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Contents

List of abbreviations and symbols

$\mathbb{K}$	The field of real or complex numbers.
p^*	The conjugate of the number p $(1 \leq p < \infty)$, that is $\frac{1}{p} + \frac{1}{p^*} = 1$.
B_E	The closed unit ball of Banach space E.
E^*	The topological dual of Banach space E.
$X^{\#}$	The Lipschitz dual of the pointed metric space X.
$m_{xx'}$	The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$, where χ_A is the characteristic function of the set A.
$\mathcal{M}(X)$	The vector space of all molecule son the metric space X.
$\mathcal{F}(X)$	The Lipschitz free space of X.
Æ(X)	The Arens-Eells space of X.
T^*	The adjoint operator of T.
T_L	The linearization of the operator T.
T_B	The bi-linearization of the operator T.
$\mathfrak{m}$	Vector measure.
$\lvert\mathfrak{m}\rvert$	Variation of vectors measure $\mathfrak{m}$.
$\lVert\mathfrak{m}\rVert$	Semivariation of vectors measure $\mathfrak{m}$.
δ_X	The isometrically embedding from X to Æ(X).
j_p	The canonical inclusion map defined between $C(K)$ and $L_p(\mu)$.
j_∞	The canonical inclusion map from $C(B_{X^\#})$ to $L_\infty(\mu)$.
i_p	The canonical inclusion map defined between $L_\infty(\mu)$ and $L_p(\mu)$.
ι_X	The Lipschitz isometric embedding, $\iota_X : X \longrightarrow C(B_{E^*})$ given by $\iota_X(x) = \langle x, . \rangle$.
$\mathcal{L}_f$	The class of all finite rank linear operators.
Lip_0	The class of all Lipschitz operators that vanish at 0.
$\mathcal{I}_{Lip}$	Lipschitz operator ideals.

Π_p^L	The class of all Lipschitz p-summing operators.
$\mathcal{L}_{as,(p;p_1,p_2)}$	The class of absolutely $(p;p_1,p_2)$-summing bilinear operators.
$S\mathcal{N}p$	The class of all strongly p-nuclear linear operators.
$\mathcal{PI}_p$	The class of all linear Pietsch-p-integral operators.
$\mathcal{PI}_\infty$	The class of all linear Pietsch-∞-integral operators.
$\mathcal{PI}_p^L$	The class of all Lipschitz Pietsch-p-integral operators.
$\mathcal{PI}_\infty^L$	The class of all Lipschitz Pietsch-∞-integral operators.
$\mathcal{D}_p$	The class of all linear strongly p-summing operators.
$\mathcal{D}_p^2$	The class of all bilinear Cohen strongly p-summing operators.
$\mathcal{D}_p^L$	The class of all Lipschitz strongly p-summing operators.
$\mathcal{R}_p$	The class of all linear p-representable operators.
$\mathcal{R}_p^L$	The class of all Lipschitz p-representable operators.
$\mathcal{L}_{si,p}$	The class of all bilinear p-semi-integral operators.
$\mathcal{L}_{f,p}^2$	The class of all bilinear factorable p-dominated operators.
$BLip$	The class of all two-Lipschitz operators.
$BLip_0$	The class of all two-Lipschitz operators that vanish at 0.
$BLip_{0\mathcal{F}}$	The class of two-Lipschitz finite rank operators.
Im_{BLip}	The two-Lipschitz image .
$\Gamma(A)$	Absolutely convex hull of A.
$\mathcal{I}_{BLip}$	Two-Lipschitz operator ideals.
$BLip_{0\mathcal{K}}$	The class of two-Lipschitz compact operators.
$BLip_{0\mathcal{W}}$	The class of two-Lipschitz weakly compact operators.
$\mathcal{D}_p^{BL}$	The class of two-Lipschitz absolutely p-summing operators.
$BL_{as,(p;p_1,p_2)}$	The class of two-Lipschitz $(p;p_1,p_2)$-summing operators.
$BL_{f,p}$	The class of two-Lipschitz factorable p-dominated operators.

List of publications

1- Khaled Hamidi, Elhadj Dahia, Dahmane Achour, Abdelhamid Tallab. Two-Lipschitz operator ideals. J. Math. Anal. Appl. 491 (2020), 124346 .

2- Khaled Hamidi, Elhadj Dahia, Lipschitz integral operators represented by vector measures, (Submitted).

0.1 Introduction

The main topic treated in this thesis is the study of some non-linear ideals, with emphasis mainly on Lipschitz and two-Lipschitz operator ideals. The theory of operator ideals was introduced by Pietsch in the linear case and is nowadays well established. The reader can find a lot of information about it in the excellent monograph [43]. The theory of operator ideals has proved to be a strong tool for the investigation and classification of linear operators between Banach spaces. Nowadays, it has become a usual tool for analyzing some classical problems of the Functional Analysis -as summability of series-. The linear theory has spread to multilinear operators and to Lipschitz operators, leading to the notions of multi-ideals and Lipschitz ideals. An axiomatic theory of multi-ideals for Banach spaces-valued m-linear mappings was given by Braunss in 1984 (see [14]). These new multi-ideals can be considered as a new area in non-linear functional analysis. Some results about the tensoriel representation of multi-ideal can be found in [23]. Since the work of Farmer and Johnson [29], where the notion of p-summing Lipschitz functions was introduced, there was an increasing interest in the study of different classes of Lipschitz functions between (pointed) metric spaces and Banach spaces. Most of them, can be seen as a generalization of different linear operators ideals between Banach spaces. This approach turned out to be very successful and a number of operator ideals have been fruitfully generalized to the Lipschitz setting in recent years by several authors (see [2], [3], [4], [11], [15], [20], [33], [38]....).

In 2016, Achour et al. [6] introduced the notion of Lipschitz operator ideals in the same spirit of linear operator ideals (see also [52]). An ideal of Lipschitz mappings $\mathcal{I}_{Lip}$ is a subclass of the class of all Lipschitz mappings between pointed metric spaces and Banach spaces such that for a metric space X and Banach space E, the components

$$\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip},$$

is a vector subspace of $Lip_0(X, E)$ that is invariant by the composition of a linear operator on the right and Lipschitz operator on the left and which contains the Lipschitz finite rank mappings type.

Some properties that hold in the linear case remain true in the Lipschitz case, and some of them are not true any more. Regarding the general construction of ideal of Lipschitz mappings, following [6], there is a way to construct a Lipschitz operator ideal starting from a given linear operator ideal, called composition method.

In 2009, Dubei et al. introduced in [27] the definition of two-Lipschitz maps which is defined on the cartesian product of two pointed metric spaces with values in a Banach space, that is Lipschitz separately in each variable. Under some difficult adequate requirements it is shown that every two-Lipschitz mapping is associated with continuous bilinear mapping from product of two suitable free Banach spaces to another Banach space (see [27]). This idea worked successfully without any conditions in [50]. There, Sánchez-Pérez present a suitable definition of real-valued two-Lipschitz mappings (under the name of Lipschitz bi-forms) that admits a good continuous bi-linearization form.

The main goal of this thesis is to study the new concept of two-Lipschitz operator ideals between pointed metric spaces and Banach spaces. We extend to the two-Lipschitz mappings setting a linear procedure for creating ideals of two-Lipschitz operators from a given linear operator ideal. We apply our results to the well known operator ideal of strongly p-summing operators ideals. We present and characterize the notion of compactness for the two-Lipschitz mappings. As far as we know, this is the first attempt in this regard.

A second purpose of the thesis is to introduce and study the Lipschitz version of Pietsch-p-integral operators. We define the Lipschitz Pietsch-p-integral operator $(1 \leq p \leq \infty)$ as a Lipschitz mapping between a pointed metric space and a Banach space by an integral representation with respect to a vector measure on the Borel σ-algebra of a compact Hausdorff space K. Special attention is paid to the factorization of these mappings and we compare our class with some well known Lipschitz operators defined by a factorization schema or by summability of series. Note that the class of Lipschitz Pietsch-1-integral operators is studied in [16]. In this case, the authors use only factorization schemes to define this concept without using vector measure theory.

The thesis consists of four chapters. In the preliminaries (Chapter 1) we establish the notation of the thesis. We introduce basic notions in Banach space theory and we recall the main definitions and properties of the Lipschitz operator theory that we will use later.

Also, we recall the most important results for the linear operator ideals, specially the ideal of compact linear operators, weakly compact linear operators and p-summing linear operators. The fourth section is devoted to the study of the bilinear operator ideals. After this, we recall the basic concepts on the theory of Lipschitz operator ideals. Finally, we recall some preliminaries on vector measures.

In Chapter 2 of this thesis we introduce the new class of Lipschitz Pietsch-p-integral operator ($1 \leq p \leq \infty$), that is defined using a factorization schema that provides the Lipschitz version of the p-integral linear operators. We give a characterization of these Lipschitz mappings by an integral representation with respect to a vector measure and a relationship between the Lipschitz Pietsch-p-integral operator and its linearization showing that this type of operators fits in the theory of composition Banach Lipschitz operator ideal.

In the next chapter (Chapter 3) we introduce the new notion of Lipschitz p-representable operators. We show that they can be seen as a natural extension of the p-representable linear operators of Roshdi Khalil ([35]) and we transfer some properties of the linear case into the Lipschitz setting.

In the last chapter (Chapter 4) We establish the basics of the theory of two-Lipschitz operator ideals between pointed metric spaces and Banach spaces. We present the composition method to produce an ideal of two-Lipschitz operators from a given operator ideal $\mathcal{I}$. We show that a two-Lipschitz operator T belongs to the resulting ideal if it can be written as $T = u \circ S$ with u belonging to $\mathcal{I}$ and S is a two-Lipschitz operator. Finally, we introduce the ideal of two-Lipschitz compact operators from the product of two pointed metric spaces X, Y into a Banach space E and we study a two-Lipschitz version of strongly p-summing linear operators in order to apply the technique of composition previously developed.

Chapter 1

Preliminaries

In this chapter we expose the concepts and results used throughout the thesis on some basic notions in Banach space theory, linear operator ideals, bilinear operator ideals, Lipschitz operator ideals and some elementary properties of vector measures.

1.1 Some basic notions in Banach space theory

We will write $\mathbb{K}$ for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For $1 \leq p \leq \infty$, by p^* we denote the conjugate of p, that is $\frac{1}{p} + \frac{1}{p^*} = 1$. Along this thesis letters, E, F, H and G denote Banach spaces. Given a Banach space E, B_E is its closed unit ball and S_E is the unit sphere of E. By $\mathcal{L}(E, F)$ we denote the Banach space of all continuous linear operators between E and F with the norm

$$\|T\| = \sup_{x \in B_E} \|T\,(x)\|\,.$$

Throughout the thesis, we call operator to a continuous linear mapping.
The set of all functionals of a normed space E (that is, the continuous linear mappings from E into the scalars) is a Banach space, denoted by E^* and called the dual of E. For $x \in E$, we shall write $\langle x, x^* \rangle$ (or $\langle x^*, x \rangle$) for the action of the functional x^* on x. The norm of $x^* \in E^*$ is given by

$$\|x^*\| = \sup \left\{ |\langle x, x^* \rangle| : x \in B_E \right\}.$$

A linear operator $T : Z_1 \longrightarrow Z_2$ between two normed spaces Z_1 and Z_2 is an isomorphism if T is a continuous bijection its inverse is also continuous. In such case the spaces Z_1 and Z_2 are said to be isomorphic. T is an isometric isomorphism when $\|T(x)\| = \|x\|$ for all $x \in Z_1$. In particular, any linear operator between normed spaces of the same finite dimension is an isomorphism.

A linear operator T is an embedding of E into F if T is an isomorphism onto its image $T(E)$. In this case we say that E embeds in F. If $T : E \longrightarrow F$ is an embedding such that $\|T(x)\| = \|x\|$ for all $x \in E$, then T is said to be isometric embedding.

Given the continuous linear operator $T : E \longrightarrow F$, the continuous linear operator $T^* : F^* \longrightarrow E^*$ defined as

$$T^*(y^*)(x) = y^*(T(x)),$$

for every $y^* \in F^*$ and $x \in E$ is called the adjoint of T and has the property that $\|T^*\| = \|T\|$.

If K is a topological space, then by $C(K)$ we mean the space of all scalar valued (*i.e.*, real or complex valued), bounded, continuous functions on K. This is a Banach space with the norm

$$\|f\|_\infty = \sup_{x\in K} |f(x)| .$$

Clearly, if K is a compact space then $C(K)$ consists of all continuous, scalar valued functions.

If K is compact, the dual of the space $C(K)$ equals the space $M(K)$ of all regular Borel measures (scalar valued, but obviously not necessarily positive) on K. The duality is defined as

$$\langle f, \mu\rangle = \mu(f) = \int_K f d\mu, \quad f \in C(K), \quad \mu \in M(K).$$

Let $1 \leq p < \infty$ and $n \in \mathbb{N}^*$. We denote by $\ell_p^n(E)$ the space of all sequences $(x_i)_{i=1}^n$ in the Banach space E with the norm

$$\left\|(x_i)_{i=1}^n\right\|_p = (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}},$$

and by $\ell_{p,\omega}^n(E)$ the space of all sequences $(x_i)_{i=1}^n$ in E with the norm

$$\left\|(x_i)_{i=1}^n\right\|_{p,\omega} = \sup_{\|x^*\|_{E^*}\leq 1} \left\|(\langle x_i, x^*\rangle)_{i=1}^n\right\|_p .$$

If $E = \mathbb{K}$, we simple write ℓ_p^n.

Let E, F, G be Banach spaces over $\mathbb{K}$. A mapping $T : E \times F \longrightarrow G$ is bilinear if it is linear separately in each coordinate. It is bounded if there exists a constant $C > 0$ such that for any $x \in E$ and $y \in F$, we have

$$\|T(x,y)\| \leq C\|x\|\|y\| . \tag{1.1}$$

We denote by $\mathcal{L}(E, F; G)$ the Banach space of all (bounded) continuous bilinear mappings from $E \times F$ into G with the norm

$$\|T\| = \sup_{\|x\|_E\leq 1, \|y\|_F\leq 1} \|T(x,y)\| = \inf\{C \geq 0 \text{ satisfying (1.1)}\} . \tag{1.2}$$

The projective norm on $E \otimes F$, the tensor product of E and F, is defined by

$$\pi(u) = \inf \sum_{i=1}^n \|x_i\| \|y_i\| , \tag{1.3}$$

where the infimum is taken over all possible representations of $u \in E\otimes F$ of the form $\sum_{i=1}^n x_i\otimes y_i$.

The tensor product $E \otimes F$ endowed with the projective norm π is denoted by $E \otimes_\pi F$ and its completion by $E\widehat{\otimes}_\pi F$. Which call the projective tensor product.

We now recall the linearization of continuous bilinear mappings. Consider the canonical continuous bilinear mapping

$$\sigma_2 : E \times F \longrightarrow E\hat{\otimes}_\pi F,$$

defined by

$$\sigma_2\left(x, y\right) = x \otimes y.$$

Theorem 1.1.1.

Let E, F and G be Banach spaces. For every continuous bilinear mapping $T : E \times F \longrightarrow G$ there exists a unique continuous linear operator $T_L : E\widehat{\otimes}_\pi F \longrightarrow G$ satisfying

$$T_L \circ \sigma_2 = T,$$

i.e.,

$$T_L\left(x \otimes y\right) = T\left(x, y\right),$$

for every $(x, y) \in E \times F$. That is, the following diagram commutes

$$\begin{array}{ccc} E \times F & \xrightarrow{\quad T \quad} & G \\ & {\scriptstyle \sigma_2} \searrow \quad \nearrow {\scriptstyle T_L} & \\ & E\hat{\otimes}_\pi F & \end{array} \quad . \tag{1.4}$$

Furthermore, $\|T_L\| = \|T\|$.

The previous theorem gives the canonical identification

$$\mathcal{L}(E, F; G) = \mathcal{L}(E\widehat{\otimes}_\pi F, G).$$

For more details refer to [47] and [24].

1.2 Lipschitz operators

As usual, X,Y,Z and W will be pointed metric spaces with a base point denoted by 0 and metric that will be denoted by d. We denote by B_X the closure of the ball centered at 0 with radius 1. A Banach space E will be considered as a pointed metric space with a base point 0 and distance $d(x,x') = \|x - x'\|$.

With $Lip_0(X,Y)$ we denote the set of all Lipschitz mappings from X to Y that maps 0 to 0. In particular, $Lip_0(X,E)$ is the Banach space of all Lipschitz mappings T from X to E that vanish at 0, under the Lipschitz norm

$$Lip\,(T) = \inf\{C > 0 : \|T(x) - T(x')\| \leq C.d(x,x');\ \forall x, x' \in X\}.$$

When $E = \mathbb{K}$, $Lip_0(X,\mathbb{K})$ is denoted by $X^{\#}$ and it is called the Lipschitz dual of X. It is clear that $\mathcal{L}(E,F)$ is a subspace of $Lip_0(E,F)$ and, in particular, E^* is a subspace of $E^{\#}$. Along the thesis we consider $B_{X^{\#}}$ endowed with the pointwise topology.

One of the main tools that we will use is the Arens–Ells space Æ(X) (see [9]). (Also known as the Lipschitz-free Banach space $\mathcal{F}(X)$ of a metric space X [31]).

Definition 1.2.1.

Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum\limits_{x\in X} m\,(x) = 0$. We denote by $\mathcal{M}(X)$ the vector space of all molecules on X.

For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A.

For $m \in \mathcal{M}(X)$ we can write $m = \sum\limits_{j=1}^{n} \lambda_j m_{x_j x'_j}$ for some suitable scalars λ_j and we write

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{j=1}^{n} |\lambda_j|\, d(x_j, x'_j),\ m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}\right\},$$

where the infimum is taken over all representations of the molecule m. Denote by Æ$\,(X)$ the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$.

The canonical Lipschitz injection map $\delta_X : X \longrightarrow$ Æ(X) defined by $\delta_X(x) = m_{x0}$ isometrically embeds X in Æ$\,(X)$.

Theorem 1.2.2. *[34, Lemma 3.1]*

Let $T \in Lip_0(X,Y)$, there exists a unique linear operator $\widehat{T} \in \mathcal{L}($Æ$\,(X)\,,$ Æ$\,(Y))$ such that

$$\widehat{T} \circ \delta_X = \delta_Y \circ T, \tag{1.5}$$

that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & Y \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ Æ(X) & \xrightarrow[\widehat{T}]{} & Æ(Y) \end{array} \tag{1.6}$$

commutes. Furthermore, $\left\|\widehat{T}\right\| = Lip(T)$.

Theorem 1.2.3. *[51, Theorem 2.2.4 (b)]*

Let $T \in Lip_0(X, E)$, there is a unique bounded linear map $T_L : Æ(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$ that is, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & E \\ & {\scriptstyle \delta_X}\searrow \quad \nearrow{\scriptstyle T_L} & \\ & Æ(X) & \end{array} \tag{1.7}$$

Furthermore $\|T_L\| = Lip(T)$.

The operator T_L is referred to as the linearization of T. The correspondence $T \longleftrightarrow T_L$ establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}(Æ(X), E)$.
In particular, the spaces $X^{\#}$ and $Æ(X)^*$ are isometrically isomorphic via the linearization $R(f) := f_L$, where $f_L(m) = \sum_{x \in X} f(x)m(x)$ (see [51, Theorem 2.2.2]).

1.3 Linear operator ideals

Recall that, from [43], a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only if has the form $x^* \otimes y : x \longmapsto \langle x, x^* \rangle y$. Then if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$.

Definition 1.3.1.

An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between

Banach spaces such that for all Banach spaces E and F its components

$$\mathcal{I}(E,F) := \mathcal{L}(E,F) \cap \mathcal{I}$$

satisfy:

(i) $\mathcal{I}(E,F)$ is a vector subspace of $\mathcal{L}(E,F)$ which contains the mappings of the form $x^* \otimes y$ where $x^* \in E^*$ and $y \in F$.

(ii) The ideal property: if $v \in \mathcal{L}(E,H)$, $u \in \mathcal{I}(H,G)$ and $w \in \mathcal{L}(G,F)$, then the composition $w \circ u \circ v$ is in $\mathcal{I}(E,F)$.

An operator ideal $\mathcal{I}$ is a normed (Banach) operator ideal if there is $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \longrightarrow [0,+\infty[$ satisfies

(i') $(\mathcal{I}(E,F), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') If $v \in \mathcal{L}(E,H)$, $u \in \mathcal{I}(H,G)$ and $w \in \mathcal{L}(G,F)$, then

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|u\|_{\mathcal{I}} \, \|v\| \, .$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one (see [43]).

The operator ideal $\mathcal{I}$ is said to be closed if each $\mathcal{I}(E,F)$ is closed for the sup norm.

1.3.1 The ideal of compact and weakly compact linear operators

1- Compact linear operators

A linear operator $u : E \longrightarrow F$ is said to be compact if $u(B)$ is a precompact subset of F for every bounded subset B of E. An equivalent formulation is that u is compact if and only if every bounded sequence $(x_i)_{i=1}^{\infty}$ in E has a subsequence $(x_{i_k})_{k=1}^{\infty}$ such that $(u(x_{i_k}))_{k=1}^{\infty}$ converges in F. We denote by $\mathcal{K}(E,F)$ the vector space of all compact linear operators from E into F.

2- Weakly compact linear operators

A linear operator $u : E \longrightarrow F$ is said to be weakly compact, in symbols $u \in \mathcal{W}(E, F)$, if u maps B_E onto a relatively weakly compact subset of F. This is equivalent to say that $(u(x_i))_{i=1}^{\infty}$ has a weakly convergent subsequence for every bounded sequence $(x_i)_{i=1}^{\infty}$ in E.

Proposition 1.3.2. *[43]*

The classes $\mathcal{K}$ and $\mathcal{W}$ are closed Banach operator ideals, where the ideal norm is the operator norm.

1.3.2 The ideal of p-summing linear operators

The theory of p-summing operators is based on a crucial criterion due to A. Pietsch [42]. Let $1 \leq p < \infty$. A linear operator $u : E \longrightarrow F$ between Banach spaces is said to be p-summing if there exists a constant $C \geq 0$ such that

$$\left\| (u(x_i))_{i=1}^{n} \right\|_p \leq C \left\| (x_i)_{i=1}^{n} \right\|_{p,\omega},$$

for every finite family $(x_i)_{i=1}^{n} \subset E$. The infimum of all such constants C is denoted by $\pi_p(u)$. The set of all p-summing operators between E and F is denoted by $\Pi_p(E, F)$. We mention that $(\Pi_p, \pi_p(\cdot))$ is Banach operator ideal (*see*[25]).

The nowadays classical Pietsch's domination theorem characterizes the p -summability of an operator by means of a norm domination uniform inequality. Concretely, it says that the mapping $u \in \mathcal{L}(E, F)$ is p-summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{E^*} (with the weak star topology) so that

$$\|u(x)\| \leq C \left(\int_{B_{E^*}} |\langle x, x^* \rangle|^p \, d\mu \right)^{\frac{1}{p}}, \qquad x \in E. \tag{1.8}$$

In this case, $\pi_p(u)$ is the least of all the constants C such that (1.8) holds.

Proposition 1.3.3. *[25, Page 39]*

If $1 \leq p \leq q < \infty$, then $\Pi_p(E, F) \subset \Pi_q(E, F)$. Moreover, for all $u \in \Pi_p(E, F)$ we have $\pi_q(u) \leq \pi_p(u)$.

In [25, Examples 2.9] we find some canonical linear p-summing operators that will be used in the sequel.

1- Let (Ω, Σ, μ) be a finite measure space, and $1 \leq p < \infty$, the formal inclusion map $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is p-summing, with $\pi_p(i_p) = \mu(\Omega)^{\frac{1}{p}}$.

2- Let K be a compact Hausdorff space and ν a positive regular Borel measure on K, and $1 \leq p < \infty$, the canonical map $j_p : C(K) \longrightarrow L_p(\nu)$ is p-summing, with $\pi_p(j_p) = \nu(K)^{\frac{1}{p}}$.

Corollary 1.3.4. *[25, Corollary 2.15]*

Let K be a compact Hausdorff space. An operator $u : C(K) \longrightarrow E$ is p-summing if and only if there exist a regular Borel probability measure μ on K and an operator $\tilde{u} \in \mathcal{L}(L_p(K, \mu), E)$ such that $T = \tilde{u} \circ j_p$ that is, the following diagram commutes

$$\begin{array}{ccc} C(K) & \xrightarrow{\quad u \quad} & F \\ & {\scriptstyle j_p} \searrow \quad \nearrow {\scriptstyle \tilde{u}} & \\ & L_p(K, \mu) & \end{array} \tag{1.9}$$

Moreover, we may arrange that $\pi_p(u) = \|\tilde{u}\|$.

1.4 Bilinear operator ideals

Let E_1, E_2, F be Banach spaces. Consider non-zero $\varphi^j \in E_j^*(j = 1, 2)$, and $y \in F$. Define the bilinear mapping

$$\varphi^1 \otimes \varphi^2 \otimes y : E_1 \times E_2 \longrightarrow F,$$

by

$$\varphi^1 \otimes \varphi^2 \otimes y(x^1, x^2) := \varphi^1(x^1)\varphi^2(x^2)y. \tag{1.10}$$

It is clear that $\varphi^1 \otimes \varphi^2 \otimes y \in \mathcal{L}(E_1, E_2; F)$ the vector space of continuous bilinear operators and

$$\left\|\varphi^1 \otimes \varphi^2 \otimes y\right\| = \left\|\varphi^1\right\| \left\|\varphi^2\right\| \|y\|.$$

We denote by $\mathcal{L}_f(E_1, E_2; F)$, the vector space of finite type bilinear operators, which is generated by the mappings of the form (1.10). All elements T of this space a finite representation of the form

$$T = \sum_{i=1}^{n} \lambda_i \varphi_i^1 \otimes \varphi_i^2 \otimes y_i$$

where $(\lambda_i)_{i=1}^n \subset \mathbb{K}, (\varphi_i^j)_{i=1}^n \subset E_j^*(j = 1, 2)$ and $(y_i)_{i=1}^n \subset F$.

Definition 1.4.1.

An ideal of bilinear mappings (or bilinear ideal) is a subclass $\mathcal{M}$ of all continuous bilinear mappings between Banach spaces such that for all Banach spaces E_1, E_2 and F, the components

$$\mathcal{M}(E_1, E_2; F) := \mathcal{L}(E_1, E_2; F) \cap \mathcal{M}$$

satisfy:

(i) $\mathcal{M}(E_1, E_2; F)$ is a vector subspace of $\mathcal{L}(E_1, E_2; F)$ which contains the bilinear mappings of finite type.

(ii) The ideal property: If $T \in \mathcal{M}(G_1, G_2; H)$, $u_j \in \mathcal{L}(E_j, G_j)$ for $j = 1, 2$ and $v \in \mathcal{L}(H, F)$, then $v \circ T \circ (u_1, u_2)$ is in $\mathcal{M}(E_1, E_2; F)$.

If $\|.\|_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathbb{R}^+$ satisfies

(i') $(\mathcal{M}(E_1, E_2; F), \|\cdot\|_{\mathcal{M}})$ is a normed (Banach) space for all Banach spaces E_1, E_2, F.

(ii') The bilinear form $T^2 : \mathbb{K}^2 \longrightarrow \mathbb{K}$ given by $T^2(x^1, x^2) = x^1 x^2$ satisfies $\|T^2\|_{\mathcal{M}} = 1$.

(iii') If $T \in \mathcal{M}(G_1, G_2; H), u_j \in \mathcal{L}(E_j, G_j)$ for $j = 1, 2$ and $v \in \mathcal{L}(H, F)$, then

$$\|v \circ T \circ (u_1, u_2)\|_{\mathcal{M}} \leq \|v\| \, \|T\|_{\mathcal{M}} \, \|u_1\| \, \|u_2\| ,$$

we say that $(\mathcal{M}, \|.\|_{\mathcal{M}})$ is a normed (Banach) bilinear ideal.

Of course the Banach spaces considered in this definition are all over the same fixed scalar field.

The bilinear ideal $\mathcal{M}$ is said to be closed if each $\mathcal{M}(E_1, E_2; F)$ is a closed subspace of $\mathcal{L}(E_1, E_2; F)$ for the sup norm.

Note that $\mathcal{L}$, the class of all continuous bilinear mappings between arbitrary Banach spaces, is the largest bilinear ideal.

Proposition 1.4.2. *[30]*

Let $\mathcal{M}$ be normed bilinear ideal and E_1, E_2, F be Banach spaces.

1- $\|T\| \leq \|T\|_{\mathcal{M}}$ *for all* $T \in \mathcal{M}(E_1, E_2; F)$.

2- $\|\varphi^1 \otimes \varphi^2 \otimes y\|_{\mathcal{M}} = \|\varphi^1\| \|\varphi^2\| \|y\|$ *for any* $\varphi^j \in E_j^*(j = 1, 2)$ *and* $y \in F$.

1.5 Lipschitz operator ideals

The notion of Lipschitz operator ideal was introduced by Achour, Rueda, Sánchez-Pérez and Yahi [6]. This can be seen as an extension of the linear operator ideal.

Definition 1.5.1. [6, Definition 2.1]
A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components

$$\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip}$$

satisfy:

(i) $\mathcal{I}_{Lip}(X, E)$ is a vector subspace of $Lip_0(X, E)$.

(ii) $eg \in \mathcal{I}_{Lip}(X, E)$ for $e \in E$ and $g \in X^{\#}$.

(iii) The ideal property: if $S \in Lip_0(X, Y)$, $T \in \mathcal{I}_{Lip}(Y, F)$ and $u \in \mathcal{L}(F, E)$, then the composition $u \circ T \circ S$ is in $\mathcal{I}_{Lip}(X, E)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is a function $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0, +\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{I}_{Lip}(X, E), \|.\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X, E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(X, Y)$, $T \in \mathcal{I}_{Lip}(Y, F)$ and $u \in \mathcal{L}(F, E)$, then

$$\|u \circ T \circ S\|_{\mathcal{I}_{Lip}} \leq Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|u\| .$$

Following [6, Definition 3.1], there is a way to construct a (Banach) Lipschitz operator ideal from a (Banach) linear operator ideal, called composition method. Let $\mathcal{I}$ be a (Banach) linear operator ideal. A Lipschitz mapping $T \in Lip_0(X, E)$ belongs to the composition Lipschitz operator ideal $\mathcal{I} \circ Lip_0$ if there exists a Banach space F, a Lipschitz operator $S \in Lip_0(X, F)$ and a linear operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a Banach operator ideal we write

$$\|T\|_{\mathcal{I}\circ Lip_0} = \inf \|u\|_{\mathcal{I}} \, Lip(S),$$

where the infimum is taken over all u and S as above.

In [6], the authors establish a criterion to decide whenever a Lipschitz operator ideal is of composition or not.

Proposition 1.5.2. *[6, Proposition 3.2]*

Let X be a pointed metric space, E a Banach space and $\mathcal{I}$ an operator ideal. A Lipschitz operator $T \in Lip_0(X, E)$ belongs to $\mathcal{I} \circ Lip_0(X, E)$ if and only if its linearization T_L belongs to $\mathcal{I}(Æ(X), E)$.

Furthermore, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a Banach operator ideal then $(\mathcal{I}\circ Lip_0, \|\cdot\|_{\mathcal{I}\circ Lip_0})$ is Banach Lipschitz operator ideal with

$$\|T\|_{\mathcal{I}\circ Lip_0} = \|T_L\|_{\mathcal{I}}.$$

1.6 Elementary properties of vector measures

Let us also recall some preliminaries on vector measures. For the general theory of vector measures we refer the reader to the classical monograph (see [26]).

Definition 1.6.1.

A function $\mathfrak{m}$ from a σ-algebra Σ of subsets of a set Ω to a Banach space E is called a finitely additive vector measure, or simply a vector measure, if whenever A_1 and A_2 are disjoint members of Σ then

$$\mathfrak{m}(A_1 \cup A_2) = \mathfrak{m}(A_1) + \mathfrak{m}(A_2).$$

If, in addition,

$$\mathfrak{m}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathfrak{m}(A_i),$$

in the norm topology of E for all sequences (A_i) of pairwise disjoint members of Σ such that $\cup_{i=1}^{\infty} A_i \in \Sigma, i.e.,$

$$\lim_{n \longrightarrow \infty} \left\| \mathfrak{m}\left(\cup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{n} \mathfrak{m}(A_i) \right\|_E = 0.$$

Then $\mathfrak{m}$ is termed a countably additive vector measure or simply, $\mathfrak{m}$ is countably additive.

Definition 1.6.2.

A vector measure $\mathfrak{m} : \Sigma \longrightarrow E$ is said to be monatomic if for every set $A \in \Sigma$ with $\mathfrak{m}(A) \neq 0$ there exist $A_i \in \Sigma, A_i \neq \emptyset (i = 1, 2)$ such that $A_i \cup A_j = A$ and $\mathfrak{m}(A_i) \neq 0$ $(i = 1, 2)$.

Definition 1.6.3.

Let $\mathfrak{m} : \Sigma \longrightarrow E$ be a vector measure.

• The variation of $\mathfrak{m}$ is the extended nonnegative function $|\mathfrak{m}|$ whose value on a set $A \in \Sigma$ is given by

$$|\mathfrak{m}|(A) = \sup \sum_{i=1}^{n} \|\mathfrak{m}(A_i)\|,$$

where the supremum is taken over all partitions $(A_i)_{i=1}^{n}$ of A into a finite number of pairwise disjoint members of Σ. If $|\mathfrak{m}|(\Omega) < \infty$, then will be called a measure of bounded variation.

• The semivariation of $\mathfrak{m}$ is the extended nonnegative function $\|\mathfrak{m}\|$ whose value on a set $A \in \Sigma$ is given by

$$\|\mathfrak{m}\|(A) = \sup \{|e^*\mathfrak{m}|(A) : e^* \in E^*, \|e^*\| \leq 1\},$$

where $|e^*\mathfrak{m}|$ is the variation of the real-valued measure $e^*\mathfrak{m}$. If $\|\mathfrak{m}\|(\Omega) < \infty$, then $\mathfrak{m}$ will be called a measure of bounded semivariation.

Remark 1.6.4.

Let $\mathfrak{m} : \Sigma \longrightarrow E$ be a vector measure.

1- The variation of scalar measure is a finite positive measure, however the variation of vector measure is not finite valued.

2- If $\mathfrak{m}$ has be a bounded variation, then $|\mathfrak{m}|$ is a finite positive measure on Σ.

3- The set of all vector measure $\mathcal{V}(E) = \{\mathfrak{m} : \Sigma \longrightarrow E\}$ is a Banach space with the norm $\|\mathfrak{m}\| = |\mathfrak{m}|(\Omega)$.

Example 1.6.5.

Let $u : L_1[0,1] \longrightarrow E$ be a continuous linear operator. Define the function $\mathfrak{m}$ by $\mathfrak{m}(A) = u(\chi_A)$ for each Lebesgue measurable set $A \subseteq [0,1]$, then

1- The function $\mathfrak{m}$ is evidently finitely additive vector measure. Moreover, for each A, one has $\|\mathfrak{m}(A)\| \leq \|u\| \lambda(A)$.

2- The function $\mathfrak{m}$ is countably additive vector measure:

Let λ Lebesgue measure and let $(A_i)_{i=1}^{\infty}$ is a sequence of disjoint Lebesgue measurable subsets of $[0,1]$, then

$$\begin{aligned}
\lim_{n\to\infty} \left\| \mathfrak{m}\left(\cup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{n} \mathfrak{m}(A_i) \right\| &= \lim_{n\to\infty} \left\| \sum_{i=n+1}^{\infty} \mathfrak{m}(A_i) \right\| \\
&= \lim_{n\to\infty} \left\| \sum_{i=n+1}^{\infty} u\left(\chi_{A_i}\right) \right\| \\
&\leq \lim_{n\to\infty} \sum_{i=n+1}^{\infty} \left\| u\left(\chi_{A_i}\right) \right\| \\
&\leq \lim_{n\to\infty} \|u\| \sum_{i=n+1}^{\infty} \left\| \chi_{A_i} \right\| \\
&\leq \lim_{n\to\infty} \|u\| \sum_{i=n+1}^{\infty} \lambda\left(A_i\right) = 0.
\end{aligned}$$

Then

$$\lim_{n\to\infty} \left\| \mathfrak{m}\left(\cup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{n} \mathfrak{m}(A_i) \right\| = 0.$$

That is, for all partition $(A_i)_{i=1}^{\infty}$ of $[0,1]$, with $A_i \cap A_j = \emptyset$, $\forall i \neq j$

$$\mathfrak{m}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathfrak{m}(A_i).$$

3- The vector measure $\mathfrak{m}$ is bounded variation:

$$\begin{aligned}
|\mathfrak{m}|\left([0,1]\right) &= \sup \left\{ \sum_{i=1}^{n} \left\| \mathfrak{m}(A_i) \right\|, (A_i)_{i=1}^{n} \text{ partition of } [0,1] \right\} \\
&= \sup \sum_{i=1}^{n} \left\| u(\chi_{A_i}) \right\| \\
&\leq \|u\| \sup \sum_{i=1}^{n} \left\| \chi_{A_i} \right\|_1 = \|u\| \sup \sum_{i=1}^{n} \int_{A_i} d\lambda \\
&= \|u\| \lambda\left([0,1]\right) = \|u\| < \infty.
\end{aligned}$$

Proposition 1.6.6.

A vector measure of bounded variation is countably additive if and only if its variation is also countably additive.

The next proposition presents two basic facts about the semivariation of a vector measure.

Proposition 1.6.7.

Let $\mathfrak{m} : \Sigma \longrightarrow E$ *be a vector measure. Then*

1- *For* $A \in \Sigma$,

$$\|\mathfrak{m}\| (A) = \sup \left\{ \left\| \sum_{A_i \in \pi} \varepsilon_i \mathfrak{m} (A_i) \right\| \right\},$$

where the supremum is taken over all partitions π *of* A *into a finite number of pairwise disjoint members of* Σ *and all finite collections* $\{\varepsilon_i\}$ *satisfying* $|\varepsilon_i| \leq 1$.

2- *For* $A \in \Sigma$,

$$\sup \{\|\mathfrak{m}(B)\| : A \supseteq B \in \Sigma\} \leq \|\mathfrak{m}\| (A) \leq 4 \sup \{\|\mathfrak{m}(A)\| : A \supseteq B \in \Sigma\}.$$

Consequently a vector measure is of bounded semivariation on Ω *if and only if its range is bounded in* E.

We recall the definition of the integral of a bounded measurable function with respect to a bounded vector measure. To this end, let Σ be a algebra of subsets of Ω and $\mathfrak{m} : \Sigma \longrightarrow E$ be a bounded vector measure. If f is a scalar-valued simple function on Ω, say $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where α_i are nonzero scalars and $A_1, ..., A_n$ are pairwise disjoint members of Σ, define

$$u (f) = \sum_{i=1}^{n} \alpha_i \mathfrak{m} (A_i).$$

It is easy to show that this formula defines a linear map u from the space of simple functions of the above form into E.

Moreover, if f is as above and $\beta = \sup \{f (w) : \quad w \in \Omega\}$, then

$$\|u (f)\| = \left\| \sum_{i=1}^{n} \alpha_i \mathfrak{m} (A_i) \right\| = \beta \left\| \sum_{i=1}^{n} \frac{\alpha_i}{\beta} \mathfrak{m} (A_i) \right\| \leq \beta \|\mathfrak{m}\| (\Omega), \tag{1.11}$$

by Proposition1.6.7 (1). Thus, if the space of simple functions over Σ is given the supremum norm, u acts on this space as a continuous linear operator with $\|u\| \leq \|\mathfrak{m}\| (\Omega)$. Another look at Proposition 1.6.7 (1) and (1.11) shows that $\|u\| = \|\mathfrak{m}\| (\Omega)$.

Next note that since u is continuous and linear from the simple functions modeled on Σ to E, u has a unique continuous linear extension, still denoted by u, to $B (\Sigma)$, the space of all scalar-valued functions on Ω that are uniform limits of simple functions modeled on Σ.

Note that in the case Σ is a σ-algebra, $B(\Sigma)$ is precisely the familiar space of bounded Σ-measurable scalar-valued functions defined on Ω.

Definition 1.6.8.

Let Σ be a σ-algebra of subsets of a set Ω and $\mathfrak{m} : \Sigma \longrightarrow E$ be a bounded vector measure. For each $f \in B(\Sigma)$, the integral of f with respect to $\mathfrak{m}$ is defined by

$$\int f d\mathfrak{m} = u(f),$$

where u is as above. (we say u is representing by $\mathfrak{m}$).

It is easy to see that this integral is linear in f (and also in $\mathfrak{m}$) and satisfies

$$\left\| \int f d\mathfrak{m} \right\| \leq \|f\|_{\infty} \|\mathfrak{m}\| (\Omega).$$

Moreover, if $e^* \in E^*$, then $e^* \int f d\mathfrak{m} = \int f d(e^*\mathfrak{m})$ holds; indeed, for simple functions f this equality is trivial and density of simple functions in $B(\Sigma)$ proves the identity for all $f \in B(\Sigma)$.

Chapter 2

Factorization of Lipschitz operators represented by vector measures

In this chapter we introduce the concept of Lipschitz Pietsch-p-integral mappings, $(1 \leq p \leq \infty)$, between a metric space and a Banach space. We represent these mappings by an integral with respect to a vector measure defined on a suitable compact Hausdorff space, obtaining in this way a rich factorization theory through the classical Banach spaces $C(K)$, $L_p(\mu, K)$ and $L_\infty(\mu, K)$. Also we show that this type of operators fits in the theory of composition Banach Lipschitz operator ideals. For $p = \infty$, we characterize the Lipschitz Pietsch-∞-integral mappings by a factorization schema through a weakly compact operator. Finally, the relationship between these mappings and some well known Lipschitz operators is given.

2.1 Linear Pietsch-p-integral operators

For $1 \leq p \leq \infty$, linear Pietsch-p-integral operators were introduced by Persson and Pietsch [41] and deeply studied in [19, 25] among others.

Definition 2.1.1.

The linear operator $u : E \longrightarrow F$, between Banach spaces, is Pietsch-p-integral $(1 \leq p < \infty)$ if there are a regular Borel countably additive vector measure $\mathfrak{m}$ of bounded semivariation on $\mathcal{B}(B_{E^*})$, (where $\mathcal{B}(B_{E^*})$ is the Borel σ-algebra of B_{E^*}), and a positive regular Borel measure μ on B_{E^*} such that

$$u(x) = \int_{B_{E^*}} \langle x, x^* \rangle \, d\mathfrak{m}(x^*), \qquad x \in E,$$

and

$$\left\| \int_{B_{E^*}} f d\mathfrak{m} \right\| \leq \left(\int_{B_{E^*}} |f|^p \, d\mu \right)^{\frac{1}{p}}, \qquad \forall f \in C(B_{E^*}).$$

The Banach space of these operators is denoted by $\mathcal{PI}_p(E, F)$ under the norm defined by $\|u\|_{\mathcal{PI}_p} = \inf \mu(B_{E^*})^{\frac{1}{p}}$, where the infimum is taken over all measures μ satisfying the above inequality.

Remark 2.1.2.

The Definition 2.1.1 is equivalent to say that there are a regular Borel probability measure space (Ω, Σ, μ), two linear operators $A \in \mathcal{L}(L_p(\mu), F)$ and $B \in \mathcal{L}(E, L_\infty(\mu))$ such that

$$u = A \circ i_p \circ B : E \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} F, \tag{2.1}$$

where $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the canonical mapping. In this case we have $\|T\|_{\mathcal{PI}_p} = \inf \|A\| \, \|B\|$, where the infimum is taken over all μ, A and B as above (see [41, Satz 18 1.8], [19, Theorem 1.8]).

Definition 2.1.3.

Let $p = \infty$, the linear operator $u : E \longrightarrow F$ is called Pietsch-∞-integral if there is a regular Borel countably additive vector measure $\mathfrak{m} : \mathcal{B}(B_{E^*}) \longrightarrow F$ of bounded semivariation such that

$$u(x) = \int_{B_{E^*}} \langle x, x^* \rangle \, d\mathfrak{m}(x^*), \qquad x \in E.$$

In this case, $\|T\|_{\mathcal{PI}_\infty^L} = \inf \|\mathfrak{m}\| (B_{E^*})$, taking the infimum over all $\mathfrak{m}$ that satisfying the above equality.

In [41, Satz 15 and Satz 17] we find some canonical linear Pietsch-p-integral operators that will be used in the sequel. Let K be a compact Hausdorff space and ν be a positive regular Borel measure on K. Let (Ω, Σ, μ) be a finite measure space and j_p, i_p be the inclusions of $C(K)$ into $L_p(K, \nu)$ and of $L_\infty(\mu)$ into $L_p(\mu)$ respectively for $1 \leq p < \infty$. Then $j_p \in \mathcal{PI}_p(C(K), L_p(K, \nu))$ and $i_p \in \mathcal{PI}_p(L_\infty(\mu), L_p(\mu))$ with $\|j_p\|_{\mathcal{PI}_p} = \|j_p\| = \nu(K)^{\frac{1}{p}}$ and $\|i_p\|_{\mathcal{PI}_p} = \|i_p\| = \mu(\Omega)^{\frac{1}{p}}$.

Remark 2.1.4.

Note that by using [21, Theorem 2.5], $u \in \mathcal{PI}_p(E, F)$ if and only if there are a compact Hausdorff space K, an embedding $h : E \longrightarrow C(K)$, a regular Borel countably additive vector measure $\mathfrak{m} : \mathcal{B}(K) \longrightarrow F$ of bounded semivariation and a positive regular Borel measure μ on K such that for all $x \in E$,

$$u(x) = \int_K h(x)\, d\mathfrak{m},$$

and

$$\left\| \int_K f d\mathfrak{m} \right\| \leq \left(\int_K |f|^p \, d\mu \right)^{\frac{1}{p}},$$

for all $f \in C(K)$. In this case

$$\|u\|_{\mathcal{PI}_p} = \inf \|h\| \, \mu(K)^{\frac{1}{p}},$$

where the infimum is taken over all K, $\mathfrak{m}$ and h as above.

2.2 Lipschitz Pietsch-p-integral operators

Definition 2.2.1.

Let X be a pointed metric space, E a Banach space and let $T \in Lip_0(X, E)$. For $1 \leq p < \infty$, the mapping T is said to be Lipschitz Pietsch-p-integral operator if there are a regular Borel probability measure space (Ω, Σ, μ), a linear operator $A \in \mathcal{L}(L_p(\mu), E)$ and a Lipschitz operator $B \in Lip_0(X, L_\infty(\mu))$ giving rise to the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ {\scriptstyle B}\downarrow & & \uparrow{\scriptstyle A} \\ L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu), \end{array} \tag{2.2}$$

where $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the canonical mapping. The set of all Lipschitz Pietsch-p-integral mappings from X to E is denoted by $\mathcal{PI}_p^L(X, E)$. With each $T \in \mathcal{PI}_p^L(X, E)$ we associate its Lipschitz Pietsch-p-integral quantity, $\|T\|_{\mathcal{PI}_p^L} = \inf \|A\| \, Lip(B)$, where the infimum is taken over all μ, A and B as above.

Remark 2.2.2.

As an easy consequence of the Definition 2.2.1, if $T \in \mathcal{PI}_p^L(X, E)$ we have $Lip(T) \leq \|T\|_{\mathcal{PI}_p^L}$. In order to see this, for all $\varepsilon > 0$ choose (A, B, μ) a Lipschitz Pietsch-p-integral factorization of T such that $T = A \circ i_p \circ B$ and $\|A\| \, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. Then, for all $x, y \in X$ we have

$$\begin{aligned} \|T(x) - T(y)\| &\leq \|A\| \, \|i_p\| \, \|B(x) - B(y)\|_\infty \\ &\leq \|A\| \, Lip(B) \, d(x, y), \end{aligned}$$

this implies that $T \in Lip_0(X, E)$ and

$$\begin{aligned} Lip(T) &\leq \|A\| \, Lip(B) \\ &\leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon, \end{aligned}$$

since this holds for all $\varepsilon > 0$ we arrive at $Lip(T) \leq \|T\|_{\mathcal{PI}_p^L}$.

Notice that the Definition 2.2.1 is the same if we consider a finite regular Borel measure space (Ω, Σ, μ), in this case, for $T \in \mathcal{PI}_P^L(X, E)$ we have

$$\|T\|_{\mathcal{PI}_P^L} = \inf \|A\| \, Lip(B) \mu(\Omega)^{\frac{1}{p}},$$

where the infimum is taken over all μ, A and B in (2.2).

We don't know if Lipschitz Pietsch-p-integrability implies Pietsch-p-integrability when the mapping T is linear. The converse is of course true as we see in the following proposition.

Proposition 2.2.3.

If X and E are Banach spaces and $T : X \longrightarrow E$ is linear Pietsch-p-integral then T is Lipschitz Pietsch-p-integral and $\|T\|_{\mathcal{PI}_P^L} \leq \|T\|_{\mathcal{PI}_P}$.

Proof. For $\varepsilon > 0$ choose a regular Borel probability measure space (Ω, Σ, μ), two linear operators $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in \mathcal{L}(X, L_\infty(\mu))$ satisfies (2.1) and $\|A\| \, \|B\| \leq \varepsilon + \|T\|_{\mathcal{PI}_P}$. The operator B is belongs to $Lip_0(X, L_\infty(\mu))$ with $\|B\| = Lip(B)$. This means that $T \in \mathcal{PI}_P^L(X, E)$ and

$$\|T\|_{\mathcal{PI}_P^L} \leq \|A\| \, Lip(B) = \|A\| \, \|B\| \leq \varepsilon + \|T\|_{\mathcal{PI}_P}.$$

Since this holds for all $\varepsilon > 0$, we obtain $\|T\|_{\mathcal{PI}_P^L} \leq \|T\|_{\mathcal{PI}_P}$. □

We have the following immediate consequence of the Definition 2.2.1.

Proposition 2.2.4. *(Inclusion Theorem).*
Let $1 \leq p \leq q < \infty$. Then $\mathcal{PI}_p^L(X, E) \subset \mathcal{PI}_q^L(X, E)$ and $\|T\|_{\mathcal{PI}_q^L} \leq \|T\|_{\mathcal{PI}_p^L}$ for all $T \in \mathcal{PI}_p^L(X, E)$.

Proof. For all $\varepsilon > 0$ choose the typical factorization of $T \in \mathcal{PI}_p^L(X, E)$, $T = A \circ i_p \circ B$ and $\|A\| \, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. In the other hand we have

$$i_p = i_{q,p} \circ i_q : L_\infty(\mu) \xrightarrow{i_q} L_q(\mu) \xrightarrow{i_{q,p}} L_p(\mu),$$

where $i_{q,p} : L_q(\mu) \longrightarrow L_p(\mu)$ is the canonical inclusion map. Then we obtain the factorization

$$T = A \circ i_{q,p} \circ i_q \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_q} L_q(\mu) \xrightarrow{D} E,$$

where $D = A \circ i_{q,p} \in \mathcal{L}(L_q(\mu), E)$, which implies that $T \in \mathcal{PI}_q^L(X, E)$ and

$$\begin{aligned}
\|T\|_{\mathcal{PI}_q^L} &\leq \|D\| \, Lip(B) \\
&\leq \|A\| \, \|i_{q,p}\| \, Lip(B) \\
&\leq \|A\| \, Lip(B) \\
&\leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$ we get $\|T\|_{\mathcal{PI}_q^L} \leq \|T\|_{\mathcal{PI}_p^L}$. □

In order to prove the factorization theorem for the class of Lipschitz Pietsch-p-integral operators, $(1 \leq p \leq \infty)$ we need the following technical lemma.

Lemma 2.2.5.

Let $J : \mathcal{M}(X) \longrightarrow C(B_{X^{\#}})$ be the operator defined by

$$J(m)(f) = \sum_{i=1}^{n} \lambda_i (f(x_i) - f(x_i')),$$

for all $m = \sum_{i=1}^{n} \lambda_i m_{x_i x_i'} \in \mathcal{M}(X)$ and $f \in B_{X^{\#}}$. Then this operator is an isometric embedding.

Proof. Since $\text{Æ}(X)^*$ and $X^{\#}$ are isometrically isomorphic via the linearization, for all $f \in X^{\#}$ there is $m^* \in \text{Æ}(X)^*$ such that $f_L = m^*$. For all $m \in \mathcal{M}(X)$ we have

$$\begin{aligned} \|J(m)\|_{C(B_{X^{\#}})} &= \sup_{f \in B_{X^{\#}}} |J(m)(f)| = \sup_{\|f_L\| \leq 1} \left| \sum_{i=1}^{n} \lambda_i (f_L(m_{x_i x_i'}) \right| \\ &= \sup_{\|m^*\| \leq 1} |\langle m, m^* \rangle| = \|m\|_{\text{Æ}(X)} = \|m\|_{\mathcal{M}(X)}, \end{aligned}$$

and the proof follows. □

For $x \in X$, we denote by δ_x the functional $\delta_x : X^{\#} \longrightarrow \mathbb{R}$ defined as $\delta_x(f) = f(x)$, $f \in X^{\#}$.

Let $\iota_X : X \longrightarrow C(B_{X^{\#}})$ the natural Lipschitz isometric embedding such that $\iota_X(x)$ is the restriction of δ_x to $B_{X^{\#}}$, for all $x \in X$.

The following theorem gives a parallel development of the factorization schemes concerning Lipschitz Pietsch-p-integral operators that highlights the role of the space $C(B_{X^{\#}})$.

Theorem 2.2.6.

Let $1 \leq p < \infty$ and let $T \in Lip_0(X, E)$. Then T is Lipschitz Pietsch-p-integral if and only if there exist a regular Borel probability measure ν on $B_{X^{\#}}$ and an operator $\widetilde{A} \in \mathcal{L}(L_p(\nu), E)$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle \iota_X}\downarrow & & \uparrow{\scriptstyle \widetilde{A}} \\ C(B_{X^{\#}}) & \xrightarrow{\ j_p\ } & L_p(\nu), \end{array} \tag{2.3}$$

where j_p is the canonical map. Moreover,

$$\|T\|_{\mathcal{PI}_p^L} = \inf \left\{ \left\| \widetilde{A} \right\| : T = \widetilde{A} \circ j_p \circ \iota_X \right\}.$$

Proof. We write $\triangle$ for the proposed infimum. Suppose that T admits a factorization (2.3). If j_∞ is the canonical inclusion map from $C(B_{X^\#})$ to $L_\infty(\nu)$, we have the factorization

$$T = \tilde{A} \circ i_p \circ j_\infty \circ \iota_X : X \xrightarrow{\iota_X} C(B_{X^\#}) \xrightarrow{j_\infty} L_\infty(\nu) \xrightarrow{i_p} L_p(\nu) \xrightarrow{\tilde{A}} E.$$

Denoting by $B = j_\infty \circ \iota_X$, it follows that $B \in Lip_0(X, L_\infty(\nu))$ and $Lip(B) \leq 1$, which implies that T is Lipschitz Pietsch-p-integral and

$$\|T\|_{\mathcal{PI}_p^L} \leq \left\|\tilde{A}\right\| Lip(B) \leq \left\|\tilde{A}\right\|.$$

Passing to the infimum we get $\|T\|_{\mathcal{PI}_p^L} \leq \triangle$.

Conversely, suppose that $T \in \mathcal{PI}_p^L(X, E)$. Fix $\varepsilon > 0$, there are a regular Borel probability measure space (Ω, Σ, μ), an operator $A \in \mathcal{L}(L_p(\mu), E)$ and a Lipschitz mapping $B \in Lip_0(X, L_\infty(\mu))$ such that

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

and $\|A\|\, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. Let $B_L \in \mathcal{L}(\text{Æ}(X), L_\infty(\mu))$ be the linearization of the Lipschitz mapping B, that is $B = B_L \circ \delta_X$ and $\|B_L\| = Lip(B)$. Consider the natural extension of the isometric embedding J, mentioned in Lemma 2.2.5, to Æ(X) which we denote also by J. The injectivity of $L_\infty(\mu)$ assures the existence of an operator $\widetilde{B_L} \in \mathcal{L}(C(B_{X^\#}), L_\infty(\mu))$ that extends B_L with $\left\|\widetilde{B_L}\right\| = \|B_L\|$, that is $B_L = \widetilde{B_L} \circ J$ or the following diagram commutes

$$\begin{array}{ccc} \text{Æ}(X) & \xrightarrow{B_L} & L_\infty(\mu) \\ {\scriptstyle J}\downarrow & \nearrow_{\widetilde{B_L}} & \\ C(B_{X^\#}). & & \end{array}$$

The operator $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is p-summing with p-summing norm one then $i_p \circ \widetilde{B_L}$ is too with $\pi_p(i_p \circ \widetilde{B_L}) \leq \left\|\widetilde{B_L}\right\|$. By Corollary 1.3.4 there exist a regular Borel probability measure ν on $B_{X^\#}$ and an operator $S \in \mathcal{L}(L_p(\nu), L_p(\mu))$ such that

$$i_p \circ \widetilde{B_L} = S \circ j_p : C(B_{X^\#}) \xrightarrow{j_p} L_p(\nu) \xrightarrow{S} L_p(\mu),$$

and $\pi_p(i_p \circ \widetilde{B_L}) = \|S\|$. Then

$$T = (A \circ S) \circ j_p \circ (J \circ \delta_X) : X \xrightarrow{J \circ \delta_X} C(B_{X^\#}) \xrightarrow{j_p} L_p(\nu) \xrightarrow{A \circ S} E.$$

Easy calculations prove that $J \circ \delta_X = \iota_X$, which implies that T admits a factorization of the form (2.3) with $\tilde{A} = A \circ S$ and we have

$$\begin{aligned} \triangle &\leq \left\| \tilde{A} \right\| \leq \|A\| \, \pi_p(i_p \circ \widetilde{B_L}) \\ &\leq \|A\| \left\| \widetilde{B_L} \right\| = \|A\| \, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$ we arrive at $\triangle \leq \|T\|_{\mathcal{PI}_p^L}$. □

The next theorem is the main result of this section and provides a characterization of the class of Lipschitz Pietsch-p-integral operators, that is an integral representation with respect to a vector measure.

Theorem 2.2.7.

Let $1 \leq p < \infty$ and let $T \in Lip_0(X, E)$. Then T is Lipschitz Pietsch-p-integral if and only if there are a compact Hausdorff space K, a Lipschitz embedding $\phi : X \longrightarrow C(K)$ with $\phi(0) = 0$, a regular Borel countably additive vector measure $\mathfrak{m} : \mathcal{B}(K) \longrightarrow E$ of bounded semivariation and a positive regular Borel measure μ on K such that

$$T(x) = \int_K \phi(x) \, d\mathfrak{m}, \quad x \in X, \tag{2.4}$$

and

$$\left\| \int_K f d\mathfrak{m} \right\| \leq \left(\int_K |f|^p \, d\mu \right)^{\frac{1}{p}}, \tag{2.5}$$

for all $f \in C(K)$. In this case

$$\|T\|_{\mathcal{PI}_p^L} = \inf Lip(\phi) \mu(K)^{\frac{1}{p}},$$

where the infimum is taken over all K, ϕ, $\mathfrak{m}$ and μ satisfying (2.4) and (2.5).

Proof. Suppose that $T \in \mathcal{PI}_p^L(X, E)$, and fix $\varepsilon > 0$. There are a regular Borel probability measure ν on $B_{X^\#}$ and $\tilde{A} \in \mathcal{L}(L_p(\nu), E)$ such that

$$T = \tilde{A} \circ j_p \circ \iota_X : X \xrightarrow{\iota_X} C(B_{X^\#}) \xrightarrow{j_p} L_p(\nu) \xrightarrow{\tilde{A}} E,$$

and $\left\| \tilde{A} \right\| \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. The linear operator $\tilde{A} \circ j_p : C(B_{X^\#}) \longrightarrow E$ is Pietsch-p-integral with

$$\left\| \tilde{A} \circ j_p \right\|_{\mathcal{PI}_p} \leq \left\| \tilde{A} \right\| \|j_p\|_{\mathcal{PI}_p} \leq \left\| \tilde{A} \right\| \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon.$$

By Remark 2.1.4, there are a compact Hausdorff space K, an embedding $h : C(B_{X^{\#}}) \longrightarrow C(K)$, a regular Borel countably additive vector measure $\mathfrak{m} : \mathcal{B}(K) \longrightarrow E$ of bounded semi-variation and a positive regular Borel measure μ on K such that for all $x \in X$,

$$T(x) = \tilde{A} \circ j_p(\iota_X(x)) = \int_K h(\iota_X(x))\, d\mathfrak{m},$$

$\left\| \int_K f d\mathfrak{m} \right\| \leq \left(\int_K |f|^p \, d\mu \right)^{\frac{1}{p}}$ for all $f \in C(K)$ and $\|h\| \, \mu(K)^{\frac{1}{p}} \leq \left\| \tilde{A} \circ j_p \right\|_{\mathcal{PI}_p} + \varepsilon$. Which means that (2.4) and (2.5) are true by taking into account that $\phi = h \circ \iota_X$ is a Lipschitz embedding from X to $C(K)$ vanishing at 0 with $Lip(\phi) \leq \|h\|$. Moreover

$$Lip(\phi)\mu(K)^{\frac{1}{p}} \leq \left\| \tilde{A} \circ j_p \right\|_{\mathcal{PI}_p} + \varepsilon \leq \|T\|_{\mathcal{PI}_p^L} + 2\varepsilon.$$

Since this holds for every $\varepsilon > 0$, it follows that $Lip(\phi)\mu(K)^{\frac{1}{p}} \leq \|T\|_{\mathcal{PI}_p^L}$.

Conversely, suppose that T satisfies the conditions (2.4) and (2.5). By Theorem VI.2.1 in [26] there exists $u \in \mathcal{L}(C(K), E)$ such that $u(f) = \int_K f d\mathfrak{m}$, $f \in C(K)$. Consider the canonical mapping

$$j_p = i_p \circ j_\infty : C(K) \xrightarrow{j_\infty} L_\infty(K, \mu) \xrightarrow{i_p} L_p(K, \mu),$$

and define $R : j_p(C(K)) \longrightarrow E$ by $R(j_p(f)) := u(f)$. The linear mapping R is well-defined and continuous with norm ≤ 1. Since for all $f \in C(K)$, we have

$$\|R(j_p(f))\| = \left\| \int_K f d\mathfrak{m} \right\| \leq \left(\int_K |f|^p \, d\mu \right)^{\frac{1}{p}} = \|j_p(f)\|.$$

By [28, Lemma IV.8.19] we have $\overline{j_p(C(K))} = L_p(K, \mu)$. So R can be extended to a continuous linear operator $\widetilde{R} : L_p(K, \mu) \longrightarrow E$ with $\left\| \widetilde{R} \right\| \leq 1$. If we put $B = j_\infty \circ \phi$, we obtain $B \in Lip_0(X, L_\infty(K, \mu))$ and $Lip(B) \leq Lip(\phi)$. On the other hand,

$$\widetilde{R} \circ i_p \circ B(x) = \widetilde{R} \circ j_p \circ \phi(x) = u(\phi(x)) = \int_K \phi(x) d\mathfrak{m} = T(x),$$

and therefore T factors as in (2.2), that is $T \in \mathcal{PI}_p^L(X, E)$ with

$$\|T\|_{\mathcal{PI}_p^L} \leq Lip(B) \left\| \widetilde{R} \right\| \mu(K)^{\frac{1}{p}} \leq Lip(\phi)\mu(K)^{\frac{1}{p}}.$$

□

Now we present a relationship between the Lipschitz Pietsch-p-integral operator and its linearization.

Theorem 2.2.8.

Let $T \in Lip_0(X,E)$ and $1 \leq p < \infty$. Then $T \in \mathcal{PI}_p^L(X,E)$ if and only if $T_L \in \mathcal{PI}_p(Æ(X),E)$. Moreover, we have

$$\|T\|_{\mathcal{PI}_p^L} = \|T_L\|_{\mathcal{PI}_p}. \tag{2.6}$$

Proof. Suppose that $T_L \in \mathcal{PI}_p(Æ(X),E)$. According to [41, Satz 18], for every $\varepsilon > 0$ we can choose a typical factorization of T_L

$$T_L = A \circ i_p \circ R : Æ(X) \xrightarrow{R} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

such that $A \in \mathcal{L}(L_p(\mu), E)$ and $R \in \mathcal{L}(Æ(X), L_\infty(\mu))$ with $\|A\| \|R\| \leq \|T_L\|_{\mathcal{PI}_p} + \varepsilon$. It is clear that the mapping $B := R \circ \delta_X$ belongs to $Lip_0(X, L_\infty(\mu))$ and $Lip(B) \leq \|R\|$. The factorization $T = T_L \circ \delta_X = A \circ i_p \circ B$ implies that $T \in \mathcal{PI}_p^L(X,E)$ and

$$\|T\|_{\mathcal{PI}_p^L} \leq \|A\| \, Lip(B) \leq \|T_L\|_{\mathcal{PI}_p} + \varepsilon.$$

Conversely, if $T \in \mathcal{PI}_p^L(X,E)$, for $\varepsilon > 0$ choose the following factorization of T

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

such that $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in Lip_0(X, L_\infty(\mu))$ with $\|A\| \, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. The uniqueness of the linearization maps gives that

$$T_L = (A \circ i_p \circ B)_L = A \circ i_p \circ B_L.$$

Then, we have that $T_L \in \mathcal{PI}_p(Æ(X),E)$ with

$$\|T_L\|_{\mathcal{PI}_p} \leq \|A\| \|B_L\| = \|A\| \, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon.$$

The proof finish. □

By Theorem 2.2.8 and Proposition 1.5.2, we have the following.

Proposition 2.2.9.

$\left(\mathcal{PI}_p^L, \|\cdot\|_{\mathcal{PI}_p^L}\right)$ is the Banach Lipschitz operator ideal generated by the composition method from the Banach operator ideal $\mathcal{PI}_p$. In other words

$$\mathcal{PI}_p^L(X,E) = \mathcal{PI}_p \circ Lip_0(X,E) \quad \textit{isometrically}$$

for every pointed metric space X and every Banach space E.

We say that a pointed metric space W is 1-injective (or an absolute Lipschitz retract) if for every metric space X, every subset X_0 of X and every Lipschitz mapping $T \in Lip_0(X_0, W)$ there is a Lipschitz mapping $\widetilde{T} \in Lip_0(X, W)$ extending T with $Lip(T) = Lip(\widetilde{T})$.
The real Banach space $L_\infty(\mu)$ for a finite measure μ is 1-injective (see [12, Chapter 1]).

By the typical Pietsch-p-integral factorization of a Lipschitz mapping T, we can find a Pietsch-p-integral extension $\widetilde{T}$.

Proposition 2.2.10.

Let X and Z be pointed metric spaces with $X \subset Z$ and let E be a Banach space. Each Lipschitz Pietsch-p-integral operator $T : X \longrightarrow E$ admits a Lipschitz Pietsch-p-integral extension $\widetilde{T} : Z \longrightarrow E$ with $\|T\|_{\mathcal{PI}_p^L} = \left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L}$.

Proof. If $T \in \mathcal{PI}_p^L(X,E)$, then for all $\varepsilon > 0$ there are a regular Borel probability measure space (Ω, Σ, μ), $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in Lip_0(X, L_\infty(\mu))$ such that

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

and $Lip(B)\,\|A\| \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. Since $L_\infty(\mu)$ is 1-injective, B admits an extension $\widetilde{B} \in Lip_0(Z, L_\infty(\mu))$ with $Lip(\widetilde{B}) = Lip(B)$.*i.e.*, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{B} & L_\infty(\mu) \\ {\scriptstyle i}\downarrow & \nearrow_{\widetilde{B}} & \\ Z, & & \end{array}$$

where $i \in Lip_0(X,Z)$ is the natural isometric embedding. This creates a Pietsch-p-integral extension $\widetilde{T} : Z \longrightarrow E$ of T having the following factorization

$$\widetilde{T} = A \circ i_p \circ \widetilde{B} : Z \xrightarrow{\widetilde{B}} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E.$$

Furthermore,

$$\left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L} \leq Lip(\widetilde{B})\left\|A\right\| = Lip(B)\left\|A\right\| \leq \left\|T\right\|_{\mathcal{PI}_p^L} + \varepsilon.$$

Since this holds for all $\varepsilon > 0$ we get $\left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L} \leq \left\|T\right\|_{\mathcal{PI}_p^L}$. For the reverse inequality, note that

$$\left\|T\right\|_{\mathcal{PI}_p^L} = \left\|\widetilde{T}\circ i\right\|_{\mathcal{PI}_p^L} \leq \left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L}.$$

□

2.3 Lipschitz Pietsch-∞-integral operators

In this section we extend the definition of the class of Pietsch-∞-integral linear operators to the case of Lipschitz operators and we will show a factorization theorem that characterizes these mappings.

Definition 2.3.1.

We say that a Lipschitz operator $T \in Lip_0(X,E)$ is Pietsch-∞-integral if there is a regular Borel countably additive vector measure $\mathfrak{m} : \mathcal{B}(B_{X^\#}) \longrightarrow E$ of bounded semivariation such that

$$T(x) = \int_{B_{X^\#}} f(x)\, d\mathfrak{m}(f), \quad x \in X. \tag{2.7}$$

We denote by $\mathcal{PI}_\infty^L(X,E)$ the set of all these mappings and we put

$$\left\|T\right\|_{\mathcal{PI}_\infty^L} = \inf \left\|\mathfrak{m}\right\|(B_{X^\#}),$$

taking the infimum over all $\mathfrak{m}$ such that (2.7) holds.

Remark 2.3.2.

If $T \in \mathcal{PI}_\infty^L(X,E)$ then $Lip(T) \leq \left\|T\right\|_{\mathcal{PI}_\infty^L}$. In order to see this, for $\varepsilon > 0$ choose $\mathfrak{m}$ such that $\left\|\mathfrak{m}\right\|(B_{X^\#}) \leq \left\|T\right\|_{\mathcal{PI}_\infty^L} + \varepsilon$ and for all $x, y \in X, we have$

$$\begin{aligned} \left\|T(x) - T(y)\right\| &\leq \int_{B_{X^\#}} \left|f(x) - f(y)\right| d\mathfrak{m}(f) \\ &\leq \left\|\mathfrak{m}\right\|(B_{X^\#}) d(x,y) \\ &\leq (\varepsilon + \left\|T\right\|_{\mathcal{PI}_\infty^L}) d(x,y). \end{aligned}$$

Hence, $Lip(T) \leq \left\|T\right\|_{\mathcal{PI}_\infty^L}$.

Now we prove the main result of this section. We characterize the Pietsch-∞-integral Lipschitz operators by means of a factorization scheme through a weakly compact linear operator.

Theorem 2.3.3.

For a Lipschitz operator $T \in Lip_0(X, E)$, the following statements are equivalent.

1. *T is Pietsch-∞-integral.*

2. *There are a compact Hausdorff space K, a Lipschitz embedding $\varphi \in Lip_0(X, C(K))$ and a weakly compact linear operator $S \in \mathcal{L}(C(K), E)$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & E \\ {\scriptstyle \varphi}\big\downarrow & \nearrow_{S} & \\ C(K). & & \end{array} \tag{2.8}$$

3. *There are a regular Borel finite measure space (Ω, Σ, μ) , a weakly compact operator $R \in \mathcal{L}(L_\infty(\mu), E)$ and a Lipschitz embedding $\phi \in Lip_0(X, L_\infty(\mu))$ giving rise to the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & E \\ {\scriptstyle \phi}\big\downarrow & \nearrow_{R} & \\ L_\infty(\mu). & & \end{array} \tag{2.9}$$

In addition,

$$\|T\|_{\mathcal{PI}^L_\infty} = \inf \|S\| \, Lip(\varphi) = \inf \|R\| \, Lip(\phi).$$

Where the first infimum is taken over all S and φ as in (2.8) and the second is taken over all R and ϕ as in (2.9).

Proof. (1)$\Longrightarrow$(2). Take $T \in \mathcal{PI}^L_\infty(X, E)$. For every $\varepsilon > 0$ choose $\mathfrak{m}$ satisfying (2.7) and $\|\mathfrak{m}\| (B_{X^\#}) \leq \|T\|_{\mathcal{PI}^L_\infty} + \varepsilon$. Consider the linear operator $S : C(B_{X^\#}) \longrightarrow E$ defined by $S(h) = \int_{B_{X^\#}} h d\mathfrak{m}$ for all $h \in C(B_{X^\#})$ and the natural Lipschitz isometric embedding $\iota_X : X \longrightarrow C(B_{X^\#})$. In this case, for all $x \in X$ we can write

$$S \circ \iota_X(x) = \int_{B_{X^\#}} \iota_X(x)(f) d\mathfrak{m}(f) = \int_{B_{X^\#}} f(x) d\mathfrak{m}(f) = T(x).$$

Theorem VI.2.5 in [26] asserts that S is weakly compact with norm $\|S\| = \|\mathfrak{m}\|(B_{X^\#})$ and then

$$\|S\|\, Lip(\iota_X) = \|\mathfrak{m}\|(B_{X^\#}) \leq \|T\|_{\mathcal{PI}^L_\infty} + \varepsilon.$$

(2)$\Longrightarrow$(3). There is a a regular Borel countably additive vector measure $\mathfrak{m} : \mathcal{B}(K) \longrightarrow E$ of bounded semivariation such that $S(f) = \int_K f d\mathfrak{m}$ for all $f \in C(K)$ and $\|\mathfrak{m}\|(K) = \|S\|$ (see [26, Theorem VI.2.1, Theorem VI.2.5 and Corollary VI.2.14]). It follows that $T(x) = S \circ \varphi(x) = \int_K \varphi(x) d\mathfrak{m}, x \in X$. On the other hand, [26, Corollary I.2.6 and Theorem I.2.1] assures the existence of a regular Borel finite measure μ on $\mathcal{B}(K)$ such that $\mathfrak{m}(A) = 0$ for all $A \in \mathcal{B}(K)$ which satisfy that $\mu(A) = 0$. Define the operator $R \in \mathcal{L}(L_\infty(\mu), E)$ by $R(f) = \int_K f d\mathfrak{m}, f \in L_\infty(\mu)$ with $\|R\| = \|\mathfrak{m}\|(K)$ (see [26, Theorem I.1.13]). This operator is weakly compact (see [26, Definition I.1.14 and Theorem VI.1.1]). Consequently,

$$R \circ (j_\infty \circ \varphi) = \int_K j_\infty \circ \varphi d\mathfrak{m} = \int_K \phi d\mathfrak{m} = T.$$

(3)$\Longrightarrow$(1). As in the proof of the second implication of Theorem 2.2.6, starting from the diagram (2.9), consider the linearization ϕ_L of $\phi \in Lip_0(X, L_\infty(\mu))$ and let $\widetilde{\phi_L} \in \mathcal{L}(C(B_{X^\#}), L_\infty(\mu))$ be the extension of $\phi_L, i.e.$, the following diagram commutes

$$\begin{array}{ccc} \text{Æ}(X) & \xrightarrow{\phi_L} & L_\infty(\mu) \\ {\scriptstyle J}\downarrow & \nearrow_{\widetilde{\phi_L}} & \\ C(B_{X^\#}). & & \end{array}$$

The linear operator $R \circ \widetilde{\phi_L} : C(B_{X^\#}) \longrightarrow E$ is weakly compact. Let $\mathfrak{m}$ be the representing vector measure of $R \circ \widetilde{\phi_L}$, that is $R \circ \widetilde{\phi_L}(f) = \int_{B_{X^\#}} f d\mathfrak{m}$ for all $f \in C(B_{X^\#})$ and $\|\mathfrak{m}\|(B_{X^\#}) = \left\| R \circ \widetilde{\phi_L} \right\|$. It follows that

$$\begin{aligned} T(x) &= R \circ \phi(x) = R \circ \widetilde{\phi_L} \circ J \circ \delta_X(x) \\ &= \int_{B_{X^\#}} J \circ \delta_X(x) d\mathfrak{m}, \end{aligned}$$

for all $x \in X$, and then $T \in \mathcal{PI}^L_\infty(X, E)$ and

$$\|T\|_{\mathcal{PI}^L_\infty} \leq \|\mathfrak{m}\|(B_{X^\#}) \leq \|R\| \left\| \widetilde{\phi_L} \right\| = \|R\|\, Lip(\phi).$$

Since this is true for every factorization as (2.9), we have $\|T\|_{\mathcal{PI}^L_\infty} \leq \|R\|\, Lip(\phi)$.

In order to show the reverse inequality, take $T \in \mathcal{PI}^L_\infty(X, E)$ and $\varepsilon > 0$. Then there is

$\mathfrak{m} : \mathcal{B}(B_{X^\#}) \longrightarrow E$ (as in Definition 2.3.1) such that (2.7) is true and $\|\mathfrak{m}\| (B_{X^\#}) \leq \varepsilon + \|T\|_{\mathcal{PI}^L_\infty}$. Following the proof of (2)$\Longrightarrow$(3), we can find a regular Borel finite measure μ on $B_{X^\#}$ and a weakly compact operator $R \in \mathcal{L}(L_\infty(\mu), E)$ represented by $\mathfrak{m}$ such that

$$\|R\| \, Lip(\phi) = \|R\| = \|\mathfrak{m}\| (B_{X^\#}) \leq \varepsilon + \|T\|_{\mathcal{PI}^L_\infty},$$

where $\phi \in Lip_0(X, L_\infty(\mu))$, is the Lipschitz embedding defined by $\phi = j_\infty \circ \iota_X$.
The required inequality follows and the second equality follows in a similar way. □

2.4 Some relations of Lipschitz Pietsch-p-integral operators with other Lipschitz operator ideals

2.4.1 Lipschitz p-summing operators

Farmer and Johnson introduced the Banach Lipschitz operator ideal of Lipschitz p-summing mappings (see [29]), extending Π_p, the operator ideal of p-summing linear operators, to the Lipschitz case.

Definition 2.4.1.
For a pointed metric space X and a Banach space E, the mapping $T \in Lip_0(X, E)$ is called Lipschitz p-summing, $1 \leq p < \infty$, if there exists a constant $C > 0$ such that for all $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ in X,

$$\sum_{i=1}^{n} \|T(x_i) - T(x'_i)\|^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} |f(x_i) - f(x'_i)|^p. \tag{2.10}$$

In this case we put $\pi^L_p(T) = \inf \{C : \text{satisfying } (2.10)\}$. The set of all Lipschitz p-summing operators from X to E is denoted by $\Pi^L_p(X, E)$.

It is well known that $(\Pi^L_p, \pi^L_p(\cdot))$ is a Banach Lipschitz operator ideal (see [6, Proposition 2.5]).

We can establish the following comparison between the classes of Lipschitz Pietsch-p-integral operators and Lipschitz p-summing operators.

Proposition 2.4.2.

Let $1 \leq p < \infty$. Every Lipschitz Pietsch-p-integral operator $T : X \longrightarrow E$ is Lipschitz p-summing with $\pi_p^L(T) \leq \|T\|_{\mathcal{PI}_p^L}$.

Proof. If $T \in \mathcal{PI}_p^L(X, E)$, for $\varepsilon > 0$ we choose a typical Lipschitz Pietsch-p-integral factorization

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

with $\|A\| \, Lip(B) \leq \varepsilon + \|T\|_{\mathcal{PI}_p^L}$. The mapping i_p is linear p-summing with $\pi_p(i_p) = 1$ (see [25, Page 40]). Then it is Lipschitz p -summing with $\pi_p^L(i_p) = 1$ (see [29, Theorem 2]). By the ideal property concerning the Lipschitz operator ideal Π_p^L, we have that $T \in \Pi_p^L(X, E)$ and $\pi_p^L(T) \leq \|A\| \, Lip(B) \leq \varepsilon + \|T\|_{\mathcal{PI}_p^L}$. □

2.4.2 Lipschitz Grothendieck-p-integral operators

The notion of Lipschitz Grothendieck-p-integral operators ($p \geq 1$) from a pointed metric space X into a Banach space E was introduced by Jiménez-Vargas et al. in [33] (under the name of strongly Lipschitz p-integral operators).

The mapping $T \in Lip_0(X, E)$ is Lipschitz Grothendieck-p-integral (in symbols $T \in \mathcal{GI}_p^L(X, E)$) if $k_E \circ T \in \mathcal{PI}_p^L(X, E^{**})$, where $k_E : E \longrightarrow E^{**}$ is the canonical injection. The class $(\mathcal{GI}_p^L, \|\cdot\|_{\mathcal{GI}_p^L})$ is a Banach Lipschitz operator ideal where $\|T\|_{\mathcal{GI}_p^L} = \|k_E \circ T\|_{\mathcal{PI}_p^L}$ (see [11, Remark 4.3 and Proposition 4.8]). It is immediate that $\mathcal{PI}_p^L(X, E) \subset \mathcal{GI}_p^L(X, E)$ and $\|T\|_{\mathcal{GI}_p^L} \leq \|T\|_{\mathcal{PI}_p^L}$ for all $T \in \mathcal{PI}_p^L(X, E)$.

2.4.3 Strongly Lipschitz p-nuclear operators

Chen and Zheng in [20] introduced the concept of strongly Lipschitz p-nuclear operators. For a pointed metric space X and a Banach space E, a mapping $T \in Lip_0(X, E)$ is strongly Lipschitz p-nuclear ($1 \leq p < \infty$) if there exist $B \in Lip_0(X, \ell_\infty)$ and $A \in \mathcal{L}(\ell_p, E)$ and a diagonal operator $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)$ induced by $\lambda = (\lambda_i)_{i\geq1} \in \ell_p$ (*i.e.*, $M_\lambda((\xi_i)_{i\geq1}) = (\lambda_i \xi_i)_{i\geq1}$) such that

$$T = A \circ M_\lambda \circ B : X \xrightarrow{B} \ell_\infty \xrightarrow{M_\lambda} \ell_p \xrightarrow{A} E. \tag{2.11}$$

The Banach space of all these mappings is denoted by $\mathcal{SN}_p^L(X,E)$ and the norm is defined by $\|T\|_{\mathcal{SN}_p^L} = \inf \|A\| \, \|M_\lambda\| \, Lip(B)$, where the infimum is taken over all the above factorizations.

Proposition 2.4.3.

Every strongly Lipschitz p-nuclear operator is Lipschitz Pietsch-p-integral. Moreover, $\|T\|_{\mathcal{PI}_p^L} \leq \|T\|_{\mathcal{SN}_p^L}$ *for all* $T \in \mathcal{SN}_p^L(X,E)$.

Proof. Given $\varepsilon > 0$, take $T \in \mathcal{SN}_p^L(X,E)$ with the factorization (2.11) such that

$$\|A\| \, \|M_\lambda\| \, Lip(B) \leq \varepsilon + \|T\|_{\mathcal{SN}_p^L}.$$

In this case, ℓ_∞ and ℓ_p are the spaces $L_\infty(\mu)$ and $L_p(\mu)$ with μ the counting measure on $\mathbb{N}$ respectively and $M_\lambda : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the multiplication operator induced by $\lambda \in L_p(\mu)$ (*i.e.*, $M_\lambda(f) = \lambda.f$). Use [25, Page 111] to see that M_λ is a Pietsch-p-integral linear operator and $\|M_\lambda\|_{\mathcal{PI}_p} = \|M_\lambda\|$ and then it is Lipschitz Pietsch-p-integral with $\|M_\lambda\|_{\mathcal{PI}_p^L} \leq \|M_\lambda\|$ (by Proposition 2.2.3). In view of the ideal property of $\mathcal{PI}_p^L$, we are done. □

2.4.4 Lipschitz weakly compact operators

The definition of Lipschitz weakly compact operators is due to Jiménez-Vargas et al. (see [33]).

Definition 2.4.4.

Let X be a pointed metric space and let E be a Banach space. The mapping $T \in Lip_0(X,E)$ is called Lipschitz weakly compact if the set

$$\left\{ \frac{T(x) - T(x')}{d(x,x')} : x, x' \in X, x \neq x' \right\},$$

is relatively weakly compact in E.

The set of all Lipschitz weakly compact operators from X into E is denoted by $Lip_{0\mathcal{W}}(X,E)$.

Proposition 2.8 in [33] asserts that every Lipschitz Grothendieck-p-integral operator is Lipschitz weakly compact. So, according to the comments above we have that $\mathcal{PI}_p^L(X,E) \subset Lip_{0\mathcal{W}}(X,E)$.

Chapter 3

Lipschitz p-representable operators

In this chapter we introduce the notion of Lipschitz p-representable operators. We show that they can be seen as a natural extension of the p-representable linear operators of Roshdi Khalil ([35]) and we transfer some properties of the linear case into the Lipschitz setting.

3.1 Linear p-representable operators

The Banach operator ideal of p-representable linear operators were introduced in [35] by Roshdi Khalil. These operators are in some sense kernel integral operators and is a sub-class of Pietsch p-integral operators.

Definition 3.1.1.

Let $1 \leq p < \infty$. A linear operator $T \in \mathcal{L}(E, F)$ is called p-representable if there exists a finite Borel measure μ on B_{E^*} and a function $g : B_{E^*} \longrightarrow F$ such that

$$g \in L^{p^*}(B_{E^*}, \mu, F), \tag{3.1}$$

and

$$T(x) = \int_{B_{E^*}} \langle x, x^* \rangle \, g(x^*) d\mu(x^*), \quad \text{for all } x \in E. \tag{3.2}$$

The Banach space of these operators is denoted by $\mathcal{R}_p(E, F)$ under the norm defined by

$$\|T\|_{\mathcal{R}_p} = \inf \|g\|_{p^*},$$

where the infimum is taken over all measures μ and g satisfying the equality (3.2).

Lemma 3.1.2. *For $T \in \mathcal{R}_p(E, F)$ then $\|T\| \leq \|T\|_{\mathcal{R}_p}$.*

Proof. Let $T(x) = \int_{B_{E^*}} \langle x, x^* \rangle \, g(x^*) d\mu(x^*)$ for some μ and g as in the Definition 3.1.1. Choose g and μ such that

$$\left(\int_{B_{E^*}} \|g(x^*)\|^{p^*} d\mu(x^*) \right)^{\frac{1}{p^*}} \leq \|T\|_{\mathcal{R}_p} + \epsilon,$$

for a given small $\epsilon > 0$. Then, using Holder's inequality

$$\begin{aligned}
\|T(x)\| &\leq \int_{B_{E^*}} \|\langle x, x^* \rangle \, g(x^*)\| \, d\mu(x^*) \\
&\leq \left(\int_{B_{E^*}} \|\langle x, x^* \rangle\|^p \, d\mu(x^*) \right)^{\frac{1}{p}} \left(\int_{B_{E^*}} \|g(x^*)\|^{p^*} d\mu(x^*) \right)^{\frac{1}{p^*}} \\
&\leq \left(\int_{B_{E^*}} \|g(x^*)\|^{p^*} d\mu(x^*) \right)^{\frac{1}{p^*}} \\
&\leq \|T\|_{\mathcal{R}_p} + \epsilon.
\end{aligned}$$

Hence $\|T(x)\| \leq \|T\|_{\mathcal{R}_p} + \epsilon$. Since ϵ is arbitrary the result follows $\|T(x)\| \leq \|T\|_{\mathcal{R}_p}$. Passing the supremum $\sup_{\|x\| \leq 1} \|T(x)\| \leq \|T\|_{\mathcal{R}_p}$ then $\|T\| \leq \|T\|_{\mathcal{R}_p}$. □

Lemma 3.1.3. *Every element $T \in \mathcal{R}_p(E, F)$ is an approximable operator in $\mathcal{L}(E, F)$.*

Proof. Let $T(x) = \int_{B_{E^*}} \langle x, x^* \rangle\, g(x^*) d\mu(x^*)$, for some finite measure on B_{E^*} and some $g \in L^{p^*}(B_{E^*}, \mu, F)$. Choose μ and g such that

$$\left(\int_{B_{E^*}} \|g(x^*)\|^{p^*} d\mu(x^*) \right)^{\frac{1}{p^*}} \leq \|T\|_{\mathcal{R}_p} + \epsilon$$

Let g_n be a sequence of simple functions in $L^{p^*}(B_{E^*}, \mu, F)$ such that

$$\int_{B_{E^*}} \|g(x^*) - g_n(x^*)\|^{p^*} d\mu(x^*) \longrightarrow 0.$$

Define

$$T_n(x) = \int_{B_{E^*}} \langle x, x^* \rangle\, g_n(x^*) d\mu(x^*).$$

Then each T_n is a finite rank operator, and

$$\|T - T_n\|_{\mathcal{R}_p} \leq \left(\int_{B_{E^*}} \|g(x^*) - g_n(x^*)\|^{p^*} d\mu(x^*) \right)^{\frac{1}{p^*}} \to 0.$$

Then by definition of approximable operators, Pietsch [43] T is approximable. This ends the proof. □

The proof of the following results can be found in [35, Theorem 2.5 and Theorem 2.8].

Theorem 3.1.4.

$(\mathcal{R}_p, \|\cdot\|_{\mathcal{R}_p})$ *is a Banach operator ideal.*

Definition 3.1.5. [35, Definition 2.6]

Let (Ω, μ) be a measure space and F a Banach space. A linear mapping $T : L^p(\Omega, \mu) \longrightarrow F$ is called B-vector integral operator if there exists a function $g \in L^{p^*}(\Omega, \mu, F)$ such that

$$T(f) = \int_{\Omega} f(t)\, g(t) d\mu(t), \qquad \text{for all } f \in L^p(\Omega, \mu). \tag{3.3}$$

The next lemma and its proof are similar to [26, Lemma III.1.4].

Lemma 3.1.6.

Every B-vector integral operator $T : L^p(\mu) \longrightarrow F$ is continuous with norm $\|T\| = \|g\|_{p^*}$.

The next factorization theorem, for the p-representable linear operators, will be used in the sequel.

Theorem 3.1.7. *[35, Theorem 2.7]*

Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$. The following are equivalent.

(i) $T \in \mathcal{R}_p(E, F)$,

(ii) There exists operators $T_1 \in \mathcal{L}(E, L^P(\Omega, \mu))$ and $T_2 \in \mathcal{L}(L^P(\Omega, \mu), F)$ for some finite measure space (Ω, μ) such that T_2 is B-vector integral operator and $T = T_2 \circ T_1$.

Proof. $(i) \Rightarrow (ii)$. Let $T \in \mathcal{R}_p(E, F)$ and $T(x) = \int_{B_{E^*}} \langle x, x^* \rangle g(x^*) d\mu(x^*)$ for some finite measure μ on B_{E^*} and $g \in L^{p^*}(B_{E^*}, \mu, F)$. Define

$$\begin{aligned} T_1 : \quad & E \to L^P(B_{E^*}, \mu) \\ & T_1(x)(x^*) = \langle x, x^* \rangle, \end{aligned}$$

and

$$\begin{aligned} T_2 : \quad & L^P(B_{E^*}, \mu) \to F \\ & T_2(f) = \int_{B_{E^*}} f(x^*) g(x^*) d\mu(x^*), \end{aligned}$$

Then T_2 is a B-vector integral operator and

$$\begin{aligned} T(x) &= \int_{B_{E^*}} \langle x, x^* \rangle g(x^*) d\mu(x^*) \\ &= \int_{B_{E^*}} T_1(x)(x^*) g(x^*) d\mu(x^*) \\ &= T_2 T_1(x). \end{aligned}$$

$(ii) \Rightarrow (i)$. Let $T = T_2 T_1$, $T_1 \in \mathcal{L}(E, L^P(\Omega, \mu))$, and T_2 is a B-vector integral operator in $\mathcal{L}(L^P(\Omega, \mu), F)$. Then $T_2 \in \mathcal{R}_p(L^p(\Omega, \mu), F)$. Using Theorem 3.1.4, we have $T_2 T_1 \in \mathcal{R}_p(E, F)$. This ends the proof. $\square$

3.2 Lipschitz p-representable operators

Definition 3.2.1.

Let $1 \leq p < \infty$. A Lipschitz operator $T \in Lip_0(X, E)$ is called Lipschitz p-representable if there exists a regular Borel probability measure μ on $B_{X^\#}$ and a function $g \in L^{p^*}(B_{X^\#}, \mu, E)$, such that

$$T(x) = \int_{B_{X^\#}} f(x)\, g(f) d\mu(f), \qquad \text{for all } x \in X, f \in B_{X^\#}. \tag{3.4}$$

We denote by $\mathcal{R}_p^L(X, E)$ the space of all Lipschitz p-representable operators from X into E. Moreover, if $T \in \mathcal{R}_p^L(X, E)$, then we set

$$\|T\|_{\mathcal{R}_p^L} = \inf \|g\|_{p^*},$$

where the infimum is taken over all measures μ and g satisfying (3.4).

We don't know if being Lipschitz p-representability implies p-representability whenever $T \in \mathcal{L}(X, E)$ between Banach spaces. The converse is of course true as we see in the next proposition.

Proposition 3.2.2.

If X and E are Banach spaces and $T : X \longrightarrow E$ is linear p-representable then T is Lipschitz p-representable and $\|T\|_{\mathcal{R}_p^L} \leq \|T\|_{\mathcal{R}_p}$.

Proof. For each $\varepsilon > 0$ we can choose a regular Borel probability measure μ on B_{X^*} and a function $g \in L^{p^*}(B_{X^*}, \mu, E)$ satisfying (3.2) and

$$\|g\|_{p^*} \leq \varepsilon + \|T\|_{\mathcal{R}_p}.$$

Let $\widetilde{\mu}$ an extension of μ to $B_{X^\#}$ and $\widetilde{g}$ an extension of the function g defined by

$$\widetilde{g}(f) = \begin{cases} g(f), & \text{if } f \in B_{X^*}. \\ 0, & \text{otherwise.} \end{cases}$$

According to [32, Page 54], $\widetilde{\mu}$ is a finite Borel measure on $B_{X^\#}$. We can write

$$\int_{B_{X^\#}} f(x)\, \widetilde{g}(f) d\widetilde{\mu}(f) = \int_{B_{X^*}} f(x)\, g(f) d\mu(f) = T(x), \qquad \text{for all } x \in X,$$

and $\|\widetilde{g}\|_{p^*} = \|g\|_{p^*} < \infty$. Hence, $T \in \mathcal{R}_p^L(X, E)$ and

$$\|T\|_{\mathcal{R}_p^L} \leq \|\widetilde{g}\|_{p^*} = \|g\|_{p^*} \leq \varepsilon + \|T\|_{\mathcal{R}_p}.$$

Since this holds for every $\varepsilon > 0$, we obtain $\|T\|_{\mathcal{R}_p^L} \leq \|T\|_{\mathcal{R}_p}$. □

We have the following immediate consequence of the Definition 3.2.1.

Proposition 3.2.3. *(Inclusion Theorem)*

Let $1 \leq p \leq q < \infty$. Then $\mathcal{R}_p^L(X, E) \subset \mathcal{R}_q^L(X, E)$ and $\|T\|_{\mathcal{R}_q^L} \leq \|T\|_{\mathcal{R}_p^L}$ for all $T \in \mathcal{R}_p^L(X, E)$.

Proof. Let $T \in \mathcal{R}_p^L(X, E)$, for every $\varepsilon > 0$ there exists a regular Borel probability measure μ defined on $B_{X^\#}$ and a function $g \in L^{p^*}(B_{X^\#}, \mu, E)$ such that for all $x \in X$ we have

$$T(x) = \int_{B_{X^\#}} f(x)\, g(f) d\mu(f) \quad \text{and} \quad \|g\|_{p^*} \leq \|T\|_{\mathcal{R}_p^L} + \varepsilon. \tag{3.5}$$

By monotonicity of the L^p-norms, we have

$$\|g\|_{q^*} \leq \|g\|_{p^*} \leq \|T\|_{\mathcal{R}_p^L} + \varepsilon.$$

Then $T \in \mathcal{R}_q^L(X, E)$ and $\|T\|_{\mathcal{R}_q^L} \leq \|T\|_{\mathcal{R}_p^L} + \varepsilon$, and the result follows. □

The following theorem gives a characterization of Lipschitz p-representable operators by a factorization schemes that highlights the role of the spaces $C(B_{X^\#})$ and $L^p(B_{X^\#}, \mu)$.

Theorem 3.2.4.

Let $1 \leq p < \infty$ and let $T \in Lip_0(X, E)$. Then T is Lipschitz p-representable if and only if there exists a regular Borel probability measure μ defined on $B_{X^\#}$ and a B-vector integral operator $S \in \mathcal{L}(L_p(\mu), E)$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle \iota_X}\big\downarrow & & \big\uparrow{\scriptstyle S} \\ C(B_{X^\#}) & \xrightarrow{\quad j_p \quad} & L_p(\mu), \end{array} \tag{3.6}$$

where j_p is the canonical map and ι_X is the natural Lipschitz isometric embedding. Moreover,

$$\|T\|_{\mathcal{R}_p^L} = \inf\left\{\|S\| : T = S \circ j_p \circ \iota_X\right\}.$$

Proof. Suppose that $T \in \mathcal{R}_p^L(X, E)$. For $\varepsilon > 0$, there exists a regular Borel probability measure μ defined on $B_{X^\#}$ and a function $g \in L^{p^*}(B_{X^\#}, \mu, E)$ such that (3.5) holds. Consider the B-vector integral operator $S : L_p(\mu) \longrightarrow E$ defined by

$$S(h) = \int_{B_{X^\#}} h(f)\, g(f)\, d\mu(f), \qquad h \in L_p(\mu).$$

Then $T = S \circ j_p \circ \iota_X$ since for all $x \in X$ we have

$$\begin{aligned} S \circ j_p \circ \iota_X (x) &= S (j_p \circ \iota_X (x)) \\ &= \int_{B_{X^\#}} j_p \circ \iota_X (x) (f) g (f) \, d\mu (f) \\ &= \int_{B_{X^\#}} f (x) \, g(f) d\mu (f) \\ &= T(x). \end{aligned}$$

Moreover, $\|S\| = \|g\|_{p^*} \leq \|T\|_{\mathcal{R}_p^L} + \varepsilon$. Since this holds for every $\varepsilon > 0$, we obtain $\|S\| \leq \|T\|_{\mathcal{R}_p^L}$.
Conversely, suppose that T admits a factorization of the form (3.6). Then there exists a regular Borel probability measure μ on $B_{X^\#}$ and $g \in L^{p^*}(B_{X^\#}, \mu, E)$ such that

$$S (h) = \int_{B_{X^\#}} h (f) \, g (f) \, d\mu (f), \qquad h \in L_p (\mu).$$

For all $x \in X$, as above we have

$$T(x) = S \circ j_p \circ \iota_X (x) = \int_{B_{X^\#}} f (x) \, g(f) d\mu (f).$$

We arrive at $T \in \mathcal{R}_p^L(X, E)$ and $\|T\|_{\mathcal{R}_p^L} \leq \|g\|_{p^*} = \|S\|$.
This ends the proof. □

According to the previous theorem and Theorem 2.2.6 we give a relationship between the Lipschitz p-representable operator and Lipschitz Pietsch-p-integral operators.

Corollary 3.2.5.
We have $\mathcal{R}_p^L (X, E) \subset \mathcal{PI}_p^L (X, E)$ *and* $\|T\|_{\mathcal{PI}_p^L} \leq \|T\|_{\mathcal{R}_p^L}$ *for every* $T \in \mathcal{R}_p^L (X, E)$.

Proof. Let $T \in \mathcal{R}_p^L (X, E)$. For $\varepsilon > 0$, there are μ and S as in Theorem 3.2.4 such that $\|S\| \leq \varepsilon + \|T\|_{\mathcal{R}_p^L}$. Then it follows from Theorem 2.2.6 that $T \in \mathcal{PI}_p^L (X, E)$ and

$$\|T\|_{\mathcal{PI}_p^L} \leq \|S\| \leq \varepsilon + \|T\|_{\mathcal{R}_p^L}.$$

□

As a consequence of the previous corollary we have the following.

Corollary 3.2.6.
If $T \in \mathcal{R}_p^L (X, E)$ *then*

$$Lip(T) \leq \|T\|_{\mathcal{R}_p^L}. \tag{3.7}$$

Proof. Since $\mathcal{PI}_p^L$ is Banach Lipschitz operator ideal we obtain

$$Lip(T) \leq \|T\|_{\mathcal{PI}_p^L} \leq \|T\|_{\mathcal{R}_p^L}.$$

□

The next theorem and its proof are similar to the linear case. We write the proof for the aim of completeness.

Theorem 3.2.7.
Let X be a pointed metric space, E a Banach space and let $p \geq 1$. Then $\left(\mathcal{R}_p^L(X,E), \|\cdot\|_{\mathcal{R}_p^L}\right)$ is a vector normed space.

Proof. It is clear that $T = 0 \in \mathcal{R}_p^L(X,E)$. By (3.7) if $\|T\|_{\mathcal{R}_p^L} = 0$ then $T = 0$.
Let $T_1, T_2 \in \mathcal{R}_p^L(X,E)$. For each $\varepsilon > 0$ choose a regular Borel probability measures μ_i on $B_{X^\#}$ and $g_i \in L^{p^*}(B_{X^\#}, \mu_i, E)$, $i = 1, 2$, such that

$$T_i\left(x\right) = \int_{B_{X^\#}} f\left(x\right) g_i(f) d\mu_i\left(f\right) \quad \text{and} \quad \|g_i\|_{p^*} \leq \|T_i\|_{\mathcal{R}_p^L} + \tfrac{\varepsilon}{2}.$$

Consider the measure $\mu = \mu_1 + \mu_2$. Then $\mu_i \ll \mu$. By Radon–Nikodym theorem then there is a μ-measurable function $h_i : B_{X^\#} \longrightarrow [0, +\infty]$ such that for any Borel sets $K \in \mathcal{B}(B_{X^\#})$,

$$\mu_i(K) = \int_K h_i(f) d\mu(f), \qquad i = 1, 2.$$

Further, since $\mu_i\left(K\right) < \mu\left(K\right)$ for all $K \in \mathcal{B}(B_{X^\#})$, we have

$$0 \leq \mu_i\left(K\right) = \int_K h_i(f) d\mu(f) < \mu\left(K\right) = \int_K d\mu(f),$$

it follows that $0 \leq h_i(f) \leq 1$ holds μ-almost everywhere, $i = 1, 2$. Now, consider the function $\widetilde{g}$ defined by

$$\widetilde{g}(f) = g_1(f) h_1(f) + g_2(f) h_2(f), \quad f \in B_{X^\#}.$$

Therefore $\widetilde{g} \in L^{p^*}(B_{X^\#}, \mu, E)$ because

$$\|\widetilde{g}\|_{p^*} \leq \|g_1 h_1\|_{p^*} + \|g_2 h_2\|_{p^*} \leq \|g_1\|_{p^*} + \|g_2\|_{p^*} < \infty.$$

On the other hand, for all $x \in X$ we have

$$\begin{aligned}(T_1 + T_2)(x) &= \int_{B_{X^\#}} f(x)\, g_1(f) d\mu_1(f) + \int_{B_{X^\#}} f(x)\, g_2(f) d\mu_2(f) \\ &= \int_{B_{X^\#}} f(x)\, g_1(f) h_1(f) d\mu(f) + \int_{B_{X^\#}} f(x)\, g_2(f) h_2(f) d\mu(f) \\ &= \int_{B_{X^\#}} f(x)\, \tilde{g}(f) d\mu(f).\end{aligned}$$

This means that $T_1 + T_2 \in \mathcal{R}_p^L(X, E)$. Furthermore,

$$\|T_1 + T_2\|_{\mathcal{R}_p^L} \leq \|\widetilde{g}\|_{p^*} \leq \|g_1\|_{p^*} + \|g_2\|_{p^*} \leq \|T_1\|_{\mathcal{R}_p^L} + \|T_2\|_{\mathcal{R}_p^L} + \varepsilon.$$

It follows that $\|T_1 + T_2\|_{\mathcal{R}_p^L} \leq \|T_1\|_{\mathcal{R}_p^L} + \|T_2\|_{\mathcal{R}_p^L}$.

By a similar argument, let $\alpha \in \mathbb{K}$, $\alpha \neq 0$, and $T \in \mathcal{R}_p^L(X, E)$. For $\varepsilon > 0$ there are μ and g as above such that (3.5) holds. Also, for all $x \in X$ we have $\alpha T(x) = \int_{B_{X^\#}} f(x)\, \alpha g(f) d\mu(f)$. Since $\alpha g \in L^{p^*}(B_{X^\#}, \mu, E)$, we obtain $\alpha T \in \mathcal{R}_p^L(X, E)$ and

$$\|\alpha T\|_{\mathcal{R}_p^L} \leq |\alpha| \, \|g\|_{p^*} \leq |\alpha| \left(\|T\|_{\mathcal{R}_p^L} + \varepsilon \right).$$

Hence $\|\alpha T\|_{\mathcal{R}_p^L} \leq |\alpha| \, \|T\|_{\mathcal{R}_p^L}$. For the reverse inequality, we have $T = \frac{1}{\alpha}(\alpha T)$ and then

$$\left\| \frac{1}{\alpha}(\alpha T) \right\|_{\mathcal{R}_p^L} \leq \left| \frac{1}{\alpha} \right| \|\alpha T\|_{\mathcal{R}_p^L}.$$

This means $\|\alpha T\|_{\mathcal{R}_p^L} \geq |\alpha| \, \|T\|_{\mathcal{R}_p^L}$. The case $\alpha = 0$ is evident by (3.7). To summarize, we have proved that $\left(\mathcal{R}_p^L(X, E), \|\cdot\|_{\mathcal{R}_p^L} \right)$ is a vector normed space. □

Following the idea of [35, Theorem 2.5] we present the following result.

Proposition 3.2.8.

Let X be a pointed metric space, E, F Banach spaces, $T \in \mathcal{R}_p^L(X, E)$ and $A \in \mathcal{L}(E, F)$. Then $A \circ T \in \mathcal{R}_p^L(X, F)$.

Proof. For all $\varepsilon > 0$, choose a regular Borel probability measure μ defined on $B_{X^\#}$ and a function $g \in L^{p^*}(B_{X^\#}, \mu, E)$ such that for all $x \in X$ we have $T(x) = \int_{B_{X^\#}} f(x)\, g(f) d\mu(f)$ and $\|g\|_{p^*} \leq \|T\|_{\mathcal{R}_p^L} + \varepsilon$. By [26, Theorem II.2.6] we have

$$A \circ T(x) = \int_{B_{X^\#}} f(x)\, A(g(f)) d\mu(f).$$

Also we have

$$\|A(g)\|_{p^*} \leq \|A\| \left(\int_{B_{X^\#}} \|g(f)\|^{p^*} d\mu(f) \right)^{\frac{1}{p^*}} = \|A\| \, \|g\|_{p^*} \leq \|A\| \left(\|T\|_{\mathcal{R}_p^L} + \varepsilon \right) < \infty,$$

this means that $A \circ T \in \mathcal{R}_p^L(X, F)$ with $\|A \circ T\|_{\mathcal{R}_p^L} \leq \|A\| \, \|T\|_{\mathcal{R}_p^L}$. □

3.3 Open problems

Problem 1.

Is $\left(\mathcal{R}_p^L, \|\cdot\|_{\mathcal{R}_p^L}\right)$ a Banach Lipschitz operator ideal?

In Theorem 3.2.7 we prove that $\left(\mathcal{R}_p^L(X, E), \|\cdot\|_{\mathcal{R}_p^L}\right)$ is a vector normed subspace of $Lip_0(X, E)$. Also the Proposition 3.2.8 asserts that the class $\mathcal{R}_p^L$ satisfies the left ideal property.

Problem 2.

When the Banach space E has the Radon Nikodym property, do we have the coincidence $\mathcal{R}_1^L(X, E) = \mathcal{PI}_1^L(X, E)$ as the linear case?

The inclusion $\mathcal{R}_1^L(X, E) \subset \mathcal{PI}_1^L(X, E)$ is proved in Theorem 3.2.5 without the hypothesis of Radon Nikodym property.

Problem 3.

Does $T \in Lip_0(X, E)$ Lipschitz p-representable equivalent to $T_L \in \mathcal{R}_p(\text{Æ}(X), E)$?

Chapter 4

Two-Lipschitz operator ideals

We introduce and investigate the concept of two-Lipschitz operator ideal between pointed metric spaces and Banach spaces. We show the basics of this new theory and we give a procedure for create a two-Lipschitz operator ideal from a linear operator ideal. We apply our result to the ideals of strongly p-summing and compact linear operator to obtain their corresponding two-Lipschitz operator ideal. Also, we establish a natural relation between two-Lipschitz and bilinear maps and show that the two-Lipschitz factorable p-dominated operators are those which are associated to the well-known p-semi-integral bilinear operators.

4.1 Bi-linearization of two-Lipschitz operators

Definition 4.1.1. [50, Section 3.4]

Let (X, d_X) and (Y, d_Y) be pointed metric spaces and let E be a Banach space, we say that a map $T : X \times Y \longrightarrow E$ is a two-Lipschitz operator if there is a constant $C > 0$ such that for each $x, x' \in X$ and $y, y' \in Y$,

$$\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\| \leq C.d_X(x, x')\, d_Y(y, y'). \tag{4.1}$$

By $BLip_0(X, Y; E)$ we denote the set of all two-Lipschitz operators from $X \times Y$ to E such that

$$T(x, 0) = T(0, y) = 0, \tag{4.2}$$

for all $x \in X$ and $y \in Y$. For $T \in BLip_0(X, Y; E)$ we set

$$BLip(T) = \inf C = \sup_{x \neq x', y \neq y'} \frac{\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\|}{d_X(x, x')\, d_Y(y, y')}. \tag{4.3}$$

For the mapping $T : X \times Y \longrightarrow E$, consider $A_y : X \longrightarrow E$ and $A_x : Y \longrightarrow E$ such that $A_y(x) = T(x, y)$ for every fixed $y \in Y$ and $A_x(y) = T(x, y)$ for every fixed $x \in X$. According to Dubei et al. in [27], T is said to be two-Lipschitz if A_x is Lipschitz for every fixed $x \in X$ and A_y is Lipschitz for every fixed $y \in Y$.

In the following proposition, we show that this definition with a requirement on the operator $x \longrightarrow A_x$ or $y \longrightarrow A_y$ is equivalent to our definition (Definition 4.1.1).

Proposition 4.1.2.

For a mapping $T : X \times Y \longrightarrow E$, the following statements are equivalent.

(i) $T \in BLip_0(X, Y; E)$.

(ii) $A_x \in Lip_0(Y, E)$ *for every fixed* $x \in X$ *and* $G : x \longrightarrow A_x$ *belongs to* $Lip_0(X, Lip_0(Y, E))$.

(iii) $A_y \in Lip_0(X, E)$ *for every fixed* $y \in Y$ *and* $H : y \longrightarrow A_y$ *belongs to* $Lip_0(Y, Lip_0(X, E))$.

Proof. $(i) \Longrightarrow (ii)$ For every fixed $x \in X$, starting from (4.1) take $x' = 0$ we obtain

$$\begin{aligned} \|A_x(y) - A_x(y')\| &= \|T(x, y) - T(x, y') - T(0, y) + T(0, y')\| \\ &\leq BLip(T) d(x, 0) d(y, y'). \end{aligned}$$

By (4.2) we have $A_x(0) = 0$. It follows that, $A_x \in Lip_0(Y, E)$ and $Lip(A_x) \leq BLip(T)d(x, 0)$. Now, for all $x, x' \in X$ we have

$$\begin{aligned} \|G(x) - G(x')\| &= Lip(A_x - A_{x'}) \\ &= \sup_{y \neq y'} \frac{\|(A_x - A_{x'})(y) - (A_x - A_{x'})(y')\|}{d(y, y')} \\ &= \sup_{y \neq y'} \frac{\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\|}{d(y, y')} \\ &\leq BLip(T)d(x, x'), \end{aligned}$$

and $G(0) = 0$ because for all $y \in Y$ we have $A_0(y) = 0$, showing that $G \in Lip_0(X, Lip_0(Y, E))$.

$(ii) \implies (i)$ The equalities (4.2) follow easily from that $A_x(0) = 0$. The assumption $\|G(x) - G(x')\| \leq Lip(G)d(x, x')$ for all $x, x' \in X$, implies that T satisfies the inequality (4.1).

The equivalence $(i) \iff (iii)$ is proved in a similar way. □

By using simple calculation, we prove the following result.

Proposition 4.1.3.

Let X, Y, Z, W be pointed metric spaces and let E, F be Banach spaces. If $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in BLip_0(X, Y; E)$ and $u \in \mathcal{L}(E, F)$ then $u \circ T \circ (f, g) \in BLip_0(Z, W; F)$, where $(f, g)(z, w) := (f(z), g(w))$, $z \in Z, w \in W$. Moreover,

$$BLip(u \circ T \circ (f, g)) \leq \|u\| \, BLip(T)Lip(f)Lip(g). \tag{4.4}$$

Proof. Take $S = u \circ T \circ (f, g)$. For any $z, z' \in Z$ and $w, w' \in W$ we have

$$\begin{aligned} &\|S(z, w) - S(z, w') - S(z', w) + S(z', w')\| \\ &\leq \|u\| \, \|T(f(z), g(w)) - T(f(z), g(w')) - T(f(z'), g(w)) + T(f(z'), g(w'))\| \\ &\leq \|u\| \, BLip(T) \, d(f(z), f(z')) \, d(g(w), g(w')) \\ &\leq \|u\| \, BLip(T) \, Lip(f) \, Lip(g) \, d(z, z') \, d(w, w'), \end{aligned}$$

it is clear that for all $z \in Z, w \in W$ we obtain $S(z, 0) = S(0, w) = 0$. Then, $u \circ T \circ (f, g) \in BLip_0(Z, W; F)$ and

$$BLip(u \circ T \circ (f, g)) \leq \|u\| \, BLip(T) \, Lip(f) \, Lip(g).$$

□

Remark 4.1.4.

If X and Y be Banach spaces. Then every bilinear operator $T : X \times Y \longrightarrow E$ is two-Lipschitz. Moreover, we have $BLip\,(T) = \|T\|$. In order to see this, for each $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} &\|T\,(x,y) - T\,(x,y') - T\,(x',y) + T\,(x',y')\| \\ &= \|T\,(x - x', y - y')\| \le \|T\| \, \|x - x'\| \, \|y - y'\| \,. \end{aligned}$$

Therefore, $BLip\,(T) \le \|T\|$.

For the reverse inequality, we will write (4.3) for $x' = y' = 0$,

$$BLip\,(T) \ge \sup_{x \neq 0, y \neq 0} \frac{\|T\,(x,y)\|}{d_X\,(x,0)\, d_Y\,(y,0)} = \|T\| \,.$$

The next theorem and its proof are similar to the Lipschitz case (see [51, Proposition 1.6.2]).

Theorem 4.1.5.

$BLip_0\,(X, Y; E)$ is a Banach space under the norm $BLip\,(\cdot)$ defined by (4.3).

In what follows, let (X, d_X) and (Y, d_Y) be metric spaces and consider the product metric space $X \times Y$ equipped with the metric

$$d\,((x,y)\,,(x',y')) = d_X\,(x,x') + d_Y\,(y,y')\,,$$

for all $x, x' \in X$ and $y, y' \in Y$.

Also if E, F be Banach spaces, the product Banach space $E \times F$ is equipped with the norm

$$\|(x,y)\|_{E \times F} = \|x\|_E + \|y\|_F\,,$$

for all $x \in E$ and $y \in F$.

Proposition 4.1.6.

The mapping $(\delta_X, \delta_Y) : X \times Y \longrightarrow Æ(X) \times Æ(Y)$ defined by

$$(\delta_X, \delta_Y)(x,y) = (\delta_X(x), \delta_Y(y)) = (m_{x0}, m_{y0}),$$

isometrically embeds $X \times Y$ in $Æ(X) \times Æ(Y)$.

Proof. For any $x, x' \in X$ and $y, y' \in Y$ we have,

$$\begin{aligned}
&\|(\delta_X, \delta_Y)(x, y) - (\delta_X, \delta_Y)(x', y')\|_{Æ(X)\times Æ(Y)} \\
&= \|(m_{x0} - m_{x'0}, m_{y0} - m_{y'0})\|_{Æ(X)\times Æ(Y)} \\
&= \|(m_{xx'}, m_{yy'})\|_{Æ(X)\times Æ(Y)} \\
&= \|m_{xx'}\|_{Æ(X)} + \|m_{yy'})\|_{Æ(Y)} \\
&= d_X(x, x') + d_Y(y, y') \\
&= d((x, y), (x', y')),
\end{aligned}$$

and the result follows. □

For all two-Lipschitz operator $T : X \times Y \longrightarrow E$, we define a bilinear mapping $T_B : \mathcal{M}(X) \times \mathcal{M}(Y) \longrightarrow E$ by

$$T_B(m_{xx'}, m_{yy'}) = T(x, y) - T(x, y') - T(x', y) + T(x', y'), \tag{4.5}$$

for all $x, x' \in X$ and $y, y' \in Y$. Thus the two-Lipschitz operator T is associated with the bilinear mapping T_B.

Theorem 4.1.7.

For every two-Lipschitz operator $T \in BLip_0(X, Y; E)$ there exists a unique continuous bilinear mapping $T_B : Æ(X) \times Æ(Y) \longrightarrow E$ satisfying (4.5) and

$$T = T_B \circ (\delta_X, \delta_Y) : X \times Y \overset{(\delta_X,\delta_Y)}{\longrightarrow} Æ(X) \times Æ(Y) \overset{T_B}{\longrightarrow} E.$$

Furthermore $BLip(T) = \|T_B\|$. The continuous bilinear mapping T_B is called bi-linearization of the two-Lipschitz operator T.

Proof. Let $(m_1, m_2) \in \mathcal{M}(X) \times \mathcal{M}(Y)$ and let $\varepsilon > 0$. Choose representations of m_1 and m_2 of the form

$$m_1 = \sum_{i=1}^{n} \alpha_i m_{x_i x_i'}, \quad m_2 = \sum_{j=1}^{r} \beta_j m_{y_j y_j'}$$

such that

$$\sum_{i=1}^{n} |\alpha_i| \, d(x_i, x_i') \leq \varepsilon + \|m_1\|_{\mathcal{M}(X)} \quad \text{and} \quad \sum_{j=1}^{r} |\beta_j| \, d(y_j, y_j') \leq \varepsilon + \|m_2\|_{\mathcal{M}(Y)}.$$

Then

$$\begin{aligned}\|T_B(m_1, m_2)\| &= \left\| \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j \left(T(x_i, y_j) - T(x_i, y_j') - T(x_i', y_j) + T(x_i', y_j') \right) \right\| \\ &\leq BLip(T) \sum_{i=1}^{n} |\alpha_i| \, d(x_i, x_i') \sum_{j=1}^{r} |\beta_j| \, d(y_j, y_j') \\ &\leq BLip(T) \left(\varepsilon + \|m_1\|_{\mathcal{M}(X)} \right) \left(\varepsilon + \|m_2\|_{\mathcal{M}(Y)} \right).\end{aligned}$$

Since this holds for every $\varepsilon > 0$ we obtain

$$\|T_B(m_1, m_2)\| \leq BLip(T) \, \|m_1\|_{\mathcal{M}(X)} \, \|m_2\|_{\mathcal{M}(Y)}.$$

Therefore, T_B is continuous and satisfies $\|T_B\| \leq BLip(T)$.

On the other hand, by the bilinearity of T and taking into account that $T = T_B \circ (\delta_X, \delta_Y)$ we get

$$\begin{aligned}&\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\| \\ &= \|T_B(\delta_X(x) - \delta_X(x'), \delta_Y(y) - \delta_Y(y'))\| \\ &\leq \|T_B\| \, \|\delta_X(x) - \delta_X(x')\| \, \|\delta_Y(y) - \delta_Y(y')\| \\ &= \|T_B\| \, d(x, x') \, d(y, y').\end{aligned}$$

It follows that $BLip(T) \leq \|T_B\|$. Therefore, $BLip(T) = \|T_B\|$. Now, the continuous bilinear mapping T_B has a unique extension to

$$\overline{\mathcal{M}(X)} \times \overline{\mathcal{M}(Y)} = \text{Æ}(X) \times \text{Æ}(Y),$$

denoted also by T_B, with $\|T_B\| = BLip(T)$.

For the uniqueness of the bi-linearization, suppose $S : \mathcal{M}(X) \times \mathcal{M}(Y) \longrightarrow E$ is a continuous bilinear mapping such that $T = S \circ (\delta_X, \delta_Y)$. Thus, for any $m_1 = \sum_{i=1}^{n} \alpha_i m_{x_i x_i'} \in \mathcal{M}(X)$ and $m_2 = \sum_{j=1}^{r} \beta_j m_{y_j y_j'} \in \mathcal{M}(Y)$

$$
\begin{aligned}
S(m_1, m_2) &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j S\left(m_{x_i x_i'}, m_{y_j y_j'}\right) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j S\left(\delta_X(x_i) - \delta_X(x_i'), \delta_Y(y_j) - \delta_Y(y_j')\right) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j T_B\left(\delta_X(x_i) - \delta_X(x_i'), \delta_Y(y_j) - \delta_Y(y_j')\right) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{r} \beta_j T_B\left(m_{x_i x_i'}, m_{y_j y_j'}\right) \\
&= T_B(m_1, m_2).
\end{aligned}
$$

This proves that $S = T_B$. $\square$

Remark 4.1.8.

Note that the bilinear operator T_B admits a linearization

$$(T_B)_L : Æ(X) \widehat{\otimes}_\pi Æ(Y) \longrightarrow E$$

satisfies

$$T = T_B \circ (\delta_X, \delta_Y) = (T_B)_L \circ \sigma_2 \circ (\delta_X, \delta_Y),$$

where $\sigma_2 : Æ(X) \times Æ(Y) \longrightarrow Æ(X) \widehat{\otimes}_\pi Æ(Y)$ is the canonical bilinear operator defined by $\sigma_2(m_{x0}, m_{y0}) = m_{x0} \otimes m_{y0}$. In addition we have

$$BLip(T) = \|T_B\| = \|(T_B)_L\|.$$

The linear operator $(T_B)_L$ is referred to as the linearization of the two-Lipschitz operator T. For the simplification, write T_L instead of $(T_B)_L$.

Next we give a simple but crucial example of a two-Lipschitz operator. Let X, Y be pointed metric spaces and let E be Banach space.

Example 4.1.9.

Consider non-zero Lipschitz functions $f \in X^\#$, $g \in Y^\#$ and $e \in E$. Define the mapping $f \cdot g \cdot e : X \times Y \longrightarrow E$ by

$$f \cdot g \cdot e(x, y) = f(x)g(y)e. \tag{4.6}$$

Then, an easy computation shows that this mapping is two-Lipschitz and

$$BLip(f \cdot g \cdot e) = Lip(f)Lip(g) \|e\| . \tag{4.7}$$

Indeed, for any $x, x' \in X$ and $y, y' \in Y$ we have

$$\begin{aligned}
&\|f.g.e(x,y) - f.g.e\,(x,y') - f.g.e\,(x',y) + f.g.e\,(x',y')\| \\
&= \|f(x)g(y)e - f(x)g(y')e - f(x')g(y)e + f(x')g(y')e\| \\
&= \|(f(x) - f(x'))\,(g(y) - g(y'))\,e\| \\
&\leq \|e\|\, Lip(f)Lip(g)d\,(x,x')\,d\,(y,y') .
\end{aligned}$$

Then $f.g.e \in BLip_0(X, Y; E)$ and

$$BLip\,(f.g.e) \leq Lip\,(f)\,Lip\,(g)\,\|e\| .$$

For the reverse inequality, let $\varepsilon > 0$ and choose $x_0, x_0' \in X$ and $y_0, y_0' \in Y$ such that

$$Lip\,(f) - \varepsilon \leq \frac{|f\,(x_0) - f\,(x_0')|}{d\,(x_0, x_0')} \quad \text{and} \quad Lip\,(g) - \varepsilon \leq \frac{|g\,(y_0) - g\,(y_0')|}{d\,(y_0, y_0')}.$$

Then, we have

$$\begin{aligned}
BLip\,(f.g.e) &\geq \frac{\|f.g.e\,(x_0,y_0) - f.g.e\,(x_0,y_0') - f.g.e\,(x_0',y_0) + f.g.e\,(x_0',y_0')\|}{d\,(x_0,x_0')\,d\,(y_0,y_0')} \\
&= \frac{|f\,(x_0) - f\,(x_0')|}{d\,(x_0,x_0')}\frac{|g\,(y_0) - g\,(y_0')|}{d\,(y_0,y_0')}\|e\| \\
&\geq (Lip\,(f) - \varepsilon)\,(Lip\,(g) - \varepsilon)\,\|e\| .
\end{aligned}$$

Since this holds for every $\varepsilon > 0$ we obtain

$$Lip\,(f)\,Lip\,(g)\,\|e\| \leq BLip\,(f.g.e) .$$

Definition 4.1.10.

We denote by $BLip_{0\mathcal{F}}(X, Y; E)$, the vector subspace of all two-Lipschitz operators generated by the mappings of the special form (4.6). All elements T of this space are called of finite type. So, any $T \in BLip_{0\mathcal{F}}(X, Y; E)$ admits a finite representation of the form

$$T = \sum_{i=1}^{n} f_i \cdot g_i \cdot e_i$$

where $(f_i)_{i=1}^n \subset X^\#, (g_i)_{i=1}^n \subset Y^\#$ and $(e_i)_{i=1}^n \subset E$.

4.2 Two-Lipschitz operator ideals

We will follow the spirit of the definitions of multilinear operator ideals ([44] or [30]) and Lipschitz operator ideals [6], for defining the concept of two-Lipschitz operator ideals.

Definition 4.2.1.

A two-Lipschitz operator ideal between pointed metric spaces and Banach spaces, $\mathcal{I}_{BLip}$, is a subclass of $BLip_0$ such that for every pointed metric spaces X, Y and every Banach space E the components

$$\mathcal{I}_{BLip}(X, Y; E) := BLip_0(X, Y; E) \cap \mathcal{I}_{BLip}$$

satisfy:

(i) $\mathcal{I}_{BLip}(X, Y; E)$ is a vector subspace of $BLip_0(X, Y; E)$.

(ii) For any $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$, the map $f \cdot g \cdot e$ belongs to $\mathcal{I}_{BLip}(X, Y; E)$.

(iii) The ideal property: if $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$, then the composition $u \circ T \circ (f, g)$ is in $\mathcal{I}_{BLip}(Z, W; F)$.

A two-Lipschitz operator ideal $\mathcal{I}_{BLip}$ is a normed (Banach) two-Lipschitz operator ideal if there is $\|\cdot\|_{\mathcal{I}_{BLip}} : \mathcal{I}_{BLip} \longrightarrow [0, +\infty[$ that satisfies

(i') For every pointed metric spaces X, Y and every Banach space E, the pair $(\mathcal{I}_{BLip}(X, Y; E), \|\cdot\|_{\mathcal{I}_{BLip}})$ is a normed (Banach) space and $BLip(T) \leq \|T\|_{\mathcal{I}_{BLip}}$ for all $T \in \mathcal{I}_{BLip}(X, Y; E)$.

(ii') $\|Id_{\mathbb{K}^2} : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K} : Id_{\mathbb{K}^2}(\alpha, \beta) = \alpha\beta\|_{\mathcal{I}_{BLip}} = 1$.

(iii') If $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$, the inequality $\|u \circ T \circ (f, g)\|_{\mathcal{I}_{BLip}} \leq \|u\| \, \|T\|_{\mathcal{I}_{BLip}} Lip(f) Lip(g)$ holds.

Of course the Banach spaces considered in this definition are all over the same fixed scalar field.

The two-Lipschitz operator ideal $\mathcal{I}_{BLip}$ is said to be closed if each $\mathcal{I}_{BLip}(X, Y; E)$ is a closed subspace of $BLip(X, Y; E)$ with the norm $BLip(\cdot)$.

Proposition 4.2.2.

Let $\mathcal{I}_{BLip}$ be a normed two-Lipschitz operator ideal, X, Y be pointed metric spaces and E be Banach space. Then

$$\|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} = \|e\| \, Lip(f) Lip(g),$$

for any $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$.

Proof. Let $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$. We can write $f \cdot g \cdot e$ in the following way

$$f \cdot g \cdot e = id_{\mathbb{K}} e \circ Id_{\mathbb{K}^2} \circ (f, g).$$

By (i'),(ii'), (iii') and (4.7), we obtain directly

$$\begin{aligned} \|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} &\leq \|id_{\mathbb{K}} e\| \, \|Id_{\mathbb{K}^2}\|_{\mathcal{I}_{BLip}} Lip(f) Lip(g) \\ &= \|e\| \, Lip(f) Lip(g) = BLip(f \cdot g \cdot e) \\ &\leq \|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}}, \end{aligned}$$

this gives, $\|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} = \|e\| \, Lip(f) Lip(g)$. □

Remark 4.2.3.

By the above definition, the class $BLip_{0\mathcal{F}}$ is the smallest two-Lipschitz operator ideal and the class of all two-Lipschitz operators between arbitrary pointed metric spaces and Banach spaces, is the largest two-Lipschitz operator ideal.

We use techniques inspired by [13], we give a method (composition method) to build a two-Lipschitz operator ideal starting from a given operator ideal. The properties enjoyed by the linear operators in this ideal can be generalized to the two-Lipschitz case and the resulting classes of two-Lipschitz mappings happen to be a two-Lipschitz ideal called ideal of composition type.

Definition 4.2.4.

Let $\mathcal{I}$ be an operator ideal. A two-Lipschitz operator $T \in BLip_0(X, Y; E)$ belongs to the composition two-Lipschitz operator ideal $\mathcal{I} \circ BLip_0$, in this case we write $T \in \mathcal{I} \circ BLip_0(X, Y; E)$, if there is a Banach space F, a two-Lipschitz operator $S \in BLip_0(X, Y; F)$ and a linear operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal we write

$$\|T\|_{\mathcal{I} \circ BLip_0} = \inf \|u\|_{\mathcal{I}} \, BLip(S),$$

where the infimum is taken over all u, S as above.

Theorem 4.2.5.

Let $\mathcal{I}$ be an operator ideal. A two-Lipschitz operator $T \in BLip_0(X,Y;E)$ belongs to $\mathcal{I} \circ BLip_0(X,Y;E)$ if and only if its linearization T_L belongs to $\mathcal{I}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$. Furthermore, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I}\circ BLip_0} = \|T_L\|_{\mathcal{I}}, \tag{4.8}$$

and we have the isometric identification

$$(\mathcal{I} \circ BLip_0(X,Y;E), \|\cdot\|_{\mathcal{I}\circ BLip_0}) = \left(\mathcal{I}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E), \|\cdot\|_{\mathcal{I}}\right). \tag{4.9}$$

Proof. For the "if" part, if $T_L \in \mathcal{I}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$, consider the factorization of T given by $T = T_L \circ \sigma_2 \circ (\delta_X, \delta_Y)$. Since the canonical bilinear mapping σ_2 is also two-Lipschitz with $BLip(\sigma_2) = \|\sigma_2\| = 1$, then $T \in \mathcal{I} \circ BLip_0(X,Y;E)$. By (4.4), we get that

$$\|T\|_{\mathcal{I}\circ BLip_0} \leq \|T_L\| \leq \|T_L\|_{\mathcal{I}}.$$

To prove the "only if" part, take $T \in \mathcal{I} \circ BLip_0(X,Y;E)$ and $\varepsilon > 0$. Choose a Banach space F, a two-Lipschitz operator $S \in BLip_0(X,Y;E)$ and a linear operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$ with $\|u\|_{\mathcal{I}} BLip(S) \leq \varepsilon + \|T\|_{\mathcal{I}\circ BLip_0}$. The uniqueness of the linearization maps gives that $T_L = u \circ S_L$, so $T_L \in \mathcal{I}(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$ by the ideal property. Furthermore,

$$\|T_L\|_{\mathcal{I}} \leq \|u\|_{\mathcal{I}} \|S_L\| = \|u\|_{\mathcal{I}} BLip(S) \leq \varepsilon + \|T\|_{\mathcal{I}\circ BLip_0}.$$

□

To show identification (4.9), just consider the correspondence $\Psi : T \longmapsto T_L$.

Proposition 4.2.6.

If $\mathcal{I}$ is a (normed, closed, Banach) operator ideal then, $\mathcal{I} \circ BLip_0$ is a (respectively normed, closed, Banach) two-Lipschitz operator ideal.

Proof. Let us check that $\mathcal{I} \circ BLip_0$ is a closed two-Lipschitz operator ideal whenever $\mathcal{I}$ is a closed operator ideal. Thanks to an argument detailed in [6, Corollary 3.3], we prove that

$\mathcal{I} \circ BLip_0(X, Y; E)$ is a closed vector subspace of $BLip_0(X, Y; E)$ with the norm $BLip(\cdot)$ and that the ideal property holds. If $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$, we can write

$$T = f \cdot g \cdot e = id_{\mathbb{K}} \otimes e \circ (f \cdot g) \in \mathcal{I} \circ BLip_0(X, Y; E).$$

An application of (4.8) reveals that $(\mathcal{I} \circ BLip_0(X, Y; E), \|\cdot\|_{\mathcal{I} \circ BLip_0})$ is a normed space. Also, for all $T \in \mathcal{I} \circ BLip_0(X, Y; E)$,

$$BLip(T) = \|T_L\| \leq \|T_L\|_{\mathcal{I}} = \|T\|_{\mathcal{I} \circ BLip_0}.$$

Since $Id_{\mathbb{K}^2} = id_{\mathbb{K}} \circ Id_{\mathbb{K}^2}$ and $id_{\mathbb{K}} \in \mathcal{I}(\mathbb{K}, \mathbb{K})$, it follows that $Id_{\mathbb{K}^2} \in \mathcal{I} \circ BLip_0(\mathbb{K}, \mathbb{K}; \mathbb{K})$ and

$$1 = BLip(Id_{\mathbb{K}^2}) \leq \|Id_{\mathbb{K}^2}\|_{\mathcal{I} \circ BLip_0} \leq \|id_{\mathbb{K}}\|_{\mathcal{I}} BLip(Id_{\mathbb{K}^2}) = 1.$$

Now, let $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$. Let $\widehat{f} \in \mathcal{L}(Æ(Z), Æ(X))$ and $\widehat{g} \in \mathcal{L}(Æ(W), Æ(Y))$ be the associated linear operators of f and g defined in (1.5). By [48, Proposition 2.3] we take $\widehat{f} \otimes \widehat{g}$ the unique linear operator defined from $Æ(Z) \widehat{\otimes}_\pi Æ(W)$ to $Æ(X) \widehat{\otimes}_\pi Æ(Y)$ by $\widehat{f} \otimes \widehat{g}(m \otimes m') = \widehat{f}(m) \otimes \widehat{g}(m')$, for all $m \in Æ(Z)$ and $m' \in Æ(W)$ with $\left\|\widehat{f} \otimes \widehat{g}\right\| = \left\|\widehat{f}\right\| \|\widehat{g}\|$ and consider the canonical bilinear mappings

$$\begin{aligned}
\sigma_2 &: Æ(X) \times Æ(Y) \longrightarrow Æ(X) \otimes Æ(Y), \\
\sigma_2' &: Æ(Z) \times Æ(W) \longrightarrow Æ(Z) \otimes Æ(W).
\end{aligned}$$

We have

$$\sigma_2 \circ (\delta_X, \delta_Y) \circ (f, g) = \widehat{f} \otimes \widehat{g} \circ \sigma_2' \circ (\delta_Z, \delta_W).$$

Since $T = T_L \circ \sigma_2 \circ (\delta_X, \delta_Y)$,

$$\begin{aligned}
u \circ T \circ (f, g) &= u \circ T_L \circ \sigma_2 \circ (\delta_X, \delta_Y) \circ (f, g) \\
&= u \circ T_L \circ \widehat{f} \otimes \widehat{g} \circ [\sigma_2' \circ (\delta_Z, \delta_W)].
\end{aligned}$$

The uniqueness of the linearization maps gives that

$$(u \circ T \circ (f, g))_L = u \circ T_L \circ \widehat{f} \otimes \widehat{g}.$$

By the ideal property concerning the operator ideal $\mathcal{I}$ and (4.8) we obtain

$$\begin{aligned}
\|u \circ T \circ (f, g)\|_{\mathcal{I} \circ BLip_0} &= \left\|u \circ T_L \circ \widehat{f} \otimes \widehat{g}\right\|_{\mathcal{I}} \\
&\leq \|u\| \, \|T_L\|_{\mathcal{I}} \left\|\widehat{f} \otimes \widehat{g}\right\| \\
&= \|u\| \, \|T\|_{\mathcal{I} \circ BLip_0} Lip(f) Lip(g).
\end{aligned}$$

Finally, it easily follows from the isometric identification (4.9) that if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a Banach operator ideal then, $(\mathcal{I} \circ BLip_0, \|\cdot\|_{\mathcal{I}\circ BLip_0})$ is a Banach two-Lipschitz operator ideal. $\square$

Let $\mathcal{I}$ be an operator ideal and Y be a pointed metric space. In the next result we give a necessary and sufficient condition for assuring that every two-Lipschitz mappings $T : X_1 \times X_2 \longrightarrow Æ(Y)$ belongs to the composition ideal $\mathcal{I} \circ BLip_0(X_1, X_2; Æ(Y))$, for all pointed metric spaces X_1 and X_2. In order to prove this result we need the following lemma.

Lemma 4.2.7.

Let $\mathcal{I}_1, \mathcal{I}_2$ be operator ideals, X_1, X_2 be pointed metric spaces and F be Banach space. If $\mathcal{I}_1 \circ BLip_0(X_1, X_2; F) \subset \mathcal{I}_2 \circ BLip_0(X_1, X_2; F)$, then $\mathcal{I}_1(Æ(X_i), F) \subset \mathcal{I}_2(Æ(X_i), F)$, for every $i = 1, 2$. In particular, if $\mathcal{I}_1 \circ BLip_0(X_1, X_2; F) = \mathcal{I}_2 \circ BLip_0(X_1, X_2; F)$, then $\mathcal{I}_1(Æ(X_i), F) = \mathcal{I}_2(Æ(X_i), F)$, for every $i = 1, 2$.

Proof. Let $u \in \mathcal{I}_1(Æ(X_2), F)$. Fix $a^1 \in X_1$ and $\varphi_1 \in Æ(X_1)^*$ with $a^1 \neq 0$ and $\varphi_1(m_{a^10}) = 1$. For all $(x^1, x^2) \in X_1 \times X_2$ take $T(x^1, x^2) = \varphi_1(m_{x^10})\, u(m_{x^20})$. It is clear that $T = u \circ R$ where $R(x^1, x^2) = \varphi_1(m_{x^10})\, m_{x^20}$. An easy calculation shows that $R \in BLip_0(X_1, X_2; Æ(X_2))$ and hence $T \in \mathcal{I}_1 \circ BLip_0(X_1, X_2; F)$. Therefore, there is a Banach space G, a linear operator $v \in \mathcal{I}_2(G, F)$ and a two-Lipschitz operator $S \in BLip_0(X_1, X_2; G)$ such that $T = v \circ S$. Now if we consider S_B the bi-linearization of S, for all $x^2, x^{2\prime} \in X_2$ we obtain

$$\begin{aligned} u(m_{x^2x^{2\prime}}) &= u(m_{x^20}) - u(m_{x^{2\prime}0}) \\ &= T(a^1, x^2) - T(a^1, x^{2\prime}) \\ &= v \circ S(a^1, x^2) - v \circ S(a^1, x^{2\prime}) \\ &= v \circ S_B(m_{a^10}, m_{x^2x^{2\prime}}). \end{aligned}$$

By linearity of u we conclude that $u(m) = v \circ w(m)$ for all $m \in \mathcal{M}(X_2)$, where $w : \mathcal{M}(X_2) \longrightarrow G$ is the bounded linear mapping defined by $w(m) = S_B(m_{a^10}, m)$ for all $m \in \mathcal{M}(X_2)$. Now, the operator w has a unique extension to a bounded linear mapping from $\overline{\mathcal{M}(X_2)} = Æ(X_2)$ to G, denoted also by w with $u = v \circ w$. Consequently, $u \in \mathcal{I}_2(Æ(X_2), F)$ by the ideal property. $\square$

Proposition 4.2.8.

Let $\mathcal{I}$ be an operator ideal and Y be a pointed metric space. For all pointed metric spaces X_1, X_2, we have

$$\mathcal{I} \circ BLip_0(X_1, X_2; Æ(Y)) = BLip_0(X_1, X_2; Æ(Y)),$$

if and only if the identity operator on $Æ(Y)$ belongs to $\mathcal{I}$.

Proof. For the sufficient condition, suppose that $id_{Æ(Y)} \in \mathcal{I}(Æ(Y), Æ(Y))$. If $T \in BLip_0(X_1, X_2; Æ(Y))$ then $T = id_{Æ(Y)} \circ T \in \mathcal{I} \circ BLip_0(X_1, X_2; Æ(Y))$. For the necessary condition, take $X_1 = X_2 = Y$ and applying the Lemma 4.2.7 for $\mathcal{I}_1 = \mathcal{I}$, $\mathcal{I}_2 = \mathcal{L}$ and $F = Æ(Y)$. $\square$

4.3 Applications: Some examples of two-Lipschitz operators ideals

4.3.1 Ideal of compact two-Lipschitz operators

We introduce the compactness concept for the two-Lipschitz operators. By showing that the new class of these operators is an ideal of the composition type, we see that the nature of this extension allows us to transfer some properties of the bilinear compact operators (and also the linear compact operators) to the two-Lipschitz case. Many papers were devoted to the concept of compactness for the bilinear mappings between Banach spaces (see [45], [46], [10], [36]).

Let E, F, G be Banach spaces and $T : E \times F \longrightarrow G$ be a bilinear operator. We call T compact, in symbols $T \in \mathcal{L}_{\mathcal{K}}(E, F; G)$ if $T(B_E \times B_F)$ is a relatively compact subset of G. This is equivalent to saying that T takes bounded sets into relatively compact sets.

Now we present the definition of compact two-Lipschitz operators. Let X, Y be pointed metric spaces, E be a Banach space and $T \in BLip_0(X, Y; E)$. As in the Lipschitz case ([33]), the two-Lipschitz image of T is the subset $Im_{BLip}(T) \subset E$ that consists of all elements of the form

$$\frac{T(x, y) - T(x', y) - T(x, y') + T(x', y')}{d(x, x')d(y, y')},$$

where $x, x' \in X, y, y' \in Y$ with $x \neq x'$ and $y \neq y'$.

It is easy to see that if $Im_{BLip}(T)$ is a bounded subset of E, then $T : X \times Y \longrightarrow E$ is a two-Lipschitz mapping, which motivates the following concept.

Definition 4.3.1.

A two-Lipschitz operator $T \in BLip_0(X, Y; E)$ is said to be compact if $Im_{BLip}(T)$ is relatively compact in E. The vector space of these mappings is indicated by $BLip_{0\mathcal{K}}(X, Y; E)$.

Remark 4.3.2.

Observe that the two-Lipschitz compact operators can be seen as an extension of the bilinear compact operators. Indeed, if X, Y, E are Banach spaces and $T : X \times Y \longrightarrow E$ is bilinear compact, it follows from $Im_{BLip}(T) = T(S_X \times S_Y)$ that T is two-Lipschitz compact.

The next result provides a shortcut for showing that the class of the two-Lipschitz compact operators is a closed two-Lipschitz operator ideal.

Recall that the absolutely convex hull of the subset A of a Banach space E is defined to be

$$\Gamma(A) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}, x_i \in A, \alpha_i \in \mathbb{R}, \sum_{i=1}^{n} |\alpha_i| \leq 1 \right\}.$$

Theorem 4.3.3.

Let X_1, X_2 be pointed metric spaces, E be a Banach space. For $T \in BLip_0(X_1, X_2; E)$, the following statements are equivalent.

(i) T is two-Lipschitz compact.

(ii) $T_B : Æ(X_1) \times Æ(X_2) \longrightarrow E$ is bilinear compact.

(iii) $T_L : Æ(X_1) \widehat{\otimes}_\pi Æ(X_2) \longrightarrow E$ is linear compact.

Proof. Take $M_{X_i} = \left\{ \frac{m_{x_i 0} - m_{x'_i 0}}{d(x_i, x'_i)} : x_i, x'_i \in X_i, x_i \neq x'_i \right\}, i = 1, 2$. By [33, Lemma 1.1] we have $B_{Æ(X_i)} = \overline{\Gamma}(M_{X_i})$, $i = 1, 2$. The equivalence between (i) and (ii) follows from the inclusions

$$Im_{BLip}(T) \subset T_B \left(\overline{\Gamma}(M_{X_1}) \times \overline{\Gamma}(M_{X_2}) \right) \subset \overline{\Gamma} \left(Im_{BLip}(T) \right),$$

and the fact that the closed absolutely convex hull of a relatively compact subset of a Banach space is compact.

The equivalence $(ii) \Longleftrightarrow (iii)$ is proved in [36]. □

Since the class $\mathcal{K}$ of compact linear operators between Banach spaces is a closed Banach operator ideal (see [43]) and using the preceding theorem, Theorem 4.2.5 and Proposition 4.2.6 we obtain the following corollary.

Corollary 4.3.4.

The class $BLip_{0\mathcal{K}}$ is the closed Banach two-Lipschitz operator ideal generated by the composition method from $\mathcal{K}$, i.e.,

$$BLip_{0\mathcal{K}}(X, Y; E) = \mathcal{K} \circ BLip_0(X, Y; E),$$

for all pointed metric spaces X, Y and Banach space E.

Remark 4.3.5.

The same technique that we have shown above should provide also the corresponding result for the class of two-Lipschitz weakly compact operators. The mapping $T \in BLip_0(X, Y; E)$ is two-Lipschitz weakly compact if $Im_{BLip}(T)$ is relatively weakly compact in E.

4.3.2 Ideal of strongly two-Lipschitz operators

Cohen in [22] introduced, $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$, the operator ideal of strongly p-summing linear operators. For $1 < p \leq \infty$, recall that a linear operator $u : E \longrightarrow F$ belongs to $\mathcal{D}_p(E, F)$ if there is a positive constant C such that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E$ and $y_1^*, \ldots, y_n^* \in F^*$ we have

$$\|(\langle u(x_i), y_i^* \rangle)_{i=1}^n\|_1 \leq C \|(x_i)_{i=1}^n\|_p \|(y_i^*)_{i=1}^n\|_{p^*,\omega}. \tag{4.10}$$

In this case,

$$\|u\|_{\mathcal{D}_p} = \inf \{C > 0 : \text{satisfying (4.10)}\}.$$

The definition of Cohen strongly p-summing m-linear operators is due to Achour and Mezrag (see [5]) in order to generalize the concept of strongly p-summing linear operators.

Definition 4.3.6.

For $1 < p \leq \infty$, a mapping $T \in \mathcal{L}(E, F; G)$ is Cohen strongly p-summing if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in E$, $y_1, ..., y_n \in F$, and any $g_1^*, ..., g_n^* \in G^*$, we have

$$\|(\langle T(x_i, y_i), g_i^* \rangle)_{i=1}^n\|_1 \leq C \left(\sum_{i=1}^n \|x_i\|^p \|y_i\|^p \right)^{\frac{1}{p}} \|(g_i^*)_{i=1}^n\|_{p^*,\omega}.$$

The vector space of these mappings is indicated by $\mathcal{D}_p^2(E,F;G)$ and the smallest C satisfying the inequality above, by $\|T\|_{\mathcal{D}_p^2}$. This defines a norm on $\mathcal{D}_p^2(E,F;G)$ and $\left(\mathcal{D}_p^2, \|\cdot\|_{\mathcal{D}_p^2}\right)$ is a Banach bilinear ideal.

The next Lipschitz generalization of the concept of strongly p-summing linear operators was introduced by Yahi, Achour and Rueda in [53].

Definition 4.3.7.

A map $T \in Lip_0(X,E)$ is strongly Lipschitz p-summing $(1 < p \leq \infty)$, if there are a Banach space F and an operator $u \in \mathcal{D}_p(F,E)$ such that

$$|\langle T(x) - T(x'), y^* \rangle| \leq d(x,x') \|u^*(y^*)\| \text{ for all } x, x' \in X, y^* \in E^*.$$

The infimum of all constants $\|u\|_{\mathcal{D}_p}$ is denoted $\|T\|_{\mathcal{D}_p^L}$. This class of mappings is denoted by $\mathcal{D}_p^L(X,E)$ and with the norm $\|T\|_{\mathcal{D}_p^L}$ it is a Banach space.

Now we are going to construct a new two-Lipschitz operator ideal by the composition method starting from the operator ideal $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$.

Definition 4.3.8.

Let $1 < p \leq \infty$. A mapping $T \in BLip_0(X,Y;E)$ is strongly two-Lipschitz p-summing if there exist a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that for all $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$ we have

$$|\langle T(x,y) - T(x',y) - T(x,y') + T(x',y'), e^* \rangle| \leq d(x,x')d(y,y') \|S(e^*)\|. \tag{4.11}$$

We denoted by $\mathcal{D}_p^{BL}(X,Y;E)$ the set of all strongly two-Lipschitz p-summing mappings from $X \times Y$ to E. Moreover, if $T \in \mathcal{D}_p^{BL}(X,Y;E)$, then we set $\|T\|_{\mathcal{D}_p^{BL}} = \inf\{\pi_{p^*}(S)\}$. The infimum is taken over all Banach spaces G and operators S such that (4.11) holds.

Let us give an example of a strongly two-Lipschitz p-summing operator.

Example 4.3.9.

Let $1 < p \leq \infty$ and $S : Y \longrightarrow E$ be a strongly Lipschitz p-summing operator and $f \in X^{\#}$. The mapping

$$T : X \times Y \longrightarrow E, \quad T(x,y) = f(x)S(y),$$

is a strongly two-Lipschitz p-summing operator with $\|T\|_{\mathcal{D}_p^{BL}} \leq Lip(f)\,\|S\|_{\mathcal{D}_p^L}$. Indeed, for $\varepsilon > 0$ choose a Banach space F and $u \in \mathcal{D}_p(F, E)$ such that $\|u\|_{\mathcal{D}_p} \leq \left(\varepsilon + \|S\|_{\mathcal{D}_p^L}\right)$ and for every $x, x' \in X$, $y, y' \in Y$, $e^* \in E^*$,

$$
\begin{aligned}
&|\langle T(x,y) - T(x',y) - T(x,y') + T(x',y'), e^*\rangle| \\
&= |f(x) - f(x')|\,|\langle S(y) - S(y'), e^*\rangle| \\
&\leq Lip(f) d(x,x') d(y,y')\,\|u^*(e^*)\|\,.
\end{aligned}
$$

Since $u^* : E^* \longrightarrow F^*$ is p^*-summing with $\|u\|_{\mathcal{D}_p} = \pi_{p^*}(u^*)$ (see [22, Theorem 2.2.2]), it follows that $T \in \mathcal{D}_p^{BL}(X, Y; E)$ and

$$
\|T\|_{\mathcal{D}_p^{BL}} \leq Lip(f)\pi_{p^*}(u^*) \leq Lip(f)\left(\varepsilon + \|S\|_{\mathcal{D}_p^L}\right).
$$

The following theorem justifies that the class under study is a true extension of the bilinear notion.

Theorem 4.3.10.

If $T \in \mathcal{L}(X, Y; E)$ is a bilinear operator between Banach spaces X, Y and E, then $T \in \mathcal{D}_p^{BL}(X, Y; E)$ if and only if $T \in \mathcal{D}_p^2(X, Y; E)$. Furthermore, $\|T\|_{\mathcal{D}_p^2} = \|T\|_{\mathcal{D}_p^{BL}}$.

Proof. Suppose that $T \in \mathcal{D}_p^{BL}(X, Y; E)$. For each $\varepsilon > 0$, choose a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that (4.11) holds and $\pi_{p^*}(S) \leq \|T\|_{\mathcal{D}_p^{BL}} + \varepsilon$. Let $(x_i)_{1\leq i\leq n} \subset X$, $(y_i)_{1\leq i\leq n} \subset Y$, and $(e_i^*)_{1\leq i\leq n} \subset E^*$. Then by (4.11) and Hölder's inequality we get

$$
\begin{aligned}
\|(\langle T(x_i, y_i), e_i^*\rangle)_{i=1}^n\|_1 &\leq \sum_{i=1}^n \|x_i\|\,\|y_i\|\,\|S(e_i^*)\| \\
&\leq \left(\sum_{i=1}^n \|x_i\|^p\,\|y_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|S(e_i^*)\|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq \pi_{p^*}(S)\left(\sum_{i=1}^n \|x_i\|^p\,\|y_i\|^p\right)^{\frac{1}{p}} \|(e_i^*)_{i=1}^n\|_{p^*,\omega}\,.
\end{aligned}
$$

It follows that T is Cohen strongly p-summing and

$$
\|T\|_{\mathcal{D}_p^2} \leq \pi_{p^*}(S) \leq \|T\|_{\mathcal{D}_p^{BL}} + \varepsilon.
$$

Conversely, suppose that $T \in \mathcal{D}_p^2(X,Y;E)$. By [5, Theorem 2.4] there is a regular Borel probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$, we have

$$\begin{aligned}
&|\langle T(x,y) - T(x',y) - T(x,y') + T(x',y'), e^*\rangle| \\
&= |\langle T(x-x', y-y'), e^*\rangle| \\
&\leq \|T\|_{\mathcal{D}_p^2} \|x-x'\| \|y-y'\| \left(\int_{B_{E^{**}}} |\langle e^*, \phi\rangle|^{p^*} d\mu(\phi)\right)^{\frac{1}{p^*}}.
\end{aligned}$$

Let A be the natural isometric embedding $E^* \longrightarrow C(B_{E^{**}})$ composed with the formal identity from $C(B_{E^{**}})$ into $L_\infty(\mu)$ given by $A(e^*)(\phi) = \langle e^*, \phi\rangle$, $e^* \in E^*, \phi \in B_{E^{**}}$. The canonical mapping $i_{p^*} : L_\infty(\mu) \longrightarrow L_{p^*}(\mu)$ is p^*-summing with $\pi_{p^*}(i_{p^*}) = 1$ (see [25, Page 40]), it follows that $i_{p^*} \circ A$ is also p^*-summing with $\pi_{p^*}(i_{p^*} \circ A) \leq 1$. Therefore, $T \in \mathcal{D}_p^{BL}(X,Y;E)$ by taking $G = L_{p^*}(\mu)$ and $S = \|T\|_{\mathcal{D}_p^2} (i_{p^*} \circ A)$. In addition,

$$\|T\|_{\mathcal{D}_p^{BL}} \leq \pi_{p^*}(S) \leq \|T\|_{\mathcal{D}_p^2}.$$

□

We show in what follows that $\mathcal{D}_p^{BL}$ is the two-Lipschitz operator ideal generated by the composition method from the linear operator ideal $\mathcal{D}_p$.

Proposition 4.3.11.

Let X, Y be pointed metric spaces and E be Banach space. For $1 < p \leq \infty$, we have $T \in \mathcal{D}_p^{BL}(X,Y;E)$ if and only if its bi-linearization $T_B : Æ(X) \times Æ(Y) \longrightarrow E$ belongs to $\mathcal{D}_p^2(Æ(X), Æ(Y); E)$. In this case $\|T\|_{\mathcal{D}_p^{BL}} = \|T_B\|_{\mathcal{D}_p^2}$.

Proof. Suppose that T is strongly two-Lipschitz p-summing operator. Let $m^1 \in \mathcal{M}(X)$, $m^2 \in \mathcal{M}(Y)$, with representations, $m^1 = \sum_{k=1}^n \alpha_k m_{x_k x'_k}$ and $m^2 = \sum_{j=1}^r \beta_j m_{y_j y'_j}$ and let $e^* \in E^*$. Then there exist a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that,

$$\left|\langle T_B(m^1, m^2), e^*\rangle\right| \leq \sum_{k=1}^n |\alpha_k| \, d(x_k, x'_k) \sum_{j=1}^r |\beta_j| \, d(y_j, y'_j) \|S(e^*)\|.$$

Taking the infimum over all representations of m^1 and m^2 we get

$$\left|\langle T_B(m^1, m^2), e^*\rangle\right| \leq \|m^1\| \|m^2\| \|S(e^*)\|.$$

By using the last inequality and Hölder's inequality we obtain

$$\left\| \left(\left\langle T_B \left(m_i^1, m_i^2 \right), e_i^* \right\rangle \right)_{i=1}^n \right\|_1 \leq \pi_{p^*}(S) \left(\sum_{i=1}^n \left\| m_i^1 \right\|^p \left\| m_i^2 \right\|^p \right)^{\frac{1}{p}} \left\| (e_i^*)_{i=1}^n \right\|_{p^*,\omega},$$

for any $m_1^1, ..., m_n^1 \in \mathcal{M}(X)$, $m_1^2, ..., m_n^2 \in \mathcal{M}(Y)$, and any $e_1^*, ..., e_n^* \in E^*$. Therefore, $T_B \in \mathcal{D}_p^2(Æ(X), Æ(Y); E)$ and $\|T_B\|_{\mathcal{D}_p^2} \leq \pi_{p^*}(S)$. Passing to the infimum over all G and S as above we arrive at $\|T_B\|_{\mathcal{D}_p^2} \leq \|T\|_{\mathcal{D}_p^{BL}}$.

Conversely, suppose that T_B is Cohen strongly p-summing. Let $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$. By [5, Theorem 2.4] there is a regular Borel probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$, we have

$$\begin{aligned} &|\langle T(x,y) - T(x',y) - T(x,y') + T(x',y'), e^* \rangle| \\ &= |\langle T_B(m_{xx'}, m_{yy'}), e^* \rangle| \\ &\leq \|T_B\|_{\mathcal{D}_p^2} \|m_{xx'}\| \|m_{yy'}\| \left(\int_{B_{E^{**}}} |\langle e^*, \phi \rangle|^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}}. \end{aligned}$$

A similar analysis to that in the proof of the second implication of Theorem 4.3.10 shows that $T \in \mathcal{D}_p^{BL}(X, Y; E)$ and $\|T\|_{\mathcal{D}_p^{BL}} \leq \|T\|_{\mathcal{D}_p^2}$. □

The proof of the following corollary is a consequence of [1, Theorem 3.6] and the previous proposition.

Corollary 4.3.12.

Let X, Y be pointed metric spaces and E be a Banach space. For $1 < p \leq \infty$, we have $T \in \mathcal{D}_p^{BL}(X, Y; E)$ if and only if its linearization $T_L \in \mathcal{D}_p(Æ(X) \widehat{\otimes}_\pi Æ(Y), E)$. In this case $\|T\|_{\mathcal{D}_p^{BL}} = \|T_L\|_{\mathcal{D}_p}$.

As a consequence, we obtain the following corollary which is a straightforward consequence of the preceding corollary, Theorem 4.2.5 and Proposition 4.2.6.

Corollary 4.3.13.

The class $\mathcal{D}_p^{BL}$ is the Banach two-Lipschitz operator ideal generated by the composition method from the operator ideal $\mathcal{D}_p$, i.e.,

$$\mathcal{D}_p^{BL}(X, Y; E) = \mathcal{D}_p \circ BLip_0(X, Y; E),$$

for all pointed metric spaces X, Y and Banach space E.

4.3.3 Ideal of two-Lipschitz $(p; p_1, p_2)$-summing operators

The definition of the Lipschitz p-summing operators below was first given by Farmer and Johnson (see Definition 2.4.1).

The definition of absolutely p-summing m-linear functionals is due to Pietsch [44]. In [37], Matos presented a definition for vector-valued mappings.

Definition 4.3.14.

Let E, F, G be Banach spaces and let $1 \leq p, p_1, p_2 < \infty$, with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. A bilinear operator $T \in \mathcal{L}(E, F; G)$ is said to be absolutely $(p; p_1, p_2)$-summing if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in E$ and $y_1, ..., y_n \in F$ we have

$$\|(T(x_i, y_i))_{i=1}^n\|_p \leq C \sup_{\varphi_1 \in B_{E^*}} \|(\varphi_1(x_i))_{i=1}^n\|_{p_1} \sup_{\varphi_2 \in B_{F^*}} \|(\varphi_2(y_i))_{i=1}^n\|_{p_2}.$$

The Banach space of these mappings is denoted by $\mathcal{L}_{as,(p;p_1,p_2)}(E, F; G)$ with the norm $\|T\|_{\mathcal{L}_{as,(p;p_1,p_2)}}$, which is the smallest C satisfying the above inequality.

We extend the definition of the class of absolutely $(p; p_1, p_2)$-summing bilinear operators to the case of two-Lipschitz operators, for which the resulting vector space of two-Lipschitz $(p; p_1, p_2)$-summing operators is a Banach two-Lipschitz operator ideal.

Definition 4.3.15.

Let $1 \leq p, p_1, p_2 < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. A mapping $T \in BLip_0(X, Y; E)$ is called two-Lipschitz $(p; p_1, p_2)$-summing if there exists a constant $C > 0$ such that for any $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ in X and $y_1, \ldots, y_n, y'_1, \ldots, y'_n$ in Y we have

$$\begin{aligned} &\left\|(T(x_i, y_i) - T(x_i, y'_i) - T(x'_i, y_i) + T(x'_i, y'_i))_{i=1}^n\right\|_p \qquad (4.12) \\ &\leq C \sup_{f \in B_{X^\#}} \left\|(f(x_i) - f(x'_i))_{i=1}^n\right\|_{p_1} \sup_{g \in B_{Y^\#}} \left\|(g(y_i) - g(y'_i))_{i=1}^n\right\|_{p_2}. \end{aligned}$$

We denote this class of two-Lipschitz operators by $BL_{as,(p;p_1,p_2)}(X, Y; E)$. In this case, we define

$$\|T\|_{BL_{as,(p;p_1,p_2)}} = \inf \{C : \text{satisfying } (4.12)\}.$$

We don't know if two-Lipschitz $(p; p_1, p_2)$-summability implies $(p; p_1, p_2)$-summability whenever the mapping T is bilinear. The converse is of course clearly true. If X, Y and E are

Banach spaces and $T : X \times Y \longrightarrow E$ is bilinear $(p;p_1,p_2)$-summing, it follows from the inclusions $B_{X^*} \subset B_{X^\#}$ and $B_{Y^*} \subset B_{Y^\#}$ that T is two-Lipschitz $(p;p_1,p_2)$-summing and $\|T\|_{BL_{as,(p;p_1,p_2)}} \leq \|T\|_{\mathcal{L}_{as,(p;p_1,p_2)}}$.

Proposition 4.3.16.

The class $\left(BL_{as,(p;p_1,p_2)}, \|\cdot\|_{BL_{as,(p;p_1,p_2)}}\right)$ *is a Banach two-Lipschitz operator ideal.*

Proof. The properties (ii), (ii') and (iii') of Definition 4.2.1 may be easily verified. So we only show that (i') holds. Let X and Y be pointed metric spaces and E be Banach space. It is easily seen that $\alpha T \in BL_{as,(p;p_1,p_2)}(X,Y;E) \subset Lip_0(X,Y;E)$, $\|\alpha T\|_{BL_{as,(p;p_1,p_2)}} = |\alpha| \, \|T\|_{BL_{as,(p;p_1,p_2)}}$ and $BLip(T) \leq \|T\|_{BL_{as,(p;p_1,p_2)}}$ for every $T \in BL_{as,(p;p_1,p_2)}(X,Y;E)$ and $\alpha \in \mathbb{K}$.

Let $S, T \in BL_{as,(p;p_1,p_2)}(X,Y;E)$, and $(x_i)_{i=1}^n, (x_i')_{i=1}^n \subset X$ and $(y_i)_{i=1}^n, (y_i')_{i=1}^n \subset Y$. Then

$$\begin{aligned}
&\left\|((S+T)(x_i,y_i) - (S+T)(x_i,y_i') - (S+T)(x_i',y_i) + (S+T)(x_i',y_i'))_{i=1}^n\right\|_p \\
&\leq \left\|(S(x_i,y_i) - S(x_i,y_i') - S(x_i',y_i) + S(x_i',y_i'))_{i=1}^n\right\|_p \\
&+ \left\|(T(x_i,y_i) - T(x_i,y_i') - T(x_i',y_i) + T(x_i',y_i'))_{i=1}^n\right\|_p \\
&\leq \left(\|S\|_{BL_{as,(p;p_1,p_2)}} + \|T\|_{BL_{as,(p;p_1,p_2)}}\right) \sup_{f \in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \\
&\times \sup_{g \in B_{Y^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2}.
\end{aligned}$$

which means that $S + T$ is in $BL_{as,(p;p_1,p_2)}(X,Y;E)$ and

$$\|S+T\|_{BL_{as,(p;p_1,p_2)}} \leq \|S\|_{BL_{as,(p;p_1,p_2)}} + \|T\|_{BL_{as,(p;p_1,p_2)}}.$$

Thus, we have shown that $(BL_{as,(p;p_1,p_2)}(X,Y;E), \|\cdot\|_{BL_{as,(p;p_1,p_2)}})$ is a normed vector subspace of $BLip_0(X,Y;E)$.

To prove the completeness of the space $BL_{as,(p;p_1,p_2)}(X,Y;E)$, take a Cauchy sequence $(T_n)_n \subset BL_{as,(p;p_1,p_2)}(X,Y;E)$. Hence for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$BLip(T_n - T_k) \leq \|T_n - T_k\|_{BL_{as,(p;p_1,p_2)}} < \varepsilon, \quad \text{for all } n, k \geq n_\varepsilon,$$

which means that $(T_n)_n$ is a Cauchy sequence in the Banach space $BLip_0(X,Y;E)$. Thus, it exists $T \in BLip_0(X,Y;E)$ such that $BLip(T_n - T) \longrightarrow 0$. Now, let $x_1, \ldots, x_n, x_1', \ldots, x_n'$ in X and $y_1, \ldots, y_n, y_1', \ldots, y_n'$ in Y. Since $T_n - T_k$ is two-Lipschitz $(p;p_1,p_2)$-summing, it follows that for every $n, k \geq n_\varepsilon$, we have

$$\|((T_n - T_k)(x_i, y_i) - (T_n - T_k)(x_i, y_i') - (T_n - T_k)(x_i', y_i) + (T_n - T_k)(x_i', y_i'))_{i=1}^n\|_p$$
$$\leq \varepsilon \sup_{f \in B_{X^\#}} \|(f(x_i) - f(x_i'))_{i=1}^n\|_{p_1} \sup_{g \in B_{Y^\#}} \|(g(y_i) - g(y_i'))_{i=1}^n\|_{p_2}.$$

Since $T_n(x, y) \longrightarrow T(x, y)$ for all $x \in X, y \in Y$ and after passing to the limit for $k \longrightarrow +\infty$, we obtain that for every $n \geq n_\varepsilon$,

$$\left\|((T_n - T)(x_i, y_i) - (T_n - T)(x_i, y_i') - (T_n - T)(x_i', y_i) + (T_n - T)(x_i', y_i'))_{i=1}^n\right\|_p$$
$$< \varepsilon \sup_{f \in B_{X^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \sup_{g \in B_{Y^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2}$$

which means that $(T_n - T) \in BL_{as,(p;p_1,p_2)}(X, Y; E)$ and hence, $T \in BL_{as,(p;p_1,p_2)}(X, Y; E)$. In addition, $\|T_n - T\|_{BL_{as,(p;p_1,p_2)}} < \varepsilon$ for all $n \geq n_\varepsilon$, *i.e.*, the sequence $(T_n)_n$ is convergent to $T \in BL_{as,(p;p_1,p_2)}(X, Y; E)$ with respect to the norm, $\|\cdot\|_{BL_{as,(p;p_1,p_2)}}$. □

Let us show with an example that $BL_{as,(p;p_1,p_2)}$ is not of composition type, that is $BL_{as,(p;p_1,p_2)} \neq \Pi_p \circ BLip_0$.

Example 4.3.17.

Let $1 \leq p, p_1, p_2 < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$. Consider the two-Lipschitz mapping

$$S : \mathbb{R} \times \mathbb{R} \longrightarrow Æ(\mathbb{R}) \widehat{\otimes}_\pi Æ(\mathbb{R}), \quad S(x, y) = m_{x0} \otimes m_{y0}.$$

Then S is two-Lipschitz $(p; p_1, p_2)$-summing. In order to see this, let $x_i, x_i', y_i, y_i' \in \mathbb{R}$ $(i = 1, ..., n)$. Then, using Hölder's inequality and taking into account that the mapping $\delta_\mathbb{R} : \mathbb{R} \longrightarrow Æ(\mathbb{R})$ is Lipschitz p-summing for all $p \geq 1$, we obtain

$$\begin{aligned}
&\left\|(S(x_i, y_i) - S(x_i, y_i') - S(x_i', y_i) + S(x_i', y_i'))_{i=1}^n\right\|_p \\
&= \left(\sum_{i=1}^n \pi\left(m_{x_i x_i'} \otimes m_{y_i y_i'}\right)^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \left\|m_{x_i x_i'}\right\|^p \left\|m_{y_i y_i'}\right\|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^n \left\|m_{x_i x_i'}\right\|^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n \left\|m_{y_i y_i'}\right\|^{p_2}\right)^{\frac{1}{p_2}} \\
&= \left(\sum_{i=1}^n \|\delta_\mathbb{R}(x_i) - \delta_\mathbb{R}(x_i')\|^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n \|\delta_\mathbb{R}(y_i) - \delta_\mathbb{R}(y_i')\|^{p_2}\right)^{\frac{1}{p_2}} \\
&\leq \sup_{f \in B_{\mathbb{R}^\#}} \left\|(f(x_i) - f(x_i'))_{i=1}^n\right\|_{p_1} \sup_{g \in B_{\mathbb{R}^\#}} \left\|(g(y_i) - g(y_i'))_{i=1}^n\right\|_{p_2}.
\end{aligned}$$

On the other hand, a trivial verification shows that $S = \sigma_2 \circ (\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$. The uniqueness of the linearization maps gives that S_L is the identity map on the infinite dimensional space $Æ(\mathbb{R}) \otimes Æ(\mathbb{R})$ and so, $S_L \notin \Pi_p$ (see [25, Page 50]). Finally, the Theorem 4.2.5 asserts that $S \notin \Pi_p \circ BLip_0$

4.3.4 Ideal of two-Lipschitz factorable p-dominated operators

The p-semi-integral multilinear mappings were introduced in [39] motivated by the work of Alencar and Matos [8]. Let E, F, G be Banach spaces and $1 \leq p < \infty$. A bilinear mapping $T \in \mathcal{L}(E, F; G)$ is p-semi-integral, in symbols $T \in \mathcal{L}_{si,p}(E, F; G)$, if there is a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n} \| T(x_i, y_i) \|^p \right)^{\frac{1}{p}} \leq C \sup_{\substack{\varphi_1 \in B_{E^*} \\ \varphi_2 \in B_{F^*}}} \left(\sum_{i=1}^{n} |\varphi_1(x_i) \varphi_2(y_i)|^p \right)^{\frac{1}{p}},$$

for any $(x_i)_{i=1}^n \subset E$ and $(y_i)_{i=1}^n \subset F$. In this case, take $\|T\|_{si,p}$ the infimum of all constants C working in the above inequality.

We can found some details about this concept in [39], [18] and [17].

The concept of factorable p-dominated bilinear operators is given in [40])

Definition 4.3.18.

Let $1 \leq p < \infty$. A bilinear operator $T : E \times F \longrightarrow G$ is factorable p-dominated if there is a constant $C > 0$ such that for every $(x_i^j)_{i=1}^n \subset E, (y_i^j)_{i=1}^n \subset F$, $(\lambda_i^j)_{i=1}^n \subset \mathbb{K}, (1 \leq j \leq m)$ and all positive integers n, m we have

$$\left(\sum_{j=1}^{m} \left\| \sum_{i=1}^{n} \lambda_i^j T\left(x_i^j, y_i^j\right) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\substack{\varphi_1 \in B_{E^*} \\ \varphi_2 \in B_{F^*}}} \left(\sum_{j=1}^{m} \left| \sum_{i=1}^{n} \lambda_i^j \varphi_1\left(x_i^j\right) \varphi_2\left(y_i^j\right) \right|^p \right)^{\frac{1}{p}}.$$

The space of all factorable p-dominated operators is denoted by $\mathcal{L}_{f,p}^2(E, F; G)$ and the smallest constant C satisfying the above inequality, by $\|T\|_{\mathcal{L}_{f,p}^2}$.

In the following definition we generalize the concept of factorable p-dominated bilinear operators, to the two-Lipschitz case obtaining in this way a Banach ideal of two-Lipschitz operators.

Definition 4.3.19.

Let $1 \leq p < \infty$. A mapping $T \in BLip_0(X, Y; E)$ is called two-Lipschitz factorable p-

dominated if there exists a constant $C > 0$ such that for any $x_i^j, x_i'^j \in X$, $y_i^j, y_i'^j \in Y$, $\lambda_i^j \in \mathbb{K}, (1 \leq i \leq n, 1 \leq j \leq s)$ and all positive integers n, s we have

$$\left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n}\lambda_i^j\left(T(x_i^j,y_i^j)-T(x_i^j,y_i'^j)-T(x_i'^j,y_i^j)+T(x_i'^j,y_i'^j)\right)\right\|^p\right)^{\frac{1}{p}}$$
$$\leq C\sup_{\substack{f\in B_{X^\#}\\ g\in B_{Y^\#}}}\left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\lambda_i^j\left(f(x_i^j)-f(x_i'^j)\right)\left(g(y_i^j)-g(y_i'^j)\right)\right|^p\right)^{\frac{1}{p}}.$$

The class of all two-Lipschitz factorable p-dominated operators is denoted by $BL_{f,p}(X,Y;E)$. In this case, we define $\|T\|_{BL_{f,p}}$ as the infimum of all constants C fulfilling the above inequality.

Remark 4.3.20.

Note that if T is two-Lipschitz factorable p-dominated then taking $n = 1$, by Hölder's inequality we have

$$\left(\sum_{j=1}^{s}\|(T(x^j,y^j)-T(x^j,y'^j)-T(x'^j,y^j)+T(x'^j,y'^j))\|^p\right)^{\frac{1}{p}}$$
$$\leq \|T\|_{BL_{f,p}}\sup_{f\in B_{X^\#}}\left\|(f(x^j)-f(x'^j))_{j=1}^s\right\|_{p_1}\sup_{g\in B_{Y^\#}}\left\|(g(y^j)-g(y'^j))_{j=1}^s\right\|_{p_2},$$

for any $x^j, x'^j \in X$, $y^j, y'^j \in Y\,(1 \leq j \leq s)$, *i.e.*, T is two-Lipschitz $(p;p_1,p_2)$-summing with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$.

Now, we study the connection between a mapping belonging to $BL_{f,p}(X,Y;E)$ and its bi-linearization.

Theorem 4.3.21.

Let X, Y be pointed metric spaces and E be a Banach space. For $1 \leq p < \infty$, we have $T \in BL_{f,p}(X,Y;E)$ if and only if its bi-linearization T_B is p-semi-integral from Æ$(X)\times$Æ(Y) to E. In this case

$$\|T\|_{BL_{f,p}} = \|T_B\|_{si,p}. \tag{4.13}$$

Proof. Suppose that $T \in BL_{f,p}(X,Y;E)$. Let $(m_j^1)_{j=1}^s \subset \mathcal{M}(X)$ and $(m_j^2)_{j=1}^s \subset \mathcal{M}(Y)$, with $m_j^1 = \sum_{i=1}^n \alpha_i^j m_{x_i^j x_i'^j}$ and $m_j^2 = \sum_{k=1}^r \beta_k^j m_{y_k^j y_k'^j}$ then, we have

$$
\begin{aligned}
&\left(\sum_{j=1}^{s}\left\|T_B(m_j^1, m_j^2)\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{s}\left\|T_B\left(\sum_{i=1}^{n}\alpha_i^j m_{x_i^j x_i'^j}, \sum_{k=1}^{r}\beta_k^j m_{y_k^j y_k'^j}\right)\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n}\alpha_i^j \sum_{k=1}^{r}\beta_k^j T_B\left(m_{x_i^j x_i'^j}, m_{y_k^j y_k'^j}\right)\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n}\alpha_i^j \sum_{k=1}^{r}\beta_k^j \left(T(x_i^j, y_k^j) - T(x_i^j, y_k'^j) - T(x_i'^j, y_k^j) + T(x_i'^j, y_k'^j)\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \|T\|_{BL_{f,p}} \sup_{\substack{f\in B_{X^\#} \\ g\in B_{Y^\#}}} \left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\alpha_i^j \sum_{k=1}^{r}\beta_i^j \left(f(x_i^j) - f(x_i'^j)\right)\left(g(y_k^j) - g(y_k'^j)\right)\right|^p\right)^{\frac{1}{p}} = (*)
\end{aligned}
$$

Since $Æ(Z)^*$ and $Z^\#$ ($Z = X$ or Y) are isometrically isomorphic via the linearization, for all $h \in Z^\#$ there is $\varphi \in Æ(Z)^*$ such that

$$
\varphi(m_{zz'}) = h_L(m_{zz'}) = h(z) - h(z'),
$$

for all $z, z' \in Z$, we obtain

$$
\begin{aligned}
(*) &= \|T\|_{BL_{f,p}} \sup_{\substack{\|\varphi_1\|\leq 1 \\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\alpha_i^j \sum_{k=1}^{r}\beta_i^j \varphi_1(m_{x_i^j x_i'^j})\varphi_2(m_{y_k^j y_k'^j})\right|^p\right)^{\frac{1}{p}} \\
&= \|T\|_{BL_{f,p}} \sup_{\substack{\|\varphi_1\|\leq 1 \\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s}\left|\varphi_1(m_j^1)\varphi_2(m_j^2)\right|^p\right)^{\frac{1}{p}}
\end{aligned}
$$

Therefore, $T_B \in \mathcal{L}_{si,p}(Æ(X), Æ(Y); E)$ and $\|T_B\|_{si,p} \leq \|T\|_{BL_{f,p}}$.

Conversely, suppose that $T_B \in \mathcal{L}_{si,p}(Æ(X), Æ(Y); E)$. Let $x_i^j, x_i'^j \in X$, $y_i^j, y_i'^j \in Y$, $\lambda_i^j \in \mathbb{K}$, $(1 \leq i \leq n, 1 \leq j \leq s)$, we have

$$
\begin{aligned}
&\left(\sum_{j=1}^{s}\left\|\sum_{i=1}^{n}\lambda_i^j \left(T(x_i^j, y_i^j) - T(x_i^j, y_i'^j) - T(x_i'^j, y_i^j) + T(x_i'^j, y_i'^j)\right)\right\|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{s}\left\|T_B\left(\sum_{i=1}^{n}\lambda_i^j m_{x_i^j x_i'^j}, m_{y_i^j y_i'^j}\right)\right\|^p\right)^{\frac{1}{p}} \\
&\leq \|T_B\|_{si,p} \sup_{\substack{\|\varphi_1\|\leq 1 \\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\lambda_i^j \varphi_1(m_{x_i^j x_i'^j})\varphi_1(m_{y_i^j y_i'^j})\right|^p\right)^{\frac{1}{p}} \\
&= \|T_B\|_{si,p} \sup_{\substack{f\in B_{X^\#} \\ g\in B_{Y^\#}}} \left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\lambda_i^j \left(f(x_i^j) - f(x_i'^j)\right)\left(g\left(y_i^j\right) - g(y_i'^j)\right)\right|^p\right)^{\frac{1}{p}}.
\end{aligned}
$$

Which means that $T \in BL_{f,p}(X, Y; E)$ and $\|T\|_{BL_{f,p}} \leq \|T_B\|_{si,p}$. □

Remark 4.3.22.

If we consider $T \in Lip_0(X, E)$, we obtain a characterization of strictly Lipschitz p-summing operators that was introduced by Saadi in [49],*i.e.*,

$$\begin{aligned} & T \text{ is strictly } p\text{-summing} \\ \Longleftrightarrow\ & T_L \text{ is linear } p\text{-summing} \\ \Longleftrightarrow\ & T \text{ is factorable } p\text{-dominated,} \end{aligned}$$

where the first equivalence follows from [49, Theorem 3.5] and the second one from Theorem 4.3.21. So, our notion is really a generalization of strictly Lipschitz p-summing operators to the two-Lipschitz case.

Pellegrino in [39] proved that the class $(\mathcal{L}_{si,p}, \|\cdot\|_{si,p})$ is a Banach ideal of bilinear mappings (see also [18]). As a straightforward consequence of this result and the Theorem 4.3.21 we have the following corollary.

Corollary 4.3.23.

The class $(BL_{f,p}, \|\cdot\|_{BL_{f,p}})$ is a Banach ideal of two-Lipschitz operators.

Proof. The proof is based on the equality (4.13) and the following statements that we get directly from the uniqueness of the bi-linearization maps,

1) $(\alpha S + T)_B = \alpha S_B + T_B$, for all $S, T \in BL_{f,p}$ and $\alpha \in \mathbb{K}$.

2) $(f \cdot g \cdot e)_B = f_L \cdot g_L \cdot e$, for any $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$, where $f_L \cdot g_L \cdot e \in \mathcal{L}(Æ(X), Æ(Y); E)$ is defined by $f_L \cdot g_L \cdot e(m, m') = f_L(m) g_L(m') e$.

3) $(u \circ T \circ (f, g))_B = u \circ T_B \circ (\widehat{f}, \widehat{g})$ for all $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in BL_{f,p}(X, Y; E)$.

□

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Abstract:

The main goal of this thesis is to study a new concept of two-Lipschitz operator ideals between pointed metric spaces and Banach spaces.
A second purpose of the thesis is to introduce and study some new Lipschitz operator ideals represented by an integral with respect to a vector measure or positive measure.

Keywords: Lipschitz operators, bilinear operators, vector measures, tensor product, Arens-Eells space, linear operator ideals, multi-linear operator ideals, Lipschitz operator ideals.

Résumé:

L'objectif principal de cette thèse est l'étude d'un nouveau concept d'idéaux d'opérateurs deux- Lipschitziens entre espaces métriques pointés et espaces de Banach.
Un deuxième objectif de la thèse est d'introduire et d'étudier de nouveaux idéaux d'opérateurs Lipschitziens représentés par une intégrale par rapport à une mesure vectorielle ou une mesure positive.

Mots clés: Opérateurs Lipchitziens, opérateurs bilinéaires, mesures vectorielles, produit tensoriel, espace de Arens-Eells, idéaux d'opérateurs linéaires, idéaux d'opérateurs multilinéaires, idéaux d'opérateurs Lipchitziens.

ملخص:

الهدف الرئيسي من هذه الأطروحة هو دراسة مفهوم جديد لمثاليات مؤثرات ثنائي-ليبشتز بين الفضاءات المترية الخاصة (pointed) وفضاءات بناخ.
الهدف الثاني للأطروحة هو تقديم ودراسة امثلة جديدة للمؤثرات الليبشيتزية ممثلة بتكامل يتعلق بقياس شعاعي أو قياس موجب.
الكلمات المفتاحية:
المؤثرات الليبشيتزية, المؤثرات المتعددة الخطية, القياس ألشعاعي, فضاء Arens-Eells, مثاليات المؤثرات الخطية ,المتعددة الخطية والليبشيتزية.