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Similarity solutions for some nonlinear fractional partial differential equations

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Introduction

The field of fractional calculus is very interresting, several definitions have been proposed for fractional derivatives and fractional integrals. These definitions include in particular Riemann-Liouville definition and Caputo definition, which have been used for generalizing many existing models into

the form of fractional partial differential equations (FPDEs), for exemple the classical heat equation or nonlinear heat equation. To obtain a particular form of analytical soltuions, many methods has been used, for exemple the Laplace transform, the Fourier transform, separation of variables technique, etc.

But recently auther methods are used, in particular, symmetry methods are applicable to different types of linear and non linear fractional partial differential equations.

In this work, we study two equations, the first equation is the fractional heat equation, the second equation the generalized non linear fractional heat equation. We use the simillarity method. We use the simillarity method. we based on the work of [3] and [4].

Similarity solutions where founded by transforrmed the fractional partial differential equation to fractional differential equation.

Our work is divided into three chapters, chapter one deals with we present a reminders on the sepcial functions Gamma and Beta and we give some remider on Riemann-liouville integral and function, derivatives of Riemann-Louiville, and fractional derivative defined in Caputo sense.

In the second chapter we will study two problems of the fractional heat equation, the first problem is the classicl problem wich we find some explicite self similar solutions, the second problem concerning the fractional heat equation with variables coefficents.

Finaly, in the third chapter, we concentrate our studies on the generalized non linear fractional heat equation, where some explicite self similar solutions are founded.

Chapter 1

Reminders and Definition

In this chapter we present a reminders on the special functions Gamma and Beta and we give some reminder on Riemann-Liouville integral and function derivatives of Riemann-Liouville and its properities which we will use in the following chapters.

1.1 Gamma Function d'Euler

In mathemamtics, the Gamma function is a complex, considered as a function special in fractional calculus.

Definition 1.1.1 *We call the Gamma function the function defined by:*

$$\begin{aligned} \Gamma(z) &= \int_0^{+\infty} t^{z-1} e^{-t} dt, \ \ (z \in \mathbb{C}, \mathrm{Re}(z) > 0) \\ \textit{with } t^{z-1} &= e^{(z-1)\ln(t)} \end{aligned}$$

Example 1.1.1 *1.* $\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1.$

2. $\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t} dt = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt = 2\int_0^{+\infty} e^{-\tau^2} d\tau.$ *(Posing the change of variable* $t = \tau^2$*)*

Lemma 1.1.1 *for all* $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, $n \in \mathbb{N}$, *we have :*

$$\begin{aligned} 1)\Gamma(z+1) &= z\Gamma(z) \\ 2)\Gamma(n) &= (n-1)! \\ 3)\Gamma(n+\frac{1}{2}) &= \frac{(2n)!\sqrt{\pi}}{4^n n!} \end{aligned}$$

Proof. **1.** Let us represent $\Gamma(z+1)$ by the Euler integral and integrate by parts, we obtain

$$\begin{aligned} \Gamma(z+1) &= \int_0^{+\infty} t^z e^{-t} dt \\ &= \left[-t^z e^{-t}\right]_0^{+\infty} + z\int_0^{+\infty} t^{z-1} e^{-t} dt \\ &= z\Gamma(z) \end{aligned}$$

2.It suffices to apply (1) for $z = n-1$.

3.We will demonstrate the formula

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

By induction on $n \in \mathbb{N}$.

For $n = 0$, we have

$$\Gamma(0+\frac{1}{2}) = \frac{(0)!\sqrt{\pi}}{4^0 0!} = \sqrt{\pi}$$

Suppose the formula verified for $(n-1)$ and consider for n:

We suppose that

$$\Gamma((n-1)+\frac{1}{2}) = \frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!}$$

is verified so

$$\begin{aligned}
\Gamma(n+\frac{1}{2}) &= (n-\frac{1}{2})\Gamma(n-\frac{1}{2}) \\
&= (n-\frac{1}{2})\frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\
&= (\frac{2n-1}{2})\frac{(2n-2)!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\
&= \frac{2n}{2n}\frac{2n-1}{2}\frac{(2n-2)!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\
&= \frac{(2n)!\sqrt{\pi}}{4^n n!}
\end{aligned}$$

So the formulas is verified for n. ■

Remark 1.1.1 *The determination of the Gamma function for negative values not convered by the formula*

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

and the transition from one interval to another $(-1,0),(-2,-1)(-3,-1),...$
The Gamma function does not exist for negative integers.

1.2 The Beta Function

In mathematics, the beta function, also called the Euler integral of the first kind, is a special function that is closely related to the gamma function and to binomial coefficients. It is defined by the integral .

Definition 1.2.1 *The function of Beta is a type of Euler integral defined by:*

$$\beta(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt,\ \ (p,q\in\mathbb{C},\ \ \mathrm{Re}(p)>0,\mathrm{Re}(q)>0)$$

for everything $p,q\in\mathbb{C}$ *, with* $\mathrm{Re}(p)>0,\mathrm{Re}(q)>0$*, we have*

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

let be $D=(0,+\infty)\times(0,+\infty)$ *we have*

$$\begin{aligned}\Gamma(p)\Gamma(q) &= (\int_0^{+\infty} x^{p-1}e^{-x}dx)(\int_0^{+\infty} y^{q-1}e^{-y}dy)\\ &= \iint_D x^{p-1}y^{q-1}e^{-(x+y)}dxdy\end{aligned}$$

Using a change of cords, consider the new change

$$\left\{\begin{array}{c} u = x + y \\ v = \frac{x}{x+y} \end{array}\right. \Longrightarrow \left\{\begin{array}{c} x = uv \\ y = u(1-v) \end{array}\right.$$

$$\frac{\partial(x,y)}{\partial(x,y)} = \left|\begin{array}{cc} v & u \\ 1-v & -u \end{array}\right| = -uv - u(1-v) = -u$$

As well as domain D' *corresponding* D *in the cords* x, v *is*

$$D' = \{(u,v)/u \geq 0, 0 \leq v \leq 1\}$$

So

$$\begin{aligned} \iint\limits_{D} x^{p-1}y^{q-1}e^{-(x+y)}dxdy &= \iint\limits_{D'} (uv)^{p-1}(u(1-v))^{q-1}e^{-u}\left|-u\right|dudv \\ &= \iint\limits_{D'} u^{p+q-1}v^{p-1}(1-v)^{q-1}e^{-u}dudv \\ &= \int_0^{+\infty}\int_0^1 u^{p+q-1}v^{p-1}(1-v)^{q-1}e^{-u}dudv \\ &= \int_0^{+\infty}\left(u^{p+q-1}e^{-u}du\right)\int_0^1 v^{p-1}(1-v)^{q-1}dv \\ &= \Gamma(p+q)\beta(p,q) \end{aligned}$$

therefore

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

1.3 Riemann-Liouville Integral

1.3.1 Function Defined on $[a, b]$

Let be $y : [a, b] \to \mathbb{R}$, a function defined by $[a, b]$.

Let us denote by (I_a^1, y) the primitive of y, which canols ont in a

$$\forall t \in [a, b] \, ; (I_{a+}^1 y)(t) = \int_a^t y(\tau) d\tau.$$

the iteration of (I_a^1, y) allows to obtain the primitive second of y which cancels out in a and whose the derivative cancels out in a, moreover, according to "Fubini" theorem, we have.

$$\begin{aligned} (I_{a+}^1 y)^2(t) &= (I_{a+}^1 y) \circ (I_{a+}^1 f) = \int_a^t \left(\int_a^u y(\tau) d\tau \right) du \\ &= \int_a^t \left(\int_a^u du \right) y(\tau) d\tau \\ &= \int_a^t (t - \tau) y(\tau) d\tau \end{aligned}$$

let be $n \in \mathbb{N}^*$, noting $(I_{a+}^1 y)^n$ the nth iteration of $(I_{a+}^1 y)$ a direct recurrence shows that

$$(I_{a+}^1 y)^n(t) = \frac{1}{(n-1)!} \int_a^t (t - \tau)^{n-1} y(\tau) d\tau$$

Definition 1.3.1 *Riemann-Liouville Integral left of ordre $\alpha > 0$ of y is defined by*

$$\forall t \in [a, b] \, , (I_{a+}^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} y(\tau) d\tau$$

and we define the fractional integral of Rimann-Liouville right of ordre $\alpha > 0$ of y defined by

$$\forall t \in [a, b] \, , (I_{b-}^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} y(\tau) d\tau$$

1.3.2 Functions Defined in $\mathbb{R}^+$ and $\mathbb{R}$

$$\begin{aligned}\forall t &\in \mathbb{R}^+, (I^{\alpha}_{0+}y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau \\ \forall t &\in \mathbb{R}, (I^{\alpha}_{a+}y)(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-\tau)^{\alpha-1}y(\tau)d\tau\end{aligned}$$

Proposition 1.3.1 *For $\alpha > 0$, $\beta > 0$, we have :*

1. $(I^{\alpha}_{a+}(t-a)^{\beta-1})(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}$.
2. $(I^{\alpha}_{b-}(b-t)^{\beta-1})(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-t)^{\alpha+\beta-1}$.

Lemma 1.3.1 *If $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, then the equations*

$$\left(I^{\alpha}_{a+}I^{\beta}_{a+}y\right)(x) = \left(I^{\alpha+\beta}_{a+a}y\right)(x), \text{ and } \left(I^{\alpha}_{b-}I^{\beta}_{b-}y\right)(x) = \left(I^{\alpha+\beta}_{b-} + y\right)(x) \tag{1.3.1}$$

We satisfied at almost every point $x \in [a,b]$ for $y(x) \in L_p(a,b)$, $(1 \leq p \leq \infty)$. If $\alpha+\beta > 1$, then the relation in (1.3.1) hold at any points of $[a,b]$.
The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

1.4 Riemann-Liouville Fractional Derivatives

If $\alpha > 0$, we denote by $[\alpha]$ the whole part of α, $[\alpha]$ is the only integer verifying $[\alpha] \leq \alpha < [\alpha] + 1$.

Let be $y : [a, b] \to \mathbb{R}$, inspired by the classic relation ship

$$\frac{d}{dt} = \frac{d^2}{dt^2} \circ_a I_t^1$$

we can define a fractional derivative of ordre, $0 \leq \alpha < 1$ by

$$\frac{d^\alpha}{dt^\alpha} = \frac{d}{dt} \circ_a I_t^{1-\alpha}$$

more generally, if $\alpha > 0$ and $n = [\alpha] + 1$, we can pose

$$\frac{d^\alpha}{dt^\alpha} = \frac{d^n}{dt^n} \circ_a I_t^{n-\alpha}$$

we get exactly the Rimann-Liouville on the left.

Definition 1.4.1 *Let be $\alpha > 0$, and $n = [\alpha] + 1$ the Rimann-Liouville fractional derivative on the left of ordre of y is defined by*

$$\forall t \in [a, b]\,,\ D_{a+}^{\alpha} y(t) = \left(\frac{d}{dt}\right)^n \circ dI_{a+}^{n-\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} y(\tau) d\tau$$

Definition 1.4.2 *Let be $\alpha > 0$, and $n = [\alpha] + 1$, the Riemann-Liouville fractional derivative on the right of ordr of y is defined by*

$$\forall t \in [a, b]\,, D_{b-}^{\alpha} y(t) = \left(\frac{-d}{dt}\right)^n \circ dI_{b-}^{n-\alpha} y(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (\tau - t)^{n-\alpha-1} y(\tau) d\tau$$

corollary 1.4.1 *If* $0 \leq \Re(\alpha) < 1$, $(\alpha \neq 0)$ *and* $y(x) \in AC[a,b]$, *then*

$$\left(D^{\alpha}_{a+}y\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{y(a)}{(x-a)^{\alpha}} + \int_a^x \frac{y'(t)dt}{(x-t)^{\alpha}}\right] \tag{1.4.2}$$

and

$$\left(D^{\alpha}_{b-}y\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{y(a)}{(x-a)^{\alpha}} + \int_a^x \frac{y'(t)dt}{(t-x)^{\alpha}}\right] \tag{1.4.3}$$

Proposition 1.4.1 *If* $\Re(\alpha) \geqq 0$ *and* $\beta \in \mathbb{C}\,(\Re(\beta) > 0)$. *Then*

$$\left(I^{\alpha}_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}\,(\Re(\alpha) > 0), \tag{1.4.4}$$

$$\left(D^{\alpha}_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}\,(\Re(\alpha) \geqq 0), \tag{1.4.5}$$

and

$$\left(I^{\alpha}_{b-}(b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}\,(\Re(\alpha) > 0), \tag{1.4.6}$$

$$\left(D^{\alpha}_{b-}(b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}\,(\Re(\alpha) \geqq 0), \tag{1.4.7}$$

In particular, if $\beta = 1$ *and* $\beta \in \mathbb{C}\left(\Re\left(\beta\right) > 0\right)$, *then the Riemann-Liouville fractional derivatives of a constant are, in general, not equal to zero*

$$\left(D_{a+}^{\alpha}1\right)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \left(D_{b-}^{\alpha}1\right)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)} \left(0 < \Re\left(\alpha\right) < 1\right), \tag{1.4.8}$$

On the other hand, for $j = 1, 2, ..., \left[\Re\left(\alpha\right)\right] + 1,$

$$\left(D_{a+}^{\alpha}\left(t-a\right)^{\alpha-j}\right)(x) = 0, \left(D_{b-}^{\alpha}\left(b-t\right)^{\alpha-j}\right)(x) = 0 \tag{1.4.9}$$

Lemma 1.4.1 *If* $\Re\left(\alpha\right) > 0$ *and* $f(x) \in L_p(a,b)\,(1 \leqq p \leqq \infty)$, *then the following equalities*

$$\left(D_{a+}^{\alpha}I_{a+}^{\alpha}f\right)(x) = f(x), \text{ and } \left(D_{b-}^{\alpha}I_{b-}^{\alpha}f\right)(x) = f(x)\left(\Re\left(\alpha\right) > 0\right) \tag{1.4.10}$$

hold almost everywhere on $[a,b]$.

Proposition 1.4.2 *If* $\Re\left(\alpha\right) > \Re\left(\beta\right) > 0$, *then, for* $f(x) \in L_p(a,b)\,(1 \leqq p \leqq \infty)$, *the relations*

$$\left(D_{a+}^{\beta}I_{a+}^{\alpha}f\right)(x) = I_{a+}^{\alpha-\beta}f(x), \text{ and } \left(D_{b-}^{\beta}I_{b-}^{\alpha}f\right)(x) = I_{b-}^{\alpha-\beta}f(x) \tag{1.4.11}$$

hold almost everywhere on $[a,b]$.

In particular, when $\beta = k \in \mathbb{N}$ *and* $\Re\left(\alpha\right) > k$, *then*

$$\left(D^{k}I_{a+}^{\alpha}f\right)(x) = I_{a+}^{\alpha-k}f(x), \text{ and } \left(D^{k}I_{b-}^{\alpha}f\right)(x) = (-1)^{k}\,I_{b-}^{\alpha-k}f(x) \tag{1.4.12}$$

Lemma 1.4.2 *Let* $\Re(\alpha) > 0,$ *and* $n = [\Re(\alpha)] + 1.$ *Also let* $g_{n-\alpha} = \left(I_{b-}^{n-\alpha} g\right)(x)$ *be the fractional integral (2.1.2) if the order* $n - \alpha.$

(a) *If* $1 \leqq p \leqq \infty$ *and* $f(x) \in I_{b-}^{\alpha}(L_p),$ *then*

$$\left(I_{b-}^{\alpha} D_{b-}^{\alpha} g\right)(x) = g(z) \tag{1.4.13}$$

(b) *If* $f(x) \in L_1(a,b)$ *and* $f_{n-\alpha}(x) \in AC^n[a,b],$ *then the equality*

$$\left(I_{b-}^{\alpha} D_{b-}^{\alpha} g\right)(x) = g(z) - \sum_{j=1}^{n} \frac{(-1)^{n-j}\, g_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha - j + 1)} (b - x)^{\alpha - j}, \tag{1.4.14}$$

holds almost everywhere on $[a,b].$

In particular, if $0 < \Re(\alpha) < 1,$ *then*

$$\left(I_{b-}^{\alpha} D_{b-}^{\alpha} g\right)(x) = g(z) - \frac{g_{1-\alpha}(a)}{\Gamma(\alpha)} (b - x)^{\alpha - 1}, \tag{1.4.15}$$

where $f_{1-\alpha}(x) = \left(I_{a+}^{1-\alpha} f\right)(x),$ *while for* $\alpha = n \in \mathbb{N},$ *the following equality holds:*

$$\left(I_{b-}^{\alpha} D_{b-}^{\alpha} g\right)(x) = g(z) - \sum_{k=0}^{n} \frac{(-1)^{k}\, g^{(k)}(b)}{k!} (b - x)^{k}, \tag{1.4.16}$$

*Now we present the rules for fractional integration by parts, which were proved in Samko et al.([**?**], Carollary of theorem 3.5 and Corollary 2 of theorem 2.4).*

1.5 The fractional derivation in caputo sense

Definition 1.5.1 *A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_m^μ if $f^m \in C_\mu, m \in \mathbb{N}$.*

Definition 1.5.2 *The fractional derivative in Caputo sense of $f(t) \in C_{-1}^m, m \in \mathbb{N}, t > 0$ is defined as [4]*

$$D_t^\beta f(t) = \begin{cases} I^{m-\beta} \frac{d^m}{dt^m} f(t), m-1 < \beta < m \\ \frac{d^m}{dt^m} f(t), \beta = m \end{cases} \tag{1.5.17}$$

Some basic properties of Caputo fractional derivative are:

1.*If $m - 1 < \beta \leq m$, $m \in \mathbb{N}$ and $f \in C_\mu^m, \mu \geq -1$, then:*

$$D_t^\beta \left[I^\beta f(t) \right] = f(t) \tag{1.5.18}$$

2.

$$I^\beta \left[D_t^\beta f(t) \right] = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}, t > 0 \tag{1.5.19}$$

3.

$$D_t^\beta t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)} t^{\gamma-\beta} \tag{1.5.20}$$

For more details on Caputo fractional derivative definition and its properties see[[7], [5], [4]].

Chapter 2

Fractional Heat Conduction Equation

In this chapter we study some problems of fractional heat conduction equation, we begin with the following problem.

2.1 Problem 1

The fractional heat conduction equation in solids can be written in the from

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left[D\frac{\partial T}{\partial x}\right], \quad x \in (0, \infty), t > 0 \tag{2.1.1}$$

where T is the perturbation of the temperature and D is the thermal diffusivity, is α fractional derivate. This equation also describes heat conduction in incompressible liquids if the convective term is negligibly small compared to the conductive term and is the case when the liquid is at rest or the temperature of the liquid changes much faster than the liquid flowd.

In order to prove that they have similarity solution, we will transform (2.1.1) by introduction new independent and dependent variables denoted by $\widetilde{t}, \widetilde{x}, \widetilde{T}$ in the following way:

$$t = a\widetilde{t}, x = a^p\widetilde{x}, T = a^q\widetilde{T}(\widetilde{x},\widetilde{t}) \tag{2.1.2}$$

Where p and q are parameters.

They will be determined from the condition that (2.1.1) remains constant under this transformation, one can easily verify that the transformed equation becomes

$$\begin{aligned}
t &= a\widetilde{t}, x = a^p\widetilde{x}, T = a^q\widetilde{T}(\widetilde{x},\widetilde{t}) \\
\frac{\partial^\alpha\left(a^q\widetilde{T}\right)}{\partial\left(a\widetilde{t}\right)^\alpha} &= \frac{\partial}{\partial\left(a^p\widetilde{x}\right)}\left[D\frac{\partial\left(a^q\widetilde{T}\right)}{\partial\left(a^p\widetilde{x}\right)}\right] \\
a^{q-\alpha}\frac{\partial^\alpha\left(\widetilde{T}\right)}{\partial\widetilde{t}^\alpha} &= a^{q-2p}\frac{\partial}{\partial\widetilde{x}}\left(D\frac{\partial\widetilde{T}}{\partial\widetilde{x}}\right)
\end{aligned}$$

$$\begin{aligned}
q-\alpha &= q-2p \\
&\Longrightarrow p = \frac{\alpha}{2}
\end{aligned} \tag{2.1.3}$$

We will delete a between the transformation formulas, and get

$$\begin{aligned}
x &= a^p\widetilde{x} \Longrightarrow x^{\frac{1}{p}} = a\left(\widetilde{x}\right)^{\frac{1}{p}} \\
t &= a^p\widetilde{t} \Longrightarrow \frac{t}{x^{\frac{1}{p}}} = \frac{\widetilde{t}}{a\left(\widetilde{x}\right)^{\frac{1}{p}}}
\end{aligned}$$

and

$$\begin{aligned}
T &= a^q\widetilde{T} \Longrightarrow \frac{T}{\widetilde{T}} = a^q \\
\left(\frac{x}{\widetilde{x}}\right)^{\frac{1}{p}} &= a \Longrightarrow \frac{T}{\widetilde{T}} = \left(\frac{x}{\widetilde{x}}\right)^{\frac{q}{p}}
\end{aligned}$$

This shows that $T(x,t)$ can be experessed as:

$$T = x^{\frac{q}{p}} U(x^{-\frac{1}{p}} t)$$

Where U is a function of the combination of independent variables, $x^{-\frac{1}{p}} t$ alone satisfying an ordinary differential equation.

So the follwing equation:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\partial}{\partial x}\left[D \frac{\partial T}{\partial x}\right]$$

Has similar solutions to the from

$$T(x,t) = x^a U(\xi), \xi = x^b t \tag{2.1.4}$$

Where a and b constants. And $b = \frac{-2}{\alpha}$ while a remains arbitrary.

We will need the follwing formulas in order to derive the normal differential equation for $U(\xi)$.

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{T(x,\tau)}{(t-\tau)^\alpha} d\tau$$

We're got:

$$t = \xi x^{-b}, y = x^b \tau, T = x^q U(\xi)$$

and

$$dy = x^b d\tau \implies d\tau = x^{-b} dy$$

So

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\left(\xi x^{-b}\right)} \int_0^\xi \frac{x^a U(y)}{(\xi - y)^\alpha} x^{b\alpha} x^{-b} dy$$

We obtain

$$\frac{1}{\Gamma(1-\alpha)} x^b \frac{\partial}{\partial \xi} \int_0^\xi \frac{x^a U(y)}{(\xi - y)^\alpha} x^{b\alpha} x^{-b} dy = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial \xi} \int_0^\xi \frac{x^{a+b\alpha} U(y)}{(\xi - y)^\alpha} dy \tag{2.1.5}$$

and

$$\frac{d^\alpha U(\xi)}{d\xi^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\xi}\int_0^\xi \frac{U(y)}{(\xi-y)^\alpha}dy$$

there fore

$$\frac{\partial^\alpha T}{\partial t^\alpha} = x^{a+b\alpha}\frac{d^\alpha U(\xi)}{d\xi^\alpha} \tag{2.1.6}$$

2.1.1 Derivatives Calculation

We have similar solution to the form

$$T(x,t) = x^a U\left(\xi\right), \xi = x^b t$$

We will calculate the derviatives:

$$\begin{aligned}\frac{\partial T}{\partial t} &= a x^{a-1}U(\xi) + x^a \frac{dU(\xi)}{d\xi} b x^{b-1} t \\ &= x^{a-1}K_1(\xi)\end{aligned}$$

and

$$K_1\left(\xi\right) = aU\left(\xi\right) + b\xi\frac{dU(\xi)}{d\xi}$$

$$\frac{\partial^2 T}{\partial x^2} = x^{a-2}K_2(\xi)$$

$$\begin{aligned}K_2(\xi) &= (a-1)K_1\left(\xi\right) + b\xi\frac{dK_1\left(\xi\right)}{d\xi} \\ &= a(a-1)U(\xi) + b(2a+b-1)\xi\frac{dU(\xi)}{d\xi} \\ &\quad + b^2\xi^2\frac{d^2U(\xi)}{d\xi^2}\end{aligned} \tag{2.1.7}$$

Proposition 2.1.1 *For the linear fractional diffusion equation (2.1.1) the similarity* *solution has the form*

$$T(x,t) = x^a U(\xi), \xi = x^b t \tag{2.1.8}$$

Where

$$b = \frac{-2}{\alpha}, a \text{ remains arbitrary}$$

Also $U(\xi)$ is a solution to the following ordinary nonlinear fractional equation.

$$\frac{d^\alpha U(\xi)}{d\xi^\alpha} = D\left[a(a-1)U + b(2a+b-1)\xi\frac{dU}{d\xi} + b^2\xi^2\frac{d^2U(\xi)}{d\xi^2}\right] \tag{2.1.9}$$

Remark 2.1.1 *The similarity variable obtained in [8.4] for linear fractional equation reads $z = \frac{x}{t^{\alpha/2}}$ comparing this with (2.1.8) we conclude that the relation between our similarity variable and the similarity variable of [8.4], is $\xi = z^{\frac{-2}{\alpha}}$. We have $\alpha = 1, D = 1, b = -2$, in this case (2.1.1) becons:*

$$U^{(1)} = \varphi\xi^2 U^{(2)} + 6\xi U^{(1)} \tag{2.1.10}$$

The solution to (2.1.10) reads

$$U(\xi) = \int_{1/(2\sqrt{\xi})}^{\infty} e^{(-S^2)} ds \tag{2.1.11}$$

2.2 Problem 2

Now we consider the fractional order heat equations with variable coefficients which written as follows

$$kt^{\alpha+2}\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\partial^2 T}{\partial x^2}, 0 < \alpha \leq 1 \tag{2.2.12}$$

The fractional order heat equation with variable coefficents with the initial condition $T(x, 0) = 0$, $T(x, t)$ with respect to the time variable t in the Caputo sense and k is an arbitrary constant.

To solve equation (2.2.12), first we perform its scaling transformation using similarity methods, see [10] and [9].

They will be determined from the condition that (2.2.12) by the new independent and dependent variables denoted by $\widetilde{t}, \widetilde{x}$, and u defined in the following way

$$t = \lambda\widetilde{t}, x = \lambda^p\widetilde{x}; T = \lambda^q\widetilde{T}. \tag{2.2.13}$$

where is called the scaling parameter and p and q are arbitrary constants, to be determined such that equation (2.2.12) remains invariant under this transformation. Using Caputo definition (1.5.17), one may easily verify that

$$\begin{aligned} \frac{\partial^\alpha T}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\tau)^\alpha}\frac{\partial T(x,\tau)}{\partial \tau}d\tau \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{\widetilde{t}} \frac{\lambda^q}{\left(\lambda\widetilde{t}-\lambda\widetilde{\tau}\right)^\alpha}\frac{\partial \widetilde{u}(\widetilde{x},\widetilde{\tau})}{\partial \widetilde{\tau}}d\widetilde{\tau} \\ &= \lambda^{q-\alpha}\frac{\partial^\alpha \widetilde{T}}{\partial \widetilde{t}^\alpha}. \end{aligned} \tag{2.2.14}$$

where $\tau = \lambda\widetilde{\tau}$, also

$$\frac{\partial^2 T}{\partial t^2} = \lambda^{q-2p}\frac{\partial^2 \widetilde{T}}{\partial \widetilde{x}^2}. \tag{2.2.15}$$

Hence by substituting equations (2.2.14) and (2.2.15) into equation (2.2.12), we get

$$k\lambda^{\alpha+2}\frac{\partial^{\alpha}\widetilde{T}}{\partial\widetilde{t}^{\alpha}}=\lambda^{q-2p}\frac{\partial^{2}\widetilde{T}}{\partial\widetilde{x}^{2}},0<\alpha\leq 1 \tag{2.2.16}$$

From equation (2.2.16), it is clear that by setting $p=-1$ then equation (2.2.12) is invariant under transformation (2.2.13). The characteristic equation associated with transformation (2.2.13) is given by

$$\frac{dT}{qT}=\frac{dx}{-x}=\frac{dt}{t} \tag{2.2.17}$$

This shows that $T(x,t)$ can be expressed as

$$T(x,y)=x^{-q}f(\xi). \tag{2.2.18}$$

where $\xi=xt$.

By using formula (2.2.18) and again using Caputo definition (1.5.17) for $0<\alpha<1$, we have

$$\begin{aligned}\frac{\partial^{\alpha}T}{\partial t^{\alpha}} &= \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{1}{(t-\tau)^{\alpha}}\frac{\partial T\left(x,\tau\right)}{\partial\tau}d\tau \\ &= \frac{1}{\Gamma(1-\alpha)}\int_{0}^{\xi}\frac{x^{-q}f'(\xi)x}{\left(\frac{\xi}{x}-\frac{y}{x}\right)^{\alpha}}x^{-1}dy \\ &= \frac{1}{\Gamma(1-\alpha)}\int_{0}^{\xi}\frac{x^{-q}f'(\xi)}{x^{-\alpha}\left(\xi-y\right)^{\alpha}}dy \\ &= x^{-q+\alpha}\frac{d^{\alpha}f}{d\xi^{\alpha}}.\end{aligned} \tag{2.2.19}$$

Where $y=x\tau$. And

$$\frac{\partial^{2}T}{\partial x^{2\alpha}}=x^{-q-2}\xi^{2}\frac{d^{2}f}{d\xi^{2}}-2q\xi\frac{df}{d\xi}+(q)(q+1)f(\xi). \tag{2.2.20}$$

Putting $q=0$ and substituting equations (2.2.19) and (2.2.20) into equation (2.2.12), the resulting ordinary FDE is given by

$$k\xi^{\alpha}\frac{d^{\alpha}f}{d\xi^{\alpha}}=\frac{d^{2}f}{d\xi^{2}} \tag{2.2.21}$$

with the initial condition $f(0)=0$.

2.3 The solution

We will find the solution of equation(2.2.21), we consider the FDE of the fomula

$$k\xi^{\alpha}\frac{d^{\alpha}f}{d\xi^{\alpha}}=\frac{d^{2}f}{d\xi^{2}} \tag{2.3.22}$$

The solution is assumed in the power series form

$$f(\xi)=\sum_{n=0}^{\infty}a_{n}\xi^{n} \tag{2.3.23}$$

By substituting series solution (2.3.23) into equation (2.3.22) and equating the coefficients of similar powers in both sides, we get

$$\begin{aligned} a_{2n} &= 0, n=0,1,2,... \qquad (2.3.24)\\ a_{3} &= k\frac{\Gamma(2)}{3!\Gamma(2-\alpha)}a_{1},\\ a_{5} &= k^{2}\frac{\Gamma(4)\Gamma(2)}{5!\Gamma(4-\alpha)\Gamma(2-\alpha)}a_{1}\\ & .\\ & .\\ & . \end{aligned}$$

The resulting solution becomes

$$f(\xi) = a_1 \left[k\frac{\Gamma(2)}{3!\Gamma(2-\alpha)}\xi^3 + k^2\frac{\Gamma(4)\Gamma(2)}{5!\Gamma(4-\alpha)\Gamma(2-\alpha)}\xi^5 + ... \right] \tag{2.3.25}$$

which has the fractional order derivative

$$\frac{d^\alpha f}{d\xi^\alpha} = a_1 \left[\frac{1}{\Gamma(2-\alpha)}\xi^{1-\alpha} + k\frac{\Gamma(4)\Gamma(2)}{3!\Gamma(4-\alpha)\Gamma(2-\alpha)}\xi^{3-\alpha} + k^2\frac{\Gamma(2)\Gamma(4)\Gamma(6)}{5!\Gamma(4-\alpha)\Gamma(2-\alpha)\Gamma(6-\alpha)}\xi^{5-\alpha} + ... \right]$$

Remark 2.3.1 *Based on these results, we propose a definition for error function with generalized coefficients and discuss its convergence.*

So An error function with generalizeed coefficient is defined in the form

$$\mathrm{erf}(\xi;k;\alpha) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1} \tag{2.3.26}$$

where

$$\begin{aligned} a &= 1 \\ a_{2n+1} &= \frac{k^n}{(2n+1)!}\prod_{i=1}^{n}\frac{\Gamma(2i)}{\Gamma(2i-\alpha)}, n = 1, 2, ... \end{aligned} \tag{2.3.27}$$

By this definition, the solution to FDE (2.3.22) is given by

$$f(\xi) = c_1 + c_2\,\mathrm{erf}(\xi;k;\alpha) \tag{2.3.28}$$

where c_1 and c_2 are arbitrary constants. Hence, in the solution of FDE (2.2.21) with the initial condition $f(0) = 0$, c_1 equals zero.

Evidently, when substitute $\alpha = 1$ and $k = -\lambda^2$ in definition (2.3.26)-(2.3.27), the obtained series is the Taylor series of the classical integer-order error function erf $\left(\frac{\lambda\xi}{\sqrt{2}}\right)$.

This coincides with the fact that in this case, equation (2.3.22) becomes the ordinary differential equation in the form $-\lambda^2\xi\frac{df}{d\xi} = \frac{d^2 f}{d\xi^2}$, which has the general solution of the form $f(\xi) = c_1 + c_2\,\mathrm{erf}(\frac{\lambda\xi}{\sqrt{2}})$, with $c_1 = 0$ to satisfy the initial condition $f(0) = 0$. Also, when approaches zero and $k = -\lambda^2$, in this case the obtained series is the Taylor series of the classical trigonometric sine function $\sin(\lambda\xi)$. This coincides with the fact that in this case equation (2.3.22) becomes the ordinary differential equation with the form $-\lambda^2 f = \frac{d^2 f}{d\xi^2}$ which has the general solution of the form$f(\xi) = c_1 \sin(\lambda\xi) + c_2 \cos(\lambda\xi)$, with $c_2 = 0$ to satisfy the initial condition $f(0) = 0$.

To check the radius of convergence of the error function with generalized coefficients defined by (2.3.26)-(2.3.27), we evaluate the limit

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{2n+3}}{a_{2n+1}}\right| &= \lim_{n\to\infty}\left|\frac{\Gamma(2n+2)}{(2n+3)\,(2n+2)\,\Gamma(2n+2-\alpha)}\right| \\ &= |k|\lim_{n\to\infty}\left|\frac{\Gamma(2n+2)}{(2n+3)\,(n+1)\,\Gamma(2n+2-\alpha)}\right| \\ &= 0. \end{aligned} \tag{2.3.29}$$

Hence, the series converges for all ξ.

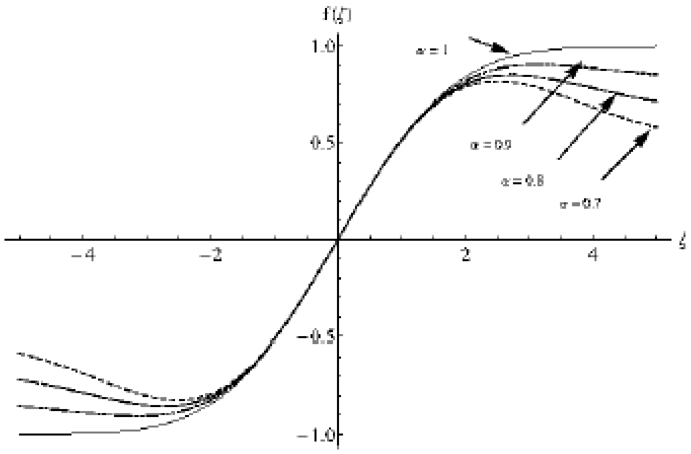

*Fig.*1.The function $\mathrm{erf}(\xi;\frac{-1}{2};\alpha)$ at different values of α

Figure 1 shows the effect of changing the order of fractional derivative on the behavior of the series representation of the solution function erf$(\xi; \frac{-1}{2}; \alpha)$at different values of (This figure is graphed with $n = 40$). Figure 1 also illustrates that as

approaches one, the graph of the solution coincides with the graph of the classical error function erf $\left(\frac{\xi}{2}\right)$ which is the solution of the integer-order differential equation corresponding to FDE (2.2.21).

Chapter 3

Generalized Heat Conduction Equation

In this chapter we will study generalization of (2.1.1) proposed by cited [3], which written under the form:

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left[\left[k + mT^{n}\right]\frac{\partial T}{\partial x}\right], x \in (0, \infty), t > 0, \tag{3.0.1}$$

Our aim is to seek exact similarity solutions.

3.1 Similarity Transformation

We have a generalized heat conduction equation, to prove it prossesses similarity solution, we will first transform its Lie-group scaling transformation (see [1]).

We will transform (3.0.1) by introducing new independent and dependent variables denoted by $\widetilde{t}, \widetilde{u}, \widetilde{T}$ in the following way

$$t = \lambda \widetilde{t}, x = \lambda^{p} \widetilde{x}, T = \lambda^{q} \widetilde{T}(\widetilde{x}, \widetilde{t}) \tag{3.1.2}$$

p, q parameters

Where we will determine p and q from the condition that (3.0.1) remains invariant under this transformation.

Where we have it

$$\begin{aligned}
\frac{\partial^\alpha T}{\partial t^\alpha} &= \frac{\partial}{\partial x}\left[[k+mT^n]\,\frac{\partial T}{\partial x}\right] \\
\frac{\partial^\alpha (\lambda^q \widetilde{T})}{\partial (\lambda \widetilde{t})^\alpha} &= \frac{\partial}{\partial (\lambda^p \widetilde{x})}\left[\left[k+m(\lambda^q \widetilde{T})^n\right]\frac{\partial (\lambda^q \widetilde{T})}{\partial\left(\lambda^p \widetilde{x}\right)}\right] \\
\lambda^{q-\alpha}\frac{\partial^\alpha \widetilde{T}}{\partial \widetilde{t}^\alpha} &= \lambda^{q-2p} k \frac{\partial^2 \widetilde{T}}{\partial \widetilde{x}^2} + \lambda^{q(1+n)-2p} m \frac{\partial}{\partial \widetilde{x}}\left(\widetilde{T}^n \frac{\partial \widetilde{T}}{\partial \widetilde{x}}\right)
\end{aligned}$$

If $k \neq 0$

The condition of invariance obriously reads:

$$\begin{aligned}
q-\alpha &= q-2p = q(1+n)-2p \\
q-\alpha &= q-2p \\
&\Longrightarrow \alpha = 2p \Longrightarrow p = \frac{\alpha}{2}
\end{aligned}$$

$$\begin{aligned}
q-2p &= q(1+n)-2p \\
&\Longrightarrow q = q(1+n) \\
&\Longrightarrow q = 0
\end{aligned}$$

$$\text{We get } q=0 \text{ and } p=\frac{\alpha}{2} \tag{3.1.3}$$

If $k = 0$

The same condition simplifies to

$$\begin{aligned} q-\alpha &= q(1+n)-2p \\ q-\alpha &= q(n+1)-2p \\ q-\alpha &= qn+q-2p \\ -\alpha &= qn-2p \\ &\Longrightarrow \quad q=\frac{2p-\alpha}{n} \end{aligned}$$

we get

$$q=\frac{2p-\alpha}{n} \tag{3.1.4}$$

Now, we can eliminate λ between the transformation formules, and get

$$\begin{aligned} x &= \lambda^p\widetilde{x} \Longrightarrow (x)^{\frac{1}{p}}=\lambda\widetilde{x} \\ t &= \lambda\widetilde{t} \Longrightarrow \frac{t}{(x)^{\frac{1}{p}}}=\frac{\widetilde{t}}{\widetilde{x}^{\frac{1}{p}}} \\ \text{and } T &= \lambda^q\widetilde{T} \Longrightarrow \frac{T}{\widetilde{T}}=\lambda^q \\ \left(\frac{x}{\widetilde{x}}\right)^{\frac{1}{p}} &= \lambda \Longrightarrow \frac{T}{\widetilde{T}}\left(\frac{x}{\widetilde{x}}\right)^{\frac{p}{q}} \end{aligned}$$

This shows that $T(x,t)$ can be expressed as

$$T=x^{\frac{q}{p}}U(x^{\frac{-1}{p}}t)$$

Where U is a function of the combination of independent variables, $x^{\frac{-1}{p}}t$ alone satisfying an ordinary differential equation.

So the following equation:

$$\frac{\partial^\alpha T}{\partial t^\alpha}=\frac{\partial}{\partial x}\left[[k+mT^n]\frac{\partial T}{\partial x}\right]$$

has similar solution to the form

$$T(x,t) = x^a U(\xi), \xi = x^b t \tag{3.1.5}$$

where a and b constants.

We will define it through (3.1.3), (3.1.4), so if $k \neq 0$, we have $a = \frac{q}{p}$ and $q = 0 \implies a = 0$ and $b = \frac{-1}{p} \implies b = \frac{-1}{1} \times \frac{2}{\alpha} \implies b = \frac{-2}{\alpha}$

If $k = 0 \implies b = \frac{(an-2)}{\alpha}$

while a remins arbitrary.

We will need the following formulas in order to derive the normal differential equation for $U(\xi)$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{T(x,\tau)}{(t-\tau)^\alpha} d\tau$$

where got

$$t = \xi x^{-b}, y = x^b \tau, \tau = x^a U(\xi)$$

and

$$dy = x^b d\tau \implies d\tau = x^{-b} dy$$

So

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial\left(\xi x^{-b}\right)} \int_0^\xi \frac{x^a U(y) x^{b\alpha}}{(\xi - y)^\alpha} x^{-b} dy$$

We obtain:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} x^b \frac{\partial}{\partial \xi} \int_0^\xi \frac{x^a U(y)}{(\xi - y)^\alpha} x^{b\alpha} x^{-b} dy \tag{3.1.6}$$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial \xi} \int_0^\xi \frac{x^{a+b\alpha} U(y)}{(\xi - y)^\alpha} dy$$

and

$$\frac{d^\alpha U(\xi)}{d\xi^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial \xi} \int_0^\xi \frac{U(y)}{(\xi - y)^\alpha} dy$$

There fore

$$\frac{\partial^\alpha T}{\partial t^\alpha} = x^{a+b\alpha}\frac{d^\alpha U(\xi)}{d\xi^\alpha} \tag{3.1.7}$$

Similarly we have:

$$\begin{aligned} T(x,t) &= x^\alpha U(\xi), \xi = x^b t \\ \frac{\partial T}{\partial t} &= ax^{a-1}U(\xi) + x^a \frac{dU(\xi)}{d\xi} bx^{b-1}t \\ &= x^{a-1}L_1(\xi) \end{aligned}$$

and

$$L_1(\xi) = aU(\xi) + b\xi\frac{dU(\xi)}{d\xi}$$

$$\frac{\partial^2 T}{\partial x^2} = x^{a-2}L_2(\xi)$$

$$\begin{aligned} L_2(\xi) &= '(a-1)L_1(\xi) + b\xi\frac{dL_1(\xi)}{d\xi} \\ &= a(a-1)U(\xi) + b(2a+b-1)\xi\frac{dU(\xi)}{d\xi} + b^2\xi^2\frac{d^2U(\xi)}{d\xi^2} \end{aligned}$$

$$\frac{\partial}{\partial x}\left[T^n\frac{\partial T}{\partial x}\right] = x^{a(1+n)-2}\left[\left[a(1+n)-1\right]L + b\xi\frac{dL}{d\xi}\right] \tag{3.1.8}$$

$$L(\xi) = U^n\left(aU + b\xi\frac{dU(\xi)}{d\xi}\right)$$

From (3.1.5) and by subsittuting (3.1.7) and (3.1.8) in (3.0.1) we obtain: for the nonlinear fractional diffusion equation (3.0.1)the similarity solution has the from:

$$T(x,t) = x^a U(\xi), \xi = x^b t \tag{3.1.9}$$

Where

$$\begin{aligned}\text{for } k &\neq 0 \\ a &= 0 \text{ and } b = \frac{-2}{\alpha} \\ \text{for } k &= 0 \\ b &= \frac{an-2}{\alpha}, a \text{ arbitrary}\end{aligned} \tag{3.1.10}$$

Also $U(\xi)$ is a solution to the following nonlinear ordinary fractional equation

$$\begin{aligned}\frac{d^{\alpha}U(\xi)}{d\xi^{\alpha}} &= k\left[a(a-1)U + b(2a+b-1)\xi\frac{dU}{d\xi} + b^2\xi^2\frac{d^2U(\xi)}{d\xi^2}\right] + \\ & m\left\{[a(1+n)-1]\,L + b\xi\frac{dL}{d\xi}\right\} \\ \text{Where } L &= U^n(aU + b\xi\frac{dU}{d\xi})\end{aligned} \tag{3.1.11}$$

3.2 Exact Solution

3.2.1 First Solution

We will now find an exact solution for (3.1.11).

Suppose $k = 0$ in (3.0.1) and find the exact solution in the following from

$$U = U_1\xi^{\beta}, T(x,t) = x^a U(\xi) \text{ and } \xi = x^b t$$

Where U_1 and β are constants.

Then, it is readily shown that the left-hand side of

$$\begin{aligned}
\frac{d^{\alpha}U(\xi)}{d\xi^{\alpha}} &= k\left[a(a-1)U + b(2a+b-1)\xi\frac{dU}{d\xi} + b^2\xi^2\frac{d^2U(\xi)}{d\xi^2}\right] + m\left\{[a(1+n)-1]\,L + b\xi\frac{dL}{d\xi}\right\} \\
\text{and } L &= U^n(aU + b\xi\frac{dU}{d\xi})
\end{aligned}$$

Transforms for $\beta > -1$ into

$$U_1\frac{B(1-\alpha, 1+\beta)}{\Gamma(1-\alpha)}\,(1-\alpha+\beta)\,\xi^{\beta-\alpha},$$

and While the right hand side becomes:

$$\begin{aligned}
U &= U_1\xi^{\beta} \\
\frac{dU}{d\xi} &= U_1\beta\xi^{\beta-1}
\end{aligned}$$

$$
\begin{aligned}
L &= U^n \left(aU + b\xi \frac{dU}{d\xi} \right) \\
&= aU^{n+1} + b\xi U^n \frac{dU}{d\xi} \\
&= aU^{n+1} + b\xi U^n \left(U_1 \beta \xi^{\beta-1} \right) \\
&= aU^{n+1} + bU^n U_1 \beta \xi^{\beta}
\end{aligned}
$$

So

$$
L = aU_1^{n+1} \xi^{\beta(n+1)} + bU_1^{n+1} \beta \xi^{\beta(n+1)}
$$

$$
\frac{\partial L}{\partial \xi} = \beta(n+1) a U_1^{n+1} \xi^{B(n+1)-1} + b\beta(n+1)\beta U_1^{n+1} \xi^{\beta(n+1)-1}
$$

$$
\begin{aligned}
b\xi \frac{\partial L}{\partial \xi} &= b\beta(n+1) a U_1^{n+1} \xi^{B(n+1)} + bb\beta(n+1)\beta U_1^{n+1} \xi^{\beta(n+1)} \\
&\Longrightarrow \\
m \left\{ \left[a(n+1) - 1 \right] L + b\xi \frac{\partial L}{\partial \xi} \right\} &= \xi^{\beta(n+1)} m U_1^{n+1} \begin{pmatrix} b\beta(n+1)a + bb\beta(n+1)\beta \\ +a(1+n)a + a(1+n)b\beta - a - b\beta \end{pmatrix} \\
&= m(a + b\beta)(U_1)^{(n+1)} \left[a(1+n) - 1 + b\beta(n+1) \right] \xi^{(n+1)\beta}, \\
U &= U_1 \xi^{\beta}.
\end{aligned}
$$

It can be a solution of (3.1.11) when

$$
\begin{aligned}
\beta &= \frac{-\alpha}{n}, n > \alpha \text{ only} \\
\text{and } B(p,q) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}
\end{aligned}
$$

(see [6]), we get the following expression for the parameter U_1 So

$$U_1 = \left[\frac{n^2\Gamma(1-\frac{\alpha}{n})}{2m(2+n)\Gamma(1-2-\frac{\alpha}{n})}\right]^{\frac{1}{n}}$$

So we have it

$$U = U_1\xi^{\beta}, T(x,t) = x^a U(\xi), \xi = x^b t$$

At that, a is eliminated and doesnot effect the final result.

So

$$\begin{aligned} T(x,t) &= x^a U_1 (x^a t)^{\beta},\ b = \frac{an-2}{\alpha}, \beta = \frac{-\alpha}{n} \\ &\Longrightarrow\ T(x,t) = U_1 x^{a+b\beta} t^{\beta} \\ T(x,t) &= U_1 x^{\frac{2}{n}} t^{\frac{-\alpha}{n}} \end{aligned}$$

So

$$T(x,t) = U_1\left(\frac{x}{t^{\frac{\alpha}{n}}}\right)^{\frac{2}{n}}$$

3.2.2 Second Solution

We can be obtained another exact solution to (3.1.2) which has the form of a wave propagating along the x-axis with certain speed c and cannot be obtained from (3.1.11).

We'll get this solution, by doing another similarity transformation (3.0.1) for $k = 0$

$$T = x^a U(\xi), \xi = \frac{ct}{x} - 1$$

Where a and c are constants. So

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{c^\alpha x^{a-\alpha}}{\Gamma(1-\alpha)} \frac{d}{d\xi} \int_{-1}^{\xi} \frac{U(\eta)}{(\xi - \eta)^\alpha} d\eta$$

and

$$\begin{aligned} \frac{\partial}{\partial x}\left[mT^n \frac{\partial T}{\partial x}\right] &= mx^{(n+1)a-2} \left\{ \left[(n+1)a - 1\right] L - (1+\xi)\frac{dL}{d\xi} \right\} \\ \text{Where } L(\xi) &= U^n \left[aU - (1+\xi)\frac{dU}{d\xi}\right] \end{aligned}$$

So the condition for existence of this similarity solution becomes:

$$\begin{aligned} a - \alpha &= (n+1)a - 2 \\ &\Longrightarrow a - \alpha = na + a - 1 \\ &\Longrightarrow 2 - \alpha = na \end{aligned}$$

$$\text{So } a = \frac{2-\alpha}{n}$$

By substitution, we find

$$\frac{c^\alpha}{\Gamma(1-\alpha)} \frac{d}{d\xi} \int_{-1}^{\xi} \frac{U(\eta)}{(\xi - \eta)^\alpha} d\eta = m \left\{ \left[(n+1)a - 1\right] L - (1+\xi)\frac{dL}{d\xi} \right\} \tag{3.2.12}$$

We seek the solution of this equation in the following from:

$$\begin{aligned} \text{If } \xi &\geq 0 (x \leq ct) \text{ so } U(\xi) = U_1 \\ \text{and if } \xi &< 0 (x > ct) \text{ so } U(\xi) = 0 \\ &\text{Where } U_1 \text{ is a canstant .} \end{aligned} \tag{3.2.13}$$

Proceeding as in the case of the previous exact solution.

From the case (3.2.12) and (3.2.13) we have

$$\begin{aligned} U_1 c^{\alpha} \frac{\Gamma(1+\beta)}{\Gamma(2-\alpha+\beta)} (1-\alpha-\beta)\, \xi^{\beta-\alpha} &= m\,(U_1)^{n+1}\,(a-\beta)\,[(n+1)(a-\beta)-1]\,\xi^{(n+1)\beta} \\ &\quad -2m(U_1)^{n+1}\beta(n+1)(a-\beta)\xi^{(n+1)\beta-1} \\ &\quad +m(U_1)^{n+1}\beta\,[(n+1)\beta-1]\,\xi^{(n+1)\beta-2} \end{aligned}$$

This equation is achieved through unique choice $\beta = a = \frac{2-\alpha}{n}$.

That provides

$$U_1 = \left\{ \frac{n c^{\alpha} \Gamma(1+\frac{2-\alpha}{n})}{m(2-\alpha)\Gamma(2-\alpha+\frac{2-\alpha}{n})} \right\}^{\frac{1}{\eta}}$$

So the find solution to

$$\begin{aligned} T &= x^{a} U(\xi), \xi = \frac{ct}{x} - 1 \\ &\text{is} \\ T(x,t) &= U_1(ct-1)^{\frac{2-\alpha}{n}}, x \leq ct \end{aligned}$$

This solution represents a wave with the amplitide U_1, which propagates along the x-axis with the speed c.

As we can get c

$$\begin{aligned} U_1 &= \left\{ \frac{nc^{\alpha}\Gamma(1+\frac{2-\alpha}{n})}{m(2-\alpha)\Gamma(2-\alpha+\frac{2-\alpha}{n})} \right\}^{\frac{1}{n}} \\ U_1^n &= \frac{nc^{\alpha}\Gamma(1+\frac{2-\alpha}{n})}{m(2-\alpha)\Gamma(2-\alpha+\frac{2-\alpha}{n})} \end{aligned}$$

$$\begin{aligned} U_1^n\left(m(2-\alpha)\Gamma(2-\alpha+\frac{2-\alpha}{n})\right) &= nc^{\alpha}\Gamma(1+\frac{2-\alpha}{n}) \\ &\Longrightarrow \quad c^{\alpha} = \frac{U_1^n\left(m(2-\alpha)\Gamma(2-\alpha+\frac{2-\alpha}{n})\right)}{n\Gamma(1+\frac{2-\alpha}{n})} \end{aligned}$$

So

$$c = \left\{ \frac{U_1^n\left(m(2-\alpha)\Gamma(2-\alpha+\frac{2-\alpha}{n})\right)}{n\Gamma(1+\frac{2-\alpha}{n})} \right\}^{\frac{1}{\alpha}}$$

We finally obtain

$$T(x,t) = U_1(ct-x)^{\frac{2-\alpha}{n}}$$

So

$$T(x,t) = U_1\left[t\left\{ \frac{U_1^n\left(m(2-\alpha)\Gamma(2-\alpha+\frac{2-\alpha}{n})\right)}{n\Gamma(1+\frac{2-\alpha}{n})} \right\}^{\frac{1}{\alpha}} - x \right]^{\frac{2-\alpha}{n}}$$

Conclusion

In this work, we are interested generalized Heat conduction Equation by the use od similarity transformation the generalization is obtained by replacing integer-order derviative by a fractional derviative we that the fractional nonlinear partial differential equations studied possess similarity solutions exactly as thier conveniently defined similarity variables these partial differential equations reduce to ordinary differential equations with fractional-order derivatuves, which are more amenable to varions analytical and numerical techniques.

When we have the order $0 < \alpha < 1$ the fractional heat conduction equation serves as the natural link between the solution of the corresponding equations emerging when the order attains its limiiting values, for the nonlinear heat conduction equation we find two of then has the from of a wave propagating in one direction with specific speed the sxact relation bet ween the speed of the wave and its amplitude is also found.

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Abstract

The aim of this reserch its to determine similarity splution where will we analyze self-similar solution to a fractional heat conduction equation abd Generalized heat conduction equation, by using Lie-group scaling transformation both problems are reduced to ordinary nonlinear fractional differential equation the exact solution to the generalized heat conduction have been presented in two special.

Key Words :

nonlinear diffusion equation, fractional derivaives, heat conduction, similarity

transformation

Résume

Le but de cette recherche est de déterminer des solutions de similarité où analyser l’équation de conduction thermique et l’équation de conduction thermique généralisée. En utilisant la transformation l'échelle de groupe de mensonges, les deux problèmes sont réduits à une équation différentielle fractionnaire non linéaire ordinaire la solution exacte aux l’équation de conduction thermique généralisée ont été présentées dans deux cas particuliers.

Mots clé:

- équation différentielle non linéaire, dérivés fractionnaires conduction thermique, transformation de similarité.

ملخص

الهدف من هذا البحث هو تحديد حل التشابه حيث سنقوم بتحليل معادلة التوصيل الحراري الجزئي ومعادلتها العامة باستخدام تحويل قياس مجموعة الكذب يتم تقليل كلتا المشكلتين الى معادلات تفاضلية كسرية غير خطية عادية . وقد تم تقديم الحل الدقيق لمعادلة التوصيل الحراري المهمة في حالتين خاصتين .

الكلمات المفتاحية

معادلة الانتشار اللاخطي , المشتقات الكسرية, معادلة التوصيل الحراري تحويل التشابه.