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SOME PROPERTIES OF VARIABLE HERZ-TYPE BESOV SPACES AND APPLICATIONS

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DEDICATE

This work is dedicated

To the spirit of my father immaculate.

To my mother.

To my wife.

To my son Mohammed Mokhtar .

To all my family.

To my friends and colleagues in M'sila University and especially to those in Department of Mathematics.

Contents

List of abbreviations and symbols

• By $\mathbb{R}^n$ we denote the n-dimensional real Euclidean space.

• By $\mathbb{N}$ we denote the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

• By $\mathbb{Z}$ we denote the set of all integer numbers.

• For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$.

• The partial differential operator of order α is denoted as,

$$\partial^\alpha f = \frac{\partial^{\alpha_1} f}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n} f}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 .... \partial^{\alpha_n} x_n}$$

• The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by

$$x \cdot y = x_1 y_1 + ... + x_n y_n.$$

• For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

• The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g).

• $f \approx g$ means $f \lesssim g \lesssim f$.

• For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we write $|x| = \sqrt{x_1^2 + ... + x_n^2}$.

• $R(u) := \{x \in \mathbb{R}^n : u/2 \leq |x| < u\}$ and $R_k := R(2^k)$.

• $C(u) := \{x \in \mathbb{R}^n : u/4 \leq |x| < 4u\}$ and $C_k := C(2^k)$.

• The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y, where X and Y are quasi-normed spaces.

• As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.

• For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x

and radius r.

• The $\triangle$ is the Laplace operator.

• "i.e." stands simply for "in other words"

• "a.e." stands simply for "almost everywhere"

• We use c for various positive constant, i.e. a constant whose value may change from appearance to appearance.

• By supp$f$ we denote the support of the function f , i.e., the closure of its non-zero set.

• If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E .

χ_E denotes its characteristic function.

• By $\mathcal{D}(\mathbb{R}^n)$ we denote the space of functions with continuous derivatives of all orders and compact support.

• By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$. The topology in the complete locally convex space $\mathcal{S}(\mathbb{R}^n)$ is generated by

$$p_N(\varphi) := \sup_{x\in\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\leq N} |\partial^\alpha \varphi(x)|, \quad N = 1, 2, 3, ...$$

• By $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

• For $0 < p \leq \infty$. The classical *Lebesgue space* $L^p(\Omega)$ is the class of all measurable functions f on Ω normed by (quasi-normed for $p < 1$)

$$\|f \mid L^p(\Omega)\| = \Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p} < \infty, \quad 0 < p < \infty$$

and

$$\|f \mid L^\infty(\Omega)\| = \operatorname*{ess-sup}_{x\in\Omega} |f(x)| < \infty.$$

• Given $p(\cdot) : \Omega \subset \mathbb{R}^n \to]c, \infty[$, we define the conjugate exponent function $p'(\cdot)$ by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \ x \in \Omega$$

with the convention that $1/\infty = 0$. Since $p(\cdot)$ is a function, the notation $p'(\cdot)$ can be mistaken for the derivative of $p(\cdot)$, but we will never use the symbol "$\prime$"

in this sense.

• The notation p' will also be used to denote the conjugate of p a constant exponent.

• We define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

• By ℓ^q, $0 < q \leq \infty$, we denote the space of all (complex) sequences $\{a_k\}_{k\in\mathbb{Z}}$ equipped with the quasi-norm

$$\left\|\{a_k\}_{k\in\mathbb{Z}} \mid \ell^q\right\| = \Big(\sum_{k=-\infty}^{\infty} |a_k|^q\Big)^{1/q}$$

(with the usual modification if $q = \infty$).

• Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$.

• Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \ \forall x \in \mathbb{R}^n.$$

Introduction

The Herz spaces initially appeared in the paper of Herz [25] to study the absolute convergence of Fourier transforms. The theory of these spaces had a remarkable development in part due to its usefulness in applications to other fields of applied mathematics. For instance, they appear in the characterization of multipliers on Hardy spaces [3], in the summability of Fourier transforms [18] and in regularity theory for elliptic equations in divergence form [45] and [46]. We refer the monograph [42] for further details and references on recent developments on Herz spaces.

Function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors. The motivation to study such function spaces comes from applications to other fields of applied mathematics, because they are applicable to the modeling for electrorheological fluids and image processing and PDE with non-standard growth conditions for instance, see [5] and [47].

Herz spaces $K^{\alpha,q}_{p(\cdot)}$ and $\dot{K}^{\alpha,q}_{p(\cdot)}$ with variable exponent p but fixed $\alpha \in \mathbb{R}$ and $q \in (0, \infty]$ were recently studied by Izuki [26, 27]. These spaces with variable exponents $\alpha(\cdot)$ and $p(\cdot)$ were studied in [1], where they gave the boundedness results for a wide class of classical operators on these function spaces. The spaces $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi in [28]. In [16] the authors gave a new equivalent norms of these function spaces. See [48] where new variable Herz spaces are given.

The variable Besov spaces $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, initially appeared in the paper of A. Almeida and P. Hästö [2]. Several basic properties were established, such as the Fourier analytical characterisation. When p, q and α are constants they coincide with the usual function spaces $B^{\alpha}_{p,q}$, studied in detail by H. Triebel in [53], [54] and [55].

Based on Besov spaces $B^{\alpha}_{p,q}$, J. Xu and D. Yang [61], [62] and [58] introduce Herz-type

Besov spaces $\dot{K}_q^{\alpha,p}B_\beta^s$. C. Shi and J. Xu [50] and [49] studied Herz-type Besov spaces $\dot{K}_{q(\cdot)}^{\alpha,p}B_\beta^s$ with variable q, but fixed α, p, s and β, where the characterization of these function spaces by the so-called Peetre maximal functions are obtained. B. Dong and J. Xu [12] also considered $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}B_\beta^s$ with variables q and α. In [41], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some applications to partial differential equations. We refer the reader to the recent paper [14] for further results for these function spaces.

Another scale of generalized Herz spaces are Herz type Hardy spaces which obtained a great development in the past few years and played important roles in harmonic analysis, see [16], [23], [35], [40] and [43].

The authors in [57] defined the Herz-type Hardy spaces with variable exponent $H\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and are studied the boundedness of some sublinear operators on this spaces. Also, in [16], the authors are proved the atomic decomposition for variable Herz-type Hardy spaces and they proved the boundedness of a wide class of sublinear operators on these spaces. If p is constant exponent, see [23], [40] and [52].

To establish the boundedness of operators in Herz-type Hardy type spaces on $\mathbb{R}^n$, one usually appeals to the atomic decomposition characterization of these spaces, which means that a function or distribution in Herz-type Hardy type spaces can be represented as a linear combination of atoms, then, the boundedness of sublinear operators in Herz-type Hardy type spaces can be deduced from their behavior on atoms in principle. It plays a fundamental role in the harmonic analysis because it is a powerful tool for dealing with duality theorems, interpolation theorems and some fundamental inequalities in harmonic analysis, see for example [23].

In addition, in [21], Loukas Grafakos, Xinwei Li and Dachun Yang are gived that bilinear operators given by finite sums of products of Calderón-Zygmund operators on $\mathbb{R}^n$ are bounded from Herz type Hardy type spaces into it self.

Our thesis consists of five chapters. In the first chapter, we give some basic properties of variable Lebesgue spaces and the mixed variable Lebesgue's-sequence spaces, after this we define new variable Herz-type function spaces where all the three parameters are variables and we also provide the atomic decomposition of this spaces.

In the second chapter, some useful lemmas that are needed have been presented to proof the main statements then we will give the main results in means the generalization of a classical Plancherel-Polya-Nikolskij inequality on variable Herz spaces.

In the chapter 3, we introduce the variable Herz-type Besov spaces and we prove the basic embedding in these spaces. In particular, we generalize the results of D. Drihem in [14].

In the chapters 4 and 5, we recall notations of variable Herz-type Hardy spaces and their central atomic decomposition characterizations. Here, we present the boundedness of some bilinear operators on this spaces given by finite sums of products of boundedness operators on variable Lebesgue spaces by using the atomic decomposition. At the end of the chapter 5, we present the boundedness of the commutators of Calderón-Zygmund operators on variable Herz spaces into itself.

Chapter 1

Variable Herz spaces

The history of Herz spaces goes back to the authors Beurling and Herz in the sixty's of the last century. In 1964, Beurling [4] first introduced some fundamental form of Herz spaces to study convolution algebras which are now called the Beurling algebras. Later in 1968, Herz [25] generalized these spaces to study the absolute convergence of Fourier transforms. These generalized spaces of functions are just the prototype of Herz spaces. Since then, the theory of these spaces had a remarkable development in part due to its usefulness in applications to other fields of applied mathematics. An interesting account with many applications for the generalized Herz spaces in some particular cases is given in [3].

Herz-type spaces have attracted many authors' interests for last three decades. In fact, many results in classical Lebesgue spaces have been generalized to Herz-type spaces.

In this chapter, we study some basic properties of semi-modular (modular) spaces which we use to define different function spaces, including classical Lebesgue, Herz spaces, variable Lebesgue spaces and variable Herz spaces. We start our first section by recalling some basic concepts and facts of semi-modular. In the next section, we recall some necessary preliminaries on variable Lebesgue spaces. In the last section of this chapter, we give some basic properties of the mixed variable Lebesgue's-sequence spaces, after this we define new Herz type function spaces with variable exponent, where all the three parameters are variables and we also provide the atomic decomposition of Herz type function spaces with variable exponent.

1.1 Basic properties of semimodular spaces

In this section, we start by recalling some basic concept about semimodular (modular) function spaces. We are mainly interested in vector spaces defined over $\mathbb{R}$.

Definition 1.1 *Let X be a real vector space. The function ρ is said to be left-continuous if the mapping $\lambda \to \rho(\lambda x)$ is left-continuous on $[0, \infty)$ for every $x \in X$, i.e.*

$$\lim_{\lambda \to 1^-} \rho(\lambda x) = \rho(x).$$

Here $a \to b^-$ means that a tends to b from below, i.e. $a < b$ and $a \to b^-$; $a \to b^+$ is defined analogously.

Definition 1.2 *Let X be a real vector space. A function $\rho : X \to [0, +\infty]$ is called a semimodular on X if it satisfies the following conditions:*

1- $\rho(0) = 0$.

2- $\rho(\lambda x) = \rho(x)$ for all $x \in X$, and for all scalar λ with $|\lambda| = 1$.

3- ρ is quasi-convex.

4- $\rho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

5- ρ is left-continuous on $[0, \infty)$ for every $x \in X$.

A semimodular ρ is called a modular if

6- $\rho(x) = 0$ implies $x = 0$.

A semimodular ρ is called continuous if

7- the mapping $\lambda \to \rho(\lambda x)$ is continuous on $[0, \infty)$ for every $x \in X$.

We first consider a very simple examples of the semimodular (modular) given in [11, Example 2.1.4].

Example 1.3 *Let Ω is a Lebesgue measurable subset of $\mathbb{R}^n$.*

(i)- If $1 \le p < \infty$, then

$$\rho_p(f) = \int_\Omega |f(x)|^p \, dx$$

defines a continuous modular on the space of all measurable functions on Ω.

(ii)- Let $\phi_\infty(t) := \infty . \chi(1, \infty)(t)$ for $t \ge 0$, i.e. $\phi_\infty(t) = 0$ for $t \in [0, 1]$ and $\phi_\infty(t) = \infty$ for $t \in (1, \infty)$. Then

$$\rho_\infty(f) = \int_\Omega \phi_\infty(|f(x)|) dx$$

defines a semimodular on the space of all measurable functions on Ω, *which is not continuous.*
(iii)- If $1 \le p < \infty$, *then*

$$\rho_p((x_j)) = \sum_{j=0}^{\infty} |x_j|^p$$

defines a continuous modular on $\mathbb{R}^{\mathbb{N}}$.

Definition 1.4 *If* ρ *is a semimodular or modular on* X, *then*

$$X_\rho = \{x \in X,\ \exists \lambda > 0 : \rho(\lambda x) < \infty\}$$

is called a semimodular space or modular space, respectively.

In general, the modular is not subadditive and thus it does not behave as a norm or a distance. But we can associate the modular with an quasi-norm. The following theorem is given in [2, Theorem 2.3].

Theorem 1.5 (The Luxemburg norm) *Let* ρ *be a semimodular on* X. *Then* X_ρ *is a quasi-normed* $\mathbb{R}$*-vector space. The quasi-norm, called the Luxemburg quasi-norm, is defined by*

$$\|x\|_\rho := \inf \{\lambda > 0 : \rho(x/\lambda) \le 1\}.$$

The following example illustrates all these points, see [11, Example 2.1.8].

Example 1.6 (Classical Lebesgue spaces) *Let* $1 \le p < \infty$. *Then the modular spaces corresponding the modular in Example 1.3/(i) are coincides with the classical Lebesgue space* $L^p(\Omega)$, *i.e.*

$$\|f\|_{L^p(\Omega)} := \|f\|_{\rho_p} := \Big(\int_\Omega |f(x)|^p\, dx \Big)^{1/p}.$$

Similarly, then the semimodular spaces corresponding the semimodular in Example 1.3/(ii) are coincides with the classical Lebesgue space $L^\infty(\Omega)$, *i.e.*

$$\|f\|_{L^\infty(\Omega)} := \|f\|_{\rho_\infty} := \operatorname*{ess\,sup}_{x \in \Omega}(|f(x)|).$$

We now present the relationship between the modular and its associated norm. The following lemma is of great technical importance, named the norm-modular unit ball property. This lemma and many other basic results were proven in [11, Lemma 2.1.14].

Lemma 1.7 (Norm-modular unit ball property) *Let ρ be a semi-modular on X. Then*

$$\|x\|_\rho \leq 1 \Longleftrightarrow \rho(x) \leq 1.$$

If ρ is continuous, then also

$$\|x\|_\rho < 1 \Longleftrightarrow \rho(x) < 1,$$

and

$$\|x\|_\rho = 1 \Longleftrightarrow \rho(x) = 1.$$

1.2 Variable Lebesgue spaces

The variable Lebesgue spaces are a generalization of the classical Lebesgue spaces where we allow the exponent to be a measurable function and thus the exponent may vary. The theory of these spaces play an important role in functional analysis, theory of partial differential equations and fluid dynamics. In fact, many properties in classical Lebesgue spaces have been generalized to the variable Lebesgue spaces.

1.2.1 Basic properties of variable exponents

Given an open set $\Omega \subset \mathbb{R}^n$. We put

$$\mathcal{P}_0(\Omega) := \{p \text{ mesurable: } p(\cdot) : \Omega \to [c, \infty[\text{ for some } c > 0\}.$$

The elements of $\mathcal{P}_0(\Omega)$ are called exponent functions or simply exponents. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\cdot)$.

We will consider the following example of exponent functions, see [7, Definition 2.1].

Example 1.8 *Some examples of exponent functions on $\Omega = \mathbb{R}$ include $p(x) = p$ for some constant p, $1 \leq p < \infty$, or $p(x) = 2 + \sin x$.*

Notation 1.9 *We denote by*

$$\mathcal{P}(\Omega) := \{p \textit{ mesurable: } p(\cdot) : \Omega \subset \mathbb{R}^n \to [1, \infty[\}.$$

Given $p \in \mathcal{P}_0(\Omega)$ and a set $E \subseteq \Omega$, let

$$p^-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \quad p^+(E) = \operatorname*{ess\,sup}_{x\in E} p(x).$$

If the domain $E = \Omega = \mathbb{R}^n$ we will simply write

$$p^- = p^-(\mathbb{R}^n), \quad p^+ = p^+(\mathbb{R}^n).$$

Definition 1.10 *Given Ω and $p \in \mathcal{P}_0(\Omega)$. The variable Lebesgue space $L^{p(\cdot)}(\Omega)$ to be the set of all measurable functions f such that $\rho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$.*

$$L^{p(\cdot)}(\Omega) := \left\{ f \ \textit{measurable} : \exists\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) = \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} \le 1 \right\},$$

equipped with the following quasi-norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

If the set on the right-hand side is empty we define $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$. If $\Omega = \mathbb{R}^n$, we will often write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Definition 1.11 *Given Ω and $p \in \mathcal{P}_0(\Omega)$, define $L^{p(\cdot)}_{loc}(\Omega)$ by*

$$L^{p(\cdot)}_{loc}(\Omega) := \left\{ f \ \textit{measurable} : f \in L^{p(\cdot)}(K) \ \textit{for every compact set} \ K \subset \Omega \right\}.$$

The following theorem give the structure of variable Lebesgue spaces.

Theorem 1.12 *Given Ω and $p \in \mathcal{P}_0(\Omega)$. The function $\|f\|_{L^{p(\cdot)}(\Omega)}$ defines a quasi-norm on $L^{p(\cdot)}(\Omega)$ and $L^{p(\cdot)}(\Omega)$ is quasi Banach spaces.*

For more detailed theorems and proofs, see [8], [7], [9] and [11].
The following proposition presents one of the most useful properties in variable Lebesgue spaces, see [7, Proposition 2.18] and [11, Lemma 3.2.6].

Proposition 1.13 *Let $p \in \mathcal{P}(\mathbb{R}^n)$ with $p^+ < \infty$ and $s > 0$ be such that $1/p^- \le s < \infty$. Then*

$$\||f|^s\|_{p(\cdot)} = \|f\|^s_{sp(\cdot)}.$$

1.2.2 Logarithmic Hölder continuity

In this subsection, we introduce the most important condition on the exponent in the study of variable exponent spaces, this condition has emerged as the right one to guarantee regularity.

Definition 1.14 *We say that a function $g : \Omega \to \mathbb{R}$ is* locally log-Hölder continuous *on Ω, abbreviated $g \in C_{loc}^{\log}(\Omega)$, if there exists $c_{\log}(g) > 0$ such that*

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x-y|)} \tag{1.1}$$

for all $x, y \in \Omega$. If $0 \in \Omega$ and

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \Omega$, then we say that g is log-Hölder continuous at the origin *(or has a* log decay at the origin*). If, for some $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds*

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \Omega$, then we say that g is log-Hölder continuous at infinity *(or has a* log *decay at infinity), abbreviated $g \in C^{\log}(\Omega)$.*

Here is an example of a function $g : \mathbb{R} \to \mathbb{R}$ is locally log-Hölder continuous on $\mathbb{R}$, see E. Nakai and Y. Sawano [44, Example 1.3].

Example 1.15 *We consider*

$$g(x) = \max(1 - e^{3-|x|}, \min(6/5, \max(1/2, 3/2 - x^2)), x \in \mathbb{R}$$

then g is log*-Hölder continuous on $\mathbb{R}$.*

Remark 1.16 *We note that all functions g are* log*-Hölder at infinity always belong to L^∞.*

Notation 1.17 *The notation $\mathcal{P}^{\log}(\Omega)$ is used for all those exponents $p \in \mathcal{P}(\Omega)$ which are locally* log*-Hölder continuous and have a* log *decay at infinity. The class $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ is defined analogously. If Ω is unbounded, then we define p_∞ by*

$$p_\infty := \lim_{|x| \to \infty} p(x).$$

1.2.3 Fundamental inequalities in variable Lebesgue spaces

The following theorem is the generalization of the classical Hölder's inequality in variable Lebesgue spaces. The classical Hölder's inequality is that for all p, $1 \leq p \leq \infty$, given $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$

$$\int_\Omega |f(x)g(x)|\, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

This inequality is true for variable exponents with a constant on the right-hand side, see for example [8, Theorem 2.33] and [56, Lemma 1.1].

Theorem 1.18 *Given Ω and $p \in \mathcal{P}(\Omega)$. Then there exists a constant K such that for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq K \|f\|_{L^{p(\cdot)}(\Omega)} \|f\|_{L^{p'(\cdot)}(\Omega)}.$$

where

$$K = (1/p^- + 1 - 1/p^+).$$

The generalized Hölder's inequality in the classical Lebesgue holds in variable Lebesgue spaces. For the proof of the following corollary, see [7, Corollary 2.28].

Corollary 1.19 *Given Ω and exponent functions $p, q \in \mathcal{P}(\Omega)$, define $r \in \mathcal{P}(\Omega)$ by*

$$\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

Then there exists a constant K such that for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, $fg \in L^{r(\cdot)}(\Omega)$ and

$$\|fg\|_{L^{r(\cdot)}(\Omega)} \leq K \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{q(\cdot)}(\Omega)}.$$

We give the basic properties of convolution operators in the variable Lebesgue spaces.

Definition 1.20 *Given two locally integrable functions f and g defined on $\mathbb{R}^n$, their convolution is the function $f * g$ defined by*

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

wherever this integral is finite.

We first discuss the failure of Minkowski's inequality to hold on the variable Lebesgue spaces. It was shown in [7, Theorem 5.19] and [11, Corollary 3.6.4] that Minkowski's inequality $\|f * g\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \|g\|_1$ is never true for non-constant exponents.

Theorem 1.21 *Let $p \in \mathcal{P}(\mathbb{R}^n)$ the inequality*

$$\|f * g\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \|g\|_1$$

is true for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ if and only if p is constant.

A function, f defined on $\mathbb{R}^n$ is radial if its value only depends on the distance to the origin. In this case there exists a function, F, of a single variable so that

$$f(x) = F(|x|)$$

The following proposition is a very weak version of Minkowski's inequality, see [8, Proposition 4.7].

Proposition 1.22 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Then for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and every non-negative, radially decreasing function $g \in L^1(\mathbb{R}^n)$,*

$$\|f * g\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \|g\|_1 . \tag{1.2}$$

Young's inequality does not hold for general exponents. This example is from [7, Example 5.21].

Example 1.23 *Let $p \in \mathcal{P}(\mathbb{R})$ be a smooth function such that $p(x) = 2$ if $x \in \mathbb{R}\backslash [-2, 2]$, and $p(x) = 4$ on $[-1, 1]$. Define*

$$f(x) = |x - 3|^{-1/3} \chi_{[2,4]}, \quad g(x) = |x|^{-2/3} \chi_{[-1,1]}.$$

Since $f^2 \in L^1(\mathbb{R})$, then, $f \in L^{p(\cdot)}(\mathbb{R})$. Similarly, since $p'(x) = 4/3$ on $[-1, 1]$ and $g^{4/3} \in L^1(\mathbb{R})$, $g \in L^{p'(\cdot)}(\mathbb{R})$. However, we do not have that

$$\|f * g\|_\infty \leq \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}$$

since $f*g$ is unbounded in a neighborhood of 3. *To show this, let* $E_x = [2,4] \cap [x-1, x+1]$ *. Then by Fatou's lemma on the classical Lebesgue spaces,*

$$\begin{aligned}\liminf_{x\to 3} f*g(x) &= \liminf_{x\to 3} \int_{\mathbb{R}} |x-y|^{-2/3} |y-3|^{-1/3} \chi_{E_x}(y)\, dy \\ &\geq \int_{\mathbb{R}} \lim_{x\to 3} |x-y|^{-2/3} |y-3|^{-1/3} \chi_{E_x}(y)\, dy \\ &\geq \int_2^4 |y-3|^{-1}\, dy = \infty\end{aligned}$$

1.3 Definition and basic properties of variable Herz spaces

In this section, we give the definition of Herz spaces with variable exponent. Also, we present basic properties and useful lemmas.

1.3.1 The mixed variable Lebesgue-sequence space

In this sebsection, we present a functional spaces create by Alexander Almeida and Peter Hästö, which allows us to define Herz spaces, Besov spaces and Herz-type Besov spaces with variable exponent. This new spaces as a generalization of the iterated function space $\ell^q(L^{p(\cdot)})$ for the case of variable q.

Definition 1.24 *Let* $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. *The mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)})$ *is defined on sequences of* $L^{p(\cdot)}$*-functions by the modular*

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left((f_v)_v\right) := \sum_v \inf\left\{\lambda_v > 0 : \rho_{p(\cdot)}\left(f_v/\lambda_v^{\frac{1}{q(\cdot)}}\right) \leq 1\right\}.$$

Here we use the convention $\lambda^{\frac{1}{\infty}} = 1$. *The quasi-norm is defined from this as usual:*

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \sum_v \inf\left\{\mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \leq 1\right\}. \tag{1.3}$$

To motivate this definition, we mention that

$$\|(f_v)_v\|_{\ell^q(L^{p(\cdot)})} = \left\| \|(f_v)_v\|_{p(\cdot)} \right\|_{\ell^q}$$

if $q \in (0, \infty]$ is constant.

Remark 1.25 *If $q^+ < \infty$, then*

$$\inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(f/\lambda^{\frac{1}{q(\cdot)}}\right) \leq 1\right\} = \left\| |f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Since the right-hand side expression is much simpler, we use this notation to stand for the left-hand side even when $q^+ = \infty$. For instance, we often use the notation

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left((f_v)_v\right) := \sum_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}$$

for the modular.

The quasi-norm in $\ell^{q(\cdot)}(L^{p(\cdot)})$ is usually quite complicated to calculate. Here are some examples where it is possible to simplify its expression. This example is from A. Almeida and P. Hästö [2, Example 3.2].

Example 1.26 *Suppose that $p \equiv \infty$. Then*

$$\rho_{\ell^{q(\cdot)}(L^{\infty})}\left((f_v)_v\right) := \sum_v \inf\left\{\lambda_v > 0 : \rho_\infty\left(f_v/\lambda_v^{\frac{1}{q(\cdot)}}\right) \leq 1\right\}.$$

Now $\rho_\infty(g) \leq 1$ if and only if $|g| \leq 1$ almost everywhere. Thus $f_v/\lambda_v^{\frac{1}{q(\cdot)}} \leq 1$ a.e., hence $\lambda_v \geq \operatorname{ess\,sup}_x |f_v(x)|^{q(x)}$. It follows that

$$\rho_{\ell^{q(\cdot)}(L^{\infty})}\left((f_v)_v\right) = \sum_v \operatorname{ess\,sup}_x |f_v(x)|^{q(x)}.$$

The following theorem presents the $\ell^{q(\cdot)}(L^{p(\cdot)})$ quasi-norm, see H. Kempka and J. Vybíral [31, Theorem 1].

Theorem 1.27 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasi-norm on the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$.*
Let $p, q \in \mathcal{P}(\mathbb{R}^n)$. If $p(x) \geq 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $q \geq 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ a.e. on $\mathbb{R}^n$, or if $1 \leq q(x) \leq p(x) < \infty$ a.e. on $\mathbb{R}^n$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm on the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$.

The following lemma is a Hardy-type inequality, see [16, Lemma 2].

Lemma 1.28 *Let* $0 < a < 1$ *and* $0 < q \leq \infty$. *Let* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ *and* $\left\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ *belong to* ℓ^q, *and*

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} \leq c\, I,$$

with $c > 0$ *only depending on* a *and* q.

1.3.2 Definition of variable Herz spaces

For convenience, we set

$$B_k := B(0, 2^k)\,, \quad R_k := B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}\,, \quad k \in \mathbb{Z}.$$

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in harmonic analysis. In classical L^p spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathcal{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \approx |B|. \tag{1.4}$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B \tag{1.5}$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \leq 2^n$), and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}} \tag{1.6}$$

for large balls ($|B| \geq 1$), with constants only depending on the log-Hölder constant of p (see, for example, [11, Section 4.5]).

For characteristic functions defined on (dyadic) annuli we have similar norm estimates, without requiring the log-Hölder continuity at every point.

The following lemma plays an important role in the proof of the main results of this paper, where is a generalization of (1.4), (1.5) and (1.6) to the case of dyadic annuli, see A. Almeida and D. Drihem [1, Lemma 2.2].

Lemma 1.29 *Let $p \in \mathcal{P}(\mathbb{R}^n)$ be log-Hölder continuous at infinity, and $R = B(0,r) \backslash B(0,\frac{r}{2})$. If $|R| > 2^{-n}$, then*

$$\| \chi_R \|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x \in R$.
The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, p is log-Hölder continuous, both at the origin and at infinity.

Definition 1.30 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$. The* homogeneous Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in L_{loc}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that*

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} := \left\| \left(2^{k\alpha(\cdot)} f \chi_k\right)_{k\in\mathbb{Z}} \right\|_{\ell^{p(\cdot)}(L^{q(\cdot)})} < \infty. \tag{1.7}$$

Obviously, If α and p, q are constant, then $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n) = \dot{K}_{q}^{\alpha,p}(\mathbb{R}^n)$ are the classical Herz spaces and if $\alpha(\cdot) = 0, p(\cdot) = q(\cdot)$ then $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$ coincide with $L^{q(\cdot)}(\mathbb{R}^n)$. A detailed discussion of the properties of these spaces my be found in [23], [24], [39], [37], [42], and references therein.
By a simple consequence of the embedding $\ell^{\theta(\cdot)}(L^{q(\cdot)}) \hookrightarrow \ell^{p(\cdot)}(L^{q(\cdot)})$ we have the following proposition.

Proposition 1.31 *Let $p, q, \theta \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ be log-Hölder continuous, both at the origin and at infinity. If $(p - \theta)^- \geq 0$, then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),\theta(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}. \tag{1.8}$$

The next proposition gives some basic embeddings between Herz spaces. See [1, Proposition 3.5].

Proposition 1.32 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p_0, p_1 \in \mathcal{P}(\mathbb{R}^n)$ and $q \in (0, \infty]$. If $p_0(\cdot) \leq p_1(\cdot)$ and $1/p_0 - 1/p_1$ be log-Hölder continuous, both at the origin and at infinity, then*

$$\dot{K}_{p_1(\cdot)}^{\alpha(\cdot)+n/p_0(\cdot)-n/p_1(\cdot),q} \hookrightarrow \dot{K}_{p_0(\cdot)}^{\alpha(\cdot),q}.$$

The next proposition is very important for the proof of the main results, see D. Drihem and F. Seghiri in [16, Proposition 1].

Proposition 1.33 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are a log-Hölder continuous, both at the origin and at infinity, then*

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}} \approx \Big(\sum_{k=-\infty}^{-1} \|2^{k\alpha(0)} f\, \chi_k\|^{q(0)}_{p(\cdot)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} \|2^{k\alpha_\infty} f\, \chi_k\|^{q_\infty}_{p(\cdot)}\Big)^{1/q_\infty}.$$

Next, we give the following lemma, which is a generalization of [21, Lemma (Hölder's inequality)].

Lemma 1.34 *Let $\alpha_i \in L^\infty(\mathbb{R}^n)$ and $p_i, q_i \in \mathcal{P}(\mathbb{R}^n)$, $i = 1, 2$, $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $1/q(\cdot) = 1/q_1(\cdot) + 1/q_2(\cdot)$. If α_i and q_i are* log-*Hölder continuous, both at the origin and at infininity, Then there exists a constant C such that for all $f \in \dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in \dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$, $fg \in \dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ and*

$$\|fg\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \le C\, \|f\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \|g\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

If the exposants are constants, this result is from [21, Lemma (Hölder's inequality)].

Proof. The proof follows immediately by applying Proposition 1.33, and Hölder's inequality in $\ell^{q(0)}(\mathbb{R}^n)$, $\ell^{q_\infty}(\mathbb{R}^n)$ and in $L^{p(\cdot)}(\mathbb{R}^n)$. ■

1.3.3 Atomic decomposition of variable Herz spaces

Atomic decomposition plays a fundamental role in the harmonic analysis, it is a powerful tool for dealing with duality theorems, interpolation theorems and some fundamental inequalities in harmonic analysis.

In recent years, it turned out that atomic decomposition of some function spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces. But this applies equally to questions of mapping properties of some operators, such as Calderón-Zygmund operators, the commutator of Calderón-Zygmund operator with a BMO function and to trace problems, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. The main goal of this subsection is to prove an atomic decomposition result for variable exponent Herz spaces. First we introduce the basic notation of atomic decomposition.

Definition 1.35 *Let $\alpha \in L^{\infty}(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$ and $s \in \mathbb{N}_0$. A function a is said to be a central $(\alpha(\cdot), p(\cdot))$-atom, if*

(i) $\operatorname{supp} a \subset \overline{B}(0,r) = \{x \in \mathbb{R}^n : |x| \leq r\}, r > 0.$

(ii) $\|a\|_{p(\cdot)} \leq |\overline{B}(0,r)|^{-\alpha(0)/n}, \qquad 0 < r < 1.$

(iii) $\|a\|_{p(\cdot)} \leq |\overline{B}(0,r)|^{-\alpha_\infty/n}, \qquad r \geq 1.$

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad |\beta| \leq s.$

A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies the conditions (iii), (vi) above and $\operatorname{supp} a \subset \overline{B}(0,r), r \geq 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 1.35, then the corresponding central $(\alpha(\cdot), p(\cdot))$-atom is called a dyadic central $(\alpha(\cdot), p(\cdot))$-atom.

Now, we establish characterizations of the spaces $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ in terms of central atomic decompositions, which make it convenient to study the boundedness of operators on these spaces. One of the main results of this thesis is given in the following theorem. It generalizes Theorem 2.3 of H. B. Wang [56, Theorem 2.3] by taking q is a constant. If α, p and q are constants, this result is [38, Theorem 2.1].

Theorem 1.36 *Let $\alpha \in L^{\infty}(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are be* log*-Hölder continuous, both at the origin and at infinity with $p^+, q^+ < \infty$, the following two statements are equivalentes*

(i)- $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$

(ii)- f can be represented by

$$f(x) = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x), \tag{1.9}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in B_k and

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty}\right)^{\frac{1}{q_\infty}} \leq c\,\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Moreover, the norms $\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}$ and $\inf\left(\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty}\right)^{\frac{1}{q_\infty}}\right)$ are equivalent, where the infimum is taken all over all decompositions of f as in (1.9).

Proof. We first prove (i) implies (ii). For every $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, write

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{+\infty} f(x)\chi_k(x) \\ &= \sum_{k=-\infty}^{+\infty} \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)} \frac{f(x)\chi_k(x)}{\left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}} = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x), \end{aligned}$$

where $\lambda_k = \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}$ and $a_k(x) = \frac{f(x)\chi_k(x)}{\left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}}$. It is obvious that supp$a_k \subset B_k$ and

$$\|a_k\|_{p(\cdot)} \approx \begin{cases} 2^{-k\alpha(0)}, & \text{if } k \le -1 \\ 2^{-k\alpha_\infty}, & \text{if } k \ge 0 \end{cases}.$$

Thus, each a_k is a central $(\alpha(\cdot), p(\cdot))$- atoms with the support B_k and

$$\begin{aligned} &\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ = &\left(\sum_{k=-\infty}^{-1} \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{+\infty} \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ \approx &\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \|f\chi_k\|_{p(\cdot)}^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{+\infty} 2^{k\alpha_\infty q_\infty} \|f\chi_k\|_{p(\cdot)}^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ \approx &\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}. \end{aligned}$$

Now we prove (ii) implies (i). Let $f(x) = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x)$ be a decomposition of f which satisfies the hypothesis (ii) of Theorem 1.36. For each $j \in \mathbb{Z}$, by the Minkowski inequality

$$\left\|f\chi_j\right\|_{p(\cdot)} \le \sum_{k=j}^{\infty} |\lambda_k| \, \|a_k\|_{p(\cdot)}. \tag{1.10}$$

By Proposition 1.33 and from (1.10), it follows that $\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$\begin{aligned} &\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k}^{+\infty} |\lambda_j| \, \|a_j\|_{p(\cdot)}\right)^{q(0)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{+\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=k}^{+\infty} |\lambda_j| \, \|a_j\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ = \; & I_1 + I_2. \end{aligned}$$

For I_1, we divide the sum $\sum_{j=k}^{+\infty} \cdots$ into two parts,

$$\sum_{j=k}^{-1} \cdots + \sum_{j=0}^{+\infty} \cdots.$$

I_1 is bounded by $I_1^a + I_1^b$, where

$$I_1^a = \Big(\sum_{k=-\infty}^{-1} \Big(2^{k\alpha(0)} \sum_{j=k}^{-1} |\lambda_j| \, \|a_j\|_{p(\cdot)} \Big)^{q(0)} \Big)^{\frac{1}{q(0)}}$$

and

$$I_1^b = \Big(\sum_{k=-\infty}^{-1} \Big(2^{k\alpha(0)} \sum_{j=0}^{+\infty} |\lambda_j| \, \|a_j\|_{p(\cdot)} \Big)^{q(0)} \Big)^{\frac{1}{q(0)}}.$$

Since $0 < \alpha(0) < \infty$, then by Lemma 1.28 (with $0 < a = 2^{-\alpha(0)} < 1$), we have

$$I_1^a \leq c \Big(\sum_{k=-\infty}^{-1} \Big(\sum_{j=k}^{-1} |\lambda_j| 2^{-(j-k)\alpha(0)} \Big)^{q(0)} \Big)^{\frac{1}{q(0)}} \leq c \Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \Big)^{\frac{1}{q(0)}}.$$

Since $0 < \alpha_\infty < \infty$, then by the embedding $\ell^{q_\infty} \hookrightarrow \ell^\infty$

$$\begin{aligned} I_1^b &\leq c \Big(\sum_{k=-\infty}^{-1} \Big(2^{k\alpha(0)} \sum_{j=0}^{+\infty} |\lambda_j| 2^{-j\alpha_\infty} \Big)^{q(0)} \Big)^{\frac{1}{q(0)}} \\ &\leq c \sup_{j \geq 0} |\lambda_j| \Big(\sum_{k=-\infty}^{-1} \Big(2^{k\alpha(0)} \sum_{j=0}^{+\infty} 2^{-j\alpha_\infty} \Big)^{q(0)} \Big)^{\frac{1}{q(0)}} \leq c \Big(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty} \Big)^{\frac{1}{q_\infty}}. \end{aligned}$$

Thus, we have the desired estimate for I_1.

For I_2, we have

$$I_2 = \Big(\sum_{k=0}^{+\infty} \Big(2^{k\alpha_\infty} \sum_{j=k}^{+\infty} |\lambda_j| \, \|a_j\|_{p(\cdot)} \Big)^{q_\infty} \Big)^{\frac{1}{q_\infty}} \leq \Big(\sum_{k=0}^{+\infty} \Big(\sum_{j=k}^{+\infty} |\lambda_j| 2^{-(j-k)\alpha_\infty} \Big)^{q_\infty} \Big)^{\frac{1}{q_\infty}}.$$

Since $0 < \alpha_\infty < \infty$, then by Lemma 1.28 (with $0 < a = 2^{-\alpha_\infty} < 1$), we have

$$I_2 \leq \Big(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty} \Big)^{\frac{1}{q_\infty}} \leq \Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty} \Big)^{\frac{1}{q_\infty}}.$$

This finishes the proof. ■

Chapter 2

Some inequalities in variable Herz spaces

The classical Plancherel-Polya-Nikolskij inequality (cf. [53, 1.3.2/5, Rem. 1.4.1/4]), says that $\|f\|_q$ can be estimated by

$$c\ R^{n(1/p-1/q)}\ \|f\|_p$$

for any $0 < p \leq q \leq \infty$, $R > 0$ and any $f \in L^p(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ with supp $\mathcal{F}f \subset \overline{B}(0, R)$. The constant $c > 0$ is independent of R. This inequality plays an important role in theory of function spaces and PDE's. Our aim is to extend this result to the variable Herz spaces.

In the preliminary section of this chapter, we give some key technical lemmas needed in the proofs of the main results. In the second section, we generalize the classical Plancherel-Polya-Nikolskij inequality on $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}$-spaces instead of $L^{p(\cdot)}$ spaces and we give their proofs.

2.1 Preliminary lemmas

In this section, we present some results which are useful for the proof of our results. First we use instead so-called η-functions, which have appropriate scaling. The function which we call η is defined on $\mathbb{R}^n$ by

$$\eta_{v,m}(x) := \frac{2^{nv}}{(1 + 2^v\,|x|)^m}$$

with $v \in \mathbb{N}$ and $m > 0$. Note that $\eta_{v,m} \in L^1(\mathbb{R}^n)$ when $m > n$ and that $\|\eta_{v,m}\| = c_m$ is independent of v.

The following lemma is from [9, Lemma 6.1], see also [30, Lemma 19].

Lemma 2.1 *Let $\alpha \in C^{\log}_{\text{loc}}$ and let $M \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from (1.1) for α. Then*

$$2^{v\alpha(x)}\eta_{v,m+M}(x-y) \leq c\, 2^{v\alpha(y)}\eta_{v,m}(x-y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v, m \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{v\alpha(x)}\eta_{v,m+M} * f(x) \leq c\, \eta_{v,m} * (2^{v\alpha(\cdot)} f)(x).$$

The next lemma tells us that in most circumstances two convolutions are as good as one, is from [9, Lemma A.3].

Lemma 2.2 *For j_0 and $j_1 \geq 0$ and $m > n$, we have*

$$\eta_{j_0,m} * \eta_{j_1,m} \approx \eta_{\min(j_0,j_1),m}$$

with the constant depending only on m and n.

The next lemma often allows us to deal with exponents which are smaller than 1, see [9, Lemma A.6].

Lemma 2.3 ("The r-trick") *Let $r > 0$, $j \in \mathbb{N}_0$ and $m > n$. Then there exists $c = c(r,m,n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F}g \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{j+1}\}$, we have*

$$|g(x)| \leq c(\eta_{j,m} * |g|^r(x))^{1/r}, \ x \in \mathbb{R}^n.$$

Definition 2.4 *Let $p, q, \beta \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\beta^+ < \infty$. The space $\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})$ is defined on sequences of $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)$-functions by the modular*

$$\varrho_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})}((f_j)_j) := \sum_{j=0}^{\infty} \left\| |f_j|^{\beta(\cdot)} \right\|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}}.$$

The quasi-norm is defined from this as usual:

$$\left\| (f_j)_j \right\|_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})} := \inf \left\{ \mu > 0 : \varrho_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})} \left(\frac{1}{\mu}(f_j)_j \right) \leq 1 \right\}.$$

Let T be a sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad x \notin \operatorname{supp} f \tag{2.1}$$

for integrable and compactly supported functions f. Condition (2.1) is satisfied by several classical operators in Harmonic Analysis, such as Calderón-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator (see [32] and [51]). Various important results have been proved in the space $\dot{K}_q^{\alpha,p}$ under some assumptions on α, p and q. The conditions $-\frac{n}{q} < \alpha < n(1-\frac{1}{q}), 1 < q < \infty$ and $0 < p \leq \infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}_q^{\alpha,p}$ spaces. This fact was first realized by Li and Yang [32] with the proof of the boundedness of the maximal function. The proof of the main result of this chapter is based on the following result, see [16].

Theorem 2.5 *Let $p \in \mathcal{P}_0(\mathbb{R}^n)$, $q \in \mathcal{P}(\mathbb{R}^n)$ with $1 < q^- \leq q^+ < \infty$, and let α, p and q are* log*-Hölder continuous, both at the origin and at infinity such that*

$$-\frac{n}{q^+} < \alpha^- \leq \alpha^+ < n\Big(1 - \frac{1}{q^-}\Big).$$

Then every sublinear operator T satisfying (2.1) which is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$ is also bounded on $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$.

2.2 Plancherel-Polya-Nikolskij inequality on variable Herz spaces

In this section, we give the main results are that the generalization of a classical Plancherel-Polya-Nikolskij inequality on variable Herz spaces.

One of our main results of this section is the following lemma.

Lemma 2.6 *Let $q, \beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$. Let $p \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ are log-Hölder continuous, both at the origin and at infinity such that $\alpha^- + \frac{n}{q^+} > 0$. Let $M \geq c_{\log}(1/\beta)$, where $c_{\log}(1/\beta)$ is the constant from (1.1) for $1/\beta$. For $m > n$, there exists $c > 0$ such that*

$$\Big\| \Big\{ \big(\eta_{j,m+M} * |f_j|^r\big)^{\frac{1}{r}} \Big\}_j \Big\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} \leq c \big\| \{f_j\}_j \big\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} \tag{2.2}$$

for any $0 < r < \min(q^-, \frac{n}{\alpha^+ + n/q^-})$ whenever the quasi-norm on the right hand side is finite.

Proof. Let $\{f_j\}_j \in \ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})$ *such that* $\| \{f_j\}_j \|_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})} = 1$. By the scaling argument, it suffice to show that

$$\left\| \left\{ \left(\eta_{j,m+M} * |f_j|^r \right)^{\frac{1}{r}} \right\}_j \right\|_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})} \leq c$$

which is equivatent to

$$\varrho_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})} \left(c \left\{ \left(\eta_{j,m+M} * |f_j|^r \right)^{\frac{1}{r}} \right\}_j \right) \leq 1.$$

In particular, we will show that

$$\sum_{j=0}^{\infty} \left\| \left| c \left(\eta_{j,m+M} * |f_j|^r \right)^{\frac{\beta(\cdot)}{r}} \right\| \right\|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} \leq 1 \tag{2.3}$$

for some constant $c > 0$. Our estimate (2.3), clearly follows from the inequality

$$\left\| \left| c \left(\eta_{j,m+M} * |f_j|^r \right)^{\frac{\beta(\cdot)}{r}} \right\| \right\|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} \leq \| \, |f_j|^{\beta(\cdot)} \, \|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} + 2^{-j} =: \delta. \tag{2.4}$$

This claim can be reformulated as showing that

$$\left\| c\delta^{-1} \left(\eta_{j,m+M} * |f_j|^r \right)^{\frac{\beta(\cdot)}{r}} \right\|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} \leq 1.$$

Since β is log-Hölder continuous and $\delta \in [2^{-j}, 1 + 2^{-j}]$, we can move $\delta^{-\frac{1}{\beta(\cdot)}}$ inside the convolution by Lemma 2.1:

$$\left\| c\delta^{-1} \left(\eta_{j,m+M} * |f_j|^r \right)^{\frac{\beta(\cdot)}{r}} \right\|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} \lesssim \left\| \left(\eta_{j,m} * \delta^{-\frac{r}{\beta(\cdot)}} |f_j|^r \right)^{\frac{\beta(\cdot)}{r}} \right\|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}}.$$

Let us prove that

$$\left\| \left(\eta_{j,m} * \delta^{-\frac{r}{\beta(\cdot)}} |f_j|^r \right)^{\frac{\beta(\cdot)}{r}} \right\|_{\dot{K}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} \lesssim 1,$$

which is equivalent to

$$\sum_{k=-\infty}^{\infty} \left\| \left(2^{k\alpha(\cdot)r} \eta_{j,m} * \delta^{-\frac{r}{\beta(\cdot)}} |f_j|^r \right)^{\frac{p(\cdot)}{r}} \chi_k \right\|_{q(\cdot)/p(\cdot)} \lesssim 1.$$

Again this is equivalent to $\left\| \left(\eta_{j,m} * \delta^{-\frac{r}{\beta(\cdot)}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}} \lesssim 1$. Observe that

$$\left\| \left(\eta_{j,m} * \delta^{-\frac{r}{\beta(\cdot)}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}} = \left\| \eta_{j,m} * \delta^{-\frac{r}{\beta(\cdot)}} |f_j|^r \right\|^{1/r}_{\dot{K}^{\alpha(\cdot)r,p(\cdot)/r}_{q(\cdot)/r}},$$

thanks to Theorem 2.5, under the assumption $0 < r < \min(q^-, \frac{n}{\alpha^+ + n/q^-})$, the right-hand side is bounded by

$$c\left\|\delta^{-\frac{1}{\beta(\cdot)}} f_j\right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \lesssim 1,$$

which follows immediately from the definition of δ. This finishes the proof. ■

Remark 2.7 *Let $q \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$. Let $p \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ be log-Hölder continuous, both at the origin and at infinity such that $\alpha^- + \frac{n}{q^+} > 0$. Observe that $\eta_{j,m} * |f_j|^r$, $j \in \mathbb{N}_0$, satisfying the size condition (2.1). Using Theorem 2.5, one can find a constant $c > 0$ such that*

$$\sup_{j \geq 0} \left\| \left(\eta_{j,m} * |f_j|^r\right)^{\frac{1}{r}} \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \leq c \sup_{j \geq 0} \|f_j\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}$$

for any $m > n$ and any $0 < r < \min(q^-, \frac{n}{\alpha^+ + n/q^-})$ whenever the quasi-norm on the right hand side is finite.

Based on Lemma r-trick, we have the following result.

Lemma 2.8 *Let $\sigma, \beta \in C_{\text{loc}}^{\log}$, $p \in \mathcal{P}_0(\mathbb{R}^n)$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $R \geq \max(1, H)$. Let q be log-Hölder continuous, both at the origin and at infinity. Then there exists a constant $c > 0$ independent of R and H such that*

$$\sup_{x \in B(0,1/H)} \lambda^{-\frac{1}{\beta(x)}} R^{\sigma(x)} |f(x)| \leq c \left(\frac{R}{H}\right)^{n/d} \left\|\lambda^{-\frac{1}{\beta(\cdot)}} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f\right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \tag{2.5}$$

for all $f \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} \cap \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F}f \subset \overline{B}(0, R)$, any $0 < d < \min(q^-, n/(\alpha + n/q)^+)$ and any $\lambda \in [R^{-1}, 1 + R^{-1}]$.

If α, p, σ, β and q are constants, this result is from [14, Lemma 3.3].

Proof. By Lemma 2.3 we have for $d, R > 0, N > n$ and any $x \in B(0, 1/H)$

$$\begin{aligned} |f(x)|^\varrho &\leq c \left(\int_{\mathbb{R}^n} |f(y)|^d \eta_{R,N+M}(x-y) dy \right)^{\varrho/d} \\ &\leq c \left(\int_{B(0,2^2/H)} (\cdots) dy \right)^{\varrho/d} + c \left(\int_{\mathbb{R}^n \setminus B(0,2^2/H)} (\cdots) dy \right)^{\varrho/d}, \end{aligned}$$

where $\varrho := \min(1, d)$ and $M \geq c_{\log}(\sigma) + c_{\log}(1/\beta)$ with $c_{\log}(\sigma)$, $c_{\log}(1/\beta)$ are the constants from (1.1) for σ and $\frac{1}{\beta}$, respectively. Here

$$\eta_{R,N+M}(\cdot) := R^n (1 + R|\cdot|)^{-N-M}.$$

Using the following decomposition

$$\int_{B(0,2^2/H)} (\cdots)dy = \sum_{j=0}^{\infty} \int_{R(2^{2-j}/H)} (\cdots)dy,$$

$$\int_{\mathbb{R}^n \setminus B(0,2^2/H)} (\cdots)dy = \sum_{j=0}^{\infty} \int_{R(2^{j+3}/H)} (\cdots)dy$$

and the well-known inequality

$$\Big(\sum_{j=0}^{\infty} |a_j|\Big)^{\tau} \le \sum_{j=0}^{\infty} |a_j|^{\tau}, \quad \{a_j\}_j \subset \mathbb{C}, \ \tau \in [0,1] \tag{2.6}$$

we obtain that $|f(x)|^{\varrho}$ can be estimated by

$$c \sum_{j=0}^{\infty} \left(V^1_{j,R,H}(x) + V^2_{j,R,H}(x)\right), \tag{2.7}$$

where

$$V^1_{j,R,H}(x) := \left(\eta_{R,N+M} * |f\chi_{R(2^{2-j}/H)}|^d(x)\right)^{\varrho/d}, \quad V^2_{j,R,H}(x) := V^1_{-j-1,R,H}(x). \tag{2.8}$$

Here N is chosen large enough such that $N > \max\left(n, n/d - (n/q+\alpha)^-\right)$. Let us give the estimation of the first term in (2.7). Lemma 2.1, a simple change of variables, the Hölder inequality (with $\frac{1}{d} = \frac{1}{q(\cdot)} + \frac{1}{d} - \frac{1}{q(\cdot)}$) and Lemma 1.29, yield for any $d, R > 0$ and any $x \in B(0, 1/H)$

$$\begin{aligned}\sum_{j=0}^{\infty} R^{\sigma(x)\varrho} \lambda^{-\frac{\varrho}{\beta(x)}} V^1_{j,R,H}(x) &\lesssim R^{n\varrho/d} \sum_{k=-\infty}^{2-[\log_2 H]} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} R^{\sigma(\cdot)} f\chi_{\widetilde{R}_k} \left\|\chi_{\widetilde{R}_k}\right\|_{t(\cdot)} \right\|^{\varrho}_{q(\cdot)} \\ &\lesssim R^{n\varrho/d} \sum_{k=-\infty}^{2-[\log_2 H]} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} R^{\sigma(\cdot)} 2^{kn(1/d-1/q(\cdot))} f\chi_{\widetilde{R}_k} \right\|^{\varrho}_{q(\cdot)},\end{aligned}$$

where $\frac{1}{t(\cdot)} := \frac{1}{d} - \frac{1}{q(\cdot)}$, $\widetilde{R}_k := \{x \in \mathbb{R}^n : 2^{k-2} \le |x| < 2^k\}$ and $[a]$ is the integer part of the real number a. This term is bounded by

$$c\Big(\frac{R}{H}\Big)^{n\varrho/d} \sum_{k=-\infty}^{2-[\log_2 H]} (2^k H)^{\varrho\left(n/d-(n/q+\alpha)^+\right)} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} 2^{k\alpha(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f\chi_{\widetilde{R}_k} \right\|^{\varrho}_{q(\cdot)}.$$

Using the fact that $n/d > (n/q+\alpha)^+$, $2^{k-3}H < 1$ and the embedding $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} \hookrightarrow \dot{K}^{\alpha(\cdot),\infty}_{q(\cdot)}$ we obtain that the right-hand side of the last expression is bounded by

$$\begin{aligned} & c\Big(\frac{R}{H}\Big)^{n\varrho/d} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f \right\|^{\varrho}_{\dot{K}^{\alpha(\cdot),\infty}_{q(\cdot)}} \sum_{k=-\infty}^{2-[\log_2 H]} (2^k H)^{\varrho\left(n/d-(n/q+\alpha)^+\right)} \\ \lesssim & \Big(\frac{R}{H}\Big)^{n\varrho/d} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f \right\|^{\varrho}_{\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}}, \end{aligned} \tag{2.9}$$

where the implicit positive constant not depending on R.

Now we estimate the second term in (2.7). Notice that for any $y \in R(2^{3+j}/H)$ and any $x \in B(0,1/H)$, we have $|x-y| > 2^j/H$, so for any $d>0$, $N \in \mathbb{N}$ and $j \in \mathbb{N}_0$,

$$\eta_{R,N}(x-y) \leq R^n \left(\frac{2^j R}{H}\right)^{-N} \leq 2^{-jN} R^n.$$

Hence by Lemma 2.1, a simple change of variables, the Hölder inequality (with $\frac{1}{d} = \frac{1}{q(\cdot)} + \frac{1}{d} - \frac{1}{q(\cdot)}$) and Lemma 1.29, we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} R^{\sigma(x)\varrho} \lambda^{-\frac{\varrho}{\beta(x)}} V_{j,R,H}^2(x) \\
\lesssim\ & R^{n\varrho/d} \sum_{k=2-[\log_2 H]}^{\infty} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} \left(2^k H\right)^{-N} R^{\sigma(\cdot)} f \chi_{\widetilde{R}_k} \right\|_d^{\varrho} \\
\lesssim\ & R^{n\varrho/d} \sum_{k=2-[\log_2 H]}^{\infty} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} \left(2^k H\right)^{-N} R^{\sigma(\cdot)} f \chi_{\widetilde{R}_k} \left\|\chi_{\widetilde{R}_k}\right\|_{t(\cdot)} \right\|_{q(\cdot)}^{\varrho} \\
\lesssim\ & \left(\frac{R}{H}\right)^{n\varrho/d} \sum_{k=2-[\log_2 H]}^{\infty} \left(2^k H\right)^{(n/d-(n/q+\alpha)^- - N)\varrho} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} H^{n/q(\cdot)+\alpha(\cdot)} 2^{k\alpha(\cdot)} R^{\sigma(\cdot)} f \chi_{\widetilde{R}_k} \right\|_{q(\cdot)}^{\varrho},
\end{aligned}$$

where the implicit positive constant does not depend on R. Using again the embedding $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),\infty}$, and since $N > n/d - (n/q+\alpha)^-$ and $2^k H > 1$, for any $k \geq 2 - [\log_2 H]$, the right-hand side of the last expression is bounded by

$$\begin{aligned}
& c \left(\frac{R}{H}\right)^{n\varrho/d} \Big(\sup_{k \geq 2-[\log_2 H]} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} H^{n/q(\cdot)+\alpha(\cdot)} 2^{k\alpha(\cdot)} R^{\sigma(\cdot)} f \chi_{\widetilde{R}_k} \right\|_{q(\cdot)} \Big)^{\varrho} \\
\leq\ & c \left(\frac{R}{H}\right)^{n\varrho/d} \left\| \lambda^{-\frac{1}{\beta(\cdot)}} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}^{\varrho}. \qquad (2.10)
\end{aligned}$$

Finally, we obtain the desired estimate from (2.9) and (2.10) taking into account the decomposition (2.7). This finishes the proof. ■

We recall that the Plancherel-Polya-Nikolskij inequality in $L^{p(\cdot)}(\mathbb{R}^n)$ (cf. [2, Lemma 6.3]), says that

$$\left\|c\ \left|2^{j\alpha(\cdot)} f\right|\right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leq \left\|\left|2^{j(\alpha(\cdot)+n/p_0(\cdot)-n/p_1(\cdot))} f\right|\right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-j}$$

for any p_1, p_0 and $q \in \mathcal{P}_0(\mathbb{R}^n)$ with $\alpha - n/p_1$, $1/q$ locally log-Hölder continuous and $p_1 \geq p_0$, and any $j \in \mathbb{N}_0$, $f \in L^{p_0(\cdot)}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ with supp $\mathcal{F}f \subset \overline{B}(0, 2^j)$ such that the quasi-norm on the right-hand side is at most one. The constant $c > 0$ is independent of j.

The following lemma is the $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}$-version of the Plancherel-Polya-Nikolskij inequality.

Lemma 2.9 *Let $R \geq 1$, $p \in \mathcal{P}_0(\mathbb{R}^n)$, q, t, s, $r \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$ with p^+, $r^+ < \infty$, $\beta \in C^{\log}_{\mathrm{loc}}$ and $\alpha_1, \alpha_2 \in C^{\log}_{\mathrm{loc}}$ such that $(\alpha_2 - \alpha_1)^- > 0$ or $\alpha_1(\cdot) = \alpha_2(\cdot)$. We suppose that $q(\cdot) \leq t(\cdot)$ and $(\alpha_1 + n/t)^- > 0$. Then there exist a positive constant $c > 0$ independent of R such that*

$$\left\| \left| R^{s(\cdot)} f \right|^{\beta(\cdot)} \right\|_{\dot{K}^{\alpha_1(\cdot)\beta(\cdot), r(\cdot)/\beta(\cdot)}_{t(\cdot)/\beta(\cdot)}} \leq c \left\| \left| R^{n/q(\cdot) - n/t(\cdot) + \alpha_2(\cdot) - \alpha_1(\cdot) + s(\cdot)} f \right|^{\beta(\cdot)} \right\|_{\dot{K}^{\alpha_2(\cdot)\beta(\cdot), \theta(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} + \frac{1}{R} \quad (2.11)$$

for all $f \in \dot{K}^{\alpha_2(\cdot),p(\cdot)}_{q(\cdot)} \cap \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F}f \subset \overline{B}(0, R)$ such that the quasi-norm on the right hand side is at most one, where

$$\theta(\cdot) = \begin{cases} r(\cdot) & \text{if} \quad \alpha_1(\cdot) = \alpha_2(\cdot) \\ p(\cdot) & \text{if} \quad (\alpha_2 - \alpha_1)^- > 0. \end{cases}$$

Remark 2.10 *We would like to mention that this lemma improves the Plancherel-Polya-Nikolskij inequality of [2, Lemma 6.3] by taking $\alpha_2(\cdot) = \alpha_1(\cdot) = 0$, $r(\cdot) = t(\cdot)$, $\dot{K}^{0,t(\cdot)/\beta(\cdot)}_{t(\cdot)/\beta(\cdot)} = L^{t(\cdot)/\beta(\cdot)}$ and by using the embedding $L^{q(\cdot)/\beta(\cdot)} = \dot{K}^{0,q(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)} \hookrightarrow \dot{K}^{0,t(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}$.*

Proof of Lemma 2.9. Let $f \in \dot{K}^{\alpha_2(\cdot),p(\cdot)}_{q(\cdot)} \cap \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F}f \subset \overline{B}(0, R)$. Our estimate (2.11), clearly follows from the inequality

$$\left\| \lambda^{-1} \left| R^{s(\cdot)} f \right|^{\beta(\cdot)} \right\|_{\dot{K}^{\alpha_1(\cdot)\beta(\cdot), r(\cdot)/\beta(\cdot)}_{t(\cdot)/\beta(\cdot)}} \leq 1$$

which is equivatent to

$$\rho_{\dot{K}^{\alpha_1(\cdot)\beta(\cdot), r(\cdot)/\beta(\cdot)}_{t(\cdot)/\beta(\cdot)}} \left(\lambda^{-1} \left| R^{s(\cdot)} f \right|^{\beta(\cdot)} \right) \leq 1.$$

In particular, we will show that

$$\sum_{k=-\infty}^{\infty} \left\| \left| 2^{k\alpha_1(\cdot)} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)} f \right|^{r(\cdot)} \chi_k \right\|_{\frac{t(\cdot)}{r(\cdot)}} \lesssim 1,$$

where

$$\lambda := \left\| \left| R^{n/q(\cdot) - n/t(\cdot) + \alpha_2(\cdot) - \alpha_1(\cdot) + s(\cdot)} f \right|^{\beta(\cdot)} \right\|_{\dot{K}^{\alpha_2(\cdot)\beta(\cdot), \theta(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} + \frac{1}{R}.$$

Note that the assumption on the norm implies that $\lambda \in [R^{-1}, R^{-1} + 1]$. We divide the sum $\sum\limits_{k=-\infty}^{\infty} \cdots$ into two parts,

$$\sum_{2^k < 2/R} (\cdots) + \sum_{2^k \geq 2/R} (\cdots) =: I_R + II_R. \quad (2.12)$$

Estimate of I_R. By Lemma 2.8 we get

$$\sup_{x\in B(0,2/R)} \left(R^{s(x)-\alpha_1(x)-n/t(x)}\lambda^{-1/\beta(x)}\left|f(x)\right|\right) \lesssim \left\|R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)}\lambda^{-1/\beta(\cdot)}f\right\|_{\dot{K}^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}}, \tag{2.13}$$

where the implicit positive constant not depending on R. The norm on the right hand side is bounded by 1. To show this, we investigate the corresponding modular:

$$\begin{aligned} &\varrho_{\ell^{\theta(\cdot)}(L^{q(\cdot)})}((2^{k\alpha_2(\cdot)}R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)}\lambda^{-1/\beta(\cdot)}f\chi_k)_k) \\ =& \sum_{k=-\infty}^{\infty}\left\|\left|2^{k\alpha_2(\cdot)}R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)}\lambda^{-1/\beta(\cdot)}f\right|^{\theta(\cdot)}\chi_k\right\|_{\frac{q(\cdot)}{\theta(\cdot)}} \\ =& \sum_{k=-\infty}^{\infty}\left\|\left|\lambda^{-1}\left|2^{k\alpha_2(\cdot)}R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)}f\right|^{\beta(\cdot)}\right|^{\theta(\cdot)/\beta(\cdot)}\chi_k\right\|_{\frac{q(\cdot)}{\theta(\cdot)}}. \end{aligned}$$

This term is bounded by 1 if and only if

$$\left\|\lambda^{-1}\left|R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)}f\right|^{\beta(\cdot)}\right\|_{\dot{K}^{\alpha_2(\cdot)\beta(\cdot),\theta(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} \leq 1,$$

which follows immediately from the definition of λ. Therefore,

$$I_R \lesssim \sum_{2^k<2/R}\left(2^kR\right)^{r^-(\alpha_1+n/t)^-}\left\|\left|2^{-nk/t(\cdot)}\right|^{r(\cdot)}\chi_k\right\|_{\frac{t(\cdot)}{r(\cdot)}} \lesssim \sum_{2^k<2/R}\left(2^kR\right)^{r^-(\alpha_1+n/t)^-} \lesssim 1,$$

where the second inequality follows by the fact that $\left\|\left|2^{-nk/t(\cdot)}\right|^{r(\cdot)}\chi_k\right\|_{\frac{t(\cdot)}{r(\cdot)}} \lesssim 1$. To show this, we investigate the corresponding modular: $\varrho_{t(\cdot)/r(\cdot)}(2^{-nkr(\cdot)/t(\cdot)}) = \int_{R_k} 2^{-nk}dx = c < \infty$. In the last inequality we use the fact that $(\alpha_1+n/t)^- > 0$.

Estimation of II_R. By Lemma 2.3 we have for $R>0, N>n/\tau$ and any $x\in R_k$

$$\begin{aligned} &\lambda^{-1/\beta(x)}R^{s(x)-\alpha_1(x)-n/t(x)}\left|f(x)\right| \\ \lesssim& \left(\int_{\mathbb{R}^n}\left|\lambda^{-1/\beta(y)}R^{s(y)-\alpha_1(y)-n/t(y)}f(y)\right|^{\tau}\eta_{R,N\tau+\delta}(x-y)dy\right)^{1/\tau} \\ \lesssim& \left(\int_{B(0,2^{k-2})}(\cdots)dy\right)^{1/\tau} + c\left(\int_{\widetilde{R}_k}(\cdots)dy\right)^{1/\tau} + c\left(\int_{\mathbb{R}^n\setminus B(0,2^{k+2})}(\cdots)dy\right)^{1/\tau} \\ =& V^1_{R,k}(x)+V^2_{R,k}(x)+V^3_{R,k}(x). \end{aligned} \tag{2.14}$$

Here $\widetilde{R}_k := \left\{x\in\mathbb{R}^n : 2^{k-2}\leq|x|\leq 2^{k+2}\right\}$ and $0<\tau<q^-$ and $\delta := c_{\log}(\alpha_2)+c_{\log}(\frac{1}{r})+c_{\log}(\frac{1}{q})$. We choose N such that

$$N > \max\left(n/d+n/\tau+(\alpha_1+n/t)^+, n/d+n/\tau-(\alpha_2+n/q)^-+(\alpha_1+n/t)^+\right), \tag{2.15}$$

where d is chosen as in Lemma 2.8. It is easy to verify that if $x \in R_k$ and $y \in B(0, 2^{k-2})$, then $|x-y| > 2^{k-2}$. This estimate and Lemma 2.8 yield for any $x \in R_k$ and any $k \in \mathbb{Z}$ such that $2^k R \geq 2$

$$\begin{aligned}
& 2^{k\alpha_1(x)} V^1_{R,k}(x) \\
\lesssim\ & 2^{k\alpha_1(x)} \sup_{y \in B(0,2^{k-2})} \left| \lambda^{-1/\beta(y)} R^{s(y)-\alpha_1(y)-n/t(y)} f(y) \right| \left(\int_{2^{k-2}<|t|<2^{k+1}} \eta_{R,N\tau+\delta}(t) dt \right)^{1/\tau} \\
\lesssim\ & \left(2^k R\right)^{n/d+n/\tau-N} 2^{k\alpha_1(x)} \left\| 2^{-k(\alpha_2(\cdot)+n/q(\cdot))} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \right\|_{K^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}}. \qquad (2.16)
\end{aligned}$$

Observe that $2^k R \geq 2$. Hence

$$\begin{aligned}
& 2^{k\alpha_1(x)} V^1_{R,k}(x) \\
\lesssim\ & \left(2^k R\right)^{n/d+n/\tau-(\alpha_2+n/q)^- - N} 2^{k\alpha_1(x)} \left\| \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f \right\|_{K^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}} \\
\lesssim\ & \left(2^k R\right)^{n/d+n/\tau-(\alpha_2+n/q)^- - N} 2^{k\alpha_1(x)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{2^k \geq 2/R} \left\| \left| 2^{k\alpha_1(\cdot)} R^{\alpha_1(\cdot)+n/t(\cdot)} V^1_{R,k} \right|^{r(\cdot)} \chi_k \right\|_{\frac{t(\cdot)}{r(\cdot)}} \\
\lesssim\ & \sum_{2^k \geq 2/R} \left(2^k R\right)^{(n/d+n/\tau-(\alpha_2+n/q)^- + (\alpha_1+n/t)^+ - N) r^-} \left\| \left| 2^{-kn/t(\cdot)} \right|^{r(\cdot)} \chi_k \right\|_{\frac{t(\cdot)}{r(\cdot)}} \\
\lesssim\ & \sum_{2^k \geq 2/R} \left(2^k R\right)^{(n/d+n/\tau-(\alpha_2+n/q)^- + (\alpha_1+n/t)^+ - N) r^-} \lesssim 1,
\end{aligned}$$

by our assumption on N. Now we observe that

$$V^2_{R,k} := \left(\eta_{R,N\tau+\delta} * |\lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\widetilde{R}_k}|^\tau \right)^{1/\tau}.$$

Hölder's inequality gives

$$\begin{aligned}
& \left| 2^{k\alpha_2(x)} R^{\alpha_2(x)} V^2_{R,k}(x) \chi_k(x) \right|^\tau \\
\lesssim\ & \eta_{R,N\tau} * \left(\left| 2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\widetilde{R}_k} \right|^\tau \right)(x) \\
\lesssim\ & \left\| \eta_{R,N\tau}(x-\cdot) R^{-n\tau/q(\cdot)} \right\|_{(q(\cdot)/\tau)'} \times \\
& \left\| \left| 2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\widetilde{R}_k} \right|^\tau \right\|_{q(\cdot)/\tau} \\
\leq\ & \left\| R^{-n\tau/q(\cdot)} \eta_{R,N\tau}(x-\cdot) \right\|_{(q(\cdot)/\tau)'} \left\| R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} \lambda^{-1/\beta(\cdot)} f \right\|^\tau_{\dot{K}^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}},
\end{aligned}$$

where

$$k\bar{\alpha}_2 := \begin{cases} k(\alpha_2)_\infty & \text{if} \quad k \geq 0 \\ k\alpha_2(0) & \text{if} \quad k < 0 \end{cases}.$$

The second norm on the right hand side is bounded by 1 due to the choice of λ, see the estimation of I_R. To show that the first norm is also bounded, we investigate the corresponding modular:

$$\varrho_{(q(\cdot)/\tau)'}(\eta_{R,N\tau}(x-\cdot)R^{-n\tau/q(\cdot)}) = R^n \int_{\mathbb{R}^n} (1+R|x-y|)^{-N(q(\cdot)/\tau)'(y)\tau} dy < \infty.$$

Let us prove that

$$\left\| \left| \left(2^k R\right)^{\alpha_2(\cdot)} R^{n/t(\cdot)} V^2_{R,k} \right|^{r(\cdot)} \chi_k \right\|_{\frac{t(\cdot)}{r(\cdot)}} \lesssim 1 \tag{2.17}$$

for any $k \in \mathbb{Z}$. We have

$$\begin{aligned} & \left| \left(2^k R\right)^{\alpha_2(x)} R^{n/t(x)} V^2_{R,k}(x) \right|^{t(x)} \\ = & \left| \left(2^k R\right)^{\alpha_2(x)} V^2_{R,k}(x) \right|^{t(x)-q(x)} \left| \left(2^k R\right)^{\alpha_2(x)} R^{n/q(x)} V^2_{R,k}(x) \right|^{q(x)} \\ \lesssim & \left| \left(2^k R\right)^{\alpha_2(x)} R^{n/q(x)} V^2_{R,k}(x) \right|^{q(x)}, \end{aligned}$$

for any $x \in R_k$, where the implicit positive constant does not depend on R and k. Therefore,

$$\begin{aligned} & \int_{R_k} \left| \left(2^k R\right)^{\alpha_2(x)} R^{n/t(x)} V^2_{R,k}(x) \right|^{t(x)} dx \\ \lesssim & \int_{\mathbb{R}^n} \left| \left(2^k R\right)^{\alpha_2(x)} R^{n/q(x)} V^2_{R,k}(x) \right|^{q(x)} dx \\ \lesssim & \int_{\mathbb{R}^n} \left(\eta_{R,N\tau} * \left(\left| 2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\widetilde{R}_k} \right|^{\tau} \right)(x) \right)^{q(x)/\tau} dx. \end{aligned}$$

This term is bounded by 1 if and only if

$$\left\| \eta_{R,N\tau} * \left(\left| 2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\widetilde{R}_k} \right|^{\tau} \right) \right\|_{q(\cdot)/\tau} \leq 1.$$

Since convolution is bounded in $L^{p(\cdot)}$ when $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, see (1.2), we obtain that the left-hand side is bounded by

$$\begin{aligned} & c \left\| 2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\widetilde{R}_k} \right\|^{\tau}_{q(\cdot)} \\ \lesssim & \left\| R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} \lambda^{-1/\beta(\cdot)} f \right\|^{\tau}_{\dot{K}^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}} \lesssim 1. \end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{2^k\geq 2/R}\left\|\left|2^{k\alpha_1(\cdot)}R^{\alpha_1(\cdot)+n/t(\cdot)}V^2_{R,k}\right|^{r(\cdot)}\chi_k\right\|_{\frac{t(\cdot)}{r(\cdot)}}\\
\lesssim\ &\sum_{2^k\geq 2/R}(2^kR)^{-(\alpha_2-\alpha_1)^-r^-}\left\|\left|\left(2^kR\right)^{\alpha_2(\cdot)}R^{n/t(\cdot)}V^2_{R,k}\right|^{r(\cdot)}\chi_k\right\|_{\frac{t(\cdot)}{r(\cdot)}}\\
\lesssim\ &\sum_{2^k\geq 2/R}(2^kR)^{-(\alpha_2-\alpha_1)^-r^-}\lesssim 1,
\end{aligned}$$

if $(\alpha_2-\alpha_1)^->0$. Let us treat the case $\alpha_1(\cdot)=\alpha_2(\cdot)$ more carefully. Let recall that for $\alpha_1(\cdot)=\alpha_2(\cdot)$ we have $\theta(\cdot)=r(\cdot)$. Hence it suffices to prove that

$$\begin{aligned}
&\left\|\left|\left(2^kR\right)^{\alpha_1(\cdot)}R^{n/t(\cdot)}V^2_{R,k}\right|^{r(\cdot)}\chi_k\right\|_{\frac{t(\cdot)}{r(\cdot)}}\\
\lesssim\ &\left\|\left|2^{k\alpha_1}\lambda^{-1/\beta(\cdot)}R^{n/q(\cdot)+s(\cdot)-n/t(\cdot)}f\chi_{\widetilde{R}_k}\right|^{r(\cdot)}\chi_k\right\|_{\frac{q(\cdot)}{r(\cdot)}}+\frac{1}{2^kR}=:\mu,
\end{aligned}$$

which is equivalent to

$$\left\|\mu^{-\frac{1}{r(\cdot)}}\left(2^kR\right)^{\alpha_1(\cdot)}R^{n/t(\cdot)}V^2_{R,k}\chi_k\right\|_{t(\cdot)}\lesssim 1. \tag{2.18}$$

We have

$$\begin{aligned}
\mu^{-\frac{1}{r(x)}} &= \mu^{\frac{1}{r(y)}-\frac{1}{r(x)}}\mu^{-\frac{1}{r(y)}}\\
&\lesssim (2^kR)^{\left|\frac{1}{r(x)}-\frac{1}{r(y)}\right|}\mu^{-\frac{1}{r(y)}}, \qquad x\in R_k, y\in\widetilde{R}_k.
\end{aligned}$$

We use the log-Hölder continuity of r to get the equivalence

$$2^{k(\frac{1}{r(x)}-\frac{1}{r(y)})}\approx 1$$

and

$$R^{\frac{1}{r(x)}-\frac{1}{r(y)}}\lesssim(1+R\left|x-y\right|)^{c_{\log}(r)}$$

for any $x\in R_k$ and any $y\in\widetilde{R}_k$. Therefore,

$$\mu^{-\frac{1}{r(x)}}V^2_{R,k}(x)\lesssim\left(\eta_{R,N\tau}\ast|\mu^{-\frac{1}{r(\cdot)}}\lambda^{-1/\beta(\cdot)}R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)}f\chi_{\widetilde{R}_k}|^{\tau}\right)^{1/\tau}.$$

Now (2.18) can be obtained by repeating the same arguments used in the proof of (2.17).

We see that $\int_{\mathbb{R}^n\setminus B(0,2^{k+2})}(\cdots)dy$ can be rewritten as $\sum_{i=0}^{\infty}\int_{R_{k+i+3}}(\cdots)dy$. Then, using (2.7), we get for any $x\in R_k$

$$\left(V^3_{R,k}(x)\right)^{\varrho}\leq c\sum_{i=0}^{\infty}\left(\eta_{R,N\tau+\delta}\ast|\lambda^{-1/\beta(\cdot)}R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)}f\chi_{R_{k+i+3}}|^{\tau}(x)\right)^{\varrho/\tau}, \tag{2.19}$$

with $\varrho := \min(1,\tau)$. Since $|x-y| > 3\cdot 2^{k+i}$ for any $x \in R_k$ and any $y \in R_{k+i+3}$, the right-hand side of (2.19) is bounded by

$$\begin{aligned} & c\ R^{\varrho(n/\tau-N)}\sum_{i=0}^{\infty} 2^{-\varrho(k+i)N}\left\|\lambda^{-1/\beta(\cdot)}R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)}f\chi_{R_{k+i+3}}\right\|_{\tau}^{\varrho} \\ \leq\ & c\ R^{\varrho(n/\tau-N)}\sum_{j=k+3}^{\infty} 2^{-\varrho jN}\left\|\lambda^{-1/\beta(\cdot)}R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)}f\chi_{R_j}\right\|_{\tau}^{\varrho}. \end{aligned}$$

By Lemma 2.8 we get

$$\begin{aligned} & \sup_{x\in B(0,2^j)}\left|\lambda^{-1/\beta(x)}R^{s(x)-\alpha_1(x)-n/t(x)}f(x)\right| \\ \lesssim\ & \left(2^j R\right)^{n/d}\left\|2^{-j(n/q(\cdot)+\alpha_2(\cdot))}\lambda^{-1/\beta(\cdot)}R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)}f\right\|_{\dot{K}_{q(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)}} \\ \leq\ & \left(2^j R\right)^{n/d}\left(2^j R\right)^{-(\alpha_2+n/q)^-}\left\|\lambda^{-1/\beta(\cdot)}R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)}f\right\|_{\dot{K}_{q(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)}} \\ \lesssim\ & \left(2^j R\right)^{n/d-(\alpha_2+n/q)^-}, \end{aligned}$$

since $2^k R \geq 2$ we estimate $\left(V_{R,k}^3\right)^{\varrho}$ by

$$c\ R^{\varrho(n/\tau+n/d-N-(\alpha_2+n/q)^-)}\sum_{j=k+3}^{\infty} 2^{\varrho j(n/d+n/\tau-N-(\alpha_2+n/q)^-)} \lesssim \left(2^k R\right)^{\varrho(n/d+n/\tau-N-(\alpha_2+n/q)^-)}.$$

Hence

$$\begin{aligned} & \sum_{2^k\geq 2/R}\left\|\left|2^{k\alpha_1(\cdot)}R^{\alpha_1(\cdot)+n/t(\cdot)}V_{R,k}^3\right|^{r(\cdot)}\chi_k\right\|_{\frac{t(\cdot)}{r(\cdot)}} \\ \lesssim\ & \sum_{2^k\geq 2/R}\left(2^k R\right)^{(n/d+n/\tau-N-(\alpha_2+n/q)^-+(\alpha_1+n/t)^+)r^-}\left\|\left|2^{-kn/t(\cdot)}\right|^{r(\cdot)}\chi_k\right\|_{\frac{t(\cdot)}{r(\cdot)}} \\ \lesssim\ & \sum_{2^k\geq 2/R}\left(2^k R\right)^{(n/d+n/\tau-N-(\alpha_2+n/q)^-+(\alpha_1+n/t)^+)r^-} \lesssim 1. \end{aligned}$$

This finishes the proof. ■

In the previous lemma we have not treated the case $t(\cdot) \leq q(\cdot)$. The next lemma gives a positive answer.

Lemma 2.11 *Let $R \geq 1$, $p \in \mathcal{P}_0(\mathbb{R}^n)$, q, t, s, $r \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ with p^+, $r^+ < \infty$, $\beta \in C_{\mathrm{loc}}^{\log}$ and $\alpha_1, \alpha_2 \in C_{\mathrm{loc}}^{\log}$ such that $\alpha_2(\cdot)+n/q(\cdot) = \alpha_1(\cdot)+n/t(\cdot)$ or $(\alpha_2+\frac{n}{q}-\alpha_1-\frac{n}{t})^- > 0$. We suppose that $t(\cdot) \leq q(\cdot)$ and $(\alpha_1+n/t)^- > 0$. Then there exist a positive constant $c > 0$ independent* *of R such that for all $f \in \dot{K}^{\alpha_2(\cdot),p(\cdot)}_{q(\cdot)} \cap \mathcal{S}'(\mathbb{R}^n)$ with* supp $\mathcal{F}f \subset \overline{B}(0,R)$, *we have*

$$\left\| \left| R^{s(\cdot)} f \right|^{\beta(\cdot)} \right\|_{\dot{K}^{\alpha_1(\cdot)\beta(\cdot),r(\cdot)/\beta(\cdot)}_{t(\cdot)/\beta(\cdot)}} \leq c \ \left\| \left| R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f \right|^{\beta(\cdot)} \right\|_{\dot{K}^{\alpha_2(\cdot)\beta(\cdot),\theta(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} + \frac{1}{R}, \tag{2.20}$$

such that the qausi-norm on the right hand side is at most one, where

$$\theta(\cdot) = \begin{cases} r(\cdot) & \text{if} \quad \alpha_2(\cdot) + n/q(\cdot) = \alpha_1(\cdot) + n/t(\cdot) \\ p(\cdot) & \text{if} \quad (\alpha_2 + n/q - \alpha_1 - n/t)^- > 0. \end{cases}$$

Proof. We employ the notations I_R and II_R from (2.12). The estimate of I_R follows easily from the previous proof. We only need to estimate the part II_R with $(\alpha_2-\alpha_1+n/q-n/t)^- > 0$. Hölder's inequality gives

$$\left\| \left| 2^{k\alpha_1(\cdot)} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)} f \right|^{r(\cdot)} \chi_k \right\|_{\frac{t(\cdot)}{r(\cdot)}} \lesssim \left\| \left| 2^{k\alpha_1(\cdot)} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)} f \right|^{r(\cdot)} \chi_k \, \|\chi_k\|_{\frac{h(\cdot)}{r(\cdot)}} \right\|_{\frac{q(\cdot)}{r(\cdot)}},$$

where $\frac{1}{t(\cdot)} := \frac{1}{q(\cdot)} + \frac{1}{h(\cdot)}$. By Lemma 1.29 we get $\|\chi_k\|_{\frac{h(\cdot)}{r(\cdot)}} \approx 2^{kn(1/t(x)-1/q(x))r(x)}$ for any $x \in R_k$. Therefore,

$$\begin{aligned} II_R &\lesssim \sup_{k\in\mathbb{Z}} \left\| \left| \lambda^{-1/\beta(\cdot)} 2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f \right|^{r(\cdot)} \chi_k \right\|_{\frac{q(\cdot)}{r(\cdot)}} \\ &\quad \times \sum_{2^k \geq 2/R} \left(2^k R\right)^{-(\alpha_2-\alpha_1+n/q-n/t)^- r^-} \\ &\lesssim \sup_{k\in\mathbb{Z}} \left\| \left| \lambda^{-1/\beta(\cdot)} 2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f \right|^{r(\cdot)} \chi_k \right\|_{\frac{q(\cdot)}{r(\cdot)}}. \end{aligned}$$

The last norm is bounded by 1 if and only if

$$\left\| \lambda^{-1/\beta(\cdot)} \left| 2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f \right| \chi_k \right\|_{q(\cdot)} \lesssim 1, \qquad k \in \mathbb{Z}$$

which follows immediately from (2.13). The proof is completed. ■

Chapter 3

Variable Herz-type Besov spaces

It is well known that the function spaces play an important role in harmonic analysis. Some example of these spaces can be mentioned such as Besov spaces $B^s_{p,q}$. These spaces include many classical spaces as special cases, for example, the Hölder spaces, the Sobolev spaces, the Bessel potential spaces, the Zygmund spaces, the local Hardy spaces and the space BMO studied in detail by H. Triebel in [53], [54] and [55].

The Besov spaces of variable smoothness $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, are a generalization of the classical Besov spaces, replacing the constants exponents s, p and q with a variable exponent functions $s(\cdot)$, $p(\cdot)$ and $q(\cdot)$, initially appeared in the paper of A. Almeida and P. Hästo [2]. Several basic properties were established, such as the Fourier analytical characterisation. When s, p and q are constants they coincide with the usual function spaces $B^s_{p,q}$.

Based on variable Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, we introduce Herz-type Besov spaces with variable smoothness, which covers Herz-type Besov spaces with fixed exponents. We will give several properties of these new family of function spaces.

3.1 Variable Besov spaces

In this section, we present the Fourier analytical definition of Besov spaces of variable smoothness and we recall their basic properties which are analogy to the Besov spaces with fixed exponents. We first need the concept of a smooth dyadic resolution of unity.

Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$.

We define put φ_0 and φ_1 by $\mathcal{F}\varphi_0(x) = \Psi(x)$, $\mathcal{F}\varphi_1(x) = \Psi(x) - \Psi(2x)$ and

$$\mathcal{F}\varphi_j(x) = \mathcal{F}\varphi_1(2^{-j}x) \quad \textit{for} \quad j = 2, 3, \ldots.$$

Then $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity,

$$\sum_{j=0}^{\infty} \mathcal{F}\varphi_j(x) = 1 \text{ for all } x \in \mathbb{R}^n.$$

Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{j=0}^{\infty} \varphi_j * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

Next we give the definition of the $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$-spaces, which introduced and investigated in [2].

Definition 3.1 *Let $\{\mathcal{F}\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity, $s : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The Besov space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\|(2^{js(\cdot)}\varphi_j * f)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Taking $s \in \mathbb{R}$ and $q \in (0, \infty]$ as constants we derive the spaces $B^s_{p(\cdot),q}$ studied by Xu in [59]. We refer the reader to the recent papers [13], [31] and [30] for further details, historical remarks and more references on these function spaces. For any $p, q \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$ and $s \in C^{\log}_{\text{loc}}$, the space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Moreover, if p, q, s are constants, we re-obtain the usual Besov spaces $B^s_{p,q}$, studied in detail by H. Triebel in [53], [54] and [55].

The following theorem is from [2, Theorem 6.1], gives basic embeddings in variable Besov spaces.

Theorem 3.2 *Let $\alpha, \alpha_0, \alpha_1 \in L^\infty$ and $p, q_0, q_1 \in \mathcal{P}_0(\mathbb{R}^n)$*

(i) If $q_0 \le q_1$, then

$$B^{\alpha(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),q_1(\cdot)}$$

(ii) If $(\alpha_0 - \alpha_1)^- > 0$*, then*

$$B^{\alpha_0(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

We further recall that, the Sobolev embeddings in the usual Besov spaces $B^s_{p,q}$ is given by

$$B^{\alpha_0}_{p_0,q} \hookrightarrow B^{\alpha_1}_{p_1,q} \tag{3.1}$$

if $\alpha_0 - n/p_0 = \alpha_1 - n/p_1$, where $0 < p_0 \leq p_1 \leq \infty$, $0 < q \leq \infty$, $-\infty < \alpha_1 \leq \alpha_0 < \infty$, see [53].

The next theorem is a generalization of (3.1) to the variable Besov spaces, see [2, Theorem 6.4].

Theorem 3.3 (Sobolev embedding) *Let* $p_0, p_1, q_0, q_1 \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $\alpha_0, \alpha_1 \in L^\infty$ *with* $\alpha_0 \geq \alpha_1$ *If* $1/q$ *and*

$$\alpha_0(x) - n/p_0(x) = \alpha_1(x) - n/p_1(x) + \varepsilon(x)$$

are locally log-Hölder continuous and $\varepsilon^- > 0$ *or* $\varepsilon \equiv 0$*, then*

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}.$$

3.2 Definition and basic properties of variable Herz-type Besov spaces

In this section, we introduce the Herz-type Besov spaces of variable smoothness and we prove some basic properties in this spaces. In particular, we generalize the results of D. Drihem in [14].

Definition 3.4 *Let* $\{\mathcal{F}\varphi_j\}_{j=0}^{\infty}$ *be a resolution of unity,* $\alpha, s : \mathbb{R}^n \to \mathbb{R}$ *and* $p, q, \beta \in \mathcal{P}_0(\mathbb{R}^n)$*. The Herz-type Besov space* $K^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} B^{s(\cdot)}_{\beta(\cdot)}$ *is the collection of all distributions* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} B^{s(\cdot)}_{\beta(\cdot)}} := \left\| \left(2^{js(\cdot)}\varphi_j * f\right)_j \right\|_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})} < \infty. \tag{3.2}$$

Clearly, $\dot{K}^{0,p(\cdot)}_{p(\cdot)} B^{s(\cdot)}_{\beta(\cdot)} = B^{s(\cdot)}_{p(\cdot),\beta(\cdot)}$. Herz-type Besov spaces $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} B^{s(\cdot)}_{\beta(\cdot)}$ with variable exponent p and α but fixed s, q and β were recently studied in [12], [50] and [49]. While the first time we introduce the spaces $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}$ with the quasi-norm (3.2). When, $\beta := \infty$ the Herz-type Besov space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\infty}^{s(\cdot)}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\sup_{v\geq 0} \left\| 2^{vs(\cdot)}\varphi_v * f \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} < \infty.$$

One recognizes immediately that if α, s, p, q and β are constants, then the spaces $\dot{K}_{q}^{\alpha,p}B_{\beta}^{s}$ are just the usual Herz-type Besov spaces were first introduced by J. Xu and D. Yang [61] and [62]. See [14] and [60] for further results of these functions spaces.

C. Shi and J. Xu [50] and [49] studied Herz-type Besov spaces $\dot{K}_{q(\cdot)}^{\alpha,p}B_{\beta}^{s}$ with variable q, but fixed α, p, s and β, where the characterization of these function spaces by the so-called Peetre maximal functions are obtained. B. Dong and J. Xu [12] also considered $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}B_{\beta}^{s}$ with variables q and α. The interest in these spaces comes not only from the theoretical reasons but also from their applications to several classical problems in analysis. In [41], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some applications to partial differential equations. We refer the reader to the recent paper [14] for further results for these function spaces.

Now, we are ready to state and prove the main results of this section. The following theorem show that the definition of the spaces $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}$ is independent of the chosen resolution of unity $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$. This justifies our omission of the subscript φ in the sequel.

Theorem 3.5 *Let* $\left\{\mathcal{F}\varphi_j\right\}_{j=0}^{\infty}$, $\left\{\mathcal{F}\psi_j\right\}_{j=0}^{\infty}$ *are two resolutions of unity,* $p, q, \beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ *and* $s \in C_{\mathrm{loc}}^{\log}$. *Let* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *be log-Hölder continuous, both at the origin and at infinity such that* $\alpha^- + \frac{n}{q^+} > 0$. *Then*

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\varphi} \approx \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\psi}.$$

Proof. It is sufficient to show that for all $f \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ such that $\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\psi}$ is finite, we have

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\varphi} \leq c\,\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\psi}$$

with $c > 0$. Interchanging the roles of ψ and φ we obtain the desired result. Putting $\psi_{-1} = 0$ we see $\mathcal{F}\varphi_v = \mathcal{F}\varphi_v \sum_{k=-1}^{k=1} \mathcal{F}\psi_{v+k}$ for all $v \in \mathbb{N}_0$. By the properties of the Fourier transform

$$\varphi_v * f = \sum_{k=-1}^{k=1} \varphi_v * \psi_{v+k} * f.$$

Fix $0 < r < \min(1, \frac{n}{\alpha^+ + n/q^-})$ and $m > n + 2c_{\log}(s) + c_{\log}(1/\beta)$ large. Since $|\varphi_v| \leq c\, \eta_{v,2m/r}$, with $c > 0$ independent of v, we obtain

$$|\varphi_v * \psi_{v+k} * f| \lesssim \eta_{v,m/r} * |\psi_{v+k} * f| \lesssim \eta_{v,m/r} * \left(\eta_{v+k,m} * |\psi_{v+k} * f|^r\right)^{1/r},$$

where in the second inequality we used Lemma 2.3. By Minkowski's integral inequality the left-hand side is bounded by

$$\begin{aligned} & c\Big((\eta_{v,m/r} * \eta_{v+k,m}^{1/r})^r * |\psi_{v+k} * f|^r\Big)^{1/r} \\ = \quad & c\, 2^{n(v+k)(1/r-1)}\Big(\left(\eta_{v,m/r} * \eta_{v+k,m/r}\right)^r * |\psi_{v+k} * f|^r\Big)^{1/r}. \end{aligned}$$

By Lemma 2.2 we have $\eta_{v,m/r} * \eta_{v+k,m/r} \approx \eta_{v+k,m/r}$. Then the last expression is bounded by $c\left(\eta_{v+k,m} * |\psi_{v+k} * f|^r\right)^{1/r}$. This, together with Lemma 2.1, gives

$$\begin{aligned} & \Big\|\left(2^{vs(\cdot)}\varphi_v * f\right)_{v\geq 0}\Big\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} \\ = \quad & \Big\|\left(2^{vs(\cdot)r}|\varphi_v * f|^r\right)_{v\geq 0}\Big\|_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})}^{1/r} \\ \lesssim \quad & \sum_{k=-1}^{k=1}\Big\|\left(2^{vs(\cdot)r}\eta_{v+k,m} * |\psi_{v+k} * f|^r\right)_{v\geq 0}\Big\|_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})}^{1/r} \\ \lesssim \quad & \sum_{k=-1}^{k=1}\Big\|\Big(\eta_{v+k,m-c_{\log}(s)} * 2^{(v+k)s(\cdot)r}|\psi_{v+k} * f|^r\Big)_{v\geq 0}\Big\|_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})}^{1/r}. \end{aligned}$$

By the change of variable $v + k = i$, this expression is bounded by

$$\begin{aligned} & \sum_{k=-1}^{k=1}\Big\|\Big(\eta_{i,m-c_{\log}(s)} * 2^{is(\cdot)r}|\psi_i * f|^r\Big)_{i\geq k}\Big\|_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})}^{1/r} \\ \lesssim \quad & 3\Big\|\left(2^{is(\cdot)r}|\psi_i * f|^r\right)_{i\geq 0}\Big\|_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})}^{1/r} \leq 3\,\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\psi}, \end{aligned}$$

where in the first inequality we have used Lemma 2.6. This finishes the proof. ■

3.2.1 Embeddings

The following theorem gives basic embeddings of the $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}$-spaces .

Theorem 3.6 *Let $p, q, \beta_1, \beta_2, p_1, p_2 \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $s, s_1, s_2 \in C_{\mathrm{loc}}^{\log}$. Let α be log-Hölder continuous, both at the origin and at infinity such that $\alpha^- + \frac{n}{q^+} > 0$.*

(i) *If $\beta_1(\cdot) \leq \beta_2(\cdot)$, then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta_1(\cdot)}^{s(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta_2(\cdot)}^{s(\cdot)}.$$

(ii) *If* $p_1(\cdot) \leq p_2(\cdot)$*, then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p_1(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p_2(\cdot)} B_{\beta(\cdot)}^{s(\cdot)}.$$

(iii) *If* $(s_1 - s_2)^- > 0$*, then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_1(\cdot)}^{s_1(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_2(\cdot)}^{s_2(\cdot)}.$$

Proof. (i) is a simple consequence of the embedding

$$\ell^{\beta_1(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}) \hookrightarrow \ell^{\beta_2(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}).$$

(ii) can be deduced from the embeddings properties of the Herz-type spaces, see (1.8). Notice that

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_1(\cdot)}^{s_1(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_1^+}^{s_1(\cdot)}$$

and

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_2^-}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_2(\cdot)}^{s_2(\cdot)}.$$

Therefore, it suffices to prove (iii) for constant exponents β_1^+ and β_2^-. We have

$$\begin{aligned} \left\| \left(2^{vs_2(\cdot)} \varphi_v * f\right)_{v \geq 0} \right\|_{\ell^{\beta_2^-}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} &\leq c \sup_{v \geq 0} \left\| \left(2^{vs_1(\cdot)} \varphi_v * f\right) \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \\ &\leq c \left\| \left(2^{vs_1(\cdot)} \varphi_v * f\right)_{v \geq 0} \right\|_{\ell^{\beta_2^+}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})}, \end{aligned}$$

with $c = \left(\sum_{v \geq 0} 2^{(s_1 - s_2)^- v \beta_2^-} \right)^{1/\beta_2^-}$. ■

Theorem 3.7 *Let* $p, q, \beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ *and* $s \in C_{\text{loc}}^{\log}$*. Let* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *be log-Hölder continuous, both at the origin and at infinity such that* $\alpha^- + \frac{n}{q^+} > 0$*. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{3.3}$$

Proof. Our proof use partially some decomposition techniques already used in [14] where the constant exponent case was studied. Also, in more general spaces given by abstract definitions, see Hedberg and Netrusov [22]. By Theorem 3.6 we need only to prove (3.3) with $\beta := \infty$.

Step 1. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\{\mathcal{F}\varphi_j\}_{j=0}^{\infty}$ is a resolution of unity. If L, M and N are sufficiently large natural numbers, then

$$\begin{aligned} \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}} &= \sup_{j\geq 0} \left\| 2^{js(\cdot)} \varphi_j * f \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \\ &\leq \sup_{j\geq 0} \left\| 2^{js(\cdot)} (1+|x|^2)^{2L} \varphi_j * f \right\|_{\infty} \left\| \frac{1}{(1+|x|^2)^{2L}} \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}. \end{aligned}$$

Take L sufficiently large such that $(\alpha - 4L + n/q)^+ < 0$, we then have

$$\varrho_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \Big(\frac{1}{(1+|\cdot|^2)^{2L}} \Big) = \sum_{k=-\infty}^{+\infty} \left\| \left| \frac{2^{k\alpha(\cdot)}}{(1+|\cdot|^2)^{2L}} \right|^{p(\cdot)} \chi_k \right\|_{\frac{q(\cdot)}{p(\cdot)}}.$$

First we remark that

$$\left\| \left| \frac{2^{k\alpha(\cdot)}}{(1+|\cdot|^2)^{2L}} \right|^{p(\cdot)} \chi_k \right\|_{\frac{q(\cdot)}{p(\cdot)}} \leq c\ 2^{k(\alpha-4L+n/q)^+ p^-} \left\| \left| 2^{-kn/q(\cdot)} \right|^{p(\cdot)} \chi_k \right\|_{\frac{q(\cdot)}{p(\cdot)}} \lesssim 2^{k(\alpha-4L+n/q)^+ p^-}$$

for any $k \geq 0$ and

$$\left\| \left| \frac{2^{k\alpha(\cdot)}}{(1+|\cdot|^2)^{2L}} \right|^{p(\cdot)} \chi_k \right\|_{\frac{q(\cdot)}{p(\cdot)}} \lesssim 2^{k(\alpha+n/q)^- p^-}$$

for any $k < 0$. Therefore,

$$\sum_{k=-\infty}^{+\infty} \left\| \left| \frac{2^{k\alpha(\cdot)}}{(1+|x|^2)^{2L}} \right|^{p(\cdot)} \chi_k \right\|_{\frac{q(\cdot)}{p(\cdot)}} \lesssim \sum_{k=-\infty}^{0} 2^{k(\alpha+n/q)^- p^-} + \sum_{k=1}^{+\infty} 2^{k(\alpha-4L+n/q)^+ p^-} \lesssim 1.$$

Hence

$$\begin{aligned} \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}} &\lesssim \sup_{j\geq 0} 2^{js^+} \left\| \mathcal{F}^{-1} [1 + (-\triangle)^L] \mathcal{F}\varphi_j \mathcal{F}f \right\|_{\infty} \\ &\lesssim \sup_{j\geq 0} 2^{js^+} \left\| [1 + (-\triangle)^L] \mathcal{F}\varphi_j \mathcal{F}f \right\|_{1} \\ &\lesssim \left\| (1+|x|)^M [1 + (-\triangle)^L] \mathcal{F}f \right\|_{\infty} \\ &\lesssim p_N(\mathcal{F}f). \end{aligned}$$

Step 2. We prove the right-hand side of (3.3). Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be the smooth dyadic resolution of unity. We put $\omega_j = \sum_{i=j-1}^{i=j+1} \mathcal{F}\varphi_i$ if $j = 1, 2, ...$ (with $\mathcal{F}\varphi_{-1} = 0$). If $f \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, then $f(\psi)$ denotes the value of the functional f of $\mathcal{S}'(\mathbb{R}^n)$ for the test function

ψ. We obtain

$$\begin{aligned} |f(\psi)| &\leq \sum_{j=0}^{\infty} \left| \varphi_j * f(\mathcal{F}^{-1}\omega_j * \psi) \right| \\ &= \sum_{j=0}^{\infty} \left\| \varphi_j * f \cdot (\mathcal{F}^{-1}\omega_j * \psi) \right\|_1 \\ &= \sum_{j=0}^{\infty} \left\| \varphi_j * f \cdot (\mathcal{F}^{-1}\omega_j * \psi) \right\|_{\dot{K}_1^{0,1}} . \end{aligned}$$

Recalling the definition of $\dot{K}_1^{0,1}$ spaces, the last sum can be rewritten as

$$\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \left\| \varphi_j * f \cdot (\mathcal{F}^{-1}\omega_j * \psi)\chi_k \right\|_1 \leq \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \sup_{x \in B(0,2^k)} \left| \varphi_j * f(x) \right| \left\| (\mathcal{F}^{-1}\omega_j * \psi)\chi_k \right\|_1 .$$

We divide the last sum into two parts $\sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} \cdots + \sum_{j=0}^{\infty} \sum_{k=-j}^{\infty} \cdots$. Lemma 2.8, gives for any $0 < d < \min(q^-, n/(\alpha + n/q)^+)$

$$\begin{aligned} &\sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} \sup_{x \in B(0,2^{-j})} \left| \varphi_j * f(x) \right| \left\| (\mathcal{F}^{-1}\omega_j * \psi)\chi_k \right\|_1 \\ \lesssim\; &\sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} \left\| 2^{(n/q(\cdot)+\alpha(\cdot))j} \varphi_j * f \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \left\| (\mathcal{F}^{-1}\omega_j * \psi)\chi_k \right\|_1 \\ \lesssim\; &\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}} \sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} 2^{(n/q+\alpha-s)^+ j} \left\| (\mathcal{F}^{-1}\omega_j * \psi)\chi_k \right\|_1 \\ \lesssim\; &\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}} \|\psi\|_{B_{1,1}^{(n/q+\alpha-s)^+}} . \end{aligned}$$

Using again Lemma 2.8, we have for any $0 < d < \min(q^-, n/(\alpha + n/q)^+)$

$$\begin{aligned} &\sum_{j=0}^{\infty} \sum_{k=-j}^{\infty} \sup_{x \in B(0,2^k)} \left| \varphi_j * f(x) \right| \left\| (\mathcal{F}^{-1}\omega_j * \psi)\chi_k \right\|_1 \\ \leq\; &\sum_{j=0}^{\infty} \sum_{k=-j}^{\infty} \left\| 2^{jn/d} 2^{k(n/d - n/q(\cdot) - \alpha(\cdot))} \varphi_j * f \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \left\| (\mathcal{F}^{-1}\omega_j * \psi)\chi_k \right\|_1 \\ \lesssim\; &\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}} \left(\|\psi\|_{\dot{K}_1^{(n/d-n/q-\alpha)^+,1} B_1^{(n/d-s)^+}} + \|\psi\|_{\dot{K}_1^{(n/d-n/q-\alpha)^-,1} B_1^{(n/d-s)^+}} \right). \end{aligned}$$

Consequently

$$|f(\psi)| \leq c\mu \, \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}} ,$$

where

$$\mu = \max(\|\psi\|_{B_{1,1}^{(n/q+\alpha-s)^+}} , \|\psi\|_{\dot{K}_1^{n/d-(n/q+\alpha)^+,1} B_1^{(n/d-s)^+}} , \|\psi\|_{\dot{K}_1^{n/d-(n/q+\alpha)^-,1} B_1^{(n/d-s)^+}}).$$

By our assumption on d we have

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{K}_1^{n/d-(n/q+\alpha)^+,1} B_1^{(n/d-s)^+}$$

and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{K}_1^{n/d-(n/q+\alpha)^-,1} B_1^{(n/d-s)^+}.$$

From this and the embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{1,1}^{(n/d-n/q-\alpha)^+}$, we obtain

$$|f(\psi)| \leq c\ p_N(\psi) \, \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_\infty^{s(\cdot)}} .$$

This proves that $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_\infty^{s(\cdot)}$ is continuously embedded in $\mathcal{S}'(\mathbb{R}^n)$. This completes the proof. ■

Applying Lemmas 2.9 and 2.11, we obtain the following Sobolev embeddings.

Theorem 3.8 *Let $p \in \mathcal{P}_0(\mathbb{R}^n)$, q_1, q_2, $\beta, r \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ with p^+, $r^+ < \infty$ and s_1, $s_2 \in C_{\text{loc}}^{\log}$. Let α_1, $\alpha_2 \in C_{\text{loc}}^{\log}$ be such that $(\alpha_1 + n/q_1)^- > 0$ and $(\alpha_2 + n/q_2)^- > 0$. Assume that*

$$s_1(\cdot) - n/q_1(\cdot) - \alpha_1(\cdot) \leq s_2(\cdot) - n/q_2(\cdot) - \alpha_2(\cdot). \tag{3.4}$$

Let $\varrho(\cdot) = \alpha_2(\cdot) + n/q_2(\cdot) - \alpha_1(\cdot) - n/q_1(\cdot)$. The embedding

$$\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),r(\cdot)} B_{\beta(\cdot)}^{s_1(\cdot)}, \tag{3.5}$$

holds if $q_2(\cdot) \leq q_1(\cdot)$ with $\alpha_2(\cdot) = \alpha_1(\cdot)$ or $(\alpha_2 - \alpha_1)^- > 0$ or $q_1(\cdot) \leq q_2(\cdot)$ with

$$\varrho(\cdot) = 0 \text{ or } \varrho^- > 0, \tag{3.6}$$

where

$\theta(\cdot) = r(\cdot)$ if $\varrho(\cdot) = 0$, $q_1(\cdot) \leq q_2(\cdot)$ or $\alpha_1(\cdot) = \alpha_2(\cdot)$, $q_2(\cdot) \leq q_1(\cdot)$

and

$\theta(\cdot) = p(\cdot)$ if $\varrho^- > 0$, $q_1(\cdot) \leq q_2(\cdot)$ or $(\alpha_2 - \alpha_1)^- > 0$, $q_2(\cdot) \leq q_1(\cdot)$.

Proof. Let $f \in \dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)}$. We have to show that

$$\begin{aligned} \|f\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),r(\cdot)} B_{\beta(\cdot)}^{s_1(\cdot)}} &\leq c \, \|f\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)}} \\ &\lesssim 1, \end{aligned}$$

which equivalente to

$$\varrho_{\ell^{\beta(\cdot)}\left(\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),r(\cdot)}\right)}\left(\left(2^{js_1(\cdot)}\varphi_j * f\right)_j\right) = \sum_{j=0}^{\infty} \left\| |2^{js_1(\cdot)}\varphi_j * f|^{\beta(\cdot)} \right\|_{\dot{K}_{q_1(\cdot)/\beta(\cdot)}^{\alpha_1(\cdot)\beta(\cdot),r(\cdot)/\beta(\cdot)}} \leq c.$$

Applying Lemmas 2.9 and 2.11 with $R = 2^j$, we obtain

$$\begin{aligned} & \left\| |2^{js_1(\cdot)}\varphi_j * f|^{\beta(\cdot)} \right\|_{\dot{K}_{q_1(\cdot)/\beta(\cdot)}^{\alpha_1(\cdot)\beta(\cdot),r(\cdot)/\beta(\cdot)}} \\ \leq \; & 2^{j(\alpha_2+n/q_2-\alpha_1-n/q_1+s_1-s_2)^+} \left\| |2^{js_2(\cdot)}\varphi_j * f|^{\beta(\cdot)} \right\|_{\dot{K}_{q_2(\cdot)/\beta(\cdot)}^{\alpha_2(\cdot)\beta(\cdot),\theta(\cdot)/\beta(\cdot)}} + 2^{-j}, \end{aligned}$$

then

$$\begin{aligned} \varrho_{\ell^{\beta(\cdot)}\left(\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),r(\cdot)}\right)}\left(\left(2^{js_1(\cdot)}\varphi_j * f\right)_j\right) & \leq 2 + \sum_{j=0}^{\infty} \left\| |2^{js_2(\cdot)}\varphi_j * f|^{\beta(\cdot)} \right\|_{\dot{K}_{q_2(\cdot)/\beta(\cdot)}^{\alpha_2(\cdot)\beta(\cdot),\theta(\cdot)/\beta(\cdot)}} \\ & \leq 2 + \varrho_{\ell^{\beta(\cdot)}\left(\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),r(\cdot)}\right)}\left(\left(2^{js_2(\cdot)}\varphi_j * f\right)_j\right) \\ & \leq 2 + c \\ & \lesssim 1. \end{aligned}$$

This finishes the proof. ∎

Theorem 3.9 *Let $p \in \mathcal{P}_0(\mathbb{R}^n)$ with $p^+ < \infty$, q_1, q_2, $\beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and s_1, $s_2 \in C_{\mathrm{loc}}^{\log}$. Let $\alpha_2 \in C_{\mathrm{loc}}^{\log}$ be such that $(\alpha_2 + n/q_2)^- > 0$. Assume that*

$$s_1(\cdot) - n/q_1(\cdot) \leq s_2(\cdot) - n/q_2(\cdot) - \alpha_2(\cdot).$$

Let $\varrho(\cdot) = \alpha_2(\cdot) + n/q_2(\cdot) - n/q_1(\cdot)$. The embedding

$$\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow B_{q_1(\cdot),\beta(\cdot)}^{s_1(\cdot)},$$

holds if $q_2(\cdot) \leq q_1(\cdot)$ with $\alpha_2(\cdot) = 0$ or $\alpha_2^- > 0$ or $q_1(\cdot) \leq q_2(\cdot)$ with

$$\varrho(\cdot) = 0 \text{ or } \varrho^- > 0,$$

where

$\theta(\cdot) = q_1(\cdot)$ if $\varrho(\cdot) = 0$, $q_1(\cdot) \leq q_2(\cdot)$ or $\alpha_2(\cdot) = 0$, $q_2(\cdot) \leq q_1(\cdot)$

and

$\theta(\cdot) = p(\cdot)$ if $\varrho^- > 0$, $q_1(\cdot) \leq q_2(\cdot)$ or $\alpha_2^- > 0$, $q_2(\cdot) \leq q_1(\cdot)$.

Proof. To prove this embeddings it suffices to take $r(\cdot) = q_1(\cdot)$ and $\alpha_1(\cdot) = 0$ in Theorem 3.8. ■

Using this result, we have the following useful consequence.

Corollary 3.10 *Let* $q_1, q_2, \beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ *and* $s_1, s_2 \in C_{\text{loc}}^{\log}$, *such that* $s_1(\cdot) - n/q_1(\cdot) \leq s_2(\cdot) - n/q_2(\cdot)$ *and* $q_2(\cdot) \leq q_1(\cdot)$. *Then*

$$B_{q_2(\cdot),\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_2(\cdot)}^{0,q_1(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow B_{q_1(\cdot),\beta(\cdot)}^{s_1(\cdot)}. \tag{3.7}$$

Proof. To prove (3.7) it suffices to take in Theorem 3.9, $\theta(\cdot) = q_1(\cdot)$ and $\alpha_2(\cdot) = 0$. However the desired embeddings follow immediately from the fact that

$$B_{q_2(\cdot),\beta(\cdot)}^{s_2(\cdot)} = \dot{K}_{q_2(\cdot)}^{0,q_2(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_2(\cdot)}^{0,q_1(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)}.$$

■

Let us define

$$\sigma_{q(\cdot)} := n(\frac{1}{\min(1, q(\cdot))} - 1) \text{ and } \bar{q} := \max(1, q).$$

Proposition 3.11 *Let* $p, q, \beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ *and* $s \in C_{\text{loc}}^{\log}$. *Let* $\alpha \in C_{\text{loc}}^{\log}$ *such that* $\alpha^- > 0$. *If* $(s - \sigma_q - \alpha)^- > 0$, *then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow L^{\bar{q}(\cdot)}.$$

Proof. To prove this proposition it suffices to use the embedding

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow B_{q(\cdot),\beta(\cdot)}^{s(\cdot)-\alpha(\cdot)} \hookrightarrow L^{\bar{q}(\cdot)},$$

where the first embedding is follows from Theorem 3.9, and the second embedding is given in [2, Proposition 6.9]. ■

Let C_u be the space of all bounded uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup norm. Concerning embeddings into C_u, we have the following result.

Corollary 3.12 *Let* p, $q \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ *with* $p^+ < \infty$ *and* $\alpha \in C_{\text{loc}}^{\log}$ *such that* $\alpha^- > 0$ *or* $\alpha(\cdot) = 0$. *Then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),\theta(\cdot)} B_1^{\alpha(\cdot)+n/q(\cdot)} \hookrightarrow C_u, \tag{3.8}$$

where

$$\theta(\cdot) = \begin{cases} \infty & \textit{if} \quad \alpha(\cdot) = 0 \\ p(\cdot) & \textit{if} \quad \alpha^- > 0. \end{cases}$$

Proof. It follows from Theorem 3.9 that

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),\theta(\cdot)} B_1^{\alpha(\cdot)+n/q(\cdot)} \hookrightarrow B_{q(\cdot),1}^{n/q(\cdot)} \hookrightarrow B_{\infty,1}^0.$$

Hence the result follows by the embedding $B_{\infty,1}^0 \hookrightarrow C_u$, see [53, Proposition 2.5.7]. ∎

The following statement holds by Theorem 3.8 and the fact that

$$\dot{K}_{q(\cdot)}^{0,q(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} = B_{q(\cdot),\beta(\cdot)}^{s_2(\cdot)}.$$

Theorem 3.13 *Let $p \in \mathcal{P}_0(\mathbb{R}^n)$ with $p^+ < \infty, q_1, q_2, \beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $s_1, s_2 \in C_{\mathrm{loc}}^{\log}$. Let $\alpha_1 \in C_{\mathrm{loc}}^{\log}$ be such that $(\alpha_1 + n/q_1)^- > 0$. Assume that*

$$s_1(\cdot) - n/q_1(\cdot) - \alpha_1(\cdot) \leq s_2(\cdot) - n/q_2(\cdot).$$

Let $\varrho(\cdot) = n/q_2(\cdot) - \alpha_1(\cdot) - n/q_1(\cdot)$. The embedding

$$B_{q_2(\cdot),\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),\theta(\cdot)} B_{\beta(\cdot)}^{s_1(\cdot)}, \tag{3.9}$$

holds if $q_2(\cdot) \leq q_1(\cdot)$ with $\alpha_1(\cdot) = 0$ or $\alpha_1^+ < 0$ or $q_1(\cdot) \leq q_2(\cdot)$ with $\varrho(\cdot) = 0$ or $\varrho^- > 0$, where $\theta(\cdot) = q_2(\cdot)$ if $\varrho(\cdot) = 0$, $q_1(\cdot) \leq q_2(\cdot)$ or $\alpha_1(\cdot) = 0$, $q_2(\cdot) \leq q_1(\cdot)$
and
$\theta(\cdot) = p(\cdot)$ if $\varrho^- > 0$, $q_1(\cdot) \leq q_2(\cdot)$ or $\alpha_1^+ < 0$, $q_2(\cdot) \leq q_1(\cdot)$.

Using this result, we obtain:

Corollary 3.14 *Let q_1, q_2, $\beta \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $s_1, s_2 \in C_{\mathrm{loc}}^{\log}$, such that $s_1(\cdot) - n/q_1(\cdot) \leq s_2(\cdot) - n/q_2(\cdot)$ and $q_2(\cdot) \leq q_1(\cdot)$. Then*

$$B_{q_2(\cdot),\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_1(\cdot)}^{0,q_2(\cdot)} B_{\beta(\cdot)}^{s_1(\cdot)} \hookrightarrow B_{q_1(\cdot),\beta(\cdot)}^{s_1(\cdot)}. \tag{3.10}$$

Proof. To prove this it suffices to take in Theorem 3.13, $\theta(\cdot) = q_2(\cdot)$ and $\alpha_1(\cdot) = 0$. Then the desired embedding is an immediate consequence of the fact that

$$\dot{K}_{q_1(\cdot)}^{0,q_2(\cdot)} B_{\beta(\cdot)}^{s_1(\cdot)} \hookrightarrow \dot{K}_{q_1(\cdot)}^{0,q_1(\cdot)} B_{\beta(\cdot)}^{s_1(\cdot)} = B_{q_1(\cdot),\beta(\cdot)}^{s_1(\cdot)}.$$

∎

Chapter 4

Boundedness of some bilinear operators on variable Herz-type Hardy spaces I

The Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent is the set of all tempered distributions on $\mathbb{R}^n$ for which $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} := \|\mathcal{M}(f)\|_{p(\cdot)}$ is finite. Based on this spaces, we introduce Herz-type Hardy spaces which obtained a great development in the past few years and played important roles in harmonic analysis.

In this chapter, we study the boundedness of some bilinear operators on variable Herz-type Hardy spaces given by finite sums of products of boundedness operators on variable Lebesgue spaces. Atomic decomposition is one of the most important methods to study the boundedness of this operators.

This chapter is organized as follows. In the first section, we recall notation of variable Herz-type Hardy spaces and their central atomic decomposition characterizations.

In the second section, the main results here, we present the boundedness of some bilinear operators on variable Herz-type Hardy spaces.

4.1 Definition and basic properties of variable Herz-type Hardy spaces

In this section, we will give the definition of Herz-type Hardy spaces with variable exponent $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with supp $\varphi \subseteq B_0$, such that

$$\int_{\mathbb{R}^n} \varphi(x)dx \neq 0 \text{ and } \varphi_t(\cdot) = t^{-n}\varphi\left(\frac{\cdot}{t}\right) \text{ for any } t > 0.$$

Let $\mathcal{M}_{\varphi}(f)$ be the grand maximal function of f defined by

$$\mathcal{M}_{\varphi}(f)(x) := \sup_{t>0} |\varphi_t * f(x)|.$$

The variable Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$ are defined in the following way.

Definition 4.1 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^{\infty}(\mathbb{R}^n)$. The* homogeneous Herz-type Hardy space *$H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{M}_{\varphi}(f) \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we put*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} := \|\mathcal{M}_{\varphi}(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

It can be shown that, if p, q, and α satisfy the conditions of definition, then the qausi-norm $\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}$ does not depend, up to the equivalence of quasi-norms, on the choice of the function φ and, hence, the space $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined independently of the choice φ.

Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ with variable exponent p but fixed α, q were recently studied by Wang and Liu [57]. The spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$, where the three parameters are variables, have been first studied in [16]. Many of the results from the fixed situation have variable counterparts. If $q \in \mathcal{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}$ with $1 < p^- \leq p^+ < \infty$, and let α and q are log-Hölder continuous, both at the origin and at infinity, such that $\alpha \in L^{\infty}(\mathbb{R}^n)$ and

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n\big(1 - \frac{1}{p^-}\big),$$

then

$$H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n) \cap L_{loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) = \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n).$$

One recognizes immediately that if α, p and q are constants, then the spaces $H\dot{K}_{p}^{\alpha,q}$ are just the usual Herz-type Hardy spaces were recently studied in [17] and [40]. See [23], [35], [43] and [52] for further results.

Now we present the characterization of Herz-type Hardy spaces in term of central atomic decompositions, which we will use it to study the boundedness of some bilinear operators on these spaces, see [16, Theorem 4] for the proof.

Theorem 4.2 *Let α and q are be* log*-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with $\mathrm{supp} a \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{1/q_\infty} \leq c\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha(\cdot) \geq n(1 - \frac{1}{p^-})$ and $s \geq [\alpha^+ + n(\frac{1}{p^-} - 1)]$, and if holds, then $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \approx \inf \left\{ \left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \right)^{1/q_\infty} \right\},$$

where the infimum is taken over all the decompositions of f as above.

Remark 4.3 *The atoms in the Theorem 1.36 and Theorem 4.2 can be taken to be supported in dyadic annuli.*

Since the space $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined independently of the choice φ. We many assume in this thesis $\varphi \geq 0$.

4.2 The boundedness from $\dot{K} \times \dot{K}$ into $H\dot{K}$ and from $H\dot{K} \times \dot{K}$ into $H\dot{K}$

In this section, we shall study the boundedness of some bilinear operators given by finite sums in variable Herz-type Hardy spaces. We will consider bilinear operators define by

$$B(f,g)(x) = \sum_{\gamma=1}^{N} \left(T_\gamma^1 f\right)(x) \left(T_\gamma^2 g\right)(x), \;\; x \in \mathbb{R}^n,$$

where $N \in \mathbb{N}$, T_γ^1 and T_γ^2 are operators satisfy some suitable conditions. The boundedness of these type of operators were first considered in [6] and [19], where they proved that B are bounded from $H^p \times H^q$ into H^r for certain rang of p's and q's when $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Grafakos and Li [20] also establishing $H^p \times H^q \hookrightarrow H^r$ boundedness for B, on the entire range $0 < p, q, r < \infty$ when $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Grafakos, Li and Yang [21] extend these results to Herz-type Hardy spaces.

In this thesis, for any f in $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ (or in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$) represented by $\sum_{i=-\infty}^{\infty} \lambda_i a_i$ where $\lambda_i \geq 0$ and each a_i is a central $(\alpha(\cdot), p(\cdot))$-atom with supp$a_i \subset B_i$ (or can be taken to be supported in dyadic annuli $C_i := \{x \in \mathbb{R}^n : 2^{j-2} < |x| \leq 2^{j+2}\}$), and for any non-negative integer s_1, the operators T_γ^j are bounded from $L^{p_j(\cdot)}(\mathbb{R}^n)$ into $L^{p_j(\cdot)}(\mathbb{R}^n)$, $j = 1, 2$, $\gamma = 1, ..., N$ and satisfying the two conditions

$$\left|T_\gamma^j a_i(x)\right| \leq c \frac{2^{is_1}}{|x|^{n+s_1}} \|a_i\|_{p_j(\cdot)} \left\|\chi_{C_i}\right\|_{p_j'(\cdot)}, \quad |x| > 2^{i+3}, j = 1, 2 \tag{4.1}$$

and

$$\left|T_\gamma^j a_i(x)\right| \leq c 2^{-in} \|a_i\|_{p_j(\cdot)} \left\|\chi_{C_i}\right\|_{p_j'(\cdot)}, \quad |x| < 2^{i-3}, j = 1, 2. \tag{4.2}$$

Remark 4.4 *Let T be a Calderón-Zygmund operator which is bounded on L^2, with kernel $K(x)$, which is $\mathcal{C}^\infty$ away from the origin, satisfying*

1- $|K(x)| \leq c\,|x|^{-n}$, if $x \neq 0$;

2- $\left|\frac{\partial^\beta}{\partial x^\beta}\left(K(x-y) - K(x)\right)\right| \leq c_\beta \frac{|y|}{|x|^{n+|\beta|+1}}$ if $|x| \geq 2\,|y|$, where $\beta = (\beta_1, \beta_2 ..., \beta_n)$ is any multi-index.

Example 4.5 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. Then Calderón-Zygmund operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and satisfying (4.1) and (4.2) with supp $a_i \subset C_i$.*

Proof. Let T be a Calderón-Zygmund operator which is bounded on L^2, with kernel K, which is $\mathcal{C}^\infty$ away from the origin, define by

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy,$$

by Taylor's developpement of $K(x-\cdot)$ at the origin, we have

$$K(x-y) = \sum_{|\beta|<s-1} \frac{(-y)^\beta}{\beta!}\frac{\partial^\beta}{\partial x^\beta}K(x) + \sum_{|\beta|=s-1} \frac{(-y)^\beta}{\beta!}\frac{\partial^\beta}{\partial x^\beta}K(x-\theta y), \theta \in \left]0,1\right[.$$

If a_i is a central $(\alpha(\cdot), p(\cdot))$-atom with supp$a_i \subset C_i$, then we have

$$\begin{aligned} Ta_i(x) &= \sum_{|\beta|<s-1} \int_{C_i} \frac{(-y)^\beta}{\beta!}\frac{\partial^\beta}{\partial x^\beta}K(x)a_i(y)dy + \sum_{|\beta|=s-1} \int_{C_i} \frac{(-y)^\beta}{\beta!}\frac{\partial^\beta}{\partial x^\beta}K(x-\theta y)a_i(y)dy \\ &= \sum_{|\beta|=s-1} \int_{C_i} \frac{(-y)^\beta}{\beta!}\frac{\partial^\beta}{\partial x^\beta}\Big(K(x-\theta y) - K(x)\Big)a_i(y)dy. \end{aligned}$$

Case 1: If $|x| > 2^{i+3}$ and $y \in C_i$, then $|x| > 2^{i+3} \geq 2\,|\theta y|$. By the condition 2 in Remark 4.4, we have

$$\begin{aligned} |Ta_i(x)| &\leq \sum_{|\beta|=s-1} \int_{C_i} \frac{|y|^\beta}{\beta!}\left|\frac{\partial^\beta}{\partial x^\beta}\Big(K(x-\theta y) - K(x)\Big)\right| |a_i(y)|\,dy \\ &\leq c\sum_{|\beta|=s-1} \int_{C_i} \frac{|y|^{1+|\beta|}}{|x|^{n+|\beta|+1}} |a_i(y)|\,dy \\ &\leq c\frac{2^{is}}{|x|^{n+s}} \int_{C_i} |a_i(y)|\,dy, \end{aligned}$$

where c is independant of i. By Hölder's inequality in $L^1(\mathbb{R}^n)$, the last expression is bounded by

$$c\frac{2^{is}}{|x|^{n+s}} \|a_i\|_{p(\cdot)} \left\|\chi_{C_i}\right\|_{p'(\cdot)}.$$

Case 2: If $|x| \leq 2^{i-3}$ and $y \in C_i$, then

$$\begin{aligned} |Ta_i(x)| &\leq \int_{C_i} |K(x-y)|\,|a_i(y)|\,dy \\ &\leq c\int_{C_i} \frac{1}{|x-y|^n} |a_i(y)|\,dy \leq c2^{-ni}\int_{C_i} |a_i(y)|\,dy, \end{aligned}$$

where c is independant of i. By Hölder's inequality in $L^1(\mathbb{R}^n)$, the last expression is bounded by

$$c2^{-ni} \|a_i\|_{p(\cdot)} \left\|\chi_{C_i}\right\|_{p'(\cdot)}.$$

Then T satisfying (4.1) and (4.2). This finishes the proof. ■

Now, we present the following important proposition.

Proposition 4.6 *If $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then*
(i) $\| \chi_k \|_{p(\cdot)} \approx c2^{\frac{kn}{p(x)}} \approx c2^{\frac{kn}{p_\infty}}, x \in R_k$ for all $k \geq 0$.
(ii) $\| \chi_k \|_{p(\cdot)} \approx c2^{\frac{kn}{p(0)}}$, for all $k \leq -1$.

Proof. For (i), is a simple consequence of the Lemma 1.29 (Since $|R_k| \approx 2^{kn} \geq 1$). For (ii), it suffice to show that

$$\begin{aligned} \rho_{p(\cdot)}\left(2^{-\frac{kn}{p(0)}}\chi_k\right) &= \int_{R_k} 2^{-knp(y)/p(0)}dy \\ &= 2^{-nk}\int_{R_k} 2^{-kn(p(y)-p(0))/p(0)}dy \\ &\approx 1. \end{aligned}$$

We have

$$2^{-kn(p(y)-p(0))/p(0)} = e^{n(p(y)-p(0))\ln 2^{-k}/p(0)}.$$

Since $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then

$$\begin{aligned} |p(y)-p(0)|\ln 2^{-k} &\lesssim \frac{\ln 2^{-k}}{\ln(1+\frac{1}{|y|})}, \ y \in R_k \\ &\lesssim \frac{\ln 2^{-k}}{\ln(1+2^{-k})}, \ k \leq -1 \\ &\lesssim c, \end{aligned}$$

where $c > 0$ independent of k. Using this estimation, we obtain

$$\rho_{p(\cdot)}\left(2^{-\frac{kn}{p(0)}}\chi_k\right) \approx 2^{-nk}\int_{R_k} e^{\frac{n}{p(0)}}dy \approx c,$$

where $c > 0$ independent of k. This finishes the proof of Proposition 4.6. ■

To prove the main results in this section, we need the following lemma.

Lemma 4.7 *Let* $q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $1 < p_i^- \leq p_i^+ < \infty, i = 1, 2$, $\frac{1}{p_1^-} + \frac{1}{p_2^-} < 1$, *and* $\alpha_1, \alpha_2 \in L^\infty(\mathbb{R}^n)$. *Let* T_γ^j *are operators bounded from* $L^{p_j(\cdot)}(\mathbb{R}^n)$ *into* $L^{p_j(\cdot)}(\mathbb{R}^n)$, $j = 1, 2$, $\gamma = 1, ..., N$ *and satisfying the two conditions* (4.1) *and* (4.2). *For all* $(\alpha_1(\cdot), q_1(\cdot))$*-atoms* a *with* $\operatorname{supp} a_i \subset C_i$, *and all* $(\alpha_2(\cdot), q_2(\cdot))$*-atoms* b *with* $\operatorname{supp} b_j \subset C_j$. *We have*

$$\sup_{t > \frac{|x|}{2}} \int_{|y| \geq 2^{j-4}} \varphi_{t,x}(y) \left| B(a_i, b_j)(y) \right| dy \leq c \sum_{\gamma=1}^{N} \mathcal{M}\left(\left| T_\gamma^1 a_i \right|^{(p_2^-)'} \right)^{1/(p_2^-)'} (x)\, 2^{-jn} \left\| b_j \right\|_{p_2(\cdot)} \left\| \chi_{C_j} \right\|_{p_2'(\cdot)}.$$

Proof of Lemma. We have

$$\int_{|y| \geq 2^{j-4}} \varphi_{t,x}(y) \left| B(a_i, b_j) \right| dy$$

can be rewritten as

$$\sum_{\ell=j-3}^{\infty} \int_{R_\ell} \varphi_{t,x}(y) \left| B(a_i, b_j) \right| dy. \tag{4.3}$$

Applying Hölder's inequality with $\frac{1}{(p_2^-)'} + \frac{1}{p_2^-} = 1$, (4.3) is bounded by

$$\begin{aligned} & c \sum_{\gamma=1}^{N} \sum_{\ell=j-3}^{\infty} \Big(\int_{\mathbb{R}^n} \varphi_{t,x}(y) \left| T_\gamma^1 a_i \right|^{(p_2^-)'} dy \Big)^{1/(p_2^-)'} \Big(\int_{R_\ell} \varphi_{t,x}(y) \left| T_\gamma^2 b_j \right|^{p_2^-} dy \Big)^{1/p_2^-} \\ \leq \; & c \sum_{\gamma=1}^{N} \sum_{\ell=j-3}^{\infty} \mathcal{M}\Big(\left| T_\gamma^1 a_i \right|^{(p_2^-)'} \Big)^{1/(p_2^-)'} (x) \Big(\int_{R_\ell} \varphi_{t,x}(y) \left| T_\gamma^2 b_j \right|^{p_2^-} dy \Big)^{1/p_2^-}, \end{aligned}$$

where we used that $|\varphi_{t,x}(\cdot)| \leq ct^{-n}$ and $|x - y| \leq t$. Obseve that $|y| \leq |x| + |x - y| \leq 3t$ and again $|\varphi_{t,x}(\cdot)| \leq ct^{-n}$, we obtain

$$\Big(\int_{R_\ell} \varphi_{t,x}(y) \left| T_\gamma^2 b_j \right|^{p_2^-} dy \Big)^{1/p_2^-} \leq c 2^{-\frac{\ell n}{p_2^-}} \left\| \left(T_\gamma^2 b_j \right) \chi_\ell \right\|_{p_2^-}, \tag{4.4}$$

by Hölder's inequality and the $L^{p_2(\cdot)}(\mathbb{R}^n)$- boundedness of T_γ^2, with $\frac{1}{p_2^-} = \frac{1}{r_2(\cdot)} + \frac{1}{p_2(\cdot)}$ the right hand side of (4.4) is bounded by

$$c 2^{-\frac{\ell n}{p_2^-}} \left\| T_\gamma^2 b_j \right\|_{p_2(\cdot)} \left\| \chi_\ell \right\|_{r_2(\cdot)} \leq c 2^{-\frac{\ell n}{p_2^-}} \left\| b_j \right\|_{p_2(\cdot)} \left\| \chi_\ell \right\|_{r_2(\cdot)},$$

where c is independant of j, ℓ. Therefore, (4.3) is bounded by

$$\begin{aligned} & c \sum_{\ell=j-3}^{\infty} 2^{-\frac{\ell n}{p_2^-}} \left\| \chi_\ell \right\|_{r_2(\cdot)} \left\| b_j \right\|_{p_2(\cdot)} \sum_{\gamma=1}^{N} \mathcal{M}\Big(\left| T_\gamma^1 a_i \right|^{(p_2^-)'} \Big)^{1/(p_2^-)'} (x) \\ \lesssim \; & \left\| b_j \right\|_{p_2(\cdot)} \sum_{\ell=j-3}^{\infty} \left\| \chi_\ell \right\|_{p_2(\cdot)}^{-1} \sum_{\gamma=1}^{N} \mathcal{M}\Big(\left| T_\gamma^1 a_i \right|^{(p_2^-)'} \Big)^{1/(p_2^-)'} (x), \end{aligned} \tag{4.5}$$

where we used (1.4) with R_ℓ in place of B. Now we can distinguish three cases as follows: If $j \geq 3$ then since $n/(p_2)_\infty > 0$, we get by Lemma 1.29 and the logarithmic decay condition of p_2 at infinity

$$\sum_{\ell=j-3}^{+\infty} \|\chi_{C_j}\|_{p_2(\cdot)} \|\chi_\ell\|_{p_2(\cdot)}^{-1} \lesssim \sum_{\ell=j-3}^{+\infty} 2^{(j-\ell)n/(p_2)_\infty} \leq c.$$

If $j < 0$ then, we have again by Lemma 1.29 and the log-Hölder continuous conditions, both at the origin and at infinity

$$\begin{aligned}
\sum_{\ell=j-3}^{+\infty} \|\chi_{C_j}\|_{p_2(\cdot)} \|\chi_\ell\|_{p_2(\cdot)}^{-1} &= \sum_{\ell=j-3}^{-1} \|\chi_{C_j}\|_{p_2(\cdot)} \|\chi_\ell\|_{p_2(\cdot)}^{-1} + \sum_{\ell=0}^{+\infty} \|\chi_{C_j}\|_{p_2(\cdot)} \|\chi_\ell\|_{p_2(\cdot)}^{-1} \\
&\lesssim \sum_{\ell=j-3}^{-1} 2^{(j-\ell)n/p_2(0)} + \sum_{\ell=0}^{+\infty} 2^{-\ell n/(p_2)_\infty} \\
&\leq c.
\end{aligned}$$

The same estimate can be obtained if $0 \leq j < 3$. Hence, in any case, we obtain

$$\sum_{\ell=j-3}^{+\infty} \|\chi_{C_j}\|_{p_2(\cdot)} \|\chi_\ell\|_{p_2(\cdot)}^{-1} \leq c.$$

Therefore, we get that (4.5) is bounded by

$$c\sum_{\gamma=1}^{N} \mathcal{M}\left(\left|T_\gamma^1 a_i\right|^{(p_2^-)'}\right)^{1/(p_2^-)'}(x)\, 2^{-jn} \|b_j\|_{p_2(\cdot)} \|\chi_{C_j}\|_{p_2'(\cdot)}.$$

This finishes the proof. ∎

Now, we give the first main result of this section with T_γ^j are Calderón-Zygmund operators.

Theorem 4.8 *Let $q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $1 < p_i^- \leq p_i^+ < \infty$, $-\frac{n}{p_i^+} < \alpha_i^- \leq \alpha_i^+ < n(1 - \frac{1}{p_i^-})$, $i = 1, 2$, and $\alpha(\cdot) = \alpha_1(\cdot) + \alpha_2(\cdot)$ such that q_1, q_2, α_1 and α_2 are* log*-Hölder continuous, both at the origin and at infinity. Let $s \geq [(\alpha_1^+ + n/p_1^-) + (\alpha_2^+ + n/p_2^-) - n]$ be a non-negative integer such that*

$$\int_{\mathbb{R}^n} x^\beta B(f,g)(x)dx = 0, \tag{4.6}$$

for all multi-indices β with $|\beta| \leq s$, and all $f, g \in L^2(\mathbb{R}^n)$ with compact support. Then $B(f,g)$ can be extended to a bounded operator from $\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n) \times \dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ into $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.

If q_1, q_2 p_1, p_2, α_1 and α_2 are constants, this result is from [21, Theorem 1].

Proof. For the proof of this theorem, we can repeat arguments similar to the ones used in the proof of Theorem 1 in [21], where the constant exponent case was studied.

Let $f \in \dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in \dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$. We choose $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\eta \equiv 1$ on $B(0,2)$ and supp$\eta \subseteq B(0,4)$.

Define

$$\eta_0(y) = \eta(\tfrac{x-y}{t})$$

and

$$\eta_1(y) = 1 - \eta_0(y).$$

We split $B(f,g)$ as sum

$$B(f,g) = B(f,\eta_1 g) + B(\eta_1 f, g) - B(\eta_1 f, \eta_1 g) + B(\eta_0 f, \eta_0 g).$$

We have

$$\begin{aligned}
\left\|B(f,g)\right\|_{H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} &= \left\|\mathcal{M}_\varphi(B(f,g))\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \\
&\leq \left\|\mathcal{M}_\varphi(B(f,\eta_1 g))\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} + \left\|\mathcal{M}_\varphi(B(\eta_1 f, g))\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \left\|\mathcal{M}_\varphi(B(\eta_1 f,\eta_1 g))\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} + \left\|\mathcal{M}_\varphi(B(\eta_0 f,\eta_0 g))\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \\
&= \left\|\Lambda_1\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} + \left\|\Lambda_2\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} + \left\|\Lambda_3\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} + \left\|\Lambda_4\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For Λ_1. We have

$$\begin{aligned}
\Lambda_1 &\leq c\sum_{\gamma=1}^{N}\sup_{t>0}\int_{\mathbb{R}^n}\phi_{t,x}(y)\left|T^1_\gamma f(y)\right|\left|T^2_\gamma(\eta_1 g(y) - T^2_\gamma(\eta_1 g)(x)\right|dy \\
&\quad + c\sum_{\gamma=1}^{N}\sup_{t>0}\int_{\mathbb{R}^n}\phi_{t,x}(y)\left|T^1_\gamma f(y)\right|\left|T^2_\gamma(\eta_1 g)(x)\right|dy \\
&\leq c\sum_{\gamma=1}^{N}\mathcal{M}(T^1_\gamma f)(x)\mathcal{M}(\eta_1 g)(x) + c\sum_{\gamma=1}^{N}\mathcal{M}(T^1_\gamma f)(x)\left|T^2_\gamma(\eta_1 g)(x)\right|,
\end{aligned}$$

By Hölder's inequality in $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|\Lambda_1\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} &\leq c\sum_{\gamma=1}^{N}\|\mathcal{M}(T^1_\gamma f)(x)\mathcal{M}(\eta_1 g)(x)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \\ &\quad + c\sum_{\gamma=1}^{N}\|\mathcal{M}(T^1_\gamma f)(x)\,|T^2_\gamma(\eta_1 g)(x)|\,\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \\ &\leq c\sum_{\gamma=1}^{N}\|\mathcal{M}(T^1_\gamma f)\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\|\mathcal{M}(\eta_1 g)\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \\ &\quad + c\sum_{\gamma=1}^{N}\|\mathcal{M}(T^1_\gamma f)\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\|T^2_\gamma(\eta_1 g)\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)},\end{aligned}$$

by Theorem 2.5, the last term is bounded by

$$c\|f\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\|g\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

The estimates for terms Λ_2 and Λ_3 in $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$-norm are similar to Λ_1.

Let consider Λ_4. We have

$$\begin{aligned}&\sup_{t>0}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(\eta_0 f,\eta_0 g)(y)dy\right| \\ =&\sup_{t>0}\left|\int_{\mathbb{R}^n}\sum_{\gamma=1}^{N}(\eta_0 f)(y)\left[\left(T^1_\gamma\right)^*\left[(\varphi_{t,x}(\cdot)-P^s_y(\cdot-y))T^2_\gamma(\eta_0 g)(\cdot)\right](y)\right]dy\right|,\end{aligned}$$

where $\left(T^1_\gamma\right)^*$ is the adjoint of T^1_γ.

Applying Hölder's inequality in $L^1(\mathbb{R}^n)$ with $\frac{1}{\tau_1}+\frac{1}{\tau_1'}=1$, then the last expression can be estimated by

$$c\left\|\eta_0 f\right\|_{\tau_1}\left\|\left(T^1_\gamma\right)^*\left[(\varphi_{t,x}(\cdot)-P^s_y(\cdot-y))T^2_\gamma(\eta_0 g)(\cdot)\right]\right\|_{\tau_1'}.$$

We use the same proof technique as proof of Theorem 1.5 in [20], the last expression is bounded by

$$c\mathcal{M}(|f|^{\tau_1})^{1/\tau_1}(x)\mathcal{M}(|g|^{\tau_2})^{1/\tau_2}(x)$$

for all τ_1, τ_2 such that $1<\tau_1<p_1^-$, $1<\tau_2<p_2^-$ and $1/\tau_1+1/\tau_2=(n+s+\varepsilon)/n>1$ for some fixed $0<\varepsilon<1$.

By our assumptions on α_1 and α_2, we may choose τ_1, τ_2 as above such that

$$-n/p_1^+<\alpha_1^-\leq\alpha_1^+<(n/\tau_1-n/p_1^-)\text{ and }-n/p_2^+<\alpha_2^-\leq\alpha_2^+<(n/\tau_2-n/p_2^-).$$

Applying Hölder's inequality in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we obtain $\|\Lambda_4\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$c\left\|\mathcal{M}(|f|^{\tau_1})^{1/\tau_1}\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\left\|\mathcal{M}(|g|^{\tau_2})^{1/\tau_2}\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

By Theorem 2.5, we have

$$\begin{aligned}\left\|\mathcal{M}(|f|^{\tau_1})^{1/\tau_1}\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} &= \left\|\mathcal{M}(|f|^{\tau_1})\right\|_{\dot{K}_{p_1(\cdot)/\tau_1}^{\tau_1\alpha_1(\cdot),q_1(\cdot)/\tau_1}(\mathbb{R}^n)} \\ &\leq c\left\||f|^{\tau_1}\right\|_{\dot{K}_{p_1(\cdot)/\tau_1}^{\tau_1\alpha_1(\cdot),q_1(\cdot)/\tau_1}(\mathbb{R}^n)} = c\left\|f\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)},\end{aligned}$$

similarly, we obtain that

$$\left\|\mathcal{M}(|g|^{\tau_2})^{1/\tau_2}\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} \leq c\left\|g\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}.$$

Then, we obtain

$$\|\Lambda_4\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c\left\|f\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\left\|g\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 4.8. ■

Next, we present the second result of this section.

Theorem 4.9 *Let $q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $1 < p_i^- \leq p_i^+ < \infty$, $\frac{1}{p_1^-} + \frac{1}{p_2^-} < 1$, $\alpha_1^- + n/p_1^+ \geq n$, $0 < \alpha_2^- + n/p_2^+ \leq \alpha_2^+ + n/p_2^- < n$, and $\alpha(\cdot) = \alpha_1(\cdot) + \alpha_2(\cdot)$ such that q_1, q_2, α_1 and α_2 are* log*-Hölder continuous, both at the origin and at infinity. Let T_γ^j are operators bounded from $L^{p_j(\cdot)}(\mathbb{R}^n)$ into $L^{p_j(\cdot)}(\mathbb{R}^n)$, $j = 1, 2$, $\gamma = 1, ..., N$ and satisfying the two conditions (4.1) and (4.2). Let $s \geq [(\alpha_1^+ + n/p_1^-) + (\alpha_2^+ + n/p_2^-) - n]$ be a non-negative integer such that*

$$\int_{\mathbb{R}^n} x^\beta B(a, g)(x)dx = 0 \tag{4.7}$$

for all multi-indices β with $|\beta| \leq s$, for all $(\alpha_1(\cdot), q_1(\cdot))$-atoms a, and all $g \in L^2(\mathbb{R}^n)$ with compact support. Then $B(f, g)$ can be extended to a bounded operator from $H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n) \times \dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ into $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.

If q_1, q_2 p_1, p_2, α_1 and α_2 are constants, this result is from [21, Theorem 2].

Proof. Our proofs use partially some decomposition techniques already used in [21] where the constant exponent case was studied. Let $f \in H\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in \dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$. Using Theorem 4.2 for f and Theorem 1.36 for g, we can write $f = \sum\limits_{i=-\infty}^{+\infty} \lambda_i a_i$ and $g = \sum\limits_{j=-\infty}^{+\infty} \mu_j b_j$ where $\lambda_i, \mu_j \geq 0$, supp$a_i \subseteq C_i$ and supp$b_j \subseteq C_j$. We start observing that

$$\left\|B(f,g)\right\|_{H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \left\|\mathcal{M}_\varphi(B(f,g))\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\begin{aligned} \mathcal{M}_\varphi(B(f,g))(x) &= \sup_{t>0} |\varphi_t * B(f,g)(x)| \\ &\leq \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.8}$$

Note that to simplify the writing, we use the notation

$$\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \cdots = \sum_{i,j} \cdots .$$

For any $x \in \mathbb{R}^n$, we may split (4.8) into two functions, namely Ψ_1 and Ψ_2, where

$$\begin{aligned} \Psi_1(x) &= \sum_{i,j} \lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x)\chi_{\{|\cdot|\leq 2^{i+4}\}}(x) \\ &\text{and} \\ \Psi_2(x) &= \sum_{i,j} \lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x)\chi_{\{|\cdot|> 2^{i+4}\}}(x). \end{aligned}$$

Step 1. Estimation of Ψ_1. We can split Ψ_1 into three parts: $\Psi_1 := I + II + III$, where

$$I(x) = \sum_{i,j} \lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x)\chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\leq 2^{j-5}\}}(x),$$

$$II(x) = \sum_{i,j} \lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x)\chi_{\{|\cdot|\leq 2^{i+4},2^{j-5}<|\cdot|\leq 2^{j+4}\}}(x)$$

and

$$III(x) = \sum_{i,j} \lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x)\chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\geq 2^{j+4}\}}(x).$$

Substep 1.1. Estimation of I. Observe that $I \leq H_1 + H_2$, where

$$H_1(x) = \sum_{i,j} \lambda_i\mu_j \sup_{0<t\leq \frac{|x|}{2}} \left| \int_{\mathbb{R}^n} \varphi_{t,x}(y) B(a_i,b_j)(y)dy \right| \chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\leq 2^{j-5}\}}(x)$$

and

$$H_2(x)=\sum_{i,j}\lambda_i\mu_j\sup_{t>\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\leq 2^{j-5}\}}(x).$$

Let us consider H_1. We have in this case $|y|\leq|x|+|x-y|\leq\frac{3}{2}|x|<2^{j-4}$. Then by (4.2) for b_j, we have

$$\begin{aligned}H_1(x) &\leq c\sum_{i,j}\lambda_i\mu_j 2^{-jn}\|b_j\|_{p_2(\cdot)}\|\chi_{C_j}\|_{p_2'(\cdot)}\sum_{\gamma=1}^{N}\sup_{0<t\leq\frac{|x|}{2}}\int_{\mathbb{R}^n}\varphi_{t,x}(y)\left|T_\gamma^1 a_i(y)\right|dy\\ &\leq c\sum_{i,j}\lambda_i\mu_j 2^{-jn}\|b_j\|_{p_2(\cdot)}\|\chi_{C_j}\|_{p_2'(\cdot)}\sum_{\gamma=1}^{N}\mathcal{M}(|T_\gamma^1 a_i|)(x),\end{aligned}$$

By Hölder's inequality in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, see Lemma 1.34, we estimate H_1 in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$-norm by

$$\begin{aligned}&c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{+\infty}\lambda_i\mathcal{M}(|T_\gamma^1 a_i|)\chi_{\{|\cdot|\leq 2^{i+4}\}}\Big\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\\ &\times\Big\|\sum_{j=-\infty}^{+\infty}\mu_j 2^{-jn}\|b_j\|_{p_2(\cdot)}\|\chi_{C_j}\|_{p_2'(\cdot)}\chi_{\{|\cdot|\leq 2^{j-5}\}}\Big\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}\\ =\;&c\sum_{\gamma=1}^{N}J_1\times J_2.\end{aligned}$$

Using Proposition 1.33, J_1 is equivalent to

$$\begin{aligned}&c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{i=-\infty}^{+\infty}\lambda_i\mathcal{M}\big(|T_\gamma^1 a_i|\big)\chi_{\{|\cdot|\leq 2^{i+4}\}}\chi_k\Big\|_{p_1(\cdot)}^{q_1(0)}\right\}^{1/q_1(0)}\\ &+c\left\{\sum_{k=0}^{+\infty}2^{k(\alpha_1)_\infty(q_1)_\infty}\Big\|\sum_{i=-\infty}^{+\infty}\lambda_i\mathcal{M}\big(|T_\gamma^1 a_i|\big)\chi_{\{|\cdot|\leq 2^{i+4}\}}\chi_k\Big\|_{p_1(\cdot)}^{(q_1)_\infty}\right\}^{1/(q_1)_\infty}.\end{aligned}$$

Using the boundedness of $\mathcal{M}$ and T_γ^1 on $L^{p_1(\cdot)}(\mathbb{R}^n)$, the last expression is bounded by

$$\begin{aligned}&c\left\{\sum_{k=-\infty}^{-1}2^{k\alpha_1(0)q_1(0)}\Big(\sum_{i=k-5}^{+\infty}\lambda_i\|a_i\|_{p_1(\cdot)}\Big)^{q_1(0)}\right\}^{1/q_1(0)}\\ &+c\left\{\sum_{k=0}^{+\infty}2^{k(\alpha_1)_\infty(q_1)_\infty}\Big(\sum_{i=k-5}^{+\infty}\lambda_i\|a_i\|_{p_1(\cdot)}\Big)^{(q_1)_\infty}\right\}^{1/(q_1)_\infty}.\end{aligned}\tag{4.9}$$

For $k\leq -1$, we write

$$\sum_{i=k-5}^{+\infty}\lambda_i\|a_i\|_{p_1(\cdot)}=\sum_{i=k-5}^{-1}...+\sum_{i=0}^{+\infty}...,\tag{4.10}$$

then, since a_i's are $(\alpha_1(\cdot), p_1(\cdot))$- atoms, (4.9) is bounded by

$$c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{i=k-5}^{-1}\lambda_i 2^{(k-i)\alpha_1(0)}\Big)^{q_1(0)}\right\}^{1/q_1(0)}+c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{i=0}^{+\infty}\lambda_i 2^{(k-i)\alpha_1^-}\Big)^{q_1(0)}\right\}^{1/q_1(0)}$$
$$+c\left\{\sum_{k=0}^{+\infty}\Big(\sum_{i=k-5}^{+\infty}\lambda_i 2^{(k-i)(\alpha_1)_\infty}\Big)^{(q_1)_\infty}\right\}^{1/(q_1)_\infty}. \tag{4.11}$$

Since $\alpha_1(0), (\alpha_1)_\infty > 0$, by Lemma 1.28, the first and the third term of (4.11) are estimated by

$$c\left\{\sum_{k=-\infty}^{-1}\lambda_k^{q_1(0)}\right\}^{1/q_1(0)}+c\left\{\sum_{k=0}^{+\infty}\lambda_k^{(q_1)_\infty}\right\}^{1/(q_1)_\infty}\leq c\,\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}.$$

To estimate the middle term we use the fact that $\alpha_1^- > 0$, we get

$$\begin{aligned}
\sum_{i=0}^{+\infty}\lambda_i 2^{(k-i)\alpha_1^-} &\leq c2^{k\alpha_1^-}\sup_{i\geq 0}\lambda_i\\
&\leq c2^{k\alpha_1^-}\left\{\sum_{i=0}^{+\infty}\lambda_i^{(q_1)_\infty}\right\}^{1/(q_1)_\infty}\\
&\leq c2^{k\alpha_1^-}\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)},\quad k\leq 0.
\end{aligned}$$

Thus,

$$J_1\leq c\,\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}.$$

Now we treat the term J_2. Similar to J_1, J_2 is equivalent to

$$\left\{\sum_{k=-\infty}^{-1}\Big(2^{k\alpha_2(0)}\sum_{j=k+4}^{+\infty}\mu_j 2^{-jn}\|b_j\|_{p_2(\cdot)}\|\chi_{C_j}\|_{p_2'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\Big)^{q_2(0)}\right\}^{1/q_2(0)}$$
$$+\left\{\sum_{k=0}^{+\infty}\Big(2^{k(\alpha_2)_\infty}\sum_{j=k+4}^{+\infty}\mu_j 2^{-jn}\|b_j\|_{p_2(\cdot)}\|\chi_{C_j}\|_{p_2'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\Big)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}.$$

For $k\leq -1$, we split $\sum\limits_{j=k+4}^{+\infty}\cdots$ as in (4.10)

$$\sum_{j=k+4}^{+\infty}\cdots=\sum_{j=k+4}^{-1}\cdots+\sum_{j=0}^{+\infty}\cdots,\text{ we put }\sum_{j=k+4}^{-1}\cdots=0\text{ if }-1\leq k\leq -4,$$

using Lemma 1.29 and the fact that b_j's are $(\alpha_2(\cdot), p_2(\cdot))$- atoms, we distinguish two cases as follows :

• In the case $k \leq -5$ and $k+4 \leq j \leq -1$, we use the log-Hölder decay of p_2 at the origin to get

$$2^{-jn} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \|\chi_k\|_{p_2(\cdot)} \lesssim 2^{\frac{(k-j)n}{p_2(0)} - j\alpha_2(0)},$$

• In the case of $k \leq -5$ and $j \geq 0$ or $-4 \leq k \leq -1$ and $j \geq k+4$, we have

$$2^{-jn} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \|\chi_k\|_{p_2(\cdot)} \lesssim 2^{\frac{(k-j)n}{p_2^+} - j\alpha_2^-}.$$

For $k \geq 0$ and $j \geq k+4$, we use the logarithmic decay condition of p_2 at infinity to get

$$2^{-jn} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \|\chi_k\|_{p_2(\cdot)} \lesssim 2^{\frac{(k-j)n}{(p_2)_\infty} - j(\alpha_2)_\infty}.$$

Therefore,

$$\begin{aligned} J_2 \;&\leq\; c\left\{\sum_{k=-\infty}^{-5}\Big(\sum_{j=k+4}^{-1}\mu_j 2^{(k-j)(\alpha_2+n/p_2)(0)}\Big)^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\ &\quad + c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=0}^{+\infty}\mu_j 2^{(k-j)\left(\alpha_2^-+n/p_2^+\right)}\Big)^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\ &\quad + c\left\{\sum_{k=0}^{+\infty}\Big(\sum_{j=k+4}^{+\infty}\mu_j 2^{(k-j)(\alpha_2+n/p_2)_\infty}\Big)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}, \end{aligned}$$

as argued before, we get the estimate

$$J_2 \leq c\left\{\sum_{k=-\infty}^{-1}(\mu_k)^{q_2(0)}\right\}^{1/q_2(0)} + c\left\{\sum_{k=0}^{+\infty}(\mu_k)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \leq c\,\|g\|_{\dot K_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

This finish the estimate of H_1.

Now let us estimate H_2. Obviousely H_2 can be decomposed as follows

$$\begin{aligned} H_2(x) \;&=\; \sum_{i,j}\lambda_i\mu_j \sup_{t>\frac{|x|}{2}}\left|\int_{|y|<2^{j-4}}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\leq 2^{j-5}\}}(x) \\ &\quad + \sum_{i,j}\lambda_i\mu_j \sup_{t>\frac{|x|}{2}}\left|\int_{|y|\geq 2^{j-4}}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\leq 2^{j-5}\}}(x). \end{aligned}$$

The first term can be estimated exactly as in J_1. By Lemma 4.7, the second term in

$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$-norm is bounded by

$$\begin{aligned}
&\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{+\infty}\lambda_i\mathcal{M}\Big(\big|T^1_\gamma a_i\big|^{(p_2^-)'}\Big)^{1/(p_2^-)'}\chi_{\{|\cdot|\le 2^{i+4}\}}\Big\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\\
&\times\Big\|\sum_{j=-\infty}^{+\infty}\mu_j 2^{-jn}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\chi_{\{|\cdot|\le 2^{j-5}\}}\Big\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}\\
=\;&\sum_{\gamma=1}^{N}N_1\times N_2. \qquad (4.12)
\end{aligned}$$

By the $L^{p_1(\cdot)/(p_2^-)'}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$, the $L^{p_1(\cdot)}(\mathbb{R}^n)$-boundedness of T^1_γ and since $p_1^- > (p_2^-)'$, we have

$$\begin{aligned}
\Big\|\mathcal{M}\Big(\big|T^1_\gamma a_i\big|^{(p_2^-)'}\Big)^{1/(p_2^-)'}\Big\|_{p_1(\cdot)} &= \Big\|\mathcal{M}\Big(\big|T^1_\gamma a_i\big|^{(p_2^-)'}\Big)\Big\|^{1/(p_2^-)'}_{p_1(\cdot)/(p_2^-)'}\\
&\le c\Big\|\big|T^1_\gamma a_i\big|^{(p_2^-)'}\Big\|^{1/(p_2^-)'}_{p_1(\cdot)/(p_2^-)'}\\
&= c\big\|T^1_\gamma a_i\big\|_{p_1(\cdot)}\\
&\le c\left\|a_i\right\|_{p_1(\cdot)}.
\end{aligned}$$

Using analogous arguments as that of J_1, we estimate N_1 by

$$c\left\{\sum_{k=-\infty}^{-1}(\lambda_k)^{q_1(0)}\right\}^{1/q_1(0)}+c\left\{\sum_{k=0}^{+\infty}(\lambda_k)^{(q_1)_\infty}\right\}^{1/(q_1)_\infty}\le c\left\|f\right\|_{H\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Estimation of N_2 is the same thing as J_2.

Substep 1.2. Estimation of ***II*****.** We choose p_0 such that $1/p_0+1/p_0'=1$ and $(p_2^-)'<p_0<p_1^-$. We then have II in $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}$-norm is bounded by

$$\begin{aligned}
&c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{+\infty}\lambda_i\mathcal{M}\big(\big|T^1_\gamma a_i\big|^{p_0}\big)^{1/p_0}\chi_{\{|\cdot|\le 2^{i+4}\}}\Big\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\\
&\times\Big\|\sum_{j=-\infty}^{+\infty}\mu_j\mathcal{M}\Big(\big|T^2_\gamma b_j\big|^{p_0'}\Big)^{1/p_0'}\chi_{\{2^{j-5}<|\cdot|\le 2^{j+4}\}}\Big\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}. \qquad (4.13)
\end{aligned}$$

Since $p_0<p_1^-$, the first norm above is estimated as in N_1 by $c\left\|f\right\|_{H\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

Similarly, using Propostion 1.33, which yields that the second norm in (4.13) is equivalent to

$$\begin{aligned}
&c\Big(\sum_{k=-\infty}^{-1}\Big(2^{k\alpha_2(0)}\sum_{j=k-4}^{k+4}\mu_j\Big\|\mathcal{M}\Big(\big|T^1_\gamma b_j\big|^{p_0'}\Big)^{1/p_0'}\Big\|_{p_2(\cdot)}\Big)^{q_2(0)}\Big)^{1/q_2(0)}\\
&+c\Big(\sum_{k=0}^{+\infty}\Big(2^{k(\alpha_2)_\infty}\sum_{j=k-4}^{k+4}\mu_j\Big\|\mathcal{M}\Big(\big|T^2_\gamma b_j\big|^{p_0'}\Big)^{1/p_0'}\Big\|_{p_2(\cdot)}\Big)^{(q_2)_\infty}\Big)^{1/(q_2)_\infty}. \qquad (4.14)
\end{aligned}$$

By the $L^{p_2(\cdot)/p_0'}$- boundedness of $\mathcal{M}$ and the $L^{p_2(\cdot)}$- boundedness of T_γ^2, we get

$$\Big\|\mathcal{M}\Big(\big|T_\gamma^2 b_j\big|^{p_0'}\Big)^{1/p_0'}\Big\|_{p_2(\cdot)} \leq c\big\|T_\gamma^2 b_j\big\|_{p_2(\cdot)} \leq c\big\|b_j\big\|_{p_2(\cdot)}.$$

As before, the terms in (4.14) are bounded by

$$\begin{aligned} & c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=k-4}^{k+4}\mu_j 2^{(k-j)\alpha_2(0)}\Big)^{q_2(0)}\right\}^{1/q_2(0)} + c\left\{\sum_{k=0}^{+\infty}\Big(\sum_{j=k-4}^{k+4}\mu_j 2^{(k-j)(\alpha_2)_\infty}\Big)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \\ \leq\ & c\left\{\sum_{k=-\infty}^{-1}(\mu_k)^{q_2(0)}\right\}^{1/q_2(0)} + c\left\{\sum_{k=0}^{+\infty}(\mu_k)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \leq c\,\|g\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}, \end{aligned}$$

where in the second inequality we used Lemma 1.28.

Substep 1.3. Estimation of III**.** We also divide III into $H_3 + H_4$, where

$$\begin{aligned} H_3(x) &= \sum_{i,j}\lambda_i\mu_j \sup_{0<t\leq\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\geq 2^{j+4}\}}(x) \\ H_4(x) &= \sum_{i,j}\lambda_i\mu_j \sup_{t>\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{|\cdot|\leq 2^{i+4},|\cdot|\geq 2^{j+4}\}}(x). \end{aligned}$$

Let us estimate H_3. Observing that $|y| \geq |x| - |x-y| \geq |x|/2 > 2^{j+3}$, we can use (4.1) with $s_1 = 1$ for b_j to obtain that

$$\sup_{0<t\leq\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right| \leq c\sum_{\gamma=1}^{N}\mathcal{M}\big(\big|T_\gamma^1 a_i\big|\big)(x)\,\frac{2^j}{|x|^{n+1}}\,\|b_j\|_{p_2(\cdot)}\,\|\chi_{C_j}\|_{p_2'(\cdot)}. \quad (4.15)$$

Applying Hölder's inequality in $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we obtain $\|H_3\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$\begin{aligned} & c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i\mathcal{M}\big(\big|T_\gamma^1 a_i\big|\big)\chi_{\{|\cdot|\leq 2^{i+4}\}}\Big\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \\ & \times\Big\|\sum_{j=-\infty}^{+\infty}\mu_j\frac{2^j}{|\cdot|^{n+1}}\,\|b_j\|_{p_2(\cdot)}\,\|\chi_{C_j}\|_{p_2'(\cdot)}\chi_{\{|\cdot|>2^{j+4}\}}\Big\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

The first norm above appeared as in J_1 and it is bounded by $c\,\|f\|_{H\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$. Using Proposition 1.33 and the same of the arguments used in the estimation of J_2, to get that the second norm above is bounded by

$$
\begin{aligned}
& c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=-\infty}^{k-5}\mu_j 2^{(j-k)(n+1-(\alpha_2+n/p_2)(0))}\Big)^{q_2(0)}\right\}^{1/q_2(0)} \\
& +c\left\{\sum_{k=0}^{+\infty}\Big(\sum_{j=-\infty}^{-1}\mu_j 2^{(j-k)(n+1-(\alpha_2^+ +n/p_2^-))}\Big)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \\
& +c\left\{\sum_{k=5}^{+\infty}\Big(\sum_{j=0}^{k-5}\mu_j 2^{(j-k)(n+1-(\alpha_2+n/p_2)_\infty)}\Big)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \\
\leq \; & c\left\{\sum_{k=-\infty}^{-1}(\mu_k)^{q_2(0)}\right\}^{1/q_2(0)}+c\left\{\sum_{k=0}^{+\infty}(\mu_k)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}\leq c\left\|g\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where in the first and the third inequality we used Lemma 1.28, since $n+1-(\alpha_2^+ +n/p_2^-)>0$.

For H_4, we put

$$
\varphi_{t,x}(y)=P_s(y)+\Re_s(y),
$$

where $P_s(y)$ is sth Taylor polynomial of $\varphi_{t,x}$ at the origin and $\Re_s(y)$ is the remainder term. By (4.7), we get

$$
\begin{aligned}
\sup_{t>\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right| &= \sup_{t>\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\Re_s(y)B(a_i,b_j)(y)dy\right| \\
&\leq c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\mathbb{R}^n}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy.
\end{aligned}
$$

We split the last expression into

$$
\begin{aligned}
& c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|x-y|>4t}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy \\
& +c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|x-y|\leq 4t}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy \\
= \; & H_4^1(x)+H_4^2(x).
\end{aligned}
$$

In the case of $|x-y|>4t$, we have $|y|\geq|x-y|-|x|\geq 2t>|x|>2^{j+4}$. We can apply (4.1) with $s_1=s+2$ for b_j , we obtain

$$
\begin{aligned}
H_4^1(x) &\leq c\frac{2^{j(s+2)}}{|x|^{n+s+2}}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}t\int_{|x-y|>4t}\frac{\left|T_\gamma^1 a_i(y)\right|}{|y|^{n+1}}dy \\
&\leq c\frac{1}{|x|^n}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\sum_{\gamma=1}^{N}\mathcal{M}\big(\left|T_\gamma^1 a_i\right|\big)(x). \qquad (4.16)
\end{aligned}
$$

Applying Hölder's inequality in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and since $|x|>2^{j+4}$, then $\|H_4^1\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{\infty}\lambda_i\mathcal{M}\big(\left|T_\gamma^1 a_i\right|\big)\chi_{\{|\cdot|\leq 2^{i+4}\}}\Big\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times\Big\|\sum_{j=-\infty}^{+\infty}\frac{\mu_j}{|\cdot|^n}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\chi_{\{|\cdot|>2^{j+4}\}}\Big\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

The first norm above is the same as J_1 and the second norm can be estimated as before by

$$c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=-\infty}^{k-5}\mu_j 2^{(j-k)(n-(\alpha_2+n/p_2)(0))}\Big)^{q_2(0)}\right\}^{1/q_2(0)}$$
$$+c\left\{\sum_{k=0}^{+\infty}\Big(\sum_{j=-\infty}^{-1}\mu_j 2^{(j-k)(n-(\alpha_2^{+}+n/p_2^{-}))}\Big)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}$$
$$+c\left\{\sum_{k=5}^{+\infty}\Big(\sum_{j=0}^{k-5}\mu_j 2^{(j-k)(n-(\alpha_2+n/p_2)_\infty)}\Big)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}.$$

Since $\alpha_2^{+}+n/p_2^{-}<n$, by Lemma 1.28, the last terms are bounded by

$$c\left\{\sum_{k=-\infty}^{-1}(\mu_k)^{q_2(0)}\right\}^{1/q_2(0)}+c\left\{\sum_{k=0}^{+\infty}(\mu_k)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty}\leq c\left\|g\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

We decompose $H_4^2(x)$ into

$$\begin{aligned}&c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\substack{|x-y|\leq 4t\\|y|\leq 2^{j+3}}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy\\&+c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\substack{|x-y|\leq 4t\\|y|>2^{j+3}}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy\\=\;&E_1(x)+E_2(x).\end{aligned}$$

By Hölder's inequality, with $\frac{1}{(p_2^-)'}+\frac{1}{p_2^-}=1$, the term $E_1(x)$ is bounded by

$$\begin{aligned}&c\sum_{\gamma=1}^{N}\frac{2^{j(s+1)}}{|x|^{n/p_2^-+s+1}}\left\|\left(T_\gamma^2 b_j\right)\chi_{B_{j+3}}\right\|_{p_2^-}\sup_{t>\frac{|x|}{2}}\Big(\frac{1}{t^n}\int_{|x-y|\leq 4t}\left|T_\gamma^1 a_i(y)\right|^{(p_2^-)'}dy\Big)^{1/(p_2^-)'}\\\leq\;&c\sum_{\gamma=1}^{N}\frac{2^{j(s+1)}}{|x|^{n/p_2^-+s+1}}\left\|\left(T_\gamma^2 b_j\right)\chi_{B_{j+3}}\right\|_{p_2^-}\mathcal{M}\Big(\left|T_\gamma^1 a_i\right|^{(p_2^-)'}\Big)^{1/(p_2^-)'}(x).\end{aligned}$$

Again by Hölder's inequality with $\frac{1}{p_2^-} = \frac{1}{p_2(\cdot)} + \frac{1}{r_2(\cdot)}$ and the $L^{p_2(\cdot)}$- boundedness of T_γ^2, we obtain

$$\begin{aligned} \left\| \left(T_\gamma^2 b_j\right) \chi_{B_{j+3}} \right\|_{p_2^-} &\leq c\left\|T_\gamma^2 b_j\right\|_{p_2(\cdot)}\left\|\chi_{B_{j+3}}\right\|_{r_2(\cdot)} \\ &\leq c\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{B_{j+3}}\right\|_{r_2(\cdot)}. \end{aligned}$$

Since $p_2 \in \mathcal{P}^{\log}$, we have by (1.5) and (1.6),

$$\left\|\chi_{B_{j+3}}\right\|_{r_2(\cdot)} \approx |B_{j+3}|^{\frac{n}{r_2(x)}} \approx \left\|\chi_{C_j}\right\|_{r_2(\cdot)} \quad \text{if } j<0,\ x \in C_j \subset B_{j+3}$$

and

$$\left\|\chi_{B_{j+3}}\right\|_{r_2(\cdot)} \approx |B_{j+3}|^{\frac{n}{(r_2)_\infty}} \approx \left\|\chi_{C_j}\right\|_{r_2(\cdot)} \quad \text{if } j \geq 0.$$

Now using (1.4), we obtain

$$\left\|\chi_{B_{j+3}}\right\|_{r_2(\cdot)} \leq c 2^{j\frac{n}{p_2^-}} \left\|\chi_{C_j}\right\|_{p_2(\cdot)}^{-1}. \tag{4.17}$$

Then $E_1(x)$ is bounded by

$$c\frac{2^{j(s+1+\frac{n}{p_2^-})}}{|x|^{n/p_2^-+s+1}} \left\|b_j\right\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2(\cdot)}^{-1} \sum_{\gamma=1}^{N} \mathcal{M}\Big(\left|T_\gamma^1 a_i\right|^{(p_2^-)'} \Big)^{1/(p_2^-)'}(x). \tag{4.18}$$

Applying Hölder's inequality in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we obtain $\|E_1\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$\begin{aligned} &c\sum_{\gamma=1}^{N}\left\| \sum_{i=-\infty}^{\infty} \lambda_i \mathcal{M}\left(\left|T_\gamma^1 a_i\right|^{(p_2^-)'} \right)^{1/(p_2^-)'} \chi_{\{|\cdot|\leq 2^{i+4}\}} \right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} \\ &\times \left\| \sum_{j=-\infty}^{+\infty} \mu_j \frac{2^{j(s+1+\frac{n}{p_2^-})}}{|\cdot|^{n/p_2^-+s+1}} \left\|b_j\right\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2(\cdot)}^{-1} \chi_{\{|\cdot|>2^{j+4}\}} \right\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{4.19}$$

The first norm above is the same as N_1. Obviousely the second term is bounded by

$$\begin{aligned} &c\left\{ \sum_{k=-\infty}^{-1} \Big(\sum_{j=-\infty}^{k-5} \mu_j 2^{(j-k)(s+1-\alpha_2(0))} \Big)^{q_2(0)} \right\}^{1/q_2(0)} \\ &+c\left\{ \sum_{k=0}^{+\infty} \Big(\sum_{j=-\infty}^{-1} \mu_j 2^{(j-k)(s+1-\alpha_2^+)} \Big)^{(q_2)_\infty} \right\}^{1/(q_2)_\infty} \\ &+c\left\{ \sum_{k=5}^{+\infty} \Big(\sum_{j=0}^{k-5} \mu_j 2^{(j-k)(s+1-(\alpha_2)_\infty)} \Big)^{(q_2)_\infty} \right\}^{1/(q_2)_\infty}. \end{aligned}$$

Since $s \geq [(\alpha_1^+ + n/p_1^-) + (\alpha_2^+ + n/p_2^-) - n]$ and $(\alpha_1^+ + n/p_1^-) \geq (\alpha_1^- + n/p_1^+) \geq n$, we have $s + 1 > \alpha_2^+$, then by Lemma 1.28 the second term of (4.19) is bounded by

$$c\left\{\sum_{k=-\infty}^{-1} (\mu_k)^{q_2(0)}\right\}^{1/q_2(0)} + c\left\{\sum_{k=0}^{+\infty} (\mu_k)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \leq c \|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

Using (4.1) with $s_1 = s+1$ for b_j, we obtain that E_2 is dominated by

$$\begin{aligned}
& c\sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \int_{\substack{|x-y|\leq 4t \\ |y|>2^{j+3}}} \frac{|y|^{s+1}}{t^{n+s+1}} \left|T_\gamma^1 a_i(y)\right| \frac{2^{j(s+1)}}{|y|^{n+s+1}} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} dy \\
\leq\ & c\sum_{\gamma=1}^{N} \frac{2^{j(s+1)}}{|x|^{n/p_2^-+s+1}} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \sup_{t>0}\left(\frac{1}{t^n}\int_{|x-y|\leq 4t} \left|T_\gamma^1 a_i(y)\right|^{(p_2^-)'} dy\right)^{1/(p_2^-)'} \\
& \times\left(\int_{|y|>2^{j+3}} |y|^{-np_2^-} dy\right)^{1/p_2^-} \\
\leq\ & c\frac{2^{j(s+1-n+n/p_2^-)}}{|x|^{n/p_2^-+s+1}} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \sum_{\gamma=1}^{N} \mathcal{M}\left(\left|T_\gamma^1 a_i\right|^{(p_2^-)'}\right)^{1/(p_2^-)'}(x),
\end{aligned}$$

which is similar to (4.18) and then we can estimate E_2. Therefore, we conclude that

$$\left\|H_4\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} \|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

Step 2. Estimation of Ψ_2. We also split Ψ_2 into $H_5 + H_6 + H_7$, where

$$H_5(x) = \sum_{i,j} \lambda_i \mu_j \mathcal{M}_\varphi(B(a_i, b_j))(x) \chi_{\{|\cdot|>2^{i+4}, |\cdot|\leq 2^{j-5}\}}(x),$$

$$H_6(x) = \sum_{i,j} \lambda_i \mu_j \mathcal{M}_\varphi(B(a_i, b_j))(x) \chi_{\{|\cdot|>2^{i+4}, 2^{j-5}<|\cdot|\leq 2^{j+4}\}}(x)$$

and

$$H_7(x) = \sum_{i,j} \lambda_i \mu_j \mathcal{M}_\varphi(B(a_i, b_j))(x) \chi_{\{|\cdot|>2^{i+4}, |\cdot|>2^{j+4}\}}(x).$$

Substep 2.1. Estimation of H_5. We split $\mathcal{M}_\varphi(B(a_i, b_j))(x)$ into

$$\begin{aligned}
& \sum_{\gamma=1}^{N} \sup_{0<t\leq\frac{|x|}{2}} \left|\int_{\mathbb{R}^n} \varphi_{t,x}(y) T_\gamma^1 a_i(y) T_\gamma^2 b_j(y) dy\right| \\
& + \sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \left|\int_{\mathbb{R}^n} \varphi_{t,x}(y) T_\gamma^1 a_i(y) T_\gamma^2 b_j(y) dy\right| \\
=\ & V_1(x) + V_2(x). \qquad (4.20)
\end{aligned}$$

Let us estimate V_1. We decompose $V_1(x)$ into two functions as follow:

$$\begin{aligned}
& \sum_{\gamma=1}^{N} \sup_{0<t\leq \frac{|x|}{2}} \int_{|y|<2^{j-4}} \varphi_{t,x}(y) \left|T_\gamma^1 a_i(y)\right| \left|T_\gamma^2 b_j(y)\right| dy \\
& + \sum_{\gamma=1}^{N} \sup_{0<t\leq \frac{|x|}{2}} \int_{|y|\geq 2^{j-4}} \varphi_{t,x}(y) \left|T_\gamma^1 a_i(y)\right| \left|T_\gamma^2 b_j(y)\right| dy \\
= \; & V_1^1(x) + V_1^2(x) .
\end{aligned}$$

Observe that in either case $|y| \geq |x| - |x-y| \geq |x|/2 > 2^{i+3}$. Using (4.1), (4.2) for a_i, b_j respectively, we get

$$V_1^1(x) \leq c \frac{2^{is_1}}{|x|^{n+s_1}} \|a_i\|_{p_1(\cdot)} \left\|\chi_{C_i}\right\|_{p_1'(\cdot)} 2^{-jn} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} . \tag{4.21}$$

Since

$$\begin{aligned}
& \left\| \sum_{i=-\infty}^{+\infty} \lambda_i \frac{2^{is_1}}{|\cdot|^{n+s_1}} \|a_i\|_{p_1(\cdot)} \left\|\chi_{C_i}\right\|_{p_1'(\cdot)} \chi_{\{|\cdot|>2^{i+4}\}} \right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} \\
\leq \; & c \left\{ \sum_{k=-\infty}^{-1} \Big(\sum_{i=-\infty}^{k-5} \lambda_i 2^{(i-k)(n+s_1-(\alpha_1+n/p_1)(0))} \Big)^{q_1(0)} \right\}^{1/q_1(0)} \\
& + c \left\{ \sum_{k=0}^{+\infty} \Big(\sum_{i=-\infty}^{-1} \lambda_i 2^{(i-k)(n+s_1-(\alpha_1^+ + n/p_1^-))} \Big)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty} \\
& + c \left\{ \sum_{k=5}^{+\infty} \Big(\sum_{i=0}^{k-5} \lambda_i 2^{(i-k)(n+s_1-(\alpha_1+n/p_1)_\infty)} \Big)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty} .
\end{aligned}$$

If $s_1 > \left(\alpha_1^+ + n/p_1^-\right) - n$, then by Lemma 1.28 the last term is bounded by

$$c \left\{ \sum_{k=-\infty}^{-1} (\lambda_k)^{q_1(0)} \right\}^{1/q_1(0)} + c \left\{ \sum_{k=0}^{+\infty} (\lambda_k)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty} \leq c \, \|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} .$$

Using a combination of the arguments used in the estimation of J_2 we obtain that

$$\left\| \sum_{j=-\infty}^{+\infty} \mu_j 2^{-jn} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \right\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}$$

is bounded by $c \, \|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}$. This finish the estimate of V_1^1.

Let us estimate V_1^2. Using (4.1) for a_i, we obtain that $V_1^2(x)$ is bounded by

$$\begin{aligned}
&\sum_{\gamma=1}^{N}\sup_{0<t\leq\frac{|x|}{2}}\int_{|y|\geq 2^{j-4}}\varphi_{t,x}(y)\frac{2^{is_1}}{|y|^{n+s_1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\left|T_\gamma^2 b_j(y)\right|dy\\
\leq\ & c\frac{2^{is_1}}{|x|^{n+s_1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\sup_{0<t\leq\frac{|x|}{2}}\Big(\int_{\mathbb{R}^n}\varphi_{t,x}(y)dy\Big)^{1/(p_2^-)'}\\
&\times\sum_{\gamma=1}^{N}\Big(\int_{|y|\geq 2^{j-4}}\varphi_{t,x}(y)\left|T_\gamma^2 b_j(y)\right|^{p_2^-}dy\Big)^{1/p_2^-}.
\end{aligned}\tag{4.22}$$

Since $2^{j-4}\leq|y|\leq 2|x-y|\leq 2t$, the term $\Big(\int_{|y|\geq 2^{j-4}}\varphi_{t,x}(y)\left|T_\gamma^2 b_j(y)\right|^{p_2^-}dy\Big)^{1/p_2^-}$ can be rewritten as

$$\sum_{\ell=j-3}^{\infty}\Big(\int_{R_\ell}\varphi_{t,x}(y)\left|T_\gamma^2 b_j(y)\right|^{p_2^-}dy\Big)^{1/p_2^-},$$

repeating the same arguments used in (4.4), we obtain (4.22) is bounded by

$$c\frac{2^{is_1}}{|x|^{n+s_1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}2^{-nj}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},$$

which is the same expression as in (4.21). Then we can obtain the desired estimate for V_1^2.

Let us estimate V_2. We put

$$\varphi_{t,x}(y)=P_s(y)+\Re_s(y).$$

By (4.7), we get

$$V_2(x)\leq c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\mathbb{R}^n}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy.$$

We split the last expression into

$$\begin{aligned}
&\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|y|>2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy\\
&+\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|y|\leq 2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy\\
=\ & V_2^1(x)+V_2^2(x).
\end{aligned}$$

Observe that,

$$\begin{aligned}
V_2^1(x)\ \leq\ & c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\substack{|y|>2^{i+3}\\|y|\leq 2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy\\
&+c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\substack{|y|>2^{i+3}\\|y|>2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy.
\end{aligned}\tag{4.23}$$

Using (4.1) with $s_1 = s+2$ for a_i and (4.2) for b_j, we obtain that the first term in the last expression is bounded by

$$c\frac{2^{i(s+1)}}{|x|^{n+s+1}}\left\|a_i\right\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}2^{-jn}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},$$

which is the same expression as in (4.21) when $s+1 > \left(\alpha_1^+ + n/p_1^-\right) - n$. Using (4.1) for a_i, the second term in (4.23) is dominated by

$$\begin{aligned}
& c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\substack{|y|>2^{i+3}\\|y|>2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}\frac{2^{i(s+2)}}{|y|^{n+s+2}}\left\|a_i\right\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\left|T_\gamma^2 b_j(y)\right|dy \\
\leq\ & c\sum_{\gamma=1}^{N}\frac{2^{i(s+2)}}{|x|^{n+s+1}}\left\|a_i\right\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\left(\int_{|y|>2^{i+3}}\frac{dy}{|y|^{(n/(p_2^-)'+1)(p_2^-)'}}\right)^{1/(p_2^-)'} \\
& \times\left(\int_{|y|>2^{j-4}}\frac{\left|T_\gamma^2 b_j(y)\right|^{p_2^-}}{|y|^n}dy\right)^{1/p_2^-}.
\end{aligned}$$

We see that

$$\begin{aligned}
\left(\int_{|y|>2^{j-4}}\left|T_\gamma^2 b_j(y)\right|^{p_2^-}|y|^{-n}\,dy\right)^{1/p_2^-} &\leq \sum_{\ell=j-3}^{\infty}2^{-\frac{\ell n}{p_2^-}}\left\|\left(T_\gamma^2 b_j\right)\chi_\ell\right\|_{p_2^-} \\
&\leq c\sum_{\ell=j-3}^{\infty}2^{-\frac{\ell n}{p_2^-}}\left\|T_\gamma^2 b_j\right\|_{p_2(\cdot)}\left\|\chi_\ell\right\|_{r_2(\cdot)} \\
&\leq c\sum_{\ell=j-3}^{\infty}2^{-\frac{\ell n}{p_2^-}}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_\ell\right\|_{r_2(\cdot)},
\end{aligned}$$

by Hölder's inequality and the $L^{p_2(\cdot)}(\mathbb{R}^n)$- boundedness of T_γ^2, with $\frac{1}{p_2^-} = \frac{1}{p_2(\cdot)} + \frac{1}{r_2(\cdot)}$. Repeating the same arguments used in (4.3), the second term in the right hand side of (4.23) is dominated by

$$c\frac{2^{i(s+1)}}{|x|^{n+s+1}}\left\|a_i\right\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}2^{-nj}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},$$

which again is the same expression as in (4.21).

Let us estimate V_2^2. We have

$$\begin{aligned}
V_2^2(x) \leq\ & \sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\substack{|y|\leq 2^{i+3}\\|y|\leq 2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy \\
& +\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{\substack{|y|\leq 2^{i+3}\\|y|>2^{j-4}}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\left|T_\gamma^2 b_j(y)\right|dy.
\end{aligned}$$

Using (4.2) for b_j and the Hölder's inequality with $\frac{1}{(p_1^-)'} + \frac{1}{p_1^-} = 1$, we obtain that the first term in the estimaton of $V_2^2(x)$ is dominated by

$$c\frac{2^{-jn}}{|x|^{n+s+1}} \|b_j\|_{p_2(\cdot)} \|\chi_{C_j}\|_{p_2'(\cdot)} \| |\cdot|^{s+1} \chi_{B_{i+3}} \|_{(p_1^-)'} \sum_{\gamma=1}^{N} \| (T_\gamma^1 a_i) \chi_{B_{i+3}} \|_{p_1^-}. \tag{4.24}$$

Applying Hölder's inequality with $\frac{1}{p_1^-} = \frac{1}{p_1(\cdot)} + \frac{1}{r_1(\cdot)}$ and the $L^{p_1(\cdot)}(\mathbb{R}^n)$- boundedness of T_γ^1, we get

$$\| (T_\gamma^1 a_i) \chi_{B_{i+3}} \|_{p_1^-} \leq c\|a_i\|_{p_1(\cdot)} \|\chi_{B_{i+3}}\|_{r_1(\cdot)}.$$

Since $p_1 \in \mathcal{P}^{\log}$, we have by (1.5) and (1.6),

$$\|\chi_{B_{i+3}}\|_{r_1(\cdot)} \approx |B_{i+3}|^{\frac{n}{r_1(x)}} \quad \text{if } i < 0,\ x \in C_i \subset B_{i+3}$$

and

$$\|\chi_{B_{i+3}}\|_{r_1(\cdot)} \approx |B_{i+3}|^{\frac{n}{(r_1)_\infty}} \quad \text{if } i \geq 0.$$

Then we get that (4.24) is bounded by

$$c\frac{2^{i(s+1)}}{|x|^{n+s+1}} \|a_i\|_{p_1(\cdot)} \|\chi_{C_i}\|_{p_1'(\cdot)} 2^{-jn} \|b_j\|_{p_2(\cdot)} \|\chi_{C_j}\|_{p_2'(\cdot)},$$

which again is the same expression as in (4.21). By the Hölder's inequality with $\frac{1}{(p_2^-)'} + \frac{1}{p_2^-} = 1$, the second term in the estimation of $V_2^2(x)$ is bounded by

$$\begin{aligned}
& c\sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \int_{2^{j-4}<|y|\leq 2^{i+3}} \frac{|y|^{s+1}}{t^{n+s+1}} |T_\gamma^1 a_i(y)| \, |T_\gamma^2 b_j(y)| \, dy \\
\leq\ & c\frac{1}{|x|^{n+s+1}} \sum_{\gamma=1}^{N} \Big(\int_{|y|\leq 2^{i+3}} \left(|y|^{(s+1+n/p_2^-)} |T_\gamma^1 a_i(y)| \right)^{(p_2^-)'} dy \Big)^{1/(p_2^-)'} \\
& \times \Big(\int_{|y|>2^{j-4}} |y|^{-n} |T_\gamma^2 b_j(y)|^{p_2^-} dy \Big)^{1/p_2^-} \\
\leq\ & c\frac{2^{i(s+1+n/p_2^-)}}{|x|^{n+s+1}} \| (T_\gamma^1 a_i) \chi_{B_{i+3}} \|_{(p_2^-)'} \sum_{\ell=j-3}^{+\infty} 2^{-\ell n/p_2^-} \| (T_\gamma^2 b_j) \chi_\ell \|_{p_2^-}.
\end{aligned}$$

Again by the Hölder's inequality, the $L^{p_1(\cdot)}$-boundedness of T_γ^1 and $(p_2^-)' < p_1^-$, we get

$$\begin{aligned}
\| (T_\gamma^1 a_i) \chi_{B_{i+3}} \|_{(p_2^-)'} &\leq c\|T_\gamma^1 a_i\|_{p_1(\cdot)} \|\chi_{B_{i+3}}\|_{r_1(\cdot)} \\
&\leq c\|a_i\|_{p_1(\cdot)} \|\chi_{B_{i+3}}\|_{r_1(\cdot)} \\
&\leq c2^{-\frac{in}{p_2^-}} \|a_i\|_{p_1(\cdot)} \|\chi_{C_i}\|_{p_1'(\cdot)},
\end{aligned} \tag{4.25}$$

with $\frac{1}{(p_2^-)'} = \frac{1}{p_1(\cdot)} + \frac{1}{r_1(\cdot)}$, where we have used the same arguments as in (4.17). Repeating the same arguments used in (4.4), we obtain the second term in $V_2^2(x)$ is bounded by

$$c\frac{2^{i(s+1)}}{|x|^{n+s+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}2^{-jn}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},$$

which is satisfy the desired estimate when $s+1>\left(\alpha_1^+ + n/p_1^-\right)-n$. The estimate of V_1 and V_2, gives

$$\left\|H_5\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \le c\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

Substep 2.2. Estimation of H_6. We write as in (4.20)

$$\mathcal{M}_\varphi(B(a_i,b_j))(x) \le V_1(x) + V_2(x).$$

Let us consider V_1. We have $|y| \ge |x|/2 > 2^{i+3}$ and then we used (4.1) for a_i, we obtain $V_1(x)$ is bounded by

$$\begin{aligned} & c\frac{2^{is_1}}{|x|^{n+s_1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\sum_{\gamma=1}^{N}\Big(\int_{\mathbb{R}^n}\varphi_{t,x}(y)dy\Big)^{1/p_0}\mathcal{M}\Big(\left|T_\gamma^2 b_j\right|^{p_0'}\Big)^{1/p_0'}(x) \\ \le\ & c\frac{2^{is_1}}{|x|^{n+s_1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\sum_{\gamma=1}^{N}\mathcal{M}\Big(\left|T_\gamma^2 b_j\right|^{p_0'}\Big)^{1/p_0'}(x), \end{aligned}$$

where p_0 is chosen as in the estimation for II_2. Now V_1 can be estimated as in (4.21) for i and as in II_2 for j.

Let use estimate V_2. We used only the case when $2^{i+4} < |x| \le 2^{j+4}$, which it is sufficient for the estimate of V_2 (it similar to the previous case i.e, V_2 in H_5). We obtain that

$$\left\|H_6\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \le c\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

Substep 2.3. Estimation of H_7. As in (4.20), we write

$$\mathcal{M}_\varphi(B(a_i,b_j))(x) \le V_1(x) + V_2(x).$$

Let us estimate V_1. Since $|y| \ge |x| - |x-y| \ge |x|/2 \ge \max(2^{i+3}, 2^{j+3})$, using (4.1) for a_i with $s_1 = \ell + 1$ and for b_j with $s_1 = 0$ respectively, we obtain

$$V_1(x) \le c\frac{2^{i(\ell+1)}}{|x|^{n+\ell+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{1}{|x|^n}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}. \tag{4.26}$$

It follows from (4.16) and (4.21) that V_1 satisfies the required estimate.

Now let us estimate V_2. we decompose the integral into four parts as follow:

$$\begin{aligned} V_2(x) &\leq c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|y|\geq\max(2^{i+4},2^{j+4})}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^2a_i(y)\right|\left|T_\gamma^2b_j(y)\right|dy \\ &\quad +c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|y|\leq\min(2^{i+4},2^{j+4})}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^2a_i(y)\right|\left|T_\gamma^2b_j(y)\right|dy \\ &\quad +c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{2^{j+4}<|y|\leq2^{i+4}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^2a_i(y)\right|\left|T_\gamma^2b_j(y)\right|dy \\ &\quad +c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{2^{i+4}<|y|\leq2^{j+4}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^2a_i(y)\right|\left|T_\gamma^2b_j(y)\right|dy \\ &= Q_1(x)+Q_2(x)+Q_3(x)+Q_4(x). \end{aligned}$$

Let us consider Q_1. Put $s=m+\ell$ where m,ℓ be non-negative integers. Using (4.1) for a_i with $s_1=m+1$ and for b_j with $s_1=s+1$, we have $Q_1(x)$ is bounded by

$$\frac{2^{i(m+1)}}{|x|^{n+s+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}2^{j\ell}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\int_{|y|\geq\max(2^{i+4},2^{j+4})}\frac{dy}{|y|^{2n}}.$$

For any i and j, we have

$$\int_{|y|\geq\max(2^{i+4},2^{j+4})}\frac{2^{nj/(p_2^-)'+ni/p_2^-}\,dy}{|y|^{2n}}\leq c.$$

Then we obtain $Q_1(x)$ is dominated by

$$c\frac{2^{i\left(m+1-n/p_2^-\right)}}{|x|^{n/\left(p_2^-\right)'+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j\left(\ell-n+n/p_2^-\right)}}{|x|^{n/p_2^-+\ell}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}. \tag{4.27}$$

The second term of (4.27) it follow from (4.18) and is bounded by $c\,\|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}$, if

$$\ell>\alpha_2^+. \tag{4.28}$$

For the first term of (4.27), we have

$$
\begin{aligned}
&\Big\| \sum_{i=-\infty}^{+\infty} \lambda_i \frac{2^{i\left(m+1-n/p_2^-\right)}}{|\cdot|^{n/\left(p_2^-\right)'+m+1}} \|a_i\|_{p_1(\cdot)} \left\|\chi_{C_i}\right\|_{p_1'(\cdot)} \chi_{\{|\cdot|>2^{i+4}\}} \Big\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \\
\leq\ & c \left\{ \sum_{k=-\infty}^{-1} \Big(\sum_{i=-\infty}^{k-4} \lambda_i 2^{(i-k)(n+m+1-n/p_2^- -(\alpha_1+n/p_1)(0))} \Big)^{q_1(0)} \right\}^{1/q_1(0)} \\
&+ \left\{ \sum_{k=0}^{+\infty} \Big(\sum_{i=-\infty}^{-1} \lambda_i 2^{(i-k)(n+m+1-n/p_2^- -(\alpha_1^+ +n/p_1^-))} \Big)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty} \\
&+ \left\{ \sum_{k=4}^{+\infty} \Big(\sum_{i=0}^{k-4} \lambda_i 2^{(i-k)(n+m+1-n/p_2^- -(\alpha_1+n/p_1)_\infty)} \Big)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty} .
\end{aligned}
$$

If

$$
m+1 > (\alpha_1^+ + n/p_1^-) + n(1/p_2^- - 1), \tag{4.29}
$$

we obtain that the first term of (4.27) is bounded by

$$
c \left\{ \sum_{k=-\infty}^{-1} (\lambda_k)^{q_1(0)} \right\}^{1/q_1(0)} + c \left\{ \sum_{k=0}^{+\infty} (\lambda_k)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty} \leq c\, \|f\|_{H\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} .
$$

Now, (4.28) and (4.29) give

$$
s+1 = m+\ell+1 > (\alpha_1^+ + n/p_1^-) + (\alpha_2^+ + n/p_2^-) - n.
$$

Let us consider Q_2. By Hölder's inequality with $\frac{1}{(p_2^-)'} + \frac{1}{p_2^-} = 1$, we obtain

$$
\begin{aligned}
Q_2(x) \ &\leq\ c \frac{2^{i(m+1)} 2^{j\ell}}{|x|^{n+s+1}} \sum_{\gamma=1}^{N} \left\| \left(T_\gamma^1 a_i\right) \chi_{B_{i+4}} \right\|_{(p_2^-)'} \left\| \left(T_\gamma^2 b_j\right) \chi_{B_{j+4}} \right\|_{p_2^-} \\
&\leq\ c \frac{2^{i(m+1)} 2^{j\ell}}{|x|^{n+s+1}} \sum_{\gamma=1}^{N} \left\|T_\gamma^1 a_i\right\|_{p_1(\cdot)} \left\|\chi_{B_{i+4}}\right\|_{r_1(\cdot)} \left\|T_\gamma^2 b_j\right\|_{p_2(\cdot)} \left\|\chi_{B_{j+4}}\right\|_{r_2(\cdot)},
\end{aligned}
$$

where we used again Hölder's inequality, with $\frac{1}{p_1(\cdot)} + \frac{1}{r_1(\cdot)} = \frac{1}{(p_2^-)'}$ and $\frac{1}{p_2(\cdot)} + \frac{1}{r_2(\cdot)} = \frac{1}{p_2^-}$. Similar arguments used to prove (4.17) give that $Q_2(x)$ is bounded by

$$
c \frac{2^{i(m+1-n/p_2^-)}}{|x|^{n/\left(p_2^-\right)'+m+1}} \|a_i\|_{p_1(\cdot)} \left\|\chi_{C_i}\right\|_{p_1'(\cdot)} \frac{2^{j(\ell-n+n/p_2^-)}}{|x|^{n/p_2^-+\ell}} \|b_j\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)},
$$

which is the same term of (4.27).

Let us consider Q_3. In this case $2^{j+4} < |y| \leq 2^{i+4}$ (i.e $j < i$) then using (4.1) for b_j, we get

$$\begin{aligned}
Q_3(x) &\leq c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{2^{j+4}<|y|\leq 2^{i+4}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\frac{2^{j(\ell+m+2)}}{|y|^{n+\ell+m+2}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}dy \\
&\leq c\frac{2^{i(m+1)}}{|x|^{n/\left(p_2^-\right)'+m+1}}\frac{2^{j(\ell+1)}}{|x|^{n/p_2^-+\ell}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\sum_{\gamma=1}^{N}\left\|\left(T_\gamma^1 a_i\right)\chi_{B_{i+4}}\right\|_{\left(p_2^-\right)'} \\
&\quad\times\left(\int_{|y|>2^{j+4}}|y|^{-(n+1)p_2^-}\,dy\right)^{1/p_2^-} \\
&\leq c\frac{2^{i(m+1)}}{|x|^{n/\left(p_2^-\right)'+m+1}}\sum_{\gamma=1}^{N}\left\|\left(T_\gamma^1 a_i\right)\chi_{B_{i+4}}\right\|_{\left(p_2^-\right)'}\frac{2^{j\left(\ell-n+n/p_2^-\right)}}{|x|^{n/p_2^-+\ell}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}.
\end{aligned}$$

As before, we have

$$Q_3(x) \leq c\frac{2^{i\left(m+1-n/p_2^-\right)}}{|x|^{n/\left(p_2^-\right)'+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j\left(\ell-n+n/p_2^-\right)}}{|x|^{n/p_2^-+\ell}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},$$

which is equivalent to (4.27). Finally, let us consider Q_4. In this case $2^{i+4} < |y| \leq 2^{j+4}$ (i.e $i < j$). Then using (4.1) for a_i, we get

$$\begin{aligned}
Q_4(x) &\leq c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{2^{i+4}<|y|\leq 2^{j+4}}\frac{|y|^{s+1}}{t^{n+s+1}}\frac{2^{i(\ell+m+2)}}{|y|^{n+\ell+m+2}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\left|T_\gamma^2 b_j(y)\right|dy \\
&\leq c\frac{2^{i(m+2)}}{|x|^{n/\left(p_2^-\right)'+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j\ell}}{|x|^{n/p_2^-+\ell}}\sum_{\gamma=1}^{N}\left\|\left(T_\gamma^2 b_j\right)\chi_{B_{j+4}}\right\|_{p_2^-} \\
&\quad\times\left(\int_{|y|>2^{i+4}}|y|^{-(n+1)\left(p_2^-\right)'}\,dy\right)^{1/\left(p_2^-\right)'}.
\end{aligned}$$

Again as before, we have $Q_4(x)$ is bounded by

$$c\frac{2^{i\left(m+1-n/p_2^-\right)}}{|x|^{n/\left(p_2^-\right)'+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j\left(\ell-n+n/p_2^-\right)}}{|y|^{n/p_2^-+\ell}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},$$

which is desired.

Combining (4.26) and the estimates for Q_1, Q_2, Q_3 and Q_4, we obtain

$$\left\|H_7\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c\,\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

This is finish the proof of Theorem 4.9. ■

Remark 4.10 *The minimum value of s given in (4.7) can be taken as*

$$[(\alpha_1^+ + n/p_1^-) + (\alpha_2^+ + n/p_2^-) - n].$$

Chapter 5

Boundedness of some bilinear operators on variable Herz-type Hardy spaces II

We conclude our thesis by giving another application of the boundedness of some bilinear operators on variable Herz-type Hardy spaces in the from $X \times Y$ into Z, where X, Y and Z are variable Herz-type Hardy spaces, this study is completed the previous chapter.

5.1 Technical lemmas

We employ the notation $\mathcal{M}_{\varphi}(f)$ and $B(f,g)$, i.e., for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with supp $\varphi \subseteq B_0$, such that

$$\int_{\mathbb{R}^n} \varphi(x)dx \neq 0 \text{ and } \varphi_t(\cdot) = t^{-n}\varphi\left(\frac{\cdot}{t}\right) \text{ for any } t > 0,$$

we define

$$\mathcal{M}_{\varphi}(f)(x) := \sup_{t>0} |\varphi_t * f(x)|,$$

and $B(f,g)$ as in the previous chapter, define by

$$B(f,g)(x) := \sum_{\gamma=1}^{N} \left(T_{\gamma}^{1} f\right)(x) \left(T_{\gamma}^{2} g\right)(x), \; x \in \mathbb{R}^n$$

where the operators T_γ^j are bounded from $L^{p_j(\cdot)}(\mathbb{R}^n)$ into $L^{p_j(\cdot)}(\mathbb{R}^n), j=1,2,\ \gamma=1,...,N$ and satisfying the two conditions for each central $(\alpha(\cdot),p(\cdot))$-atom a_i with supp$a_i\subset C_i$

$$\left|T_\gamma^j a_i(x)\right| \leq c\frac{2^{is_1}}{|x|^{n+s_1}}\left\|a_i\right\|_{p_j(\cdot)}\left\|\chi_{C_i}\right\|_{p_j'(\cdot)},\ \ |x|>2^{i+3}, j=1,2 \tag{5.1}$$

and

$$\left|T_\gamma^j a_i(x)\right| \leq c2^{-in}\left\|a_i\right\|_{p_j(\cdot)}\left\|\chi_{C_i}\right\|_{p_j'(\cdot)},\ \ |x|<2^{i-3}, j=1,2. \tag{5.2}$$

Note that to simplify the writing, we use the notation

$$\sum_{j\in\mathbb{Z}}\sum_{i\geq j}\cdots=\sum_{i\geq j}\cdots.$$

Let us now state technical lemmas which we will use in the proof of main theorem in this chapter.

Lemma 5.1 *Let $q_1,q_2\in\mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}$, $p_1,p_2\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, $1<p_i^-\leq p_i^+<\infty$, $\frac{1}{p_1^-}+\frac{1}{p_2^-}<1$, $\alpha_i^-+\frac{n}{p_i^+}\geq n$, $i=1,2$, and $\alpha(\cdot)=\alpha_1(\cdot)+\alpha_2(\cdot)$ such that q_1,q_2,α_1 and α_2 are* log*-Hölder continuous, both at the origin and at infinity. Let T_γ^j are operators bounded from $L^{p_j(\cdot)}(\mathbb{R}^n)$ into $L^{p_j(\cdot)}(\mathbb{R}^n), j=1,2,\ \gamma=1,...,N$ and satisfying the two conditions (5.1) and (5.2). Let $s\geq[(\alpha_1^++n/p_1^-)+(\alpha_2^++n/p_2^-)-n]$ be a non-negative integer such that*

$$\int_{\mathbb{R}^n}x^\beta B(a,b)(x)dx=0 \tag{5.3}$$

for all multi-indices β with $|\beta|\leq s$, for all $(\alpha_1(\cdot),p_1(\cdot))$-atoms a, and all $(\alpha_2(\cdot),p_2(\cdot))$-atoms b. Then

$$\sum_{i\geq j}\lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))\chi_{\{|\cdot|\leq 2^{i+5}\}} \tag{5.4}$$

can be extended to a bounded operator from $H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)\times H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ into $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ for all $i,j\in\mathbb{Z}$ and all $\lambda_i,\mu_j\geq 0$.

Proof. We split (5.4) into two functions D_1+D_2, where

$$\begin{aligned} D_1(x) &= \sum_{i\geq j}\lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x)\,\chi_{\{|\cdot|\leq 2^{j+5}\}}(x) \\ D_2(x) &= \sum_{i\geq j}\lambda_i\mu_j\mathcal{M}_\varphi(B(a_i,b_j))(x)\,\chi_{\{2^{j+5}<|\cdot|\leq 2^{i+5}\}}(x). \end{aligned}$$

Substep 1.1. Estimation of D_1. By Hölder's inequality with $\frac{1}{p_0}+\frac{1}{p_0'}=1$ where $(p_2^-)'<p_0<p_1^-$, we obtain

$$\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\leq c\sum_{\gamma=1}^{N}\mathcal{M}\left(\left|T_\gamma^1 a_i\right|^{p_0}\right)^{1/p_0}(x)\,\mathcal{M}\left(\left|T_\gamma^2 b_j\right|^{p_0'}\right)^{1/p_0'}(x).$$

Applying Hölder's inequality in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, then $\|D_1\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{+\infty}\lambda_i\mathcal{M}\left(\left|T_\gamma^1 a_i\right|^{p_0}\right)^{1/p_0}\chi_{\{|\cdot|\leq 2^{i+5}\}}\Big\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times\Big\|\sum_{j=-\infty}^{+\infty}\mu_j\mathcal{M}\left(\left|T_\gamma^2 b_j\right|^{p_0'}\right)^{1/p_0'}\chi_{\{|\cdot|\leq 2^{j+5}\}}\Big\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

Both terms above can be estimated as N_1 in (4.12). Therefore, we have

$$\|D_1\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}\leq c\,\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\|g\|_{H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

Substep 1.2. Estimation of D_2. we split D_2 into $D_2^1+D_2^2$, where

$$D_2^1(x)=\sum_{i\geq j}\lambda_i\mu_j\sup_{0<t\leq\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{2^{j+5}<|\cdot|\leq 2^{i+5}\}}(x)$$

and

$$D_2^2(x)=\sum_{i\geq j}\lambda_i\mu_j\sup_{t>\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{2^{j+5}<|\cdot|\leq 2^{i+5}\}}(x).$$

First, let us estimate D_2^1. Observing that $|y|\geq|x|-|x-y|\geq|x|/2\geq 2^{j+4}$. Applying Hölder's inequality with $\frac{1}{(p_2^-)'}+\frac{1}{p_2^-}=1$ and using (5.1) for b_j, we obtain

$$\begin{aligned}
&\sup_{0<t\leq\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\\
\leq\ & c\sum_{\gamma=1}^{N}\sup_{t\leq\frac{|x|}{2}}\left(\int_{|y|>2^{j+3}}\varphi_{t,x}(y)\left|T_\gamma^1 a_i(y)\right|^{(p_2^-)'}dy\right)^{\frac{1}{(p_2^-)'}}\left(\int_{|y|>2^{j+3}}\varphi_{t,x}(y)\left|T_\gamma^2 b_j(y)\right|^{p_2^-}dy\right)^{\frac{1}{p_2^-}}\\
\leq\ & c\sum_{\gamma=1}^{N}\mathcal{M}\left(\left|T_\gamma^1 a_i\right|^{(p_2)'}\right)^{1/(p_2)'}(x)\,\frac{2^{j(\ell+1)}}{|x|^{n+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}.
\end{aligned}\tag{5.5}$$

By Hölder's inequality in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, then $\|D_2^1\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$c\sum_{\gamma=1}^{N}\Big\|\sum_{i=-\infty}^{+\infty}\lambda_i\mathcal{M}\left(\left|T_\gamma^1 a_i\right|^{p_2'}\right)^{1/p_2'}\chi_{\{|\cdot|\leq 2^{i+5}\}}\Big\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times\Big\|\sum_{j=-\infty}^{+\infty}\mu_j\frac{2^{j(\ell+1)}}{|\cdot|^{n+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\chi_{\{|\cdot|>2^{j+5}\}}\Big\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.\tag{5.6}$$

Since $p_1^- > \left(p_2^-\right)'$, the first term in (5.6) can be estimated as N_1 in (4.12). If $\ell+1 > (\alpha_2^+ + n/p_2^-) - n$, the second term can be estimated similarly to (4.15). Therefore, we have

$$\left\|D_2^1\right\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \le c\left\|f\right\|_{H\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \left\|g\right\|_{H\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

Second, let us estimate D_2^2. We put $\varphi_{t,x}(y) = P_\ell(y) + \Re_\ell(y)$ as in H_4 (**Substep1.3** in the previous chapter). By (5.3), we get

$$D_2^2(x) \le \sum_{i\ge j}\lambda_i\mu_j \sup_{t>\frac{|x|}{2}} \int_{\mathbb{R}^n} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} \left|B(a_i,b_j)(y)\right| dy \chi_{\{2^{j+5}<|\cdot|\le 2^{i+5}\}}(x).$$

We split the last expression into $D_2^{21}(x) + D_2^{22}(x)$, where

$$D_2^{21}(x) = \sum_{i\ge j}\lambda_i\mu_j \sup_{t>\frac{|x|}{2}} \int_{|x-y|>4t} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} \left|B(a_i,b_j)(y)\right| dy \chi_{\{2^{j+5}<|\cdot|\le 2^{i+5}\}}(x)$$

and

$$D_2^{22}(x) = \sum_{i\ge j}\lambda_i\mu_j \sup_{t>\frac{|x|}{2}} \int_{|x-y|\le 4t} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} \left|B(a_i,b_j)(y)\right| dy \chi_{\{2^{j+5}<|\cdot|\le 2^{i+5}\}}(x).$$

• In the case of $|x-y| > 4t$, we have $|y| \ge |x-y| - |x| \ge 2t > |x| > 2^{j+5}$ and $|x-y| \le 2|y|$. Using (5.1) with $s_1 = \ell+3$ for b_j, we obtain

$$\begin{aligned}
&\sup_{t>\frac{|x|}{2}} \int_{|x-y|>4t} \frac{|y|^{\ell+1}}{t^{n+\ell+1}} \left|B(a_i,b_j)(y)\right| dy \\
\le\ & c\sum_{\gamma=1}^{N} \frac{1}{|x|^{n+\ell+2}} \sup_{t>\frac{|x|}{2}} \int_{\substack{|x-y|>4t\\ |y|>2^{j+5}}} t\left|T_\gamma^1 a_i(y)\right| \frac{2^{j(\ell+3)}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}}{|y|^{n+2}} dy.
\end{aligned}$$

Since $2t \le |y|$, the last expression is bounded by

$$\begin{aligned}
& c\frac{2^{j(\ell+3)}}{|x|^{n+\ell+2}} \left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \int_{\substack{|x-y|>4t\\ |y|>2^{j+5}}} \frac{t^{1/(p_2^-)'}\left|T_\gamma^1 a_i(y)\right|}{|x-y|^{(n+1)/(p_2^-)'}} \frac{dy}{|y|^{n/p_2^-+1}} \\
\le\ & c\frac{2^{j(\ell+3)}}{|x|^{n+\ell+2}} \left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \sum_{\gamma=1}^{N} \sup_{t>0}\Big(\int_{|x-y|>4t} \frac{t\left|T_\gamma^1 a_i(y)\right|^{(p_2^-)'}}{|x-y|^{n+1}} dy\Big)^{1/(p_2^-)'} \\
& \times\Big(\int_{|y|>2^{j+5}} \frac{dy}{|y|^{n+p_2^-}}\Big)^{1/p_2^-} \\
\le\ & c\sum_{\gamma=1}^{N} \mathcal{M}\Big(\left|T_\gamma^1 a_i\right|^{(p_2)'}\Big)^{1/(p_2)'}(x)\, \frac{2^{j(\ell+2)}}{|x|^{n+\ell+2}} \left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},
\end{aligned}$$

which is similar to (5.5) and satisfy the desired estimate if $\ell+1>(\alpha_2^+ + n/p_2^-)-n-1$.

• In the case of $|x-y|\le 4t$, we split $D_2^{22}(x)$ into

$$\begin{aligned}
&\sum_{i\ge j}\lambda_i\mu_j\sup_{t>\frac{|x|}{2}}\int_{\substack{|x-y|\le 4t\\ |y|\le 2^{j+3}}}\frac{|y|^{\ell+1}}{t^{n+\ell+1}}|B(a_i,b_j)(y)|\,dy\chi_{\{2^{j+5}<|\cdot|\le 2^{i+5}\}}\\
&+\sum_{i\ge j}\lambda_i\mu_j\sup_{t>\frac{|x|}{2}}\int_{\substack{|x-y|\le 4t\\ |y|>2^{j+3}}}\frac{|y|^{\ell+1}}{t^{n+\ell+1}}|B(a_i,b_j)(y)|\,dy\chi_{\{2^{j+5}<|\cdot|\le 2^{i+5}\}}\\
=\;& F_1(x)+F_2(x).
\end{aligned}$$

Let us consider F_1. If $|x|>2^{j+5}$ and $|y|\le 2^{j+3}$ then $\frac{1}{t}\le\frac{4}{|x-y|}<\frac{c}{|x|}$. Then by Hölder's inequality with $1=\frac{1}{(p_2^-)'}+\frac{1}{p_2^-}$, we have

$$\begin{aligned}
&\sup_{t>\frac{|x|}{2}}\int_{\substack{|x-y|\le 4t\\ |y|\le 2^{j+3}}}\frac{|y|^{\ell+1}}{t^{n+\ell+1}}|B(a_i,b_j)(y)|\,dy\\
\le\;&\frac{c}{|x|^{n/p_2^-+\ell+1}}\sum_{\gamma=1}^{N}\sup_{t>0}\Big(\frac{1}{t^n}\int_{|x-y|\le 4t}\left|T_\gamma^1 a_i(y)\right|^{(p_2^-)'}dy\Big)^{\frac{1}{(p_2^-)'}}\left\||\cdot|^{\ell+1}(T_\gamma^2 b_j)\chi_{B_{j+3}}\right\|_{p_2^-},
\end{aligned}\tag{5.7}$$

again by Hölder's inequality and the $L^{p_2(\cdot)}$-boundedness of T_γ^2,

$$\begin{aligned}
\left\||\cdot|^{\ell+1}(T_\gamma^2 b_j)\chi_{B_{j+3}}\right\|_{p_2^-} &\le c2^{(\ell+1)j}\left\|T_\gamma^2 b_j\right\|_{p_2(\cdot)}\left\|\chi_{B_{j+3}}\right\|_{r_2(\cdot)}\\
&\le c2^{(\ell+1)j}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{B_{j+3}}\right\|_{r_2(\cdot)}\\
&\le c2^{j(\ell+1-n+\frac{n}{p_2^-})}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},
\end{aligned}\tag{5.8}$$

with $\frac{1}{p_2^-}=\frac{1}{p_2(\cdot)}+\frac{1}{r_2(\cdot)}$, where we have used the same arguments as in (4.17). We obtain (5.7) is bounded by

$$c\sum_{\gamma=1}^{N}\mathcal{M}\Big(\left|T_\gamma^1 a_i\right|^{(p_2^-)'}\Big)^{1/(p_2^-)'}(x)\,\frac{2^{j\left(\ell+1-n+n/p_2^-\right)}}{|x|^{n/p_2^-+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},\tag{5.9}$$

which is equivalent to (4.18) and satisfy the desired estimate if $\ell+1>\alpha_2^+$.

Let us consider F_2. We use (5.1) for b_j, we get $\sup\limits_{t>|x|/2}\int_{\substack{|x-y|\le 4t\\ |y|>2^{j+3}}}\frac{|y|^{\ell+1}}{t^{n+\ell+1}}|B(a_i,b_j)(y)|\,dy$ is bounded by

$$\begin{aligned}
&c\frac{2^{j(\ell+2)}}{|x|^{n/p_2^-+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\sum_{\gamma=1}^{N}\sup_{t>0}\Big(\frac{1}{t^n}\int_{|x-y|\le 4t}\left|T_\gamma^1 a_i(y)\right|^{(p_2^-)'}dy\Big)^{1/(p_2^-)'}\\
&\times\Big(\int_{|y|>2^{j+3}}\frac{dy}{|y|^{(n+1)p_2^-}}\Big)^{1/p_2^-}\\
\le\;& c\sum_{\gamma=1}^{N}\mathcal{M}\Big(\left|T_\gamma^1 a_i(y)\right|^{(p_2^-)'}\Big)^{1/(p_2^-)'}(x)\,\frac{2^{j\left(\ell+1-n+n/p_2^-\right)}}{|x|^{n/p_2^-+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)},
\end{aligned}\tag{5.10}$$

which is equivalent to (4.18).

Combining (5.9) and (5.10), we obtain that

$$\left\|D_2^{22}\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c\left\|f\right\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\left\|g\right\|_{H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

This finish the proof of Lemma 5.1. ■

In the previous lemma we have treated the sum if x in $\overline{B}\left(0,2^{j+5}\right)$. The next lemma gives the sum if x is not in $\overline{B}\left(0,2^{j+5}\right)$.

Lemma 5.2 *Under the same conditions of the previous lemma. Then*

$$\sum_{i\geq j}\lambda_i\mu_j\mathcal{M}_{\varphi}(B(a_i,b_j))\chi_{\{|\cdot|>2^{i+5}\}} \tag{5.11}$$

can be extended to a bounded operator from $H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)\times H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ *into* $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ *for all* $i,j\in\mathbb{Z}$ *and all* $\lambda_i,\mu_j\geq 0$.

Proof. We split (5.11) into G_1+G_2, where

$$G_1(x)=\sum_{i\geq j}\lambda_i\mu_j\sup_{t\leq\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{|\cdot|>2^{i+5}\}}(x)$$

and

$$G_2(x)=\sum_{i\geq j}\lambda_i\mu_j\sup_{t>\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\chi_{\{|\cdot|>2^{i+5}\}}(x).$$

Substep 2.1. Estimation of G_1. Observe that in this case $|y|\geq|x|-|x-y|\geq\frac{|x|}{2}>2^{i+4}$, using (5.1) for a_i and b_j respectively, we get

$$\begin{aligned}&\sup_{0<t\leq\frac{|x|}{2}}\left|\int_{\mathbb{R}^n}\varphi_{t,x}(y)B(a_i,b_j)(y)dy\right|\\ \leq\ & c\sum_{\gamma=1}^{N}\sup_{0<t\leq\frac{|x|}{2}}\int_{|y|>2^{i+3}}\varphi_{t,x}(y)\frac{2^{i(m+1)}}{|y|^{n+m+1}}\left\|a_i\right\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j(\ell+1)}}{|y|^{n+\ell+1}}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}dy.\end{aligned}$$

Since $|y|\geq\frac{|x|}{2}$, the last expression is bounded by

$$c\frac{2^{i(m+1)}}{|x|^{n+m+1}}\left\|a_i\right\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j(\ell+1)}}{|x|^{n+\ell+1}}\left\|b_j\right\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}.$$

Applying the Hölder inequality in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we obtain $\left\|G_1\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$\begin{aligned} & c\Big\| \sum_{i=-\infty}^{+\infty} \lambda_i \frac{2^{i(m+1)}}{|\cdot|^{n+m+1}} \left\|a_i\right\|_{p_1(\cdot)} \left\|\chi_{C_i}\right\|_{p_1'(\cdot)} \chi_{\{|\cdot|>2^{i+5}\}} \Big\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} \\ & \times \Big\| \sum_{j=-\infty}^{+\infty} \mu_j \frac{2^{j(\ell+1)}}{|\cdot|^{n+\ell+1}} \left\|b_j\right\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \chi_{\{|\cdot|>2^{j+5}\}} \Big\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{5.12}$$

If $m+1 > (\alpha_1^+ + n/p_1^-) - n$ and $\ell + 1 > (\alpha_2^+ + n/p_2^-) - n$, then (5.12) is estimate as from (5.6).

Substep 2.2. Estimation of G_2. We put $\varphi_{t,x}(y) = P_s(y) + \Re_s(y)$. As in H_4, by (5.3), we have

$$\begin{aligned} \sup_{t>\frac{|x|}{2}} \left| \int_{\mathbb{R}^n} \varphi_{t,x}(y) B(a_i, b_j)(y) dy \right| & \leq \sup_{t>\frac{|x|}{2}} \int_{\mathbb{R}^n} \frac{|y|^{s+1}}{t^{n+s+1}} \left|B(a_i, b_j)(y)\right| dy, \quad i \geq j \\ & \leq U_1(x) + U_2(x) + U_3(x), \end{aligned}$$

where

$$\begin{aligned} U_1(x) &= \sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \int_{|y|\leq 2^{j+3}} \frac{|y|^{s+1}}{t^{n+s+1}} \left|T_\gamma^1 a_i(y)\right| \left|T_\gamma^2 b_j(y)\right| dy \\ U_2(x) &= \sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \int_{2^{j+3}<|y|\leq 2^{i+3}} \frac{|y|^{s+1}}{t^{n+s+1}} \left|T_\gamma^1 a_i(y)\right| \left|T_\gamma^2 b_j(y)\right| dy \\ U_3(x) &= \sum_{\gamma=1}^{N} \sup_{t>\frac{|x|}{2}} \int_{|y|>2^{i+3}} \frac{|y|^{s+1}}{t^{n+s+1}} \left|T_\gamma^1 a_i(y)\right| \left|T_\gamma^2 b_j(y)\right| dy. \end{aligned}$$

We put $s = m + \ell + 1$, where m, ℓ are non-negative integers.

Let us consider U_1. By Hölder's inequality with $\frac{1}{(p_2^-)'} + \frac{1}{p_2^-} = 1$ and since $i \geq j$, we obtain $U_1(x)$ is bounded by

$$c \sum_{\gamma=1}^{N} \frac{2^{i(m+1)} 2^{j(\ell+1)}}{|x|^{n+m+\ell+2}} \left\|(T_\gamma^1 a_i)\chi_{B_{i+3}}\right\|_{(p_2^-)'} \left\|(T_\gamma^2 b_j)\chi_{B_{j+3}}\right\|_{p_2^-},$$

by (4.25) and (5.8), we get that $U_1(x)$ is bounded by

$$c\frac{2^{i(m+1-n/p_2^-)}}{|x|^{n/(p_2^-)'+m+1}} \left\|a_i\right\|_{p_1(\cdot)} \left\|\chi_{C_i}\right\|_{p_1'(\cdot)} \frac{2^{j(\ell+1-n+n/p_2^-)}}{|x|^{n/p_2^-+\ell+1}} \left\|b_j\right\|_{p_2(\cdot)} \left\|\chi_{C_j}\right\|_{p_2'(\cdot)}, \tag{5.13}$$

the last expression is similar to (4.27) and satisfy the desired estimate if

$$m + 1 > (\alpha_1^+ + n/p_1^-) + n(1/p_2^- - 1) \text{ and } \ell + 1 > \alpha_2^+.$$

Let us consider U_2. Using (5.1) for b_j, we obtain $U_2(x)$ is bounded by

$$\begin{aligned} & c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{2^{j+3}<|y|\leq 2^{i+3}}\frac{|y|^{s+1}}{t^{n+s+1}}\left|T_\gamma^1 a_i(y)\right|\frac{2^{j(\ell+m+3)}}{|y|^{n+\ell+m+3}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}dy \\ \leq\ & c\sum_{\gamma=1}^{N}\frac{2^{i(m+1)}}{|x|^{n/p_1^-+m+1}}\left\|(T_\gamma^1 a_i)\chi_{B_{i+3}}\right\|_{p_1^-}\frac{2^{j(\ell+2)}}{|x|^{n/(p_1^-)'+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)} \\ & \times\left(\int_{|y|>2^{j+4}}\frac{dy}{|y|^{(n+1)(p_1^-)'}}\right)^{1/(p_1^-)'}. \end{aligned}$$

Again by the Hölder's inequality and the $L^{p_1(\cdot)}$-boundedness of T_γ^1,

$$\begin{aligned} \left\|(T_\gamma^1 a_i)\chi_{B_{i+3}}\right\|_{p_1^-} &\leq c\left\|T_\gamma^1 a_i\right\|_{p_1(\cdot)}\left\|\chi_{B_{i+3}}\right\|_{r_1(\cdot)} \\ &\leq c2^{i(\frac{n}{p_2^-}-n)}\left\|a_i\right\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}, \end{aligned}$$

with $\frac{1}{p_1^-}=\frac{1}{p_1(\cdot)}+\frac{1}{r_1(\cdot)}$, where we have used the same arguments as in (4.17). This estimation and since

$$\left\|\,|\cdot|^{-n-1}\chi_{\{|\cdot|>2^{j+4}\}}\right\|_{(p_1^-)'}\leq c2^{-\left(1+n/p_1^-\right)j},$$

we get $U_2(x)$ is bounded by

$$c\frac{2^{i(m+1-n+n/p_1^-)}}{|x|^{n/p_1^-+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j(\ell+1-n/p_1^-)}}{|x|^{n/(p_1^-)'+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}. \tag{5.14}$$

The last expression is symmetric to (5.13) and satisfy the desired estimate if

$$m+1>\alpha_1^+ \text{ and } \ell+1>(\alpha_2^++n/p_2^-)+n(1/p_1^--1).$$

Finally, let us consider U_3. Using (5.1) for a_i and b_j respectively, we get that $U_3(x)$ is bounded by

$$\begin{aligned} & c\sum_{\gamma=1}^{N}\sup_{t>\frac{|x|}{2}}\int_{|y|>2^{i+3}}\frac{1}{t^{n/2+m+1}}\frac{2^{i(m+1)}}{|y|^{n+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)} \\ & \times\frac{1}{t^{n/2+\ell+1}}\frac{2^{j(s+1)}}{|y|^{n+s+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}|y|^{s+1}\,dy \\ \leq\ & c\frac{2^{i(m+1)}}{|x|^{n/2+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}2^{i(m+1)}\left(\int_{|y|>2^{i+3}}\frac{dy}{|y|^{2(n+m+1)}}\right)^{1/2} \\ & \times\frac{2^{j(\ell+1)}}{|x|^{n/2+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}\left(\int_{|y|>2^{j+3}}\frac{dy}{|y|^{2n}}\right)^{1/2} \\ \leq\ & c\frac{2^{i(m+1-n/2)}}{|x|^{n/2+m+1}}\|a_i\|_{p_1(\cdot)}\left\|\chi_{C_i}\right\|_{p_1'(\cdot)}\frac{2^{j(\ell+1-n/2)}}{|x|^{n/2+\ell+1}}\|b_j\|_{p_2(\cdot)}\left\|\chi_{C_j}\right\|_{p_2'(\cdot)}. \end{aligned} \tag{5.15}$$

The estimation of the both terms are similar. We choose for example the first term. We have

$$\begin{aligned}
&\left\| \sum_{i=-\infty}^{+\infty} \lambda_i \frac{2^{i(m-n/2+1)}}{|\cdot|^{n/2+m+1}} \|a_i\|_{p_1(\cdot)} \|\chi_{C_i}\|_{p_1'(\cdot)} \chi_{\{|\cdot|>2^{i+5}\}} \right\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} \\
\leq\ & c\left\{ \sum_{k=-\infty}^{-1} \Big(\sum_{i=-\infty}^{k-6} \lambda_i 2^{(i-k)(n/2+m+1-\alpha_1(0)-n/p_1(0))} \Big)^{q_1(0)} \right\}^{1/q_1(0)} \\
&+c\left\{ \sum_{k=0}^{+\infty} \Big(\sum_{i=-\infty}^{-1} \lambda_i 2^{(i-k)(n/2+m+1-(\alpha_1^+ + n/p_1^-))} \Big)^{q_1(0)} \right\}^{1/q_1(0)} \\
&+c\left\{ \sum_{k=6}^{+\infty} \Big(\sum_{i=0}^{k-6} \lambda_i 2^{(i-k)(n/2+m+1-(\alpha_1+n/p_1)_\infty)} \Big)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty}
\end{aligned}$$

If $m+1 > \left(\alpha_1^+ + n/p_1^-\right) - n/2$, then the first term of (5.15) is bounded by

$$c\left\{ \sum_{k=-\infty}^{-1} (\lambda_k)^{q_1(0)} \right\}^{1/q_1(0)} + c\left\{ \sum_{k=0}^{+\infty} (\lambda_k)^{(q_1)_\infty} \right\}^{1/(q_1)_\infty} \leq c\,\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} .$$

Similar, if $\ell+1 > (\alpha_2 + n/p_2)^+ - n/2$, then the second term of (5.15) is bounded by

$$c\left\{ \sum_{k=-\infty}^{-1} (\mu_k)^{q_2(0)} \right\}^{1/q_1(0)} + c\left\{ \sum_{k=0}^{+\infty} (\mu_k)^{(q_2)_\infty} \right\}^{1/(q_1)_\infty} \leq c\,\|g\|_{H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)} .$$

Therefore, $U_3(x)$ also satisfies the desired estimate if

$$s+1 > \left(\alpha_1^+ + n/p_1^-\right) + \left(\alpha_2^+ + n/p_1^-\right) - n.$$

Combining (5.13), (5.14) and (5.15), we obtain

$$\|G_2\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c\,\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)} \|g\|_{H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)} .$$

This finish the proof of Lemma 5.2. ■

.

5.2 Boundedness from $H\dot{K} \times H\dot{K}$ into $H\dot{K}$.

In this section, we present the boundedness from $H\dot{K} \times H\dot{K}$ into $H\dot{K}$.

Theorem 5.3 *Let $q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $1 < p_i^- \le p_i^+ < \infty$, $\frac{1}{p_1^-} + \frac{1}{p_2^-} < 1$, $\alpha_i^- + \frac{n}{p_i^+} \ge n$, $i = 1, 2$, and $\alpha(\cdot) = \alpha_1(\cdot) + \alpha_2(\cdot)$ such that q_1, q_2, α_1 and α_2 are be* log*-Hölder continuous, both at the origin and at infinity. Let T_γ^j are operators bounded from $L^{p_j(\cdot)}(\mathbb{R}^n)$ into $L^{p_j(\cdot)}(\mathbb{R}^n), j = 1, 2$, $\gamma = 1, ..., N$ and satisfying the two conditions (5.1) and (5.2). Let $s \ge [(\alpha_1^+ + n/p_1^-) + (\alpha_2^+ + n/p_2^-) - n]$ be a non-negative integer such that*

$$\int_{\mathbb{R}^n} x^\beta B(a, b)(x)dx = 0$$

for all multi-indices β with $|\beta| \le s$, for all $(\alpha_1(\cdot), p_1(\cdot))$-atoms a, and all $(\alpha_2(\cdot), p_2(\cdot))$-atoms b. Then $B(f, g)$ can be extended to a bounded operator from $H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n) \times H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ into $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.

If q_1, q_2 p_1, p_2, α_1 and α_2 are constants, this result is from [21, Theorem 3].

Proof. Suppose $f \in H\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ and $g \in H\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$. Using Theorem 4.2 for f and g, we can write $f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$ and $g = \sum_{j=-\infty}^{+\infty} \mu_j b_j$ where $\lambda_i, \mu_j \ge 0$, a_i's are $(\alpha_1(\cdot), p_1(\cdot))$-atoms, b_j's are $(\alpha_2(\cdot), p_2(\cdot))$- atoms with supp$a_i \subseteq C_i$ and supp$b_j \subseteq C_j$. We observing that

$$\begin{aligned}\|B(f,g)\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} &\le \Big\|\sum_{i\ge j} \lambda_i\mu_j \mathcal{M}_\varphi(B(a_i,b_j))\Big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \Big\|\sum_{i<j} \lambda_i\mu_j \mathcal{M}_\varphi(B(a_i,b_j))\Big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

By the symmetry of the roles in $\sum_{i\ge j}$ and $\sum_{i<j}$, we will know that the estimate of $\sum_{i<j}$ is analogous to that of $\sum_{i\ge j}$, then it sufficient to show that

$$\begin{aligned}\Big\|\sum_{i\ge j} \lambda_i\mu_j \mathcal{M}_\varphi(B(f,g))\Big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} &\le \Big\|\sum_{i\ge j} \lambda_i\mu_j \mathcal{M}_\varphi(B(f,g))\chi_{\{|\cdot|\le 2^{i+5}\}}\Big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \Big\|\sum_{i\ge j} \lambda_i\mu_j \mathcal{M}_\varphi(B(f,g))\chi_{\{|\cdot|> 2^{i+5}\}}\Big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \\ &= \|T_1\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} + \|T_2\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \qquad (5.16)\end{aligned}$$

is bounded.

The estimation of (5.16) is a consequences of Lemma 5.1 and Lemma 5.2. This finish the proof of Theorem 5.3. ■

Remark 5.4 *The minimum value of s satisfying (5.3) can be taken as*

$$[(\alpha_1^+ + n/p_1^-) + (\alpha_2^+ + n/p_1^-) - n],$$

5.3 Applications

In this section, we present the boundedness of the commutators of Calderón-Zygmund operators on variable Herz spaces into itself.

Recall that for $0 < p < \infty$, $H\dot{K}_p^{\alpha,p}(\mathbb{R}^n)$ are the usual Hardy spaces $H^p(\mathbb{R}^n)$ and we have

$$H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \text{ when } p > 1.$$

Definition 5.5 *The space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions f such that*

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx < \infty,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Remark 5.6 *The space $BMO(\mathbb{R}^n)$ is the dual space of the Hardy space $H^1(\mathbb{R}^n)$.*

We have the following duality theorems, for the proof see M. Izuki and T. Noi [29, Theorem 3].

Theorem 5.7 *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in L^\infty(\mathbb{R}^n)$. Then we have*

$$\left(\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)\right)' = \dot{K}_{p'(\cdot)}^{-\alpha(\cdot),q'(\cdot)}(\mathbb{R}^n).$$

The following result generalize Corollarie 1 of Grafakos, Li and Yang [21] by taking α, p and q are constants.

Corollary 5.8 *Let b be in $BMO(\mathbb{R}^n)$ and T be a Calderón-Zygmund operator. Then the commutator*

$$[b, T](f) = bT(f) - T(bf)$$

maps $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ into itself when $p \in \mathcal{P}^{\log}(\mathbb{R}^n), q \in \mathcal{P}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$, $-\frac{n}{p^+} < \alpha^- \le \alpha^+ < n(1 - \frac{1}{p^-})$ and α, q are log*-Hölder continuous, both at the origin and at infinity.*

Proof. Consider the bilinear operator

$$B(f,g) = (Tf)g - f(T^*g)$$

where T^* is the adjoint operator of T.

Since T^* is the adjoint operator of T, then we have

$$(Tf\,|g) = (f\,|T^*g)\,,$$

for all f, g square integrable and compactly supported functions, which give that $B(f,g)$ has integral zero, then $B(f,g)$ is satisfies the conditions of Theorem 4.8.

By Theorem 4.8, we have

$$\big\|B(f,g)\big\|_{H\dot{K}_1^{0,1}(\mathbb{R}^n)} = \big\|B(f,g)\big\|_{H^1(\mathbb{R}^n)} \leq c\big\|f\big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}\big\|g\big\|_{\dot{K}_{p'(\cdot)}^{-\alpha(\cdot),q'(\cdot)}(\mathbb{R}^n)}.$$

Using the Remark 5.6, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n}[b,T](f)(x)g(x)dx\right| &= \left|\int_{\mathbb{R}^n} b(x)\left[(Tf)(x)g(x) - f(x)(T^*g)(x)\right]dx\right| \\ &= \left|\int_{\mathbb{R}^n} b(x)B(f,g)(x)dx\right| \\ &\leq \big\|b\big\|_{BMO(\mathbb{R}^n)}\big\|B(f,g)\big\|_{H^1(\mathbb{R}^n)} \\ &\leq c\big\|b\big\|_{BMO(\mathbb{R}^n)}\big\|f\big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}\big\|g\big\|_{\dot{K}_{p'(\cdot)}^{-\alpha(\cdot),q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Now it sufficient to apply the supremum to both sides of the last expression when $\big\|g\big\|_{\dot{K}_{p'(\cdot)}^{-\alpha(\cdot),q'(\cdot)}(\mathbb{R}^n)} = 1$ and using the Theorem 5.7, we obtain

$$\big\|[b,T](f)\big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c\big\|b\big\|_{BMO(\mathbb{R}^n)}\big\|f\big\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof. ∎

Conclusion and future prospects

The study aimed to highlight the importance of variable exponent spaces, for the purpose of access to the objectives of the study the researcher used the results of classical function spaces and the log-Hölder condition to guarantee regularity. We have "translated" the results from classical function spaces to variable function spaces.

We concluded that the most important results are:

▶ We have shown that the atomic decomposition and Plancherel-Polya-Nikolskij inequality are holds for variable Herz spaces.

▶ We have shown the Sobolev embeddings is holds in variable Herz-type Besov spaces.

▶ Also, We have shown the boundedness of some bilinear operators (in particular Calderón-Zygmund operators) are holds in variable Herz-type Hardy spaces.

▶ At the end of thesis, we have shown the boundedness of the commutators of Calderón-Zygmund operators on variable Herz spaces into itself.

We provide a few prospects for future studies of variable function spaces as follow:

- Applications to PDE.
- Generalization of the results given in chapters 4 and 5 for multilinear operators under suitable conditions.
- The boundedness of fractional integrals operators.

Bibliography

[1] A. Almeida and D. Drihem, Maximal, potential and singular type operators on Herz spaces with variable exponents, J. Math. Anal. Appl. **394** (2012), 781-795.

[2] A. Almeida and P. Hästö, Besov spaces with variable smoothness and integrability, J. Funct. Anal. **258** (2010), 1628-1655.

[3] A. Baernstein II and E. T. Sawyer, Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$, Mem. Amer. Math. Soc. **53** (1985).

[4] A. Beurling, Construction and analysis of some convolution algebras, Ann. Inst. Fourier Grenoble. **14** (1964), 1-32.

[5] Y. Chen, S. Levine and R. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. **66** (2006), 1383-1406.

[6] R. Coifman and L. Grafakos, Hardy space estimates for multilinear operators I, Rev. Mat.Iber. **8** (1992), 45-62.

[7] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue spaces. Birkhäuser, Berlin, 2013.

[8] D. Cruz-Uribe, A. Fiorenza, M. Ruzhansky and J. Wirth, Variable Lebesgue space and hyperbolic systems, edited by Sergey Tikhonov. Advanced Courses in Mathematics CRM Barcelona, vol. **20**, Birkhäuser, Basel, 2013. ISBN 978-3-0348-0839-2.

[9] L. Diening, P. Hästö and S. Roudenko, Function spaces of variable smoothness and integrability, J. Funct. Anal. **256** (2009), 1731-1768.

[10] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta and T. Shimomura, Maximal functions in variable exponent spaces: limiting cases of the exponent, Ann. Acad. Sci. Fenn. Math. **34** (2009), 503-522.

[11] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, **2017**, Springer Verlag, Berlin, 2011.

[12] B. Dong and J. Xu, New Herz type Besov and Triebel-Lizorkin spaces with variable exponents, J. Funct. Spaces Appl. **2012** (2012), ID 384593, 27 pages.

[13] D. Drihem, Atomic decomposition of Besov spaces with variable smoothness and integrability, J. Math. Anal. Appl. **389** (2012), 15-31.

[14] D. Drihem, Embeddings properties on Herz-type Besov and Triebel-Lizorkin spaces, Math. Ineq and Appl. **16** (2013), 439-460.

[15] D. Drihem and R. Heraiz, Herz type Besov spaces of variable smoothness and integrability, Kodai Math. J. **40** (2017) 31–57.

[16] D. Drihem and F. Seghiri, Notes on the Herz-type Hardy spaces of variable smoothness and integrability, Math. Ineq and Appl. **19** (2016), 145-165.

[17] D.S. Fan and D.C. Yang, Herz-type Hardy spaces on Vilenkin groups and their applications, Science in China. **43** (2000), 481-491.

[18] H. G. Feichtinger and F. Weisz, Herz spaces and summability of Fourier transforms, Math. Nachr. **281** (2008), 309-324.

[19] L. Grafakos, Hardy space estimates for multilinear operators II, Rev. Mat. Iber. **8** (1992), 45-62.

[20] L. Grafakos and X. Li, Bilinear operators on homogeneous groups, J. Operator Theory **44** (2000), 63-90.

[21] L. Grafakos, X. Li and D. Yang, Bilinear operators on Herz-type Hardy spaces, Trans. Amer. Math. Soc. **350** (1998), 1249-1275.

[22] L. I. Hedberg and Y. Netrusov, An axiomatic approach to function spaces, spectral synthesis and Luzin approximation, Mem. Amer. Math. Soc. **188** (2007), no. 882.

[23] E. Hernandez and D. Yang, Interpolation of Herz-type Hardy spaces, Illinois J. Math. **42** (1998), 564-581

[24] E. Hernandez and D. Yang, Interpolation of Herz spaces and applications, Math. Nachr. **205** (1999), 69-87.

[25] C. Herz, Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms, J. Math. Mech. **18** (1968), 283-324.

[26] M. Izuki, Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system, East. J. Approx. **15** (2009), 87-109.

[27] M. Izuki, Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization, Anal. Math. **36** (2010), 33-50.

[28] M. Izuki and T. Noi, Boundedness of some integral operators and commutators on generalized Herz spaces with variable exponents, http://www.sci.osaka-cu.ac.jp/math/OCAMI/preprint/2011/11-15.pdf.

[29] M. Izuki and T. Noi, Duality of Besov, Triebel-Lizorkin and Herz spaces with variable exponents, Rend. Circ. Mat. Palermo. **63** (2014), 221-245.

[30] H. Kempka and J. Vybíral, Spaces of variable smoothness and integrability: Characterizations by local means and ball means of differences, J. Fourier Anal. Appl. **18** (2012), 852-891.

[31] H. Kempka and J. Vybíral, A note on the spaces of variable integrability and summability of Almeida and Hästö, Proc. Amer. Math. Soc. **141** (2013), 3207-3212.

[32] X. Li and D. Yang, Boundedness of some sublinear operators on Herz spaces, Illinois J. Math. **40** (1996), 484-501.

[33] X. Li, S. Lu and D. Yang, Certain bilinear operators on Herz-type Hardy spaces, Beijing Math. **2** (1996), 72-95.

[34] Y. Liang, D. Yang and C. Zhuo, Intrinsic Square Function Characterizations of Hardy Spaces with Variable Exponents, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 4, 1541–1577.

[35] L. Liu, Herz and Herz type Hardy spaces estimates of multilinear integral operators for the extreme cases, Journal of Math. of Kyoto Univ. **46** (2006), 1-20.

[36] S. Lu, K. Yabuta and D. Yang, Boundedness of some sublinear operators in weighted Herz-type spaces, Kodai Math. J. **23** (2000) 391–410.

[37] S. Lu and D. Yang, The decomposition of weighted Herz space on $\mathbb{R}^n$ and its applications, Sci. China (Ser. A). **38** (1995), 147-158.

[38] S. Lu and D. Yang, The decomposition of Herz spaces on local fields and its application, J. Math. Anal. Appl. **196** (1995), 296–313.

[39] S. Lu and D. Yang, The local versions of $H_p(\mathbb{R}^n)$ spaces at the origin, Studia Math. **116** (1995), 103-131.

[40] S. Lu and D. Yang, Some characterizations of weighted Herz-type Hardy spaces and their applications, Acta Math. Sinica (New Ser.) **13** (1997), 45-58.

[41] S. Lu and D. Yang, Herz-type Sobolev and Bessel potential spaces and their applications, Sci. in China (Ser. A). **40** (1997), 113-129.

[42] S. Lu, D. Yang and G. Hu, Herz Type Spaces and Their Applications, Beijing: Science Press, 2008.

[43] A. Miyachi, Remarks on Herz-type Hardy spaces. Acta Math. Sinica. English Series **17** (2001), 339–360.

[44] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, J. Funct. Anal. **262** (2012), no. 9, 3665-3748.

[45] M. A. Ragusa, Homogeneous Herz spaces and regularity results, Nonlinear Anal. **71** (2009), 1-6.

[46] M. A. Ragusa, Parabolic Herz spaces and their applications, Appl. Math. Lett. **25** (2012), 1270-1273.

[47] M. Růžička, Electrorheological fluids: modeling and mathematical theory, Lecture Notes in Mathematics, **1748**, Springer-Verlag, Berlin, 2000.

[48] S. Samko, Variable exponent Herz spaces, Mediterr. J. Math. **10** (2013), 2007-2025.

[49] C. Shi and J. Xu, A characterization of Herz-Besov-Triebel spaces with variable exponent, Acta Mathematica Sinica, Chinese Series. **55** (2012), 653-664.

[50] C. Shi and J. Xu, Herz type Besov and Triebel-Lizorkin spaces with variable exponent, Front. Math. China. **8** (2013), 907-921.

[51] F. Soria and G. Weiss, A remark on singular integrals and power weights, Indiana. Univ. Math. J. **43** (1994), 187-204.

[52] L. Tang and D. Yang, Boundedness of multilinear operators in Herz-type Hardy spaces, Acta Math.Sinica (English Ser.). **16** (2000), 295-306.

[53] H. Triebel, Theory of function spaces, Birkhäuser, Basel, 1983.

[54] H. Triebel, Theory of function spaces, II, Birkhäuser, Basel, 1992.

[55] H. Triebel, Theory of function spaces, III, Birkhäuser, Basel, 2006.

[56] H. B. Wang, Anisotropic Herz spaces with variable exponents, Funct. Approx. Comment. Math, **18** (2015), pp1-15.

[57] H. B. Wang and Z. G. Liu, The Herz-type Hardy spaces with variable exponent and their applications, Taiwanese J. Math. **16** (2012), 1363-1389.

[58] J. Xu, Some properties on Herz-type Besov spaces (in chinese), J. Hunan Univ. (Natural Sci). **30** (2003), 75-78.

[59] J. Xu, Variable Besov and Triebel-Lizorkin spaces, Ann. Acad. Sci. Fenn. Math. **33** (2008), 511-522.

[60] J. Xu, Decompositions of non-homogeneous Herz type Besov and Triebel-Lizorkin spaces, Sci. China. Math. **57** (2014), 315-331.

[61] J. Xu and D. Yang, Applications of Herz-type Triebel-Lizorkin spaces, Acta. Math. Sci (Ser. B). **23** (2003), 328-338.

[62] J. Xu and D. Yang, Herz-type Triebel-Lizorkin spaces I, Acta Math. Sci (English) (Ed.). **21** (2005), 643-654.

ملخص المذكرة:

في هذه المذكرة عممننا متراجحة "بلانشرال-بوليا-نيكولسكيج" في فضاءات "هارز" ذات الأدلة المتغيرة ثم قمنا بدراسة فضاءات "هارز-بيزوف" ذات الأدلة المتغيرة، حيث أعطينا بعض الخصائص الأساسية لهذه الفضاءات، وفي الأخير درسنا محدودية بعض المؤثرات ثنائية الخطية في فضاءات "هارز-هاردي" ذات الأدلة المتغيرة باستعمال التفكيك الذري لهذه الفضاءات.

الكلمات المفتاحية: فضاءات هارز، فضاءات هارز-بيزوف، فضاءات هارز-هاردي، احتواءات سوبولاف، مؤثرات تحت الخطية، أدلة متغيرة، الدالة الأعظمية، الذرة.

Résumé de thèse:

Dans cette thèse, nous avons généralisé l'inégalité de Plancheral-Polya-Nikolskij dans les espace de Herz avec des exposants variables, puis nous avons étudié les espaces Herz-Besov avec des exposants variables où nous avons donné certaines propriétés de base de ces espaces et finalement, nous avons étudié la continuité de certains opérateurs bilinéaires dans les espaces de Herz-Hardy avec des exposants variables en utilisant la décomposition atomique de ces espaces.

Mots clés : Espaces de Herz, Espaces de Herz-Besov, Espaces de Herz-Hardy, inclusions de Sobolev, Opérateurs sous linéaire, Exposants variables, La fonction maximale, Atome.

Abstract of thesis:

In this thesis, we have generalize Plancheral-Polya-Nikolskij inequality in Herz spaces with variables exponents, after that we studied the Herz-Besov spaces with variable exponents where we gave some basic properties of these spaces and finally we studied the boundedness of some bilinear operators in Herz-Hardy spaces with variables exponents using atomic decomposition of these spaces.

Key words: Herz spaces, Herz-type Besov spaces, Herz-type Hardy spaces, Sobolev type embedding’s, Sublinear operators, Variable exponent, Maximal function, Atom.