^{n}(-1)\frac{C_n^m \lambda^{n-m}}{\Gamma(2n-m\alpha)}(t-x)^{2n-1-m\alpha}$$

.Therefore, the corresponding solution of the integral equation have the form

$$u(x) = (1-x) + \int_x^1 \left(\sum_{n=1}^{\infty}\sum_{m=0}^{n}(-1)\frac{C_n^m \lambda^{n-m}}{\Gamma(2n-m\alpha)}(t-x)^{2n-1-m\alpha}\right)(1-t)dt$$

From which it follows that the eigenfunctions are

$$\widetilde{X}_j(x) = (1-x) - \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)\frac{C_n^k \lambda^{n-k}}{\Gamma(2n-k\beta+2)}(1-x)^{2n+1-k\alpha}$$

Of course the system of functions $\left\{\widetilde{X}_j(x)\right\}$ *is the system of eigenfunctios of a problem*

$$X"(x) + D_{0x}^{\alpha}X = \lambda X(x), \tag{3.3.1}$$

$$X(0) = X(1) = 0, \tag{3.3.2}$$

Which conjugate to the problem (3.1.6)-(3.1.7), and eigenvalues of these two problems, how it was noticed, are the same. Here $D_{x1}^{\alpha}X$ *- is the operator of fractional differentiation with a start at a point* X *and the and at 1,i.e.it the operator conjugate to the operator of* *fractional differentiation* D_{0x}^{α}.

Now the solution of the problem (3.1.1)-(3.1.2)-(3.3.3)-(3.1.4) is writting out by standard way

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos(\pi nt) + B_n\sin(\pi nt))\chi_n(x) \tag{3.3.3}$$

Finally, we will to declare constante A_n *and* B_n *so that the initial conditions (3.1.3)-(3.1.4) are satisfied. To find* A_n *let* $t = 0$ *in (3.3.6). Then*

$$u(x,0) = \sum_{n=1}^{\infty}A_n\chi_n(x) = \varphi(x), \tag{3.3.4}$$

Taking to account that systems $\{X_j(x)\}_{i=1}^{\infty}$ *and* $\left\{\widetilde{X}_j(x)\right\}_{i=1}^{\infty}$ *and are biorthogonal we obtain*

$$A_n = C\int_0^1 \varphi_0(x)\widetilde{X}_n(x)dx,\ (n = 1,2,3,...) \tag{3.3.5}$$

To find B_n *we differentiate both sides of (3.3.6) with respect to* t *, then obtain*

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}(\pi n)(-A_n\sin(\pi nt) + B_n\cos(\pi nt))^{*}\chi_n(x) = \Psi(x) \tag{3.3.6}$$

.Put $t = 0$ *in (3.3.9), then*

$$\Psi(x) = \pi\sum_{n=1}^{\infty}nB_n\chi_n(x) \tag{3.3.7}$$

From (3.3.10) follows that

$$\pi nB_n = \int_0^1 \Psi(x)\widetilde{X}_n(x)dx, \tag{3.3.8}$$

Or, what is the same

$$B_n = \frac{1}{\pi n}\int_0^1 \Psi(x)\widetilde{X}_n(x)d. \tag{3.3.9}$$

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Abstract

Fractional analysis is a branch of mathematical analysis which studies the possibility of defining non-integer powers of derivative and integrating operators. we define some useful functions such as Gamma function, Beta function and MittagLeffler function. These functions play a very important role in the theory of fractional order differential calculus. we will present the definition of a fractional derivative and then we study the three most popular approaches to fractional derivatives: the Grünwald-Letnikov, Riemann-Liouville and Caputo approaches as well as their properties. Finally, we study some examples of fractional derivatives. Keywords: Fractional derivation, Grünwald-Letnikov, Riemann-Liouville, Caputo. and operator theory to study the spectral properties (such as eigenvalues and eigenfunctions) of fractional differential operators.

Key words: Closed Operator, spectrum, Differential of fractional order.

ملخص

التحليل الكسري هو فرع من فروع التحليل الرياضي الذي يدرس إمكانية تحديد قوى غير صحيحة للإشتقاق والتكامل
نعرف بعض الدوال المفيدة مثل : الدالة جاما والدالة بيتا والدالة ميتاج-لافلار، هذه الدوال تلعب دورا هاما في نظرية حساب التفاضل ذو الدرجة الكسرية . بعد ذلك سوف نقوم بتعريف المشتق ذو الدرجة الكسرية ثم ندرس الطرق الأكثر شيوعا للمشتقات ذو الدرجة الكسرية طريقة ريمان-ليوفيل وطريقة كابوتو بالإضافة إلى خصائصها و ونظرية المشغل لدراسة الخصائص الطيفية (مثل القيم الذاتية والوظائف الذاتية) لعوامل التفاضل الكسري.
الكلمات المفتاحية: العامل المغلق، الطيف، تفاضل الرتبة الجزئية.

Résumé

L'analyse fractionnaire est une branche de l'analyse mathématique qui étudie la possibilité de définir des puissances non entières des opérateurs de dérivation et d'intégration. Nous définissons certaines fonctions utiles telles que la fonction Gamma, la fonction Béta et la fonction de Mittag-Leffler. Ces fonctions jouent un rôle très important dans la théorie du calcul différentiel d'ordre fractionnaire. Nous présenterons la définition d’une dérivée fractionnaire puis on étudie les trois approches des dérivés fractionnaires les plus populaires: l’approche de Grünwald-Letnikov, de Riemann-Liouville et de Caputo ainsi que leurs propriétés. Enfin, nous étudions quelques exemples des dérivés fractionnaires. Mots clé: Dérivation fractionnaire, Grünwald-Letnikov, Riemann-Liouville, Caputo. de la théorie des opérateurs pour étudier les propriétés spectrales (telles que les valeurs propres et les fonctions propres) des opérateurs différentiels fractionnaires.

Mots Clés: Opérateur fermé, spectre, Differential d'order fractionnaire.