PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

University Mohamed Boudiaf - M'sila

Faculty Of Mathematics And Informatics

Department of Mathematics

Master Dissertation

Domain:Mathematics and Informatics
Field: Mathematics
Option : Numerical analysis and Mathematical

Theme

On The Spectral Theory of Multivalued Linear Operators

Persented by:
BAALI Sara

Presented in front of the jury :

M_r *TALLAB Abdelhamid*	*MCA,*	University of M'sila	**Chair.**
M_r *HERAIZ Toufik*	*MCB*	University of M'sila	**Supervisor.**
M_r *MAAZOUZ Ahmed*	*MAA,*	University of M'sila	**Examiner.**

The college year 2020/2021

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Dedication

I dedicate this work to my father elkhier and my mother samira for their ultimate support

To my dear brothers ebdelwahab, yacine, yahya, haroune, and my sister wissam

To my uncles, and all our relatives

To these who were a means of help in a way or another

And to all those are precious parts of our lives

Table of contents

Notations

$LR(X,Y)$	Linear relations.
$D(T)$	Domain of the relation T.
$G(T)$	The graph of T.
$R(T)$	The range of M.
$\alpha(T)$	The nullity of T.
$\beta(T)$	The deficiency of T..
$\kappa(T)$	The index of T..
$c\kappa(T)$	The coindex of T.
$\gamma(T)$	The minimum modulus of T.
T'	The adjoint of T.
$\tilde{T}$	The completion of T.
$\rho(T)$	The resolvent set of T. .
$\sigma(T)$	The spectrum of T.
$\sigma_{ei}(T)$	Essential spectrum.of T.
$P_\sigma(T)$	The point spectrum of T.
$R\sigma(T)$	Residual spectrum of T.
$C\sigma(T)$	The continuous spectrum of T.
$\Phi_+(X)$	The set of upper semi-Fredholm relation on X.
$\Phi_-(X)$	The set of lower semi-Fredholm relation on X.
$PR(\Phi_+(X,Y))$	The sets of upper semi-Fredholm perturabtions .
$PR(\Phi_-(X,Y))$	The sets of lower semi-Fredholm perturabations .
$A_\alpha(X,Y)$	The classe of $\alpha -$ Atkinson type relations from X into Y
$A_\beta(X,Y)$	The classe of $\beta -$ Atkinson type relations from X into Y

Introduction

The concept of multivalued linear operators, or linear relations, one of the most exciting and influential fields of research in modern mathematics applications of this theory can be found in economic theory, noncooperative games, artificial intelligence, medicine, and it is one of the most useful objects in functional analysis. Linear relations were introduced into functional analysis by J. von Neumann motivated by the need to consider adjoints of non - densely defined linear differential operators. $[1, II.1.10]$

This work aims to study the spectral theory of multivalued linear operators, or linear relations, it is composed of three charters:

In the first chapter, we present the basic properties of multivalued linear operators, we start with relations on sets and discuss the operational rules for linear relations, the fundamental theorems of the linear relations algebra, we present the fundamental notions about the norm of a linear relation, the continuity and openness. In the final section of this chapter, we study the notion of the adjoint and its properties.

In the second chapter, we demonstrate the spectrum when we start by the section of the resolved operator which by we pass to the concepts of the spectrum, then we continue to the compactness of the augmented spectrum. Subsequently, we come to the most important section is Semi - Fredholm linear relations (lower and upper semi - fredolm). In the final section of this chapter, we study the essential spectrum (some classes of essential spectrum Preliminary perturbation results).

In the third chapter is devoted to the study of some classes of perturbations relation (Relatively boundedness and compactness, Semi Fredholm perturbation, Atkinson perturbation). In the last chapter, we study the stability of the essential spectrum.

Chapter 1

BASIC PROPERTIES

In this chapter we recall some basic informations and concepts used in the following chapter.

1.1 Relation on sets

Definition 1.1.1 *Let U.V.W.... denote arbitrary nonempty sets. A relation T from U to V is any mapping having domain $D(T)$ a nonempty subset of U, and taking values in $2v\backslash\phi$ (the collection of nonempty subsets of V). Such a mapping T is also known as a set valued (or multivalued) mapping. If T maps the points of its domain to singletons, then T is said to be a single valued relation, or function. For $u \in U$, $u \notin D(T)$ we define $Tu = \varnothing$. With this convention, we have*

$$D(T) := \{u \in U : Tu \neq \varnothing\}. \tag{1.1.1}$$

The class of all relations from U to V will be denoted by $R(U, V)$. Examples of relations are functions, inverses of functions, adjoints of linear operators, partial order relations, equivalence relations, and convex processes.

If $T \in R(U, V)$ then the **graph** of T is the subset $G(T)$ of $U \times V$ defined by

$$G(T) := \{(u, v) \in U \times V : u \in D(T), v \in T(u)\}. \tag{1.1.2}$$

A relation in $R(U, V)$ is uniquely determined by its graph, and conversely any nonempty subset of $U \times\ V$ uniquely determines a relation . (Some authors identify a relation with its graph , but we shall not do so here)

Let $T \in LR(U, V)$. The inverse of T is the relation T^{-1} ,given by

$$G(T^{-1}) := \{(v, u) \in V \times U : (u, v) \in G(T)\} \tag{1.1.3}$$

. Given a subset $M \subset U$, we write

$$T(M) := U\{T(m) : m \in M \cap D(T)\} \tag{1.1.4}$$

called the **image** of M, with

$$R(T) = T(U) = T(D(T)) \tag{1.1.5}$$

called the **range** of T. If $R(T) = Y$, then T is called **surjective** . If T^{-1} is single valued, then T is called **injective**. If T is injective then it is easy to see that the following implication holds :

$$T(u_1) = T(u_2) \Rightarrow u_1 = u_2 \quad (u_1, u_2 \in D(T)) \tag{1.1.6}$$

we shall denoted $T(\{u\}) = T(u)$ by Tu; no distinction will be made between a single valued map and a map into$V$.

Let $\phi \neq N \subset V$. Then we have

$$T^{-1}(N) = \{u \in D(T) : N \cap Tu \neq \phi\} \tag{1.1.7}$$

in particular, for $v \in R(T)$

$$T^{-1}v = \{u \in D(T) : v \in Tu\} \tag{1.1.8}$$

It is clear that $D(T^{-1}) = R(T)$, and $R(T^{-1}) = D(T)$

The **identity relation** defined on a nonempty subset E of U is denoted by I_E or simply I, when E is understood; it is the relation in $R(U, U)$ whose graph is

$$G(I_E) = \{(e, e) : e \in E\}$$

1.2 Restrictions and Extensions of Relations

Let M be a subset of U such that $M \cap D(T) \neq \emptyset$. The restriction of T to M, denoted by $T\mid_M$, is defined by

$$T\mid_M \in R(U, V),$$

$$D(T\mid_M) = D(T) \cap M,$$

$$(T\mid_M)m = Tm \text{ For } m \in M.$$

Observe that

$$G(T\mid_M) = G(T) \cap (M \times V)$$

$$= \{(u, v) \in G(T) : u \in M\}$$

We have

$$T\mid_M = T\mid_{M \cap D(T)}$$

Given two relations S and T in $R(U, V)$, we say that S is an **extension** of T if

$$S\mid_{D(T)} = T$$

1.3 Linear relations (Multivalued linear operaors)

Definition 1.3.1 *Let X and Y be vector spaces over the field $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. A relation $T \in R(X, Y)$ is called a **linear relation (or multivalued linear operator)** if for all $x, z \in D(T)$ and nonzero scalars α we have*

$$Tx + Tz = T(x + z) \tag{1.3.1}$$

$$\alpha Tx = T(\alpha x) \tag{1.3.2}$$

Evidently the domain of a linear relation is a linear subspace.

The class of linear relations in $R(X, X)$ will be denoted by $LR(X, X)$. We write

$$LR(X, X) := LR(X).$$

It is clear that if M is a linear subspace of X and $T \in LR(X, Y)$, then

$$T\mid_M \in LR(X, Y)$$

and

$$TJ_M \in LR(M, Y)$$

Proposition 1.3.1 [*1*, *II*.1.10] *Let $T \in R(X, Y)$. The following properties are equivalent:*

1) *T is a linear relation.*

2) *$G(T)$ is a linear subspace of $X \times Y$.*

3) *T^{-1} is a linear relation.*

4) *$G(T^{-1})$ is a linear subspace of $Y \times X$.*

1.4 Operational rules for a single linear relation

The rules governing the operations of T on subsets are listed in the following proposition (where 0_X denotes the zero operator on X):

Proposition 1.4.1

a) $T(\alpha M) = \alpha T(M) \qquad (M \subset X, \alpha \in \Bbbk, \alpha \neq 0)$

b) $T(M+N) \supset T(M) + T(N) \quad (M, N \subset X)$

c) $T(M+N) = T(M) + T(N) \ \ (M \subset X, N \subset D(T))$

d) $TT^{-1}(M) = M \cap R(T) + T(0) \qquad (M \subset Y)$

e) $T^{-1}T(M) = M \cap D(T) + T^{-1}(0)(M \subset X)$

f) $T^{-1}(0) \times \{0\} = G(T) \cap (X \times \{0\}) = G(T) \cap G(0_X)$

g) $\{0\} \times T(0) = G(T) \cap ((0\} \times Y)$

h) $X \times R(T) = G(T) + (X \times \{0\}) = G(T) + G(0_X)$

i) $D(T) \times Y = G(T) + (\{0\} \times Y)$.

Corollary 1.4.1 *let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$, then $ST \in LR(X,Z)$.*

1.5 The algebra of linear relations

In this section we define the operations of addition and scalar multiplication in $LR(X,Y)$ and describe the laws governing these operations combined with the operations of composition and inversion.

1.5.1 Addition and scalar multiplication in $LR(X,Y)$

Let S and T be linear relations in $LR(X,Y)$ and let α be a scalar. The relations $S+T$ and αT are defined as follows:.

(1) $(S+T)x := Sx + Tx \quad (x \in \ X)$

(2) $(\alpha T)x := \alpha(Tx) \quad (x \in X, \alpha \in \mathbb{k})$.

(3) $D(S+T) = D(S) \cap D(T)$

(4) $D(\alpha T) = D(T)$.

(5) $G(S+T) = \{(x,y) \in X \times Y : y = s+t, \text{ where } (x,s) \in G(S) \text{ and } (x,t) \in G(T)\}$

(6) $G(\alpha T) = \{(x, \alpha y) \in X \times Y : (x,y) \in G(T)\}$.

(7) $\alpha(\beta T) = (\alpha\beta)T \quad (\alpha, \beta \in \mathbb{k})$

(8) $S+T = T+S$.

and if R, S, T are in $LR(X,Y)$

(9) $R + (S+T) = (R+S) + T$.

1.6 Dimension of domain and range

Proposition 1.6.1 $[1, II.1.10]$ *Let* $T \in LR(X,Y)$ *, and let* $M \subset X$ *. then*

$$\dim R(T)\diagup T(M) \leq \dim D(T)\diagup D(T) \cap M \leq \dim X\diagup M.$$

In particular , if M is a finite codimensional subspace of $D(T)$, then $T(M)$ is a finite codimensional subspace of $R(T)$.

Notation 1.6.1 *For a given linear relation* $T \in LR(X,Y)$*, let* QT *(or* Q*, if* T *is understood) denote the natural quotient map of* Y *onto* $Y\diagup T(0)$.

Proposition 1.6.2 *The linear relation* QT *is single valued .*

Proof. Let $Qy_1, Qy_2 \in QTx$. then $Qy_1 - Qy_2 \in QTx - QTx = QT(0) = 0$.

The single valued version of the next proposition is known as " the fundamental theorem of linear algebra " . ■

Proposition 1.6.3 $\dim D(T) + \dim T(0) = \dim D(T^{-1}) + \dim T^{-1}(0)$.

Proof. We have

$$\dim R(T) = \dim R(T)\diagup T(0) + \dim T(0).$$

Clearly ,

$$\dim D(QT) = \dim N(QT) + \dim R(QT).$$

Now $D(T) = D(QT)$ and $N(QT) = \{x \in D(T) : 0 \in QTx\}$
$= \{x \in D(T) : Tx = T(0)\}$
$= N(T)$. Also $R(QT) = R(T)\diagup T(0)$.

Hence from (1) and (2),

$$\dim D(T) + \dim T(0) = \dim N(T) + \dim R(T)$$

■

Proposition 1.6.4 [*1*, II.1.10] *QT is single valued.*

Proof. Let $x \in D(T)$ and let $z_1, z_2 \in QTx$. Then $z_1 - z_2 \in QTx - QTx = QT(0) \subset Q\overline{T(0)}$ Hence $z_1 = z_2$. ■

Definition 1.6.1 *For a linear relation $T \in LR(X, Y)$ we define*

$$\alpha(T) = \dim N(T)$$

$$\beta(T) = \dim Y\diagup R(T)$$

*called the **nullity** and **deficiency** of T **respectively.***

where $R(T) := \{y \in Y \mid y \in Tx, x \in D(T)\}$.

The **index** $\kappa(T)$ of T is defined as

$$\kappa(T) = \alpha(T) - \beta(T)$$

provided $\alpha(T)$ and $\beta(T)$are not both infinite. If $\alpha(T)$ and $\beta(T)$are both infinite, then T is said to have no index.

Definition 1.6.2 *The index of T^{-1} is called the **coindex** of T and is denoted by $c\kappa(T)$.thus*

$$c\kappa(T) = \dim T(0) - \operatorname{codim} D(T).$$

All single valued, or everywhere defined, linear relations have a coindex. In case T is single valued and everywhere defined, then $c\kappa(T) = 0$.

1.7 The norm of a linear relation

Throughout the sequel $X, Y, Z....$ will denote normed vector spaces, and T a linear relation in $LR(X, Y)$, unless otherwise specified.

We write $Bx := \{x \in X : ||x|| \leq 1)$.

Definition 1.7.1 *We define*

$$\| Tx \| := \| QTx \| \ (x \in D(T)) \tag{1.7.1}$$

$$\| T \| := \| QT \| \tag{1.7.2}$$

*called the **norm** of Tx and T respectively. If $D(T) = X$ and if $||T|| < \infty$, then we shall say that T is **bounded**.*

Let U and V be nonempty subsets of a normed space. We define the **distance** between U and V by the formula

$$d(U, V) := \inf\{\| u - v \| : u \in U, v \in V\} \tag{1.7.3}$$

we shall write $d(x, V)$, or $d(V, x)$, for the destance between $\{x\}$ and V .

Proposition 1.7.1

1) *If* $y_1, y_2 \in Tx$ *,then* $d(y_1, T(0)) = d(y_2, T(0))$.

2) *We have for* $x \in D(T)$ *,*

$$\inf_{y \in Tx} d(y, T(0)) = \sup_{y \in Tx} d(y, T(0)).$$

3) $\| Tx \| = d(y, T(0))$ for any $y \in T(x)$.

4) $\| Tx \| = d(Tx, T(0)) = d(Tx, 0) \quad (x \in D(T))$.

5) For $S, T \in LR(X, Y)$ we have

$$\| Sx + Tx \| \leq \|Sx\| + \|Tx\| \qquad (x \in D(T + S)).$$

6) For all $\alpha \in \mathbb{k}$ and $x \in D(T)$we have

$$\|\alpha Tx\| = |\alpha| \, \|Tx\|$$

7) $\|T\| = \sup_{x \in Bx} \|Tx\|$.

8) For $S, T \in LR(X, Y)$ we have

$$\| S + T\| \leq \|S\| + \|T\|$$

9) $\|\alpha T\| = |\alpha| \, \|T\| \, (\alpha \in \mathbb{k})$.

The minimum modulus:

The minimum modulus of T is the quantity

$$\gamma(T) := \sup \{\lambda : \|Tx\| \geq \lambda d(x, N(T)) \text{ for } x \in D(T)\}$$

Proposition 1.7.2 *[1] we have*

$$\gamma(T) = \sup \left\{\lambda : TB_{D(T)} \supset \lambda B_{R(T)}\right\}$$

Theorem 1.7.1 *[1] we have the formula*

$$\gamma(T) = \left\|T^{-1}\right\|^{-1}$$

1.7.1 Continuity and openess

Definition 1.7.2 *Let S be an arbitrary relation from one topological space to another . Then S is said to be continuous if for each neighbor V in $R(S)$, the inverse image $S^{-1}(V)$ is a neighbor in $D(S)$. The relation S is called open if whenever U is a neighbor in $D(S)$, the image $S(U)$ is a neighbor in $R(S)$. We note that*

$$S \text{ is continuous if and only if } S^{-1} \text{ is open.}$$

It is simple to verify that the composition of continuous single valued relations is continuous. Conditions will be derived under which the composition of linear relations is continuous or open .

Let Ux denote the open unit ball of X , i.e.

$$Ux := \{: x \in X : \|x\| < 1\}$$

Let E be a closed subspace of X. A simple verification shows that

$$Q_E Ux = U_{Q_E(x)}$$

$$Q_E Bx = B_{Q_E(x)}$$

We note the following simple topological criterion for continuity :

$$T \text{ is continuous if and only if } T\overline{\Omega} \subset \overline{T\Omega} \text{ for every subset } \Omega \text{ of } D(T).$$

Proposition 1.7.3 $[1, II.1.10]$ *let $T \in LR(X,Y)$.then*

(a) *T is continuous if and only if $\|T\| < \infty$*

(b) *T is open if and only if $\gamma(T) > 0$.*

in particular:

(c) *if $\dim D(T) < \infty$,then T is continuous.*

(d) *if $\dim R(T) < \infty$,then T is open.*

Proof.

(a) let T be continuous . then there exists an open neighbourhood U in $D(T)$ such that $T^{-1}U_{R(T)} \supset U$.hence

$$T^{-1}U_{R(T)} - T^{-1}U_{R(T)} \supset U - U,$$

an open neighbourhood of $\{0\}$ in $D(T)$. also $2T^{-1}U_{R(U)} = T^{-1}(U_{R(T)} - U_{R(T)} = T^{-1}U_{R(T)})$. consequently there exists $\lambda > 0$ such that

$$T^{-1}(U_{R(T)}) \supset \lambda U_{D(T)}$$

and hence

$$T^{-1}(B_{R(T)}) \supset \frac{1}{2}\lambda B_{D(T)}$$

Therefore

$$TT^{-1}(B_{R(T)}) = B_{R(T)} + T(0) \supset \frac{1}{2}\lambda T B_{D(T)}$$

and hence $\|T\| < \infty$ by *Proposition* $[1, II.1.10]$

conversely assume that $\|T\| < \infty$.Let V be an open ball neighbourhood in $R(T)$, with center y .then

$$V_0 = V - \{y\} = \alpha U_{R(T)} \text{ for some } \alpha > 0.$$

by *Proposition* $[1, II.1.10]$ we have for some $\lambda > 0$,

$$TU_{D(T)} \subset \lambda U_{R(T)} + T(0).$$

Hence

$$U_{D(T)} + T^{-1}(0) \subset \lambda T^{-1}U_{R(T)} = \alpha^{-1}\lambda T^{-1}V_0.$$

thus

$$T^{-1}(V_0) = T^{-1}(V - y) \supset \lambda^{-1}\alpha U_{D(T)} + T^{-1}(0)$$

hence

$$U_{D(T)} + T^{-1}(0) \subset \lambda T^{-1}U_{R(T)} = \alpha^{-1}\lambda T^{-1}V_0.$$

Thus

$$T^{-1}(V_0) = T^{-1}(V - y) \supset \lambda^{-1}\alpha U_{D(T)} + T^{-1}(0)$$

Hence

$$\begin{aligned} T^{-1}V - T^{-1}y + T^{-1}y &\supset \lambda^{-1}\alpha U_{D(T)} + T^{-1}y \\ \text{i.e.}\quad T^{-1}V &\supset \theta^{-1}\alpha U_{D(T)} + T^{-1}y \end{aligned}$$

ie which is a neighbor in $D(T)$. Therefore T is continuous.

(b) Follows immediately from (a).

(c) Follows from the continuity of the linear operator QT, since $\|T\|= \|QT\|$

(d) Substitute T^{-1} for T in (c).

■

1.8 The adjoint

if X is a normed linear space, then X' will denote the norm dual of X; i.e the space of all continuous linear functionals x' defined on X, with the norm $\|x'\| = \inf\{\lambda : |x'x| \leq \lambda\|x\| \text{ for all } x \in X\}$.

we shall adopt the following notation: if $K \subset X$ and $L \subset X'$, then

$$K^{\perp} := \{x' \in X' : x'x = 0 \text{ for all } x \in K\}.$$
$$L^{\top} := \{x \in X : x'x = 0 \text{ for all } x \in L\}.$$

Clearly, $K^{\perp}$ and $L^{\top}$ are closed linear subspaces of X' and X respectively.

Definition 1.8.1 *the adjoint T' of T is defined by*

$$G(T') = G(-T^{-1})^{\perp} \subset Y' \times X' \tag{1.8.1}$$

where

$$\langle (y,x),(y',x')\rangle := \langle x,x'\rangle + \langle y,y'\rangle\,. \tag{1.8.2}$$

this means that

$$(y',x') \in G(T') \text{ if and only if } y'y - x'x = 0 \text{ for all } (x,y) \in G(T). \tag{1.8.3}$$

in 1.8.3 we have $y'y = x'x$ for all $y \in Tx$, $x \in D(T)$, Hence

$$x' \in T'y' \Longleftrightarrow y'Tx = x'x \ \text{ for all } x \in D(T) \tag{1.8.4}$$

therefore x' is an extension of $y'T$. we can thus characterize the adjoint as follows:

$$G(T') = \{(y', x') \in Y' \times X' : x' \text{ is an extension of } y'T\}. \tag{1.8.5}$$

From 1.8.5 we see that if T is densely defined , then $T\,'$will be single valued (since in that case , $y'T$ will have a unique extension to all of X).

Proposition 1.8.1 *T' is a closed linear relation in $LR(Y', X')$, and we have:*

$$D(T') = \{y' \in Y' : y'T \textit{ is continuous and single valued } \}.$$

furthermore, given $y' \in D(T'), x \in D(T)$, we have $T'y'x = y'Tx \in \Bbbk$.

(a) $\left(\overline{T}\right)' = T'$

(b) $(T')^{-1} = (T^{-1})'$

(c) $(\lambda T)' = \lambda T' \quad (\lambda \neq 0)$

Proof. (a) and (b) are immediate consequences of the definition.

(c) For $\lambda \neq 0$ we have:

$$\begin{aligned} G((\lambda T)') &= \{(y', x') : y'y = x'x \ \text{ for } (x, y) \in G(\lambda T)\} \\ &= \{(y', \lambda x') : y'(\lambda y) = (\lambda x')x, \text{ for } (x, y) \in G(T)\} \\ &= G(\lambda T'). \ \blacksquare \end{aligned}$$

Proposition 1.8.2 **a)** $N(T') = R(T)^{\perp}$

b) $T'(0) = D(T)^{\perp}$

c) $\overline{T}(0) = D(T')^{\top}$

d) $N(\overline{T}) = R(T')^{\top}$

Proposition 1.8.3 *let $S, T \in LR(X, Y)$*

a) $G(S' + T) \subset G\left((S+T)'\right)$

b) if $D(T) \subset D(S)$, and if S is continuous (for example , if $D(S') = Y'$), then

$$S' = T' + (S+T)'$$

c) $(S+T)'$ is an extension of $S' + T'$ if and only if $(D(S) \cap D(T))^{\perp} = D(S)^{\perp} = D(T)^{\perp}$

Theorem 1.8.1 **(a)** *let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. then $G(T'S') \subset G((ST)')$.Furthermore ,if either*

(i) $R(T') = X'$ and $D(S) \subset R(T)$, or

(ii) $D(S') = Z'$ and $R(T) \subset D(S)$, then

$$(ST)' = T'S' \tag{1}$$

(b) *let M be a finite codimensional subspace of $D(T)$, closed in $D(T)$. then*

$$(TJ_M^X)' = (J_M^X)'T'. \tag{2}$$

Closed graph , Open mapping and closed range theorems

Definition 1.8.2 *Let $T \in LR(X,Y)$, and let $\tilde{T}$ denote the linear relation in $LR(\tilde{X},\tilde{Y})$whose graph is the completion of $G(T)$. We call $\tilde{T}$ the **completion** (or c**ompleteclosure**) of T.*

If $G(T)$ is complete, then T is said to be **completely closed** (or **complete**).
If $\tilde{T}x = Tx$ for all $x \in D(T)$, then T is called **completely closable** (or **completable**).

Theorem 1.8.2 *[1, Theorem III.4.2] (closed graph and Open Mapping theorem)*

(1) *$\tilde{T}$ is continuous if and only if $D(\tilde{T})$ is closed .*

(2) *$\tilde{T}$ is open if and only if $R(\tilde{T})$ is closed*

Theorem 1.8.3 *[1] (closed Range Theorem).*

the following properties are equivalent:

(1) $R(\tilde{T})$ *closed.*

(2) $R(T')$ *closed.*

(3) $R(\tilde{T}')$ *weak*$^*-$ *closed.*

Chapter 2

SPECTRUM

Throughout this chapter T will denote a linear relation in $LR(X)$, where X is a normed space over the complex field $\mathbb{C}$. We shall write $\lambda - T := \lambda I_{D(T)} - T$

2.1 Spectrum and the resolvent operator

Definition 2.1.1 *Given $T \in LR(X)$ let us write for $\lambda \in \mathbb{C}$*

$$R(\lambda, T) := (\lambda - T)^{-1} \tag{2.1.1}$$

*called the **resolvent** of T (corresponding to λ) and*

$$T_\lambda = R(\lambda, T)^{\sim} \tag{2.1.2}$$

*called the **complete resolvent** of T It is simple to verify that T_λ is the resolvent of $\tilde{T}$, i.e.*

$$T_\lambda = (\lambda - \widetilde{T})^{-1} \tag{2.1.3}$$

*The **resolvent set** of T is the set*

$$\rho(T) := \{\lambda \in \mathbb{C} : T_\lambda \text{ is everywhere defined and single valued }\}. \tag{2.1.4}$$

*It is clear from the Closed Graph theorem for linear operators that T_λ is a bounded linear operator (defined on $\widetilde{X}$) if and only if $\lambda \in \rho(T)$. In that case T_λ is called a **resolvent operator.***

*The **spectrum** of T is the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$.*

Proposition 2.1.1 *(resolvent Equation)*

let $\lambda, \mu \in \rho(T)$. *then* $T_\mu - T = (\lambda - \mu)T_\mu T_\lambda$

Corollary 2.1.1 *if* $\lambda, \mu \in \rho(T)$,*then* $T_\mu T_\lambda = T_\lambda T_\mu$

we recall the following simple result.

Lemma 2.1.1 *let* B *be a bounded linear operator from a banach space* X *into itself .if* $\|B\| < 1$ *then* $I - B$ *is invertiable and*

$$\lim_{n\to\infty} \left\| (I-B)^{-1} - \sum_{k=0}^{n} B^k \right\| = 0.$$

Corollary 2.1.2 *the family* $\{T_\lambda : \lambda \in \rho(T)\}$ *of resolvent operators is holomorphic.*

Proposition 2.1.2 $\sigma(T) = \sigma(T')$.

2.1.1 Classification of the spectrum : eigenvalues

Definition 2.1.2 *A scalar* λ *such that* $N(\lambda - T) \neq \{0\}$ *is called an* ***eingenvalue*** *of* T.

let λ be an eigenvalue of T. then the non zero subspace $N(\lambda - T)$ is called the **eigenvalue** of T corresponding to λ, and the dimension of $N(\lambda - T)$ is called the **geometric multiplicity** of λ. the non zero vectors in $N(\lambda - T)$ are called **eigenvectors.**

Clearly , if λ is an eigenvalue of T, then $\lambda \in \sigma(T)$.

The set $\sigma(T)$ is decomposed into the following three disjoint sets :

- $P\sigma(T)$, called the **point spectum** of T, consisting of the eigenvalue of T
- $R\sigma(T)$, called the **residual spectrum** of T, consisting of the points $\lambda \in \mathbb{C}$ such that $\lambda - T$ is injective but does not have dense range.
- $C\sigma(T)$, called the **continuous spectrum** of T, consisting of the points $\lambda \in \mathbb{C}$ such that $\lambda - T$ is injective and has dense range but is not open.

2.1.2 Compactness of the augmented spectrum

Let $\mathbb{C}$ denote the extended complex plane, $\mathbb{C} = \mathbb{C} \cup \{\infty\}$, endowed with the usual topology. Then $\mathbb{C}_\infty$ is a compact topological space.

The Mobius transformation

$$\eta(\lambda) = (\mu - \lambda)^{-1}$$

where μ is a fixed point of $\mathbb{C}$, is a topological homeomorphism from $\mathbb{C}_\infty$ onto itself.

Definition 2.1.3 *The* ***augmented spectrum*** *of T is the set*

$\overline{\sigma}(T) = \sigma(T) \cup \{\infty\}$ if $0 \in \sigma(T^{-1}) = \sigma(T)$ otherwise

Thus $\{\infty\} \not\subset \overline{\sigma}(T)$ if and only if $\tilde{T}$ is a bounded linear operator, if and only if $0 \notin 0(T^{-1})$.

The augmented spectrum is non empty (if $X \neq \{0\}$)since if $\{\infty\} \not\subset \overline{\sigma}(T)$ then

$$\overline{\sigma}(T) = \sigma(T) = \sigma(\tilde{T}) \text{ is non empty.}$$

Theorem 2.1.1 *Let $\mu \in \rho(T), D(T) \neq \{0\}$. Then*

$$\eta(\overline{\sigma}(T)) = \sigma(T_\mu)$$

Corollary 2.1.3 *Let T have a non empty resolvent set.*

a) *If $\tilde{T}$ is a bounded linear operator, then $\sigma(T)$ is a compact subset of $\mathbb{C}$.*

b) *If $\tilde{T}$ is not a bounded linear operator, then $\{\infty\} \subset \overline{\sigma}(T)$, and $\overline{\sigma}(T)$ is a compact subset of $\mathbb{C}_\infty$.*

2.2 Semi-Fredholm linear relations

2.2.1 The main classes: preliminary results

Definition 2.2.1 *Let $T \in LR(X,Y)$. Then T is called compact, precompact, strictly singular, upper semi-Fredholm, lower semi - Fredholm, Fredholm, partially continuous (precompact, or compact), or nowhere continuous, respectively, if the linear operator QT has the corresponding property.*

In other words:

T *is* ***compact*** *if and only if* $\overline{QTB_x}$ *is compact.*

T *is* ***precompact*** *if and only if* QTB_x *is totally bounded.*

T *is strictly* ***singular*** *if and only if there is no infinite dimensional subspace* M *of* $D(T)$ *for which* $T\mid_M$ *is injective and open.*

T *is* ***upper semi - Fredholm*** *if and only if there exists a finite codimensional subspace* M *of* X *for which* $T\mid_M$*is injective and open. (This is equivalent to the corresponding property for* QT*)*

T *is l**ower semi Fredholm** if and only if* T'*(or equivalently* $(QT)'$*) is upper semi - Fredholm*

T *is* ***partially continuous*** *(respectively* ***partially precompact, partially compact****) if there exists a finite codimensional subspace* M *of* X *such that* $T\mid_M$ *is continuous (respectively precompact, compact).*

T *is* ***nowhere continuous*** *if and only if* $T\mid_M$*is discontinuous whenever* $M \in \mathrm{I}(D(T))$

Note that if $T \in LR(X,Y)$*is precompact and* Y *is complete, then* T *is compact .*

We shall denote the class of upper semi - Fred linear relations by $F_+(X,Y)$*, which we abbreviate as* F_+*. Likewise,* $F_-(X,Y)$*(or* F_-*)will denote the class of lower semi - Fredholm linear relations.*

Proposition 2.2.1 *From the definitions we have the equivalences*

a) $T \in F_- \Leftrightarrow T' \in F_+$

b) $T \in F_+ \Leftrightarrow QT' \in F_+$

c) $T \in F_- \Longleftrightarrow QT' \in F_-$

d) Y complete, T surjective $\Rightarrow T \in F_-$

The relation T is called Fredholm if it is both in F_+ and F_-, We write $F(X,Y)$ (or F) for the class of Fredholm relations.

Proposition 2.2.2 *The relation T is upper semi - Fredholm if and only if $T\mid_M$ is bounded below for some $M \in C(X)$, i.e.*

$\|Tm\| \geq c\,\|m\| \qquad (m \in M)$

where $c > 0$.

Proof. suppose the stated inequality holds. then $T\mid_M$ is clearly injective and open, i.e.

$T \in F_+$

conversely, let $T \in F_+$thenfor some $M \in C(X), T\mid_M$ is injective and open i.e.

$$\|Tm\| \geq \gamma\,(T\mid_M)\,d(m, N(T\mid_M)) = \gamma\,(T\mid_M)\,\|m\|$$

where $\gamma\,(T\mid_M) > 0$ ■

Proposition 2.2.3 $[1, II.1.10]$ *let T be a continuous linear relation with finite dimensional range .Then T is compact .*

Proof. *Let $T \in LR(X,Y)$. we have*

$$QTB_X \subset \|T\|\,B_{R(QT)}$$

where $B_{R(QT)}$ is compact subset of QY,therefore T is compact ■

Proposition 2.2.4 *Let T and T_n $(n \in N)$ be everywhere defined and single valued and such that*

$$\lim_{n\to\infty} \|T_n - T\| = 0$$

where each $T_n(n \in N)$ is precompact. Then T is precompact.

Proof. Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ such that

$$\|T - In\| < \epsilon \diagup 3 \tag{2.2.1}$$

SinceT_N is precompact, $Q_{Ty}T_N B_{D(T)}$ is totally bounded and so there exist points $x_1, x_2..., x_m$ in $B_{D(T)}$ such that for each $x \in B_{D(T)}$there exists x_i satisfying

$$\|T_N x - T_N x_i\| < \epsilon \diagup 3. \tag{2.2.2}$$

Hence from 2.2.1 and 2.2.2 ,

$$\|T_2 - Tx_i\| \leq \|T_x - T_N x\| + \|T_N x - T_N x_i\| + \|T_N x_i - Tx_i\| < \epsilon$$

showing that $QTB_{D(T)}$ is totally bounded, as required. ■

Theorem 2.2.1 *Let $T \in LR(X,Y)$. The following properties are equivalent:*

1) *T is not upper semi - Fredholm.*

2) *There is no closed finite codimensional subspace M of X for which $T\mid_M$ is injective and open.*

3) *There exists an infinite dimensional subspace M of $D(T)$ such that $T\mid_M$ is precompact. (moreover if T is closed, we may assume M to be a closed subspace of X.)*

Definition 2.2.2 *A linear relation T is called a Φ_-**-relation** if it has finite dimensional kernel and close range, and a Φ_- **-relation** if its range is closed and finite codimensional*

so when X and Y are two completes spaces we extend the classes of closed single-valued Fredholm type operators to include closed linear relations, and note that the definitions of the classes $F_+(X,Y)$ and $F_-(X,Y)$ are consistent with:

$$\Phi_+(X,Y) := \{T \in CR(X,Y) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in} Y\},$$

$$\Phi_-(X,Y) := \{T \in CR(X,Y) : \beta(T) < \infty \text{ and } R(T) \text{ is closed in} Y\}$$

Proposition 2.2.5 [*1*] *let $T \in F_+$. then any bounded sequence (x_n) in $D(T)$ such that Tx_n is cauchy has a cauchy subsequence.*

Theorem 2.2.2 *The following properties are equivalent*

1) $T \notin F_+$

2) There exists a non precompact bounded subset W of $D(T)$ such that $QT(W)$is precompact

3) T has a singular sequence, i.e. a sequence $\{x_n\}$ of norm one elements of $D(T)$ such that (x_n) has no Cauchy subsequence and $\lim_{n\to\infty} \|Tx_n\| = 0$

2.2.2 Lower semi-fredholm relations

Proposition 2.2.6 . *Let dim* $D(T) = \infty$. *Then*

(a) $\alpha(T) < \infty \Longrightarrow \gamma(T) \leq \Gamma(T)$

(b) $\overline{\beta}(T') < \infty = \gamma(T') \leq \Gamma(T)$

Proposition 2.2.7 *the following proprties are equivalent :*

1) $T \in F_-$

2) $\overline{\beta}(T) < \infty$ *and* $\gamma(T') > 0$

3) $\tilde{T} \in \Phi_-$

4) $QT \in F_-$

2.3 The essential spectrum

Throughout this chapter, except where stated otherwise, X will denote a complex normed linear space and T a relation in $LR(X)$.

2.3.1 Preliminary perturbation results

The **essential resolvent** set of T is the set

$$\rho_e(T) := \{\lambda \in \mathbb{C} : \alpha(\lambda - T) < \infty \text{ and } \beta(\lambda - T) < \infty\}$$

and the **essential resolvent sets**, $\rho_{ei}(T)$ for $i = 1, 2, 3, 4, 5$, of $T \in LR(X)$ are defined as follows:

$$\rho_{e1}(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_+ \cup \Phi_-\}$$

$$\rho_{e2}(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_+\}$$

$$\rho_{e3}(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi\}$$

$$\rho_{e4}(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi \text{ and } \kappa(\lambda - T) = 0\}$$

$$\rho_{e5}(T) \quad : \quad = \cup \rho_{e1}^{(n)}(T) \text{ where } \rho_{e1}^{(n)} \text{ is componenet of } \rho_{e1}(T)$$
$$\text{and } \ \rho_{e1}^{(n)}(T) \cap \rho_{e}^{(n)}(T) \ \neq \ \emptyset.$$

The essential spectrum of T is the complementary set

$$\sigma_e(T) := \mathbb{C} \backslash \rho_e(T)$$

We clearly have

$$\rho_e(T) = \rho_e(\tilde{T}) = \rho_e(T').$$

The **essential spectra**, $\sigma_{ei}(T), i = 1, 2, 3, 4, 5,$ of $T \in LR(X)$ are the respective complements of the essential resolvent sets:

$$\sigma_{ei}(T) := \mathbb{C} \setminus \rho_{ei}(T), \qquad i = 1, 2, 3, 4, 5.$$

We also define

$$\rho'_{e2}(T) := \{\lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_{-}\}$$

$$\sigma'_{e2}(T) := \mathbb{C} \diagdown \rho'_{e2}(T)$$

Properties of the Eessential Spectra

Proposition 2.3.1 *Clearly we have that $\rho_{ei}(T) \supset \rho_{ej}(T)$ for $i < j < 4$, and, thus, $\sigma_{ei}(T) \subset \sigma_{ej}(T)$ for $i < j < 4$. We will see later that $\rho_{e4}(T) \supset \rho_{e5}(T)$.*

Proposition 2.3.2 *If $T \in LR(X, Y)$ is continuous with finite dimensional range, then T is compact.*

Proposition 2.3.3 *The following are equivalent:*

(i) $T \notin \Phi_{+}$

(ii) *There exists a non-precompact bounded subset W of $D(T)$.*

(iii) *T has a singular sequence, i.e. there exists a sequence $\{x_n\} \subset D(T)$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $\{x_n\}$ has no convergent subsequence and $QTxn \to 0$ as $n \to \infty$.*

Proposition 2.3.4 *The following properties are equivalent for a given* $\lambda \in \mathbb{C}$

(1) $\lambda \in \rho_e(T)$

(2) $\lambda - T \in F_+ \cap F_-$

(3) The complete closure of $\lambda - T$ is a Fredholm relation.

Proposition 2.3.5 $\rho_e(T)$ *is an open set.*

Proof. *Clearly* $\rho_e(T) = \emptyset$ *or* $\mathbb{C}$ *if* $D(T)$ *is finite dimensional. Hence assume that* $D(T)$ *is infinite dimensional. Let* $\lambda \in \rho_e(T)$. *Then* $\lambda - T \in F_+ \cap F_-$.

Let $|\mu| < \Gamma(\lambda - T)$ *Then* $(\lambda + \mu) - T \in F_+$. *Also, since* $\lambda - T \in F_-$. *therefore* $\lambda - T' \in \phi_+$ *and so, excluding the trivial case when* $\dim D(T') < \infty$, *we have*

$$|\mu| < \Gamma(\lambda - T') \Rightarrow (\lambda + \mu) - T' \in \Phi_+ \Rightarrow (\lambda + \mu) - T \in F_-$$

Hence $\lambda + \mu \in \rho_e(T)$ *whenever* $|\mu| < \min\{\Gamma(\lambda - T), \Gamma(\lambda - T')\}$. *Therefore* $\rho_e(T)$ *is an open subset of* $\mathbb{C}$. ■

Proposition 2.3.6 *Let* $T \in LR(X)$. *Then*

(i) $\sigma_{ei}(T') = \sigma_{ei}(T)$ for $i = 1, 3, 4, 5$.

(ii) $\sigma_{ei}(T') = \sigma'_{ei}(T)$.

Corollary 2.3.1 *if* $\rho_{e4}(T)$ *is connected and* $\rho(T) \neq \emptyset$, *then* $\rho_{e5}(T) = \rho_{e4}(T)$

Chapter 3

STABILITY OF THE ESSENTIAL SPECTRUM

3.1 Classes of some perturbations relation

3.1.1 Relatively boundedness and compactness:

Definition 3.1.1 *[4] Let S, T be linear relations from X to Y and from X to Z, respectively.The linear relation S is called relatively bounded with respect to T (or* ***T- bounded****) if $D(S) \supset D(T)$ and there exist constants a, b for which the inequality*

$$\|Sx\| \leq a\|x\| + b\|Tx\|$$

holds for all $x \in D(T)$

Definition 3.1.2 *The relation S is called* ***T-precompact*** *(or* ***precompact relative*** *to T) if $D(S) \supset D(T)$ and SG_T is precompact. Similarly, S is called* ***T-compact*** *(or* ***compact relative to T****) if SG_T is compact.*

It is sometimes convenient to renorm the space X_T with some other equivalent norm. The following proposition is useful:

Proposition 3.1.1 *The norms $\|.\|_T$ and $\|.\|_{\lambda-T}$ are equivalent.*
Proof. *We have for $x \in D(T)$,*

$$\begin{aligned}\|x\|_{\lambda-T} &= \|x\| + \|(\lambda - T)x\| \leq \|x\| + |\lambda|\,\|x\| + \|Tx\| \\ &= (1+|\lambda|)\,\|x\| + \|Tx\| \leq (1+|\lambda|)(\|x\| + \|Tx\|) \\ &= (1+|\lambda|)\,\|x\|_T\end{aligned}$$

and

$$\begin{aligned}\|x\|_T &= \|x\| + \|Tx\| = \|x\| + \|\lambda x - (\lambda x - Tx)\| \leq (1+|\lambda|)\,\|x\| + \|(\lambda - T)x\| \\ &\leq (1+|\lambda|)(\|x\| + \|(\lambda - T)x\|) = (1+|\lambda|)\,\|x\|_{\lambda-T}\,.\end{aligned}$$

therefore $\|.\|_T$ *and* $\|.\|_{\lambda I-T}$ *are equivalent .* ■

3.1.2 Semi-Fredholm perturbation

Let $S \in LR(X,Y)$ be continuous such that $D(T) \subset D(S)$ and $T(0) \supset S(0)$.

S is called an **upper semi-Fredholm perturbation**,$T + S \in \Phi_+(X,Y)$ whenever $T \in \Phi_+(X,Y)$.

S is called a **lower semi-Fredholm perturbation**, $T + S \in \Phi_-(X,Y)$ whenever $T \in \Phi_-(X,Y)$.

The sets of upper semi-Fredholm and lower semi-Fredholm perturbations are, respectively, denoted by $PR(\Phi_+(X,Y))$, $PR(\Phi_-(X,Y))$. If $X = Y$, we have $PR(\Phi_+(X)) := PR(\Phi_+(X,X))$ and $PR(\Phi_-(X)) := PR(\Phi_-(X,X))$.

3.1.3 Atkinson perturbation

The classes of $\alpha-$Atkinson multivalued and $\beta-$Atkinson multivalued operators from X into Y are, respectively, the following:

$$A_\alpha(X,Y) := \{T \in \Phi_+(X,Y) : R(T) \text{ is topologically complemented in } Y\}$$

$$A_\beta(X,Y) := \{T \in \Phi_-(X,Y) : N(T) \text{ is topologically complemented in } X\}$$

3.2 Some perturbation of Linear Relations

Let T be a linear relation in $LR(X)$ and suppose that the complex number is in the resolvent set of T. Then if S is any precompact relation satisfying $D(S) \supset D(T)$ and $\dim S(0) < \infty$, we have :

$$\rho_e(T+S) = \rho_e(T)$$

Lemma 3.2.1 *let $T \in F_+(X,Y)$,then*

$$T \in F_- \Longleftrightarrow TG \in F_-$$

Theorem 3.2.1 *Let $T \in LR(X)$ be an arbitrary linear relation ,and let S be any* ***T-compact*** *(resp, T-precompact) relation satisfying* $\dim S(0) < \infty$.

if either S is closable (resp, completely closable), or X is reflexive and T is closable , then

$$\sigma_e(T+S) = \sigma_e(T)$$

Proof. First suppose that S is single valued. Let $\lambda \in \mathbb{C}$. Suppose that $\lambda - T \in F_+$. Then $(1-T)G_T \in F_+$. Since SG_T is compact (resp., precompact),

therefore $(\lambda - (T+S))G_T \in F_+$. Hence by the equivalence of the norms $\|.\|$ and $\|.\|_{T+S}$

$$(\lambda - (T+S))G_{T+S} \in F_+ \tag{3.2.1}$$

and by the equivalence of $\|.\|_{T+S}$ and $\|.\|_{\ \lambda-(T+S)}$,

$$(\lambda - (T+S))G_{\lambda-(T+S)} \in F_+$$

. Hence,

$$\lambda - (T+S) \in F_+.$$

Observing that each step in the above argument is reversible, we obtain the equivalence

$$\lambda - T \in F_+ \Leftrightarrow \lambda - (T+S) \in F_+. \tag{3.2.2}$$

Now suppose that $\lambda \in \rho_e(T)$. Then ,

$$(\lambda - T)G_{\lambda - T} \in F_-$$

and

$$(\lambda - T)G_T \in F_-$$

.Therefore

$$(\lambda - T)G_T - SG_T \in F_-$$

$$\text{i.e.} \qquad (\lambda - (T + S))G_T \in F_-$$

whence

$$(\lambda - (T + S))G_{\lambda - (T+S)} \in F_-$$

Hence

$$(\lambda - (T + S)) \in F_-$$

Consequently by 3.2.2, $\lambda \in \rho_e(T + S)$. We have thus shown that

$$\rho_e(T) \subset \rho_e(T + S). \tag{3.2.3}$$

(since the inclusion holds trivially if $\rho_e(T)$ is empty).

Now let $\lambda \in \rho_e(T + S)$. From the equivalence of the norms $\|\|_T$ and$\|\|_{T+S}$,

S is $(S + T) - \textit{compact}$ (resp., $(S + T) - \textit{precompact}$). Hence from 3.2.3, we conclude that

$$\rho_e(T + S) \subset \rho_e(T + S - S) = \rho(T). \tag{3.2.4}$$

Therefore $\sigma_e(T + S) =, \sigma_e(T)$

Now suppose that $\dim S(0) < \infty$. Then there exists a continuous projection P defined on $R(S)$ with kernel $S(0)$, and

$$S = PS + S - S. \tag{3.2.5}$$

Moreover PSG_T is compact (resp., Precompact) and so from what has been proved

$$\sigma_e(T) = \sigma_e(T + PS). \tag{3.2.6}$$

But$R(S - S) = S(0)$ is finite dimensional, and $\|S + S\| = 0$. Therefore by the finite rank perturbation

$$\sigma_e(T + PS) = \sigma_e(T + PS + S - S) = \sigma_e(T + S). \tag{3.2.7}$$

hence from 3.2.6 and 3.2.7,

$$\sigma_e(T) = \sigma_e(T+S)$$

■

Corollary 3.2.1 *let the relation T have a compact resolvent operator and let S be any bounded linear operator.then*

$$\sigma_e(T+S) = \sigma_e(T)$$

Proposition 3.2.1 *the norms $\|.\|_T$ and $\|.\|_{\lambda-T}$ are equivalent.*

Proposition 3.2.2 *[2] Let $T \in LR(X,Y)$ and suppose $S \in LR(X,Y)$ satisfies $D(S) \supset \overline{D(T)}$ and $S(0) \subset T(0)$, and is **T- bounded** with $a, b > 0, b < 1$ such that for $x \in D(T)$, $\|SX\| < a\|x\| + b\|Tx\|$*

(1) The norms $\|.\|_T$ and $\|.\|_{\lambda-T}$ are equivalent .

(2) If X and Y are complete and T is closed , then $T+S$ is closed.

Proposition 3.2.3 *Let $S, T \in LR(X,Y)$. If $S(0) \subset T(0)$ then $\Delta(S) < \Gamma(T) \Rightarrow T+S \in \Phi_+$,where*

$$\Gamma(T) := \inf_{M \in I(D(T))} \|T\,|_M\|, \quad \Delta(S) := \sup_{M \in I(D(S))} \Gamma(S|_M),$$

and $I(X)$ denotes the collection of infinite dimensional subspaces of X.

Proposition 3.2.4 *Let $S, T \in LR(X,Y)$, $D(S) \supset D(T)$ and let $T \in \Phi_-$.*

(1) If dim $R(S) < \infty$, then $T+S \in \Phi_-$

(2) If S is precompact , then $T+S \in \Phi_-$.

(3) If $\|S\| < \gamma(T')$, then $T+S \in \Phi_-$.

Proposition 3.2.5 **(a)** *Suppose $T \in \Phi_+(X,Y)$ and $S \in LR(X,Y)$ is strictly singular . If $\|S\| < \infty$, $D(S) \supset D(T), S(0) \subset T(0)$, then $\kappa(T+S) = \kappa(T)$.*

(b) *Suppose* $T \in \Phi_{-}(X,Y)$ *and* $S \in LR(X,Y)$ *is such that* S' *'is strictly singular . If* $\|S'\| < \infty, D(S) \supset D(T), S(0) \subset T(0)$,*then* $\kappa(T+S) = \kappa(T)$.

Lemma 3.2.2 *let the relation* F *satisfy* $D(T) \supset D(T)$ *and* $\dim R(F) < \infty$ *, then*

$$\sigma_e(T+F) = \sigma_e(T)$$

Proof. let $\lambda \in \rho_e(T)$ then $\lambda - T \in F_+$. Hence $\lambda - (T+F) \in F_+$, since F is strictly singular .Likewise , $\lambda - (T+F) \in F_-$. Hence $\rho_e(T) \subset \rho_e(T+F)$.

Now let $\lambda \in \rho_e(T+F)$. then $\lambda - (T+F-F) \in F_+ \cap F_-$ by the preceding.

Therefore $\lambda - T \in F_+$,

$Q_F(\lambda - T) = Q_F(\lambda - T + F - F) \in F_-$. and thus $\lambda - T \in F_-$ Therefore $\lambda \in \rho_e(T)$ ■

Theorem 3.2.2 *Let* $T \in LR(X)$ *be closed and suppose* $S \in LR(X)$ *is* ***T- compact*** *with* ***T-bound*** $b < 1$, $D(S) \supset \overline{D(T)}$ *and* $S(0) \subset T(0)$. *Then for* $i = 1,2,3,4$

$$\sigma_{ei}(T+S) = \sigma_{ei}(T).$$

If additionally ρ_{e4} *is connected and neither* $\rho(T)$ *nor* $\rho(T+S)$ *are empty , then*

$$\sigma_{e5}(T+S) = \sigma_{e5}(T).$$

Proof. By Proposition 3.2.1 , the norms $\|.\|_T$. and $\|.\|_{\lambda-T}$ are equivalent and hence , S is $(\lambda - T)$ **-compact** . Let $G_{\lambda-T}$ denote the graph operator from space $X_{\lambda-T} := (X, \|x\|_{\lambda-T})$ into X. Suppose $\lambda - T \in \Phi_\pm$. Clearly

$$R(TG_{\lambda-T}) = R(T)$$

and as subsets of the set X , we have $N(TG_{\lambda-T}) = N(T)$ By Proposition [2, 4.1]. $(\lambda - T)G_{\lambda-T}$ is open , and hence $(\lambda - T)G_{\lambda-T} \in \Phi_\pm$. Thus , by *Propositions* 3.2.3 and 3.2.4 , it follows that

$$(\lambda - T) - S = \lambda - (T+S) \in \Phi_\pm$$

and by *Propositions* 3.2.5 ,

$$\kappa(\lambda - (T+S) = \kappa(\lambda - T).$$

On the other hand , suppose $\lambda - (T + S) \in \Phi_{\pm}$. By the equivalence of the norms $\|.\|_T$ and $\|.\|_{\lambda-(T+S)}$ (*Proposition* 3.2.2) , it follows that S is $\lambda - (T + S)$ **- compact** . Arguing as before , it follows that $\lambda - T \in \Phi\pm$. and

$$\kappa(\lambda - T) = \kappa(\lambda - (T + S).$$

thus ,$\rho_{ei}(T + S) =, \rho_{ei}(T)$ for $i = 1, 2, 3, 4$. It follows from the additional hypothese s, *Corollary* 2.3.1 and what has just been proved that

$$\rho_{e5}(T) =, \rho_{e4}(T)-, \rho_{e4}(T)-, \rho_{e5}(T)(T + S).$$

■

Proposition 3.2.6 *Let* $T \in LR(X, Y)$ *with* $\gamma(T) > 0$.*suppose* $S \in LR(X, Y)$ *satisfies* $D(S) \supset D(T), S(0) \subset T(0)$ *and* $S < \gamma(T)$. *Then* $\alpha(T + S) \leq \alpha(T)$ *and* $\beta(T + S) \leq \beta(T)$

Semi-Fredholm perturbation Let $S \in LR(X, Y)$ be continuous such that $D(T) \subset D(S)$ and $T(0) \supset S(0)$.

(1) S is called an **upper semi-Fredholm perturbation**, $T + S \in \Phi_+(X, Y)$ whenever $T \in \Phi_+(X, Y)$.

(2) S is called a **lower semi-Fredholm perturbation,** $T + S \in \Phi_-(X, Y)$ whenever $T \in \Phi_-(X, Y)$.

The sets of upper semi-Fredholm and lower semi-Fredholm perturbations are, respectively,denoted by $PR(\Phi_+(X, Y), PR(\Phi_-(X, Y)$, If $X = Y$, we have $PR(\Phi_+(X)) := PR(\Phi_+(X, X))$, and $PR(\Phi_-(X) := PR(\Phi_-(X, X))$.

Atkinson perturbation: Let $S, T \in LR(X, Y)$ be continuous such that $D(T) \subset D(S)$ and $T(0) \supset S(0)$.

(1) S is called an α-**Atkinson perturbation** if $T+S \in A_\alpha(X, Y)$, whenever $T \in A_\alpha(X, Y)$ such that$T(0)$ is topologically complemented.

(2) S is called a β-**Atkinson perturbation** if $T+S \in A_\beta(X,Y)$, whenever $T \in A_\beta(X,Y)$ such that $T'(0)$ is topologically complemented.

The sets of $\alpha-$Atkinson perturbations and $\beta-$Atkinson perturbations are, respectively,denoted by $PR(A_\alpha(X,Y))$, $PR(A_\beta(X,Y))$. If $X = Y$, we have $PR(A_\alpha(X)) := PR(A_\alpha(X,X))$, and $PR(A_\beta(X)) := PR(A_\beta(X,X))$.

Theorem 3.2.3 *[3] Let $T \in CR(X,Y)$ and $S \in LR(X,Y)$ be continuous such that $S(0) \subset T(0)$and S is $T-$bounded with $T-$bound $\delta < 1$. Then, the following statements hold:*

(1) If $T \in A_\alpha(X,Y)$ and Si_T is $\alpha-$Atkinson perturbation, then $T+S \in A_\alpha(X,Y)$ and $\kappa(T+S) = \kappa(T)$.

(2) If $T \in A_\beta(X,Y)$ and Si_T is $\beta-$Atkinson perturbation, then $T+S \in A_\beta(X,Y)$ and $\kappa(T+S) = \kappa(T)$.

(3) If $T \in \Phi_+(X,Y)$ and Si_T is upper semi-Fredholm perturbation, then $T+S \in \Phi_+(X,Y)$ and $\kappa(T+S) = \kappa(T)$.

(4) If $T \in \Phi_-(X,Y)$ and Si_T is lower semi-Fredholm perturbation, then $T+S \in \Phi_-(X,Y)$ and $\kappa(T+S) = \kappa(T)$.

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Abstract

in this work, we present the spectral theory of multivalued linear operators where we have given a the definition and properties of the spectrum. Then we explain about the essential Spectrum and some classes of perturbations for study the stability of the Essential Spectrum .

Keywords and phrases: Linear relation, Essential Spectrum, Perturbation theory

الملخص

في هذه الأطروحة، نقدم النظرية الطيفية للمشغلين الخطيين متعددي القيم حيث قدمنا تعريفًا و خصائص للطيف. ثم نشرح الطيف الأساسية وبعض انواع الاضطرابات لدراسة استقرار الطيف الأساسي

الكلمات والعبارات الأساسية:العلاقة الخطية، الطيف الأساسي، الإضافات

Résumé

Dans ce mémoire, nous présentons la théorie spectrale des opérateurs linéaires multi-valeurs où nous avons donné en définition et les propriétés du spectre. Ensuite, nous expliquons le spectre essentiel et quelques classes de perturbations pour étudier la Stabilité du spectre essentiel.

Mots-clés et phrases: les relations linéaires, Spectre essentiel, La théorie de perturbation