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DEDICATIONS

To my whole family and all my friends,
especially my husband, my daughters,
and my mother

N. Ouagueni

NOTATIONS

$\mathbb{R}$	Real numbers $(-\infty, +\infty)$.
$\mathbb{R}^*$	Nonzero real numbers $(-\infty, 0) \cup (0, +\infty)$.
$\mathbb{R}^+$	Positive real numbers $(0, +\infty)$.
$\mathbb{C}$	Complex numbers, $z \in \mathbb{C}$, then $z = x + iy$, where $x, y \in \mathbb{R}$, and $i^2 = -1$.
$Re(\alpha)$	Real part of complex α.
$\mathbb{Z}$	Integers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
$\mathbb{Z}_0^-$	Negative integers $\{..., -3, -2, -1, 0\}$.
$\mathbb{N}$	Natural numbers $\{0, 1, 2, 3, ...\}$.
$\mathbb{N}^*$	Nonzero natural numbers $\{0, 1, 2, 3, ...\}$.
Ω	Finite closed interval of the real axis $\mathbb{R}$.
$L^p(\Omega)$	Space of all measurable functions u on Ω, where $\lvert u\rvert^p \in L^1(\Omega)$, for $1 < p < \infty$.
$L^1(\Omega)$	Space of the Lebesgue complex-valued measurable functions u on Ω, for which $\lVert u\rVert_{L^1} = \int_\Omega \lvert u(s)\rvert\, ds < \infty$.
$L^\infty(\Omega)$	Space of all measurable functions, for which $\exists C > 0, \lvert u(t)\rvert \leq C$, p.p. $t \in \Omega$.
$C(\Omega)$	The Banach space of all continuous functions from Ω into $\mathbb{R}$.
$C^n(\Omega)$	the set of mappings having n times continuously differentiable on Ω.
C_α^n, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$	$\{f(t), t > 0 : \ f(t) = t^p f_1(t) \ \text{with } p > \alpha \text{ and } f_1 \in C^n([0, \infty))\}$.
$X_c^p(\Omega)$	Space of all measurable functions u on Ω, for which $\lVert u\rVert_{X_c^p} = \underset{c\in\mathbb{R},\, 1\leq p<\infty}{\left(\int_\Omega \lvert s^c u(s)\rvert^p \frac{ds}{s}\right)^{\frac{1}{p}}} < \infty$.
$X_c^\infty(\Omega)$	Space of all measurable functions u on Ω, where $\lVert u\rVert_{X_c^\infty} = ess \sup_{a\leq t\leq b} [t^c \lvert u(t)\rvert]_{c\in\mathbb{R}} < \infty$.
$\Gamma(.)$	Euler's gamma function.
$B(.,.)$	The beta function.
$E_\alpha(.)$	The classical Mittag-Leffler function.
$E_{\alpha,\beta}(.)$	The generalized Mittag-Leffler function.
$W_{(\mu,a),(v,b)}(.)$	The generalized Wright function.
$\mathcal{I}^1 u$	Primitive of Lebesgue summable function u.
$\mathcal{I}^n u$	Cauchy formula for the n^{th} integrals, $n \in \mathbb{N}$.
$\mathcal{I}^\alpha u$	Riemann-Liouville's fractional integral of order α.

${}^{RL}\mathcal{D}^{\alpha}u$	RIEMANN-LIOUVILLE's fractional derivative of order α.
${}^{C}\mathcal{D}^{\alpha}u$	CAPUTO's fractional derivative of order α.
${}^{H}\mathcal{I}^{\alpha}u$	HADAMARD's fractional integral of order α.
${}^{H}\mathcal{D}^{\alpha}u$	HADAMARD's fractional derivative of order α.
${}^{CH}\mathcal{D}^{\alpha}u$	CAPUTO–HADAMARD's fractional derivative of order α.
$\mathcal{J}_{\beta}^{\tau,\alpha}u$	The left-sided ERDÉ LYI–KOBER's fractional integral of order α.
$\mathcal{P}_{\beta}^{\tau,\alpha}u$	The left-sided ERDÉ LYI–KOBER's fractional derivative of order α.
${}^{\rho}\mathcal{I}^{\alpha}u$	KATUGAMPOLA's fractional integral of order α.
${}^{\rho}\mathcal{D}^{\alpha}u$	KATUGAMPOLA's fractional derivative of order α.
${}^{C}\mathcal{D}^{\alpha,\rho}u$	CAPUTO-KATUGAMPOLA's fractional derivative of order α.
${}_{a}^{R}D_{b}^{\alpha}\ u$	RIESZ fractional derivative of order α.
${}_{a}^{RC}D_{b}^{\alpha}\ u$	RIESZ-CAPUTO's derivative of order α.
${}_{a}^{RG}I_{b}^{\alpha,\rho}u$	Generalized RIESZ type integral of order α.
${}_{a}^{RC}D_{b}^{\alpha,\rho}\ u$	Generalized RIESZ–CAPUTO fractional derivative of order α.
${}^{\rho}\mathcal{D}^{\alpha,\gamma}u$	HILFER-KATUGAMPOLA's fractional derivative of order α and type γ.
$\delta = tD\ (D = \frac{d}{dt})$	The δ derivative.
$\delta_x = x^{1-\rho}D\ (D = \frac{d}{dx})$	The δ_x derivative.
${}^{\rho}D_{0,t}^{\alpha,\gamma}u$	HILFER-KATUGAMPOLA's time fractional derivative of order α.
${}^{\rho}D_{0,x}^{\beta,\gamma}u$	HILFER-KATUGAMPOLA's space fractional derivative of order β.
$HKFD$	HILFER-KATUGAMPOLA's fractional derivative.
$CGFD$	CAPUTO generalized fractional derivative.
$GRCFD$	Generalized RIESZ-CAPUTO's fractional derivative.
FI and FD	Fractional integral and fractional derivative.
$\frac{\partial^{\alpha,\rho}u}{\partial\lvert x\rvert^{\alpha}}$	The GRCFD.
FDE	Fractional differential equation.
$FPDE$	Fractional partial differential equation.
BVP	Boundary Value Problem.
FC	Fractional calculus.

CONTENTS

Introduction 1
1 Background materials and preliminaries 4
1.1 A historical overview of fractional calculus . . . 4
1.2 Basic elements of functional analysis . . . 5
1.3 Special functions encountered in FC . . . 6
1.4 Basic fractional integrals and derivatives . . . 8
1.4.1 Riemann-Liouville fractional integrals and derivatives . . . 8
1.4.2 Caputo-type fractional derivatives . . . 11
1.4.3 Hadamard fractional integrals and derivatives . . . 12
1.4.4 Caputo-Hadamard-type FDs . . . 13
1.4.5 Erdélyi-Kober fractional integral and derivative . . . 13
1.5 Generalized fractional integrals and derivatives of Katugampola . . . 14
1.5.1 Caputo-type generalized fractional derivatives of Katugampola . . . 17
1.5.2 Generalized Riesz-Caputo fractional derivative . . . 18
1.5.3 Hilfer-Katugampola's fractional derivative . . . 20
1.6 Fractional-order partial differential equations . . . 21
1.6.1 Self-similar solutions of fractional equations . . . 22
1.6.2 Studies and results of FDEs' and FPDEs' solutions . . . 23
1.7 Fixed point theorems . . . 24
2 Existence and uniqueness of solution for a mixed-type fractional differential equation and Ulam-Hyers stability 26
2.1 Definition of integral solution . . . 27
2.2 Fundamental lemmas . . . 31
2.3 Existence results of at least one solution . . . 37
2.4 Uniqueness of solution . . . 40
2.5 Ulam-Hyers stability . . . 41
2.6 Examples . . . 43

3 Existence results of self-similar solutions of the space-fractional diffusion equation involving the generalized Riesz-Caputo fractional derivative 44
3.1 Statement of the problem 44
3.2 Definition of integral solution of the basic profile f 47
3.3 Fundamental lemmas 48
3.4 Existence and uniqueness results of the basic profile f 50
3.5 Existence and uniqueness results to the original problem 58
4 Self-similar solutions of time-space fractional diffusion equation involving Hilfer-Katugampola derivative 61
4.1 Determination of a symmetry group of scaling transformation 61
4.2 Statement of the problem 63
4.3 Definition of integral solution of the basic profile f 67
4.4 Determination of the explicit solution f 68
4.5 The exact form of the self-similar solutions 74
4.6 Graphical representation of the exact solution 74
Conclusion and perspectives 78
Bibliography 78

INTRODUCTION

Fractional calculus (FC) is a mathematical analysis subject which deals with different possible approaches of defining fractional-order derivatives (FDs) and integrals (FIs).

To define FIs and FDs, many approches have been proposed in the literature, including Riemann-Liouville's, Hadamard's, Caputo's, Riesz's, Erdelyi-Kober's approches,...etc. The development of each one of these approches should go through a series of stages ranging from exponential functions to special functions. Based on those fractional operators, the theory of classical (integer order) differential equations (IODEs) has been then generalized to the broader theory of fractional-order ordinary and partial differential equations (FDEs, and FPDEs). For more details on the subject, the reader may refer to [9, 21, 39, 48, 56].

FPDEs can be used for the modeling and study of many important phenomena in many different fields of science and engineering, such as diffusion processes, damping law,...etc. One can find a variety of applications in [29, 34, 38, 46, 55]. Closed-form solutions are useful when dealing with many features of natural and physical phenomena.

Generally, for PDEs, we can search for special type solutions known as group-invariant solutions. As in [17], by solving a reduced system of equations (which has fewer independent variables compared to the original problem), the group-invariant solutions can be found. These solutions are also known as self-similar solutions which are used to model many processes in mathematics and engineering mechanics. The FPDEs which have self-similar solutions can easily be reduced to ordinary differential equations (ODEs). This latter process helps to simplify one's work on FPDEs.

The idea behind solutions' self-similarity along with Lie group analysis have also been applied in FDEs. For example, Luchko and Gorenflo in [44], have been discussed the application of Lie group analysis for the equation ($\frac{\partial^\alpha u}{\partial t^\alpha} = d\frac{\partial^\beta u}{\partial x^\beta}$). They have found scale-invariant solutions for the FDE with a new independent variable $\eta = xt^{\frac{-\alpha}{\beta}}$. The left and right sided Erdélyi-Kober derivatives which depend on α, β of this equation and on the parameter γ of its scaled group are considered. They have derived a general solution in terms of the generalized Wright function. In [17], a particular case (for $\beta = 2$) of the previous method was proved to be a scale invariant solutions for a time fractional diffusion equation.

Throughout the literature, one may easily be aware of the existence of plenty of research works on fractional (space, time and time-space) diffusion equations by using the similarity method. For more details, the reader may check [3, 11, 13, 22].

The fixed point methods play a particularly major role in solving the problems modeled by FDEs. In non-linear analysis, theses tools are used to study the existence of solutions for many kinds of problems modeled by FDEs and FPDEs.

Recently, many efforts have been devoted to deal with the existence, uniqueness and multiplicity of real or positive solutions of nonlinear fractional problems. Fixed point theorems come in the heart of the non-linear analysis techniques used in those efforts.

Different variants of fixed-point theorems and other non-linear analysis techniques have been used by researchers to develop solutions and their existence for non-linear initial value problems (IVPs) and boundary value problems (BVPs) of FDEs or FPDEs (see [6–8, 10, 12, 13, 15, 16, 19, 27, 47, 58]).

The stability problem introduced by Ulam in [59] has attracted the attention and efforts of many famous researchers (see [32, 33]). Not long ago, Ulam-Hyers stability problem for FDEs has gained much research attention (see [30, 42, 61–63]).

The main objective of this thesis is to study of the existence and uniqueness of solutions of nonlinear mixed FDEs involving two different fractional derivatives and space-fractional diffusion equation with generalized Riesz Caputo FD (GRCFD). We introduce also the self-similar solutions in an explicit form of space-time fractional diffusion equation involving Hilfer-katugampola's fractional derivative by applying the successive approximation method (see [39]). The main tools we have used in our work are the Banach's contraction principle, Schauder's, Schaefer's fixed point theorems and the nonlinear alternative of Leray-Schauder's type (see [26]).

The remainder of this thesis is organized as follows. In the next chapter, we will re-call some fractional order operators from the theory of FC. To do so, we'll start by giving a historical overview of FC. Then, we'll go trough some basics of functional analysis, the special functions encountered when studying FC and elementary FIs and FDs. We'll also talk about FPDEs, their self-similar solutions, and about some studies and results of BVPs involving both FDEs and FPDEs. The content of this chapter will serve as a background for the next chapters.

In the second chapter, we will investigate the existence and uniqueness of solution for the following BVP involving a nonlinear FDE with two different fractional derivatives

$$ {}^{C}D^{\beta,\rho}_{1^-}\left({}^{\alpha}D^{\rho}_{0^+}+\lambda\right)u(t)=f(t,u(t)),\ t\in J=[0,1], $$

with the boundary conditions

$$\left({}^{\rho}I_{0^+}^{1-\alpha}u\right)(0) = u_0,$$

$$\left({}^{\alpha}D_{0^+}^{\rho} + \lambda\right)u(1) = u_1,$$

where $\alpha, \beta, \alpha+\beta \in (0,1)$, $\lambda, \rho > 0$, $u_0, u_1 \in \mathbb{R}$, ${}^{C}D_{1^-}^{\beta,\rho}$ is the right Caputo-Katugampola's FD (CKFD) of order β. ${}^{\alpha}D_{0^+}^{\rho}$ is the left Katugampola's FD of order α and ${}^{\rho}I_{0^+}^{1-\alpha}$ is the Katugampola's FI.

In the third chapter, we will discuss the existence, uniqueness of the solution of the following space-fractional diffusion equation, which is

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{\alpha,\rho} u(x,t)}{\partial |x|^{\alpha}}, \ (x,t) \in [0,X] \times [t_0, \infty[\,,\ 1 < \alpha \leq 2,$$

under the following self-similar form

$$u(x,t) = t^{\beta} f\left(\frac{x}{t^{\frac{1}{\alpha\rho}}}\right), \text{ with } (x,t) \in [0,X] \times [t_0, \infty[\,,$$

where $u(x,t)$ is a scalar function of space variables $x \in [0,X]$, and time $t \in [t_0, \infty)$, with $X, t_0 > 0$, $\frac{\partial^{\alpha,\rho}}{\partial |x|^{\alpha}}$ is the Generalized Reisz-Caputo FD (GRCFD) of order α with $\rho > 0$ and which is the main motivation of the present research. The "basic profile" f is not known in advance and is to be identified, $\beta \in \mathbb{R}$ is a constant chosen so that the solutions exist.

In the last chapter, we will introduce the self-similar solutions for the following space-time FPDE (known as time-space fractional diffusion equation)

$${}^{\rho}D_{0,t}^{\alpha,\gamma}u(x,t) = {}^{\rho}D_{0,x}^{\beta,\gamma}u(x,t)\ ,\ x > 0,\ t > 0,$$

with the following conditions

$${}^{\rho}I_{0,x}^{2-m}u(0^+,t) = t^{\mu+\frac{\alpha\rho}{\beta}(2-m)}B,\ \delta_x\, {}^{\rho}I_{0,x}^{2-m}u(0^+,t) = t^{\mu+\frac{\alpha\rho}{\beta}(1-m)}A$$

where $u(x,t)$ is a scalar function of space and time variables $x, t > 0$, $\rho > 0, m = \beta + \gamma(2-\beta)$, $0 \leq \gamma \leq 1$ and $\delta_x = \left(x^{1-\rho}\frac{\partial}{\partial x}\right)$, D is the Hilfer-Katugampola's FD (HK) of order $0 < \alpha \leq 1$ and $1 < \beta \leq 2$ respectively.

Finally, the thesis will be concluded and some future perspectives will be given.

Almost all the outcomes of this work have already been either published or submitted for publication in peer-reviewed journals. For instance, the results included in chapters 2 and 3 have been published in [53] and [54], respectively, and those included in chapter 4 have been submitted for publication.

1

BACKGROUND MATERIALS AND PRELIMINARIES

In this chapter we will re-call some fractional order operators for the theory of FC. We'll also talk about FPDEs, their self-similar solutions, and about some studies and results of BVPs involving both FDEs and FPDEs.

To do so, we'll start by giving a historical overview of FC. Then, we'll go trough some basics of functional analysis and the special functions encountered when studying FC. This section will serve as a background for the next sections and chapters.

1.1 A HISTORICAL OVERVIEW OF FRACTIONAL CALCULUS

The birth of FC can be traced back to 1695 in a letter written by L'Hôspital to Leibniz asking him about a particular notation he had used in his writings for the n^{th}-derivative of the linear function $u(t) = t$, $\frac{d^n t}{dt^n}$. L'Hôspital wondered what the result would be if $n = \frac{1}{2}$. Leibniz's response was: "An apparent paradox, from which one day useful consequences will be drawn ".

Following L'Hôspital's and Leibniz's first inquisition. Fourier, Euler and Laplace are among the many minds who tackled the FC and its mathematical consequences [50].

Several mathematicians used their own notation and methodology to introduce definitions that fit the concept of an integral or derivative of non-integer order. The most famous of these definitions in the field of FC are Riemann-Liouville and Caputo definitions. In this thesis, we'll adress these two definitions along with many others.

Most of the mathematical theory applicable to the study of FC was developed prior to the turn of the twentieth century. However, it is only during the last century that the most intriguing advances in engineering and scientific application have been achieved. In some cases, Mathematics had to change in order to meet the requirements of physical reality.

Caputo reformulated Riemann-Liouville FD in order to use integer-order initial conditions to solve his FDEs [55]. In 1996, Kolowankar reformulated again Riemann-Liouville FD [40].

1.2 BASIC ELEMENTS OF FUNCTIONAL ANALYSIS

Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$.
We denote by $L^p(\Omega)$ $(1 \leq p \leq \infty)$ the space of the LEBESGUE complex-valued measurable functions u on Ω for which $\|u\|_{L^p} < \infty$, where

$$\|u\|_{L^p} = \left(\int_\Omega |u(s)|^p \, ds \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty)$$

As a particular case, for $p = 1$, we denote by $L^1(\Omega)$ the space of the LEBESGUE complex-valued measurable functions u on Ω for which $\|u\|_{L^1} < \infty$, where

$$\|u\|_{L^1} = \int_\Omega |u(s)| \, ds.$$

For the case $p = \infty$, we denote by

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}, \text{ where } u \text{ is measurable and } \exists C > 0 \text{ such that } |u(t)| \leq C \text{ p.p on } \Omega\},$$

the space of the bounded LEBESGUE complex-valued measurable functions u on Ω, where

$$\|u\|_{L^\infty} = ess \sup_{a \leq t \leq b} |u(t)|.$$

Here $ess \sup_{a \leq t \leq b} |u(t)|$ is the essential maximum of the function $|u(t)|$ [see, for example, NIKOLSKII [49], p. 12-13].

As in [39], for $1 \leq p \leq \infty$ and $c \in \mathbb{R}$, consider the space $X_c^p(\Omega)$ of the LEBESGUE complex-valued measurable functions u on Ω as follows

$$X_c^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \|u\|_{X_c^p} = \left(\int_\Omega |s^c u(s)|^p \frac{ds}{s} \right)^{\frac{1}{p}} < \infty \right\},$$

For the case $p = \infty$,

$$\|u\|_{X_c^\infty} = ess \sup_{a \leq t \leq b} [t^c |u(t)|], \ c \in \mathbb{R}.$$

We denote by $C(\Omega)$, the Banach space of all continuous functions from Ω into $\mathbb{R}$, with the norm

$$\|u\|_\infty = \sup_{a \leq t \leq b} |u(t)|.$$

We denote also by $C^n(\Omega, \mathbb{R})$ with $n \in \mathbb{N}_0$ the set of mappings having n times continuously differentiable on Ω.

Definition 1.1 ([39]) *The space of functions C_α^n, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, consists of all functions $f(t)$, $t > 0$, that can be represented in the form $f(t) = t^p f_1(t)$ with $p > \alpha$ and $f_1 \in C^n([0, \infty))$.*

1.3 SPECIAL FUNCTIONS ENCOUNTERED IN FC

Here, we re-call the definitions and some properties of some special functions encountered when working on FC. These are the EULER gamma, the Beta, the MITTAG-LEFFLER and the generalized WRIGHT functions.

Euler's gamma function

EULER's gamma function $\Gamma(z)$ generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

Definition 1.2 (Euler's gamma function [39]) EULER's *gamma function* $\Gamma(\alpha)$ *is defined by*

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} e^{-s} ds, \left(Re(\alpha) > 0\right), \tag{1.1}$$

where $s^{\alpha-1} = e^{(\alpha-1)\ln(s)}$. *This integral converges for all complex* $\alpha \in \mathbb{C}$ $\left(Re(\alpha) > 0\right)$.

Properties of Euler's gamma function (see [39])

Here, we indicate some properties of EULER's gamma function

1. It satisfies the following relation

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), Re(\alpha) > 0, \tag{1.2}$$

2. It generalizes the factorial

$$\Gamma(n+1) = n!, \forall n \in \mathbb{N}.$$

3. Since $\Gamma(\alpha)$ is infinite for all the negative integer values of α, it satisfies the following

$$\frac{1}{\Gamma(-n)} = 0, \text{ for } n = 0, 1, 2, 3...$$

Beta function

The Beta function is very important for the computation of the FDs of the power function. It is also as EULER's integral of the first kind. It shares the form that typically resembles the FI and FD of many functions.

Definition 1.3 (Beta function[39]) *For the two parameters, p and q (*$Re(p) > 0$ *and* $Re(q) > 0$*), the Beta function is defined by*

$$B(p,q) = \int_0^1 s^{p-1} (1-s)^{q-1} ds. \tag{1.3}$$

Properties of Beta function (see [39])

1. For all $p, q \in \mathbb{C} \backslash \mathbb{Z}_0^-$, the connection between EULER's gamma function and Beta function is given by
$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.4}$$
This latter provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of EULER's gamma function.

2. The Beta function is symmetric, i.e.,
$$B(p,q) = B(q,p), \forall p, q \in \mathbb{C} \backslash \mathbb{Z}_0^-.$$

Mittag-Leffler function

The MITTAG-LEFFLER function is an important function that is widely used in the field of FC. Just as the exponential naturally arises out of the solution to integer order differential equations, the MITTAG-LEFFLER function plays an analogous role in the solution of FDEs.

Definition 1.4 (See [39, 55]) *For $\sigma, \gamma > 0$, $z \in \mathbb{R}$, the classical* MITTAG-LEFFLER *function $E_\sigma(z)$ and the generalized* MITTAG-LEFFLER *function $E_{\sigma,\gamma}(z)$ are defined by*
$$E_\sigma(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\sigma k + 1)} \tag{1.5}$$
and
$$E_{\sigma,\gamma}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\sigma k + \gamma)}, \tag{1.6}$$
respectively . It is clear from these two equations that $E_{\sigma,1}(z) = E_\sigma(z)$.

Lemma 1.1 (see [60]) *Let $\alpha \in (0,1)$, $\theta > \alpha$ be arbitrary. The functions E_α, $E_{\alpha,\alpha}$ and $E_{\alpha,\theta}$ are nonnegative and have the following properties*
i) For any $t \in J$, $E_\alpha(-\lambda t^\alpha) \leq 1$, $E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}$ and $E_{\alpha,\theta}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\theta)}$.
ii) For any t_1, $t_2 \in J$,
$$\begin{aligned} |E_\alpha(-\lambda t_2^\alpha) - E_\alpha(-\lambda t_1^\alpha)| &= O(|t_2 - t_1|^\alpha), \text{ as } t_2 \to t_1, \\ |E_{\alpha,\alpha}(-\lambda t_2^\alpha) - E_{\alpha,\alpha}(-\lambda t_1^\alpha)| &= O(|t_2 - t_1|^\alpha), \text{ as } t_2 \to t_1, \\ |E_{\alpha,\alpha+1}(-\lambda t_2^\alpha) - E_{\alpha,\alpha+1}(-\lambda t_1^\alpha)| &= O(|t_2 - t_1|^\alpha), \text{ as } t_2 \to t_1. \end{aligned}$$

Generalized Wright function

Definition 1.5 ([44]) *The generalized* WRIGHT *function is defined by the series expansion as follows*
$$W_{(\mu,a),(v,b)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a + \mu k)\Gamma(b + vk)}, \ \mu, v \in \mathbb{R}, \ a, b \in \mathbb{C}. \tag{1.7}$$

1.4 BASIC FRACTIONAL INTEGRALS AND DERIVATIVES

In this section, we introduce the methods and results used in our work. We begin by giving the definitions of the FIs and the FDs. Following that, we observe that-only-certain properties of classical derivatives can be generalized to the fractional case. The majority of the definitions in this chapter are taken from [39, 55, 56], which we refer to for a through analysis of the subject.

1.4.1 *Riemann-Liouville fractional integrals and derivatives*

As in the case with the majority of introductory works on FC, we will start by the Riemann-Liouville FI. We will focus generally on the left-sided version of the FIs and FDs. Their right-sided counterparts are rarely used, since they are anti-causal.

a) **The Riemann-Liouville's FIs**

a.1) **Functions defined on a bounded interval**

Let $[a, b]$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$, and let u be a measurable continuous function on $[a, b]$ in $\mathbb{R}$. Let's start by noting $\mathcal{I}^1_{a^+}$, the primitive of u, and we give

$$\mathcal{I}^1_{a^+} u(t) = \int_a^t u(s)\, ds. \tag{1.8}$$

The iteration of $\mathcal{I}^1_{a^+}$ allows to obtain the primitive second of u. Moreover, according to the theorem of Fubini,

$$\begin{aligned} \mathcal{I}^2_{a^+} u(t) &= \mathcal{I}^1_{a^+} \circ \mathcal{I}^1_{a^+} u(t) = \int_a^t \left(\int_a^s u(\tau)\, d\tau \right) ds = \int_a^t u(\tau) \left(\int_\tau^t ds \right) d\tau \\ &= \int_a^t (t-\tau)\, u(\tau)\, d\tau. \end{aligned}$$

The Riemann-Liouville's approach is based on the Cauchy formula (1.9) for the n^{th} integral which uses only a simple integration so as to provide a good basis for generalization

$$\mathcal{I}^n_{a^+} u(t) = \int_a^t \int_a^{t_1} \dots \int_a^{t_{n-1}} u(t_n)\, dt_n dt_{n-1} \dots dt_1 = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} u(s)\, ds. \tag{1.9}$$

Now it is clear how to get an integral of arbitrary order. We simply generalize the Cauchy formula (1.9), the integer n is substituted by a positive real number α and the Euler gamma function is used instead of the factorial.

Similarly, if we go back to the starting relationship (1.8) for a function $u : [a, b] \to \mathbb{R}$, we can notice that the integral

$$\mathcal{I}^1_{b^-} u(t) = \int_b^t u(s)\, ds = -\int_t^b u(s)\, ds,$$

is also a primitive of u, which this time involves the values to the right of u.

From the relationship

$$\int_b^t (t-s)^{n-1} u(s)\, ds = (-1)^n \int_t^b (s-t)^{n-1} u(s)\, ds,$$

we could define in the same way the right-sided integral of order n of u by

$$\forall t \in [a,b],\ \mathcal{I}_{b^-}^n u(t) = \frac{(-1)^n}{(n-1)!} \int_t^b (s-t)^{n-1} u(s)\, ds,$$

by replacing the integer n by positive real number α, we obtain the following definition.

Definition 1.6 (Riemann-Liouville's FIs [39]) *The left-sided and the right-sided* Riemann-Liouville's *FIs of order $\alpha > 0$ of a continuous function $u : [a,b] \to \mathbb{R}$ are given respectively by*

$$\mathcal{I}_{a^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\, ds,\ t \in [a,b], \tag{1.10}$$

$$\mathcal{I}_{b^-}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s)\, ds,\ t \in [a,b]. \tag{1.11}$$

a.2) **Functions defined on $\mathbb{R}$ and $\mathbb{R}^+$**

A natural direction is to extend the definitions (1.10) and (1.11) to the real axis $\mathbb{R}$ and the half axis $\mathbb{R}^+$. This will give the Liouville's FIs defined below

Definition 1.7 (Liouville's FIs on $\mathbb{R}$ [39]) *The left-sided and the right-sided* Liouville's *FIs of order $\alpha > 0$ of a continuous function $u : \mathbb{R} \to \mathbb{R}$ are given respectively by*

$$\mathcal{I}_{+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} u(s)\, ds,\ t \in \mathbb{R},$$

$$\mathcal{I}_{-}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (s-t)^{\alpha-1} u(s)\, ds,\ t \in \mathbb{R}.$$

Definition 1.8 (Liouville's FIs on $\mathbb{R}^+$ [39]) *The left-sided and the right-sided* Liouville's *FIs of order $\alpha > 0$ of a continuous function $u : \mathbb{R}^+ \to \mathbb{R}$ are given respectively by*

$$\mathcal{I}_{0^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,\ t > 0,$$

$$\mathcal{I}_{-}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (s-t)^{\alpha-1} u(s)\, ds,\ t > 0.$$

b) **The Riemann-Liouville's FDs**

While FIs are defined in a straightforward way, defining the counterpart derivatives is more complicated. Since there is no analogous formula to (1.9) in the case of FD, a commonly adopted approach is to define it through the FI.

If $\alpha > 0$, the integer part of α denoted $[\alpha]$ is the unique integer satisfying

$$[\alpha] \leq \alpha < [\alpha] + 1.$$

Let $u : [a, b] \to \mathbb{R}$. From the classic relationship $\frac{d}{dt} = \frac{d^2}{dt^2} \circ \mathcal{I}^1_{a^+}$, we can define a FD of order $0 \leq \alpha < 1$ as follows

$$\frac{d^\alpha}{dt^\alpha} = \frac{d}{dt} \circ \mathcal{I}^{1-\alpha}_{a^+}.$$

Forthermore, if $\alpha > 0$ and if $n = [\alpha] + 1$, we get

$$\frac{d^\alpha}{dt^\alpha} = \left(\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+}. \tag{1.12}$$

Definition 1.9 (Riemann-Liouville's FDs [39]) *Let $\alpha > 0$, and $n = [\alpha] + 1$. The left-sided and right-sided* RIEMANN-LIOUVILLE's *FDs of order α of a continuous function $u : [a, b] \to \mathbb{R}$ are given respectively by*

$$\mathcal{D}^\alpha_{a^+} u(t) = \left(\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

$$\mathcal{D}^\alpha_{b^-} u(t) = \left(-\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} u(s)\, ds.$$

When $u : \mathbb{R} \to \mathbb{R}$, the preceding definitions gives the LIOUVILLE's derivatives.

Definition 1.10 (Liouville FDs [39]) *Let $\alpha > 0$, and $n = [\alpha] + 1$. The left-sided and right-sided* LIOUVILLE's *FDs of order α of a continuous function $u : \mathbb{R} \to \mathbb{R}$ are given respectively by*

$$\mathcal{D}^\alpha_{+} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{-\infty}^t (t-s)^{n-\alpha-1} u(s)\, ds, \forall t \in \mathbb{R}$$

$$\mathcal{D}^\alpha_{-} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^{+\infty} (s-t)^{n-\alpha-1} u(s)\, ds, \forall t \in \mathbb{R}.$$

Properties of Riemann-Liouville fractional operators

1. By considering (1.12), when $\alpha = n \in \mathbb{N}$, all these derivatives coincide with their integer-order counterparts

$$\mathcal{D}^n_{a^+} u(t) = \left(\tfrac{d}{dt}\right)^n \circ \mathcal{I}^0_{a^+} = \left(\tfrac{d}{dt}\right)^n [u(t) - u(a)] = u^{(n)},$$
$$\mathcal{D}^n_{b^-} u(t) = \left(-\tfrac{d}{dt}\right)^n \circ \mathcal{I}^0_{b^-} = (-1)^n \left(\tfrac{d}{dt}\right)^n [u(b) - u(t)] = (-1)^n u^{(n)}.$$

2. For any $\alpha, \beta > 0$, we have

$$\mathcal{I}_{a^+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}, \tag{1.13}$$

$$\mathcal{D}_{a^+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}, \tag{1.14}$$

as a particular case. If $\beta = 1$ and $\alpha > 1$, then the Riemann-Liouville's FD of a constant is in general not equal to zero

$$\mathcal{D}_{a^+}^{\alpha} 1 = \frac{1}{\Gamma(1-\alpha)} (t-a)^{-\alpha}.$$

For more details, see [39].

1.4.2 *Caputo-type fractional derivatives*

Going back to $[a,b]$, the inversion of the compositions in the right side of (1.12) gives

$$\frac{d^{\alpha}}{dt^{\alpha}} = \mathcal{I}_{a^+}^{n-\alpha} \circ \left(\frac{d}{dt}\right)^n. \tag{1.15}$$

Equation (1.15) named as Caputo's derivative seems also a reasonable approach to define a FD. However, one shoud note that this definition is less natural than the previous one, since $\frac{d}{dt} \circ \mathcal{I}_{a^+}^{1} u(t) = u(t)$, while $\mathcal{I}_{a^+}^{1} \circ \frac{d}{dt} u(t) = u(t) - u(a)$.
This last term (here $u(a)$) is in fact very often encountered when working on FC.

Definition 1.11 ([39]) *Let $\alpha > 0$, and $n = [\alpha] + 1$. The left-sided and the right-sided* Caputo's *FDs of order α of a function $u \in C^n([a,b], \mathbb{R})$, are given respectively by*

$${}^{C}\mathcal{D}_{a^+}^{\alpha} u(t) = \mathcal{I}_{a^+}^{n-\alpha} \circ \left(\frac{d}{dt}\right)^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds, \tag{1.16}$$

$${}^{C}\mathcal{D}_{b^-}^{\alpha} u(t) = \mathcal{I}_{b^-}^{n-\alpha} \circ \left(-\frac{d}{dt}\right)^n u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds, \tag{1.17}$$

Properties of Caputo-type fractional derivatives

Here, we present some properties of the caputo's FD, see [39, 56].

1. Let $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$, such that $n = [\alpha] + 1$. If $u^{(n)}$ is a continuous function, so almost everywhere

$$\lim_{\alpha \to n^-} {}^{C}\mathcal{D}_{a^+}^{\alpha} u(t) = u^{(n)}(t),$$

$$\lim_{\alpha \to n^-} {}^{C}\mathcal{D}_{b^-}^{\alpha} u(t) = (-1)^n u^{(n)}(t).$$

2. If $\alpha \notin \mathbb{N}$ and $u(t)$ is a function for which the Caputo's FDs ${}^{C}\mathcal{D}_{a^+}^{\alpha} u(t)$ and ${}^{C}\mathcal{D}_{b^-}^{\alpha} u(t)$ of order $\alpha > 0$ exist together with the Riemann-Liouville's FDs $\mathcal{D}_{a^+}^{\alpha} u(t)$ and $\mathcal{D}_{b^-}^{\alpha} u(t)$, then we have

$$
\begin{aligned}
{}^{C}\mathcal{D}_{a^+}^{\alpha} u(t) &= \mathcal{D}_{a^+}^{\alpha} u(t) - \textstyle\sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \quad n = [\alpha]+1, \\
{}^{C}\mathcal{D}_{a^+}^{\alpha} u(t) &= \mathcal{D}_{a^+}^{\alpha} u(t), \qquad \text{if } u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0.
\end{aligned}
$$

Also

$$
\begin{aligned}
{}^{C}\mathcal{D}_{b^-}^{\alpha} u(t) &= \mathcal{D}_{b^-}^{\alpha} u(t) - \textstyle\sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \quad n = [\alpha]+1, \\
{}^{C}\mathcal{D}_{b^-}^{\alpha} u(t) &= \mathcal{D}_{b^-}^{\alpha} u(t), \qquad \text{if } u(b) = u'(b) = \cdots = u^{(n-1)}(b) = 0.
\end{aligned}
$$

3. If $\alpha, \beta > 0$, then

$$
{}^{C}\mathcal{D}_{a^+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}.
$$

In particular, if $\beta = 1$, we get

$$
{}^{C}\mathcal{D}_{a^+}^{\alpha} 1 = 0.
$$

1.4.3 *Hadamard fractional integrals and derivatives*

Here, we define Hadamard's fractional operators.

Definition 1.12 (Hadamard FIs) *(see [39]).*

The left-sided and right-sided Hadamard's *FIs of order $\alpha > 0$ of a continuous function $u : [a,b] \to \mathbb{R}$ are given respectively by*

$$
\begin{aligned}
{}^{H}\mathcal{I}_{a^+}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s}, t \in [a,b], \qquad (1.18) \\
{}^{H}\mathcal{I}_{b^-}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} u(s) \frac{ds}{s}, t \in [a,b].
\end{aligned}
$$

When $a = 0$ and $b = \infty$, the last two expresions are given respectively by

$$
\begin{aligned}
{}^{H}\mathcal{I}_{0^+}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s}, t > 0, \\
{}^{H}\mathcal{I}_{-}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^\infty \left(\log \frac{s}{t}\right)^{\alpha-1} u(s) \frac{ds}{s}, t > 0.
\end{aligned}
$$

Definition 1.13 (Hadamard FDs) *(see [39]).*

The left-sided and the right-sided Hadamard's *FDs of order $\alpha > 0$ of a continuous function $u : [a,b] \to \mathbb{R}$ are given respectively by*

$$
\begin{aligned}
{}^{H}\mathcal{D}_{a^+}^{\alpha} u(t) &= \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} u(s) \frac{ds}{s}, t \in [a,b], \qquad (1.19) \\
{}^{H}\mathcal{D}_{b^-}^{\alpha} u(t) &= \left(-t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\log \frac{t}{s}\right)^{n-\alpha-1} u(s) \frac{ds}{s}, t \in [a,b].
\end{aligned}
$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the number α.

When $a = 0$ and $b = \infty$, the last two expresions are given respectively by

$$\begin{aligned} {}^{H}\mathcal{D}^{\alpha}_{0^+} u(t) &= \left(t\frac{d}{dt}\right)^{n} \frac{1}{\Gamma(n-\alpha)} \int_0^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} u(s)\frac{ds}{s}, t>0, \\ {}^{H}\mathcal{D}^{\alpha}_{-} u(t) &= \left(-t\frac{d}{dt}\right)^{n} \frac{1}{\Gamma(n-\alpha)} \int_t^\infty \left(\log\frac{t}{s}\right)^{n-\alpha-1} u(s)\frac{ds}{s}, t>0. \end{aligned}$$

1.4.4 *Caputo-Hadamard-type FDs*

The Caputo-Hadamard's FDs are defined as follows

Definition 1.14 (Caputo-Hadamard's FDs) *(see [24]).*
The left-sided and right-sided Caputo*-type modification of* Hadamard *FDs of order $\alpha \geq 0$ of a function $u(t) \in AC^n_\delta[a,b]$ are given respectively by*

$$\begin{aligned} {}^{CH}\mathcal{D}^{\alpha}_{a^+} u(t) &= {}^{H}\mathcal{D}^{\alpha}_{a^+}\left[u(t) - \sum_{k=1}^{n-1} \frac{\left(\delta^k u\right)(a)}{k!} \log\left(\frac{t}{a}\right)^{k}\right](t), \\ {}^{CH}\mathcal{D}^{\alpha}_{b^-} u(t) &= {}^{H}\mathcal{D}^{\alpha}_{b^-}\left[u(t) - \sum_{k=1}^{n-1} \frac{\left((-1)^k \delta^k u\right)(b)}{k!} \log\left(\frac{b}{t}\right)^{k}\right](t). \end{aligned}$$

where $n = [\alpha] + 1$, and $\delta = t\frac{d}{dt}$.

1.4.5 *Erdélyi-Kober fractional integral and derivative*

Here, we define the Erdélyi–Kober FD, which is considered as a generalization of its Riemann-Liouville's FD.

Definition 1.15 (Erdélyi-Kober FI. ([45])) *Let $\alpha, \beta > 0$. The left-sided Erdélyi-Kober FI of order α of a function $u \in C_\alpha$ is defined by*

$$\left(\mathcal{J}^{\tau,\alpha}_{\beta} u\right)(t) = \frac{\beta}{\Gamma(\alpha)} t^{\beta\tau} \int_t^\infty s^{-\beta(\tau+\alpha-1)-1}\left(s^\beta - t^\beta\right)^{\alpha-1} u(s)\, ds,\ \tau \in \mathbb{R}. \tag{1.20}$$

Definition 1.16 (Erdélyi-Kober FD. ([45])) *Let $n-1 < \alpha \leq n,\ n \in \mathbb{N}$. The left-sided Erdélyi-Kober FD of order α of a function $u \in C^n_\alpha$ is defined by*

$$\left(\mathcal{P}^{\tau,\alpha}_{\beta} u\right)(t) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta} t\frac{d}{dt}\right)\left(J^{\tau+\alpha,n-\alpha}_{\beta} u\right)(t). \tag{1.21}$$

The ERDÉLYI-KOBER fractional operators of a power function are given by (see [28])

$$\mathcal{P}_{\beta}^{\tau,\alpha} t^p = \frac{\Gamma\left(\alpha+\tau-\frac{p}{\beta}\right)}{\Gamma\left(\tau-\frac{p}{\beta}\right)} t^p,\ \tau-\frac{p}{\beta}>0. \tag{1.22}$$

$$\mathcal{J}_{\beta}^{\tau,\alpha} t^p = \frac{\Gamma\left(\tau-\frac{p}{\beta}\right)}{\Gamma\left(\alpha+\tau-\frac{p}{\beta}\right)} t^p,\ \tau-\frac{p}{\beta}>0. \tag{1.23}$$

1.5 GENERALIZED FRACTIONAL INTEGRALS AND DERIVATIVES OF KATUGAMPOLA

In [35–37] KATUGAMPOLA introduced a new fractional integro-differential operator which generalized both the REIMANN-LIOUVILLE and HADAMARD operators. The generalized fractional operators of order $\alpha > 0$ of a function $u \in X_c^p[a,b]$ for $-\infty < a < b < +\infty$, known as KATUGAMPOLA's fractional operators, are defined below

Definition 1.17 (Generalized FIs. (see [4, 35])) *The left-sided and right-sided of the generalized fractional integrals of order $\alpha > 0$ and parameter $\rho > 0$ of a function $u \in X_c^p[a,b]$ are defined respectively by*

$$\left({}^{\rho}\mathcal{I}_{a^+}^{\alpha} u\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \left(t^{\rho}-s^{\rho}\right)^{\alpha-1} u(s)\, ds,\ t>a, \tag{1.24}$$

and

$$\left({}^{\rho}\mathcal{I}_{b^-}^{\alpha} u\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b s^{\rho-1} \left(s^{\rho}-t^{\rho}\right)^{\alpha-1} u(s)\, ds,\ t<b, \tag{1.25}$$

where $\Gamma(.)$ is EULER's Gamma function.

These are the fractional generalizations of the $n-$fold left-sided and right-sided integrals of the form

$$\int_a^t s_1^{\rho-1} \int_a^{s_1} s_2^{\rho-1} \ldots \int_a^{s_{n-1}} u(s_n)\, ds_n ds_{n-1} \ldots ds_1,$$

and

$$\int_t^b s_1^{\rho-1} \int_{s_1}^b s_2^{\rho-1} \ldots \int_{s_{n-1}}^b u(s_n)\, ds_n ds_{n-1} \ldots ds_1,$$

with $n \in \mathbb{N}$, respectively

The generalized FDs of KATUGAMPOLA are defined below

Definition 1.18 (Generalized FDs.(See [4, 36])) *The left-sided and right-sided generalized FDs for a differential operator of order $\alpha > 0$ with dependence on a parameter $\rho > 0$ are defined respectively as*

$$\begin{aligned}\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha}u\right)(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n {}^{\rho}\mathcal{I}_{a^+}^{n-\alpha}u(t)\\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{n-\alpha-1}u(s)\,ds,\end{aligned}$$

and

$$\begin{aligned}\left({}^{\rho}D_{b^-}^{\alpha}u\right)(t) &= \left(-t^{1-\rho}\frac{d}{dt}\right)^n {}^{\rho}\mathcal{I}_{b^-}^{n-\alpha}u(t)\\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(-t^{1-\rho}\frac{d}{dt}\right)^n \int_t^b s^{\rho-1}\left(s^{\rho}-t^{\rho}\right)^{n-\alpha-1}u(s)\,ds,\end{aligned}$$

where $n-1<\alpha<n$, $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Properties of the generalized fractional operators of Katugampola on the half-axis $\mathbb{R}^+$

Let $[a,b]$ be a finite closed interval on the half-axis $\mathbb{R}^+$, In what follows, we give some properties of Katugampola's fractional operators.

Theorem 1.1 *[35–37] Let $\alpha, \rho \in \mathbb{R}^+$, then*

$$\begin{aligned}\lim_{\rho\to 1}\left({}^{\rho}\mathcal{I}_{a^+}^{\alpha}u\right)(t) &= \mathcal{I}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\,ds,\\ \lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{I}_{a^+}^{\alpha}u\right)(t) &= {}^{H}\mathcal{I}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{u(s)}{s}ds,\\ \lim_{\rho\to 1}\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha}u\right)(t) &= \mathcal{D}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-\alpha-1}u(s)\,ds,\\ \lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha}u\right)(t) &= {}^{H}\mathcal{D}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{u(s)}{s}ds.\end{aligned}$$

Remark 1.1 *For the right-sided case, we get the same result.*

Theorem 1.2 *([35, 36])Let $\alpha, \beta, \rho > 0$. For any $u, v \in C[a,b]$, we have the following properties*

- Index property

$$\begin{aligned}{}^{\rho}\mathcal{I}_{a^+}^{\alpha}\,{}^{\rho}\mathcal{I}_{a^+}^{\beta}u(t) &= {}^{\rho}\mathcal{I}_{a^+}^{\alpha+\beta}u(t)\,,\ \text{for all } \alpha,\beta>0\\ {}^{\rho}\mathcal{D}_{a^+}^{\alpha}\,{}^{\rho}\mathcal{D}_{a^+}^{\beta}u(t) &= {}^{\rho}\mathcal{D}_{a^+}^{\alpha+\beta}u(t)\,,\ \text{for all } 0<\alpha,\beta<1\,.\end{aligned}$$

- Inverse property: for all $\beta \in (0,1)$, we have

$${}^{\rho}\mathcal{D}_{a^+}^{\beta}\,{}^{\rho}\mathcal{I}_{a^+}^{\beta}u(t) = u(t).$$

- Composition property : for $0 < \alpha < \beta < 1$ and $u \in L^p(a,b)$, we have

$$ {}^{\rho}\mathcal{D}^{\alpha}_{a^+}\, {}^{\rho}\mathcal{I}^{\beta}_{a^+} u(t) = {}^{\rho}\mathcal{I}^{\beta-\alpha}_{a^+} u(t) \quad \text{and} \quad {}^{\rho}\mathcal{D}^{\alpha}_{b^-}\, {}^{\rho}\mathcal{I}^{\beta}_{b^-} u(t) = {}^{\rho}\mathcal{I}^{\beta-\alpha}_{b^-} u(t)\, . $$

- Linearity property: for all $\beta \in (0,1)$, we have

$$ \begin{aligned} {}^{\rho}\mathcal{I}^{\beta}_{a^+}(u+v)(t) &= {}^{\rho}\mathcal{I}^{\beta}_{a^+} u(t) + {}^{\rho}\mathcal{I}^{\beta}_{a^+} v(t)\, , \\ {}^{\rho}\mathcal{D}^{\beta}_{a^+}(u+v)(t) &= {}^{\rho}\mathcal{D}^{\beta}_{a^+} u(t) + {}^{\rho}\mathcal{D}^{\beta}_{a^+} v(t)\, . \end{aligned} $$

Lemma 1.2 (see [6, 10, 11]) *For $\alpha, \rho > 0$ and $\mu > -\rho$. We have the following properties*

$$ {}^{\rho}\mathcal{I}^{\alpha}_{0^+} t^{\mu} = \frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)} t^{\mu+\alpha\rho}, \tag{1.26} $$

$$ {}^{\rho}\mathcal{D}^{\alpha}_{0^+} t^{\mu} = \frac{\Gamma\left(1+\dfrac{\mu}{\rho}\right)\rho^{\alpha-1}}{\Gamma\left(1+\dfrac{\mu}{\rho}-\alpha\right)} t^{\mu-\alpha\rho}, \tag{1.27} $$

$$ \begin{aligned} \left({}^{\rho}\mathcal{I}^{\alpha}_{0^+}\, {}^{\rho}\mathcal{D}^{\alpha}_{0^+}\right) u(t) &= u(t) - \frac{\rho^{1-\alpha}\, {}^{\rho}\mathcal{I}^{1-\alpha}_{0^+} u(0^+)}{\Gamma(\alpha)} t^{\rho(\alpha-1)} \\ &\quad - \frac{\rho^{2-\alpha}\, {}^{\rho}\mathcal{I}^{2-\alpha}_{0^+} u(0^+)}{\Gamma(\alpha-1)} t^{\rho(\alpha-2)}, \ 1 < \alpha \le 2. \end{aligned} \tag{1.28} $$

As a particular case, for $\rho = 1$ and $\mu > -1$, the relations (1.26) and (1.27) reduce to the RIEMANN-LIOUVILLE's FI (resp. FD) of the power function (1.13) and (1.14), which are given by [39, 55, 56],

$$ \begin{aligned} {}^{1}\mathcal{I}^{\alpha}_{0^+} t^{\mu} &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\alpha)} t^{\mu+\alpha}, \\ {}^{1}\mathcal{D}^{\alpha}_{0^+} t^{\mu} &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha}. \end{aligned} $$

Lemma 1.3 *([6, 10])Let $c > 0$ and $p \ge 1$, then*

$$ C[a,b] \hookrightarrow X^p_c[a,b]\, , $$

and for all $u \in C[a,b]$, we get

$$ \|u\|_{X^p_c} \le \|u\|_{\infty}\, , \forall b \le (pc)^{\frac{1}{pc}}. $$

In what follows, the parameters $\alpha, n, p, a, b, c \in \mathbb{R}^+$ satisfy

$$ n = [\alpha] + 1, p \ge 1 \text{ and } \ a < b \le (pc)^{\frac{1}{pc}}. $$

Remark 1.2 *([6, 10]) Let $\alpha, \rho > 0$. Then for any $u \in C[a,b]$, we have*

$$ {}^{\rho}\mathcal{I}^{1}_{a^+}\left(t^{1-\rho}\frac{d}{dt}\right)u(t) = u(t) - u(a), $$

and

$$ {}^{\rho}\mathcal{I}^{\alpha}_{a^+}u(t) = \left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathcal{I}^{\alpha+1}_{a^+}u(t). $$

Theorem 1.3 (See [25]) *The Cauchy problem*

$$ \begin{cases} \left({}^{\rho}D^{\alpha}_{0^+} - \lambda\right)u(t) = f(t),\ t > 0,\ 0 < \alpha \leq 1,\ \lambda \in \mathbb{R}, \\ \left({}^{\rho}I^{1-\alpha}_{0^+}u\right)(0) = k,\ k \in \mathbb{R}, \end{cases} $$

has the solution

$$ \begin{aligned} u(t) &= k\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) \\ &\quad + \int_0^t \left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha}\right)\tau^{\rho-1}f(\tau)d\tau. \end{aligned} $$

1.5.1 *Caputo-type generalized fractional derivatives of Katugampola*

Not long ago, in [5], a Caputo-type modification of the generalized FI of Katugampola was proposed. This later is the Caputo type of generalized FD (CGFD). It represents a generalization of the Caputo and Caputo-Hadamard FDs, known as CAPUTO-KATUGAMPOLA'S FD.

Definition 1.19 (CGFD. (see [4])) *Let $0 < a < b, \rho$ be a positive real number, $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ be such that $\alpha \in (n-1, n)$ and $u : [a,b] \to \mathbb{R}$ a function of class C^n. The left-sided and right-sided of CGFDs of order α and parameter ρ are defined respectively by*

$$ {}^{C}D^{\alpha,\rho}_{a^+}u(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_a^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{n-\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right)^{n}u(s)ds, \tag{1.29} $$

and

$$ {}^{C}D^{\alpha,\rho}_{b^-}u(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_t^b s^{\rho-1}\left(s^{\rho}-t^{\rho}\right)^{n-\alpha-1}\left(-s^{1-\rho}\frac{d}{ds}\right)^{n}u(s)ds. \tag{1.30} $$

The next two results justify definition 1.19, since the CGFD is an inverse operation of the generalized FI of KATUGAMPOLA.

Theorem 1.4 (see [4]) *Let $\alpha > 0$ such that $\alpha \in (n-1, n)$, $n \in \mathbb{N}$ and $\rho > 0$. Given a function $u \in C^n[a,b]$, we have*

$$\left({}^{\rho}I^{\alpha}_{a^+}\ {}^{C}D^{\alpha,\rho}_{a^+}\ u\right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{k}, \tag{1.31}$$

$$\left({}^{\rho}I^{\alpha}_{b^-}\ {}^{C}D^{\alpha,\rho}_{b^-}\ u\right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{(-1)^k u^{(k)}(b)}{k!}\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^{k}. \tag{1.32}$$

If $0 < \alpha < 1$, then

$${}^{\rho}I^{\alpha}_{b^-}\ {}^{C}D^{\alpha,\rho}_{b^-}u(t) = u(t) - u(b). \tag{1.33}$$

Lemma 1.4 (See [5]) *Let $J = [a,b]$, α, $\rho > 0$ and $u \in C(J,\mathbb{R}) \cap C^1(J,\mathbb{R})$. Then, the* Caputo-Katugampola *FDE*

$${}^{C}D^{\alpha,\rho}_{b^-}u(t) = 0,$$

has a solutions

$$u(t) = c_0 + c_1\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right) + c_2\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^2 + \dots + c_{n-1}\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$ and $n = [\alpha]+1$.

The following theorem covers the compositions of the fractional integral operators ${}^{\rho}I^{\alpha}_{a^+}$ and ${}^{\rho}I^{\alpha}_{b^-}$ with the fractional differential operators ${}^{C}D^{\alpha,\rho}_{a^+}$ and ${}^{C}D^{\alpha,\rho}_{b^-}$ respectively.

Theorem 1.5 (see [51]) *Let $\alpha, \beta, \rho \in \mathbb{R}$ such that $\alpha > \beta$ and $\beta > 0$. If $u \in X^p_c[a,b]$, then for $\rho > 0$*

$$\left({}^{C}D^{\beta,\rho}_{a^+}\ {}^{\rho}I^{\alpha}_{a^+}u\right)(t) = \left({}^{\rho}I^{\alpha-\beta}_{a^+}u\right)(t),$$
$$\left({}^{C}D^{\beta,\rho}_{b^-}\ {}^{\rho}I^{\alpha}_{b^-}u\right)(t) = \left({}^{\rho}I^{\alpha-\beta}_{b^-}u\right)(t).$$

In what follows, we define the GRCFD.

1.5.2 *Generalized Riesz-Caputo fractional derivative*

Aleem et al. in [2], presented a generalisation of the Riesz fractional operator, where this operator covers as particular cases the classical Riesz FD. In the same paper, the authors have also proposed a Caputo-type modification of this operator. This new FD was named as the generalized Riesz-Caputo FD known as Riesz-Caputo Katugampola's FD. In the same paper, some fundamental results have been introduced and proved.

First, we give the definitions of Riesz FD, Caputo-Riesz FD and the generalized Riesz type integral.

Definition 1.20 (Riesz FD [39, 55]) *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, the* RIESZ *fractional derivative of order α of a continuous function $u : [a, b] \to \mathbb{R}$ is given by*

$$ {}_a^R D_b^{\alpha}\, u(t) = -\Psi_{\alpha}(\mathcal{D}_{a^+}^{\alpha} + (-1)^n\, \mathcal{D}_{b^-}^{\alpha})u(t), $$

where $\Psi_{\alpha} = \left(\frac{1}{2}\right)$ or $\sec\left(\frac{\pi\alpha}{2}\right)$ and $\mathcal{D}_{a^+}^{\alpha}$, $\mathcal{D}_{b^-}^{\alpha}$ are the left and right-sided RIEMANN-LIOUVILLE *FDs defined in definition 1.9.*

Definition 1.21 (Caputo-Riesz FD [39, 55]) *Let $\alpha > 0$ and $n = [\alpha] + 1$. For $u(t) \in C[a, b]$, the classical Riesz-Caputo derivative is defined by*

$$ \begin{aligned} {}_a^{RC} D_b^{\alpha}\, u(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^b |t-s|^{n-\alpha-1} u^{(n)}(s)\, ds \\ &= \frac{1}{2}({}^C\mathcal{D}_{a^+}^{\alpha} + (-1)^n\, {}^C\mathcal{D}_{b^-}^{\alpha})u(t), \end{aligned} \tag{1.34} $$

where ${}^C\mathcal{D}_{a^+}^{\alpha}$ and ${}^C\mathcal{D}_{b^-}^{\alpha}$ are left and right hand sided CAPUTO'S *FDs defined in (1.16) and (1.17) respectively.*

Definition 1.22 (Generalized-Riesz FI. (see [2])) *Let $u(t) \in X_c^p(a, b)$ and $\alpha, \rho > 0$. Then, for $a \leq t \leq b$, the generalized* RIESZ *type integral is defined as*

$$ \begin{aligned} \left({}_a^{RG} I_b^{\alpha,\rho} u\right)(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^b s^{\rho-1} \left|(s^{\rho} - t^{\rho})\right|^{\alpha-1} u(s)\, ds \\ &= ({}^{\rho}I_{a^+}^{\alpha} u)(t) + ({}^{\rho}I_{b^-}^{\alpha} u)(t), \end{aligned} \tag{1.35} $$

where ${}^{\rho}I_{a^+}^{\alpha}$ and ${}^{\rho}I_{b^-}^{\alpha}$ are left and right sided generalized FIs, defined in (1.24) and (1.25) respectively.

Definition 1.23 (The GRCFD. (see [2])) *Let $\alpha, \rho \in \mathbb{R}$ with $\alpha, \rho > 0$ and $u(t) \in X_c^p(a, b)$, for $a \leq \eta \leq b$. Then the GRCFD is defined as*

$$ \begin{aligned} {}_a^{RC}\mathcal{D}_b^{\alpha,\rho} u(t) &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^b s^{\rho-1} \left|(t^{\rho} - s^{\rho})\right|^{n-\alpha-1} \left(s^{1-\rho}\frac{d}{ds}\right)^n u(s)ds \\ &= \frac{1}{2}\left({}^C\mathcal{D}_{a^+}^{\alpha,\rho} + (-1)^n\, {}^C\mathcal{D}_{b^-}^{\alpha,\rho}\right) u(t), \end{aligned} \tag{1.36} $$

where ${}^C D_{a^+}^{\alpha,\rho}$ and ${}^C D_{b^-}^{\alpha,\rho}$ are left and right sided of CGFDs which defined in (1.29) and (1.30) respectively.

Lemma 1.5 (see [2]) *Let $u \in AC_{\delta}^n[0, \mu]$ with $0 \leq t \leq \mu$. Then the following relation is true*

$$ {}_0^{RG}\mathcal{I}_{\mu}^{\alpha,\rho}\, {}_0^{RC}\mathcal{D}_{\mu}^{\alpha,\rho} u(\eta) = \frac{1}{2}\left({}^{\rho}\mathcal{I}_{0^+}^{\alpha}\, {}^C\mathcal{D}_{0^+}^{\alpha,\rho} + (-1)^n\, {}^{\rho}\mathcal{I}_{\mu^-}^{\alpha}\, {}^C\mathcal{D}_{\mu^-}^{\alpha,\rho}\right) u(t). \tag{1.37} $$

Remark 1.3 *If $1 < \alpha \leq 2$, and $\rho > 0$, then for $g(\eta) \in C[0, \mu]$, the relation illustrated in (1.37) becomes*

$$ \begin{aligned} {}_0^{RG}\mathcal{I}_{\mu}^{\alpha,\rho}\, {}_0^{RC}\mathcal{D}_{\mu}^{\alpha,\rho} g(\eta) &= g(\eta) - \frac{1}{2}\left[g(0) + g(\mu)\right] \\ &\quad -\frac{\eta^{\rho}}{2\rho}\left[g'(0) + g'(\mu)\right] + \frac{\mu^{\rho}}{2\rho} g'(\mu). \end{aligned} $$

Remark 1.4 *If $1 < \alpha \leq 2$, then, for all $g \in C[0,\mu]$, and by remark 1.3 and theorem 1.5, we have*

$$\begin{aligned} {}_0^{RG}\mathcal{I}_\mu^{\alpha-1,\rho}\, {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\eta) &= \frac{d}{d\eta}\, {}_0^{RG}\mathcal{I}_\mu^{\alpha,\rho}\, {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\eta) \\ &= \frac{d}{d\eta}\Big[g(\eta) - \frac{1}{2}[g(0)+g(\mu)] \\ &\quad + \frac{\mu^\rho}{2\rho} g'(\mu) - \frac{\eta^\rho}{2\rho}[g'(0)+g'(\mu)]\Big] \\ &= g'(\eta) - \frac{[g'(0)+g'(\mu)]}{2}\eta^{\rho-1}. \end{aligned} \tag{1.38}$$

Furthermore, if $g'(0) + g'(\mu) = 0$, then

$${}_0^{RG}\mathcal{I}_\mu^{\alpha-1,\rho}\, {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\eta) = g'(\eta). \tag{1.39}$$

In addition, for each $\eta \in [0,\mu]$, using the fact that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we obtain

$$\begin{aligned} |g'(\eta)| &= \left| {}_0^{RG}\mathcal{I}_\mu^{\alpha-1,\rho}\, {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\eta) \right| \\ &= \left| \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_0^\mu \zeta^{\rho-1} |(\zeta^\rho - \eta^\rho)|^{\alpha-2}\, {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\zeta)\, d\zeta \right| \\ &\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_0^\mu \zeta^{\rho-1} |(\zeta^\rho - \eta^\rho)|^{\alpha-2} \left| {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\zeta) \right| d\zeta \\ &\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_0^\eta \zeta^{\rho-1} (\eta^\rho - \zeta^\rho)^{\alpha-2} \left| {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\zeta) \right| d\zeta \\ &\quad + \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_\eta^\mu \zeta^{\rho-1} (\zeta^\rho - \eta^\rho)^{\alpha-2} \left| {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\zeta) \right| d\zeta \\ &\leq \sup_{0\leq\eta\leq\mu} \left| {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g(\eta) \right| \Big[\frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_0^\eta \zeta^{\rho-1} (\eta^\rho - \zeta^\rho)^{\alpha-2} d\zeta \\ &\quad + \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_\eta^\mu \zeta^{\rho-1} (\zeta^\rho - \eta^\rho)^{\alpha-2} d\zeta \Big] \\ &= \left\| {}_0^{RC}D_\mu^{\alpha,\rho} g \right\|_\infty \Big[\frac{-\rho^{2-\alpha}}{\rho(\alpha-1)\Gamma(\alpha-1)} \int_0^\eta \frac{d}{d\zeta}\left((\eta^\rho - \zeta^\rho)^{\alpha-1} \right) \\ &\quad + \frac{\rho^{2-\alpha}}{\rho(\alpha-1)\Gamma(\alpha-1)} \int_\eta^\mu \frac{d}{d\zeta}\left((\zeta^\rho - \eta^\rho)^{\alpha-1} \right) \Big] \\ &= \left\| {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g \right\|_\infty \left\{ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left[-\left[(\eta^\rho - \zeta^\rho)^{\alpha-1} \right]_0^\eta + \left[(\zeta^\rho - \eta^\rho)^{\alpha-1} \right]_\eta^\mu \right] \right\} \\ &= \frac{\left(\eta^{\rho(\alpha-1)} + (\mu^\rho - \eta^\rho)^{\alpha-1} \right)}{\rho^{\alpha-1}\Gamma(\alpha)} \left\| {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g \right\|_\infty \\ &\leq \frac{2\mu^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)} \left\| {}_0^{RC}\mathcal{D}_\mu^{\alpha,\rho} g \right\|_\infty. \end{aligned} \tag{1.40}$$

1.5.3 *Hilfer-Katugampola's fractional derivative*

Here, we define the Hilfer-Katugampola's FD (HKFD).

Definition 1.24 (HKFD. ([52])) *Let $n-1<\alpha\leq n,\ 0\leq\gamma\leq 1$ and $\rho>0$. The left-sided HKFD of order α and type γ of a function u is defined by*

$$^{\rho}\mathcal{D}_{a^{+}}^{\alpha,\gamma}u(t)={}^{\rho}\mathcal{I}_{a^{+}}^{\gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^{n}{}^{\rho}\mathcal{I}_{a^{+}}^{(1-\gamma)(n-\alpha)}u(t)\,, \tag{1.41}$$

where $^{\rho}\mathcal{I}_{a^{+}}$ is the left-sided generalized FI given in (1.24).

For $\gamma=0$ and $\rho=1$, then the HKFD is reduced to the Riemann-Liouville's FD.

Now, we recall the following property

Property 1 (see [52]) The operator $^{\rho}D_{0^{+}}^{\alpha,\gamma}$ can be written as

$$^{\rho}D_{0^{+}}^{\alpha,\gamma}={}^{\rho}I_{0^{+}}^{\gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}I_{0^{+}}^{(1-\delta)}={}^{\rho}I_{0^{+}}^{\gamma(1-\alpha)}\,{}^{\rho}D_{0^{+}}^{\delta}\,,$$

where $\delta=\alpha+\gamma\left(1-\alpha\right)$.

Lemma 1.6 (see [52]) *Let $0<\alpha\leq 1,\ 0\leq\gamma\leq 1$ and $\delta=\alpha+\gamma\left(1-\alpha\right)$. Then*

$$^{\rho}\mathcal{I}_{0^{+}}^{\delta}\,{}^{\rho}\mathcal{D}_{0^{+}}^{\delta}u={}^{\rho}\mathcal{I}_{0^{+}}^{\alpha}\,{}^{\rho}\mathcal{D}_{0^{+}}^{\alpha,\gamma}u, \tag{1.42}$$

where the term $^{\rho}\mathcal{I}_{0^{+}}^{\delta}\,{}^{\rho}\mathcal{D}_{0^{+}}^{\delta}u$ is given in equation (1.28) of lemma 1.2.

1.6 FRACTIONAL-ORDER PARTIAL DIFFERENTIAL EQUATIONS

FPDEs are generalizations of integer-order partial differential equations (PDEs). FPDEs can be used for the modeling and study of many important phenomena in many different fields of science and engineering, such as diffusion processes, damping law,...etc. One can find a variety of applications in [29, 34, 38, 46, 55]. The solutions of FPDEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural sciences and engineering.

Definition 1.25 (FDEs [39]) *A fractional differential equation is a relationship of the form*

$$F(\eta,u(\eta),\mathcal{D}^{\alpha_1}u(\eta),\mathcal{D}^{\alpha_2}u(\eta),...)=0,\ \alpha_1,\alpha_2,...>0,$$

between the variable $\eta\in\mathbb{R}$, and the FDs of orders $\alpha_1,\alpha_2,...$ of the unknown function u at the point η. Here $\mathcal{D}^{\alpha}u$ presents a fractional differential operator of order $\alpha>0$.

Definition 1.26 (FPDEs [11]) *A FPDE for the function u is a relationship between y, the independent variables* $(\eta_1, \eta_2, ...\eta_n) \in \mathbb{R}^n$ *and one or more FDs* $\mathcal{D}^{\alpha_1}_{\eta_1}y, \mathcal{D}^{\alpha_2}_{\eta_2}y, ..., \mathcal{D}^{\alpha_3}_{\eta_3}\mathcal{D}^{\alpha_4}_{\eta_4}y...$, *that we can write in the form*

$$F(y, \eta_1, ..., \mathcal{D}^{\alpha_1}_{\eta_1}y, \mathcal{D}^{\alpha_2}_{\eta_2}y, ..., \mathcal{D}^{\alpha_3}_{\eta_3}\mathcal{D}^{\alpha_4}_{\eta_4}y...) = 0, \ \alpha_1, \alpha_2, ... > 0,$$

where $\mathcal{D}^{\alpha}_{\eta_i}y$ *presents a fractional differential operator of order* α *at* η_i, $i = 1, 2, ..., n$.

1.6.1 *Self-similar solutions of fractional equations*

In general, for some FPDEs which are symmetric, we can determine the exact solutions using some (finite or infinite) transformations. For instance, a FPDE can be reduced to a FDE, so the solutions are called "self-similar solutions" (see [14, 31]). These self-similar solutions play an important role in the study of FPDEs. They are very important in physics because they model phenomena that are independent of the scale of measurement.

Let

$$\mathcal{D}^{\alpha}_{x}y = f(x, t, y, \mathcal{D}^{\alpha_1}_{t}y, \mathcal{D}^{\alpha_2}_{x}y, ...), \alpha > \alpha_1 > \alpha_2, ... > 0, \tag{1.43}$$

be a FPDE and $y = y(x, t)$ is a scalar function of space and time variables $(x, t) \in \mathbb{R}^2$. The symbol $\mathcal{D}^{\alpha}_{x}y$ represents a space-fractional differential operator of order α.

We suggest finding the solution of the equation (1.43) in the following "self-similar" form

$$y(x, t) = \lambda^b y(\lambda^a x, \lambda t), (x, t) \in \mathbb{R} \times \mathbb{R}^+, a, b \in \mathbb{R} \text{ and } \lambda > 0. \tag{1.44}$$

Based on this consideration, we search the values of a and b, for which $\lambda^b y(\lambda^a x, \lambda t)$ is a solution of equation (1.43), for all $\lambda > 0$ knowing that y is a solution of the same equation.

Definition 1.27 (Self-similar [14, 31, 44]) *A function which is invariant by a change of scale in time is called "self-similar". The principle of the search for self-similar solutions consists in replacing the form* (1.44) *in the equation* (1.43), *which makes it possible to transform the FPDE* (1.43) *into a FDE.*

Remark 1.5 ([11, 14, 18, 31]) *In general, sevral forms of self-similar solutions exist. To admit such solutions for equation* (1.43), *we must first check the so-called "similarity" conditions.*

If we take for example $\lambda = \frac{1}{t} > 0$, the self-similar solution (1.44) can be given by

$$y(x, t) = \frac{1}{t^b} y(\frac{x}{t^a}, 1) = t^{-b}u(\eta), \text{ where } \eta = \frac{x}{t^a}, (x, t) \in \mathbb{R} \times \mathbb{R}^+, a, b \in \mathbb{R}. \tag{1.45}$$

In this case, the function u called the "basic profile", is not known in advance and is to be identified.

To discuss the self-similar solutions, we shoud first deduce the equation satisfied by the function $u(\eta)$ in (1.45) which is used to find the self-similar solutions.

1.6.2 *Studies and results of FDEs' and FPDEs' solutions*

Recently, some results dealing with the existence, uniqueness and multiplicity of real or positive solutions of BVPs involving nonlinear mixed FDEs have been studied (see [1, 20, 30, 41, 42, 60]). In [57], Somia *et al.* used some fixed point theorems to study the existence, uniqueness of positive solution and Ulam-Hyers stability for the following mixed fractional BVP with integral boundary conditions:

$$\begin{cases} D^{\beta}_{1^-}\left({}^{C}D^{\alpha}_{0^+}u\right)(t) = f(t,u(t)),\ t \in J = [0,1], \\ {}^{C}D^{\alpha}_{0^+}u(1) = 0,\ u(0) = \gamma\int_0^1 u(t)dt, \end{cases}$$

where $\alpha, \beta \in (0,1], \gamma \in (0,1]$. $D^{\beta}_{1^-}$ denote the right conformable FD, ${}^{C}D^{\alpha}_{0^+}$ denote the left Caputo FD, u is the unknown function and $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

Wang and Li in [60], investigated the following BVP of the FDE with two different FDs:

$$\begin{cases} {}^{C}D^{\beta}_{1^-}\left(D^{\alpha}_{0^+} + \lambda\right)u(t) = f(t,u(t)),\ t \in J = (0,1], \\ (I^{1-\alpha}_{0^+}u)(0) + ({}^{\rho}I^{q}_{0^+}u)(1) = 0,\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u_0, \end{cases}$$

where $\alpha, \beta,\ \alpha+\beta \in (0,1),\ \lambda, \rho, q > 0,\ \alpha+\rho > 1$. ${}^{C}D^{\beta}_{1^-}$ is the right Caputo FD of order β. $D^{\alpha}_{0^+}$ is the left Riemann-Liouville's FD of order α. $I^{1-\alpha}_{0+}$ is the Riemann-Liouville's FI and ${}^{\rho}I^{q}_{0^+}$ is the Katugampola's FI.

In a recent work [13], Basti *et al.* applied the Banach's contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type to show the existence and uniqueness of self-similar solutions for the following space-fractional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^{\alpha} u}{\partial x}, \alpha \in (1,2],$$

under the self similar form

$$u(x,t) = t^{\beta} f(\eta),\ \text{with}\ \eta = xt^{\frac{-1}{\alpha}},\ \beta \in \mathbb{R},$$

where $u = u(x,t)$ is a scalar function of space variables $x \in [0, X]$, and time $t \in [t_0, \infty)$, with $X, t_0 > 0$. $\frac{\partial^{\alpha} u}{\partial x}$ is the Caputo's space FD.

In [3], Al-Musalhi and Karimov, applied the successive approximation method to obtain self-similar solutions in an explicit form for the following two different space-time fractional sub-diffusion equations

$$D_{0,t}^{\alpha,\gamma}u(x,t) = D_{0,x}^{\beta,\gamma}u(x,t)\ ,0<\alpha\leq 1, 1<\beta\leq 2,$$

and

$$\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}u(x,t) = x^{-\beta\rho}\frac{\partial^{\beta}}{\partial x^{\beta}}u(x,t)\ ,0<\alpha\leq 1, 1<\beta\leq 2,$$

under the self similar form

$$u(x,t) = t^{\delta}f(\eta)\,,\ \text{with}\ \eta = xt^{\frac{-\alpha}{\beta}}, \delta>0$$

namely, FDE involving time and space Hilfer derivatives and FDE involving Hyper-Bessel operator in time and Erdélyi-Kober FD in space variable respectively, where $u(x,t)$ is a scalar function of space and time variables $x,t>0,\ \rho>0,\ 0\leq\gamma\leq 1$.

1.7 FIXED POINT THEOREMS

In this part, we recall some famous fixed point theorems that we will use to obtain various existence results. first, we give some definitions and theorems, which have an important role in defining these fixed point theorems.

Definition 1.28 (Equicontinuous) *Let E be a Banach space. We call a part P in $C(E,\mathbb{R})$ is equicontinuous if*

$$\forall\varepsilon>0,\ \exists\delta>0, \forall u,v\in E, \forall A\in P,\ \ \|u-v\|<\delta\Rightarrow\|A(u)-A(v)\|<\varepsilon.$$

Theorem 1.6 (Arzelà-Ascoli's theorem. (see [26])) *Let E be a compact space. If A is an equicontinuous, bounded subset of $C(E,\mathbb{R})$, then A is relatively compact.*

Definition 1.29 (Completely continuous. (see [23])) *We say $A: E\rightarrow E$ is completely continuous if for any bounded subset P of E, the set $A(P)$ is relatively compact.*

Lemma 1.7 (The generalized Gronwall inequality. (see [2])) *Let $\alpha>0$, $0<\eta<\mu$, and assume that $g(\eta), u_1(\eta)$ and $u_2(\eta)$ be locally integrable, nonnegative and non-decreasing functions. Also,* *assume that* $v_1(\eta)$ *and* $v_2(\eta)$ *are a non-decreasing continuous functions such that* $0 \leq v_1(\eta), v_2(\eta) \leq L$, *where L is a constant. Furthermore, if* $g(\eta)$ *satisfies the inequality*

$$\begin{aligned} g(\eta) &\leq u_1(\eta) + \rho^{1-\alpha} v_1(\eta) \int_0^{\eta} \zeta^{\rho-1} (\eta^{\rho} - \zeta^{\rho})^{\alpha-1} g(\zeta)\, d\zeta \\ &\quad + u_2(\eta) + \rho^{1-\alpha} v_2(\eta) \int_{\eta}^{\mu} \zeta^{\rho-1} (\zeta^{\rho} - \eta^{\rho})^{\alpha-1} g(\zeta)\, d\zeta, \end{aligned}$$

then the following inequality holds true

$$\begin{aligned} g(\eta) &\leq (u_1(\eta) + u_2(\eta)) E_{\alpha,1}\left(\rho^{-\alpha} v_2(\eta) \Gamma(\alpha) (\mu^{\rho} - \eta^{\rho})^{\alpha}\right) \\ &\quad \times E_{\alpha,1}\left(\rho^{-\alpha} v_1(\eta) \Gamma(\alpha) \eta^{\rho\alpha}\right), \end{aligned}$$

where $E_{\alpha,1}(.)$ *is a Mittag-Leffler function.*

Theorem 1.7 (Banach's fixed point theorem. (see [26])) *Let E be a Banach space and* $Q : E \to E$ *is a contraction mapping. Then Q has a fixed point i.e.*

$$\exists! x \in E : \; Qx = x.$$

Theorem 1.8 (Schauder's fixed point theorem. (see [26])) *Let E be a Banach space, and let P be a closed, convex and non-empty subset of E. Let* $T : P \to P$ *be a continuous mapping such that* $T(P)$ *is a relatively compact subset of E. Then T has at least one fixed point in P.*

Theorem 1.9 (Nonlinear alternative of Leray-Schauder type. (see [26])) *Let E be a Banach space with* $P \subset E$ *be a closed and convex. U be an open subset of P with* $0 \in U$. *Assume that* $A : \overline{U} \to P$ *is a continuous, compact (that is,* $A(\overline{U})$ *is a relatively compact subset of P) map. Then either;*
(i) *A has a fixed point in* $\overline{U}$; *or*
(ii) *there is a point* $u \in \partial U$ *and* $\sigma \in (0, 1)$ *with* $u = \sigma A(u)$.

2

EXISTENCE AND UNIQUENESS OF SOLUTION FOR A MIXED-TYPE FRACTIONAL DIFFERENTIAL EQUATION AND ULAM-HYERS STABILITY

In this chapter, we have discussed a special type of BVPs for non-linear FDE which involves both the right-sided Caputo-Katugampola's and the left-sided Katugampola's FDs. Based on some new techniques and some properties of the Mittag-Leffler function, we have introduced a formula of the solution of FDE with boundary conditions. To study the existence and uniqueness results of the solution, we have applied some known fixed point theorems (i.e., Banach's contraction principle, Schauder's, Schaefer's fixed point theorems and the nonlinear alternative of Leray-Schauder type). We have also studied the Ulam-Hyers stability of this FDE. To illustrate the theoretical results in this work, we have given two examples, (see [53]).

we investigate the existence and uniqueness of solution for the following BVP involving a nonlinear FDE with two different FDs

$$ {}^{C}D_{1^-}^{\beta,\rho}\left({}^{\rho}D_{0^+}^{\alpha}+\lambda\right)u(t)=f(t,u(t)),\ t\in J=[0,1], \tag{2.1} $$

with the boundary conditions

$$ \left({}^{\rho}I_{0^+}^{1-\alpha}u\right)(0)=u_0, \tag{2.2} $$

$$ \left({}^{\rho}D_{0^+}^{\alpha}+\lambda\right)u(1)=u_1, \tag{2.3} $$

where $\alpha,\beta\in(0,1)$, $\lambda,\rho>0$, $u_0,u_1\in\mathbb{R}$, ${}^{C}D_{1^-}^{\beta,\rho}$ is the right Caputo-Katugampola's FD of order β. ${}^{\rho}D_{0^+}^{\alpha}$ is the left Katugampola's FD of order α, ${}^{\rho}I_{0+}^{1-\alpha}$ is the Katugampola's FI and f is a given function satisfying the following conditions

(H_1) $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function.

(H_2) There exists a constant $L>0$ such that

$$|f(t,u) - f(t,v)| \leq L\,|u-v|\,, \forall u,v \in \mathbb{R},\ t \in [0,1].$$

(H_3) There exists a constant $M > 0$ with

$$|f(t,u)| \leq M,$$

for each $t \in [0,1]$.

2.1 DEFINITION OF INTEGRAL SOLUTION

In this subsection, we give the formula of the solution to the problem (2.1)-(2.3) and we give some properties of the Green function.

We start by solving the following linear problem

$$ {}^{C}D_{1^-}^{\beta,\rho}\left({}^{\rho}D_{0^+}^{\alpha} + \lambda\right) u(t) = h(t),\ t \in J = [0,1], \tag{2.4}$$

with

$$\left({}^{\rho}I_{0^+}^{1-\alpha} u\right)(0) = u_0, \tag{2.5}$$

$$\left({}^{\rho}D_{0^+}^{\alpha} + \lambda\right) u(1) = u_1. \tag{2.6}$$

Lemma 2.1 *Let $\alpha,\beta,\rho,\lambda \in \mathbb{R}$ be such that $0 < \alpha,\beta < 1$ and $\lambda,\rho > 0$. For a given $h \in C([0,1],\mathbb{R})$, the solution u to the linear BVP (2.4)-(2.6) is given by*

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \\ &\quad + u_1\left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) + \int_0^1 K(t,\tau)\ h(\tau)d\tau, \end{aligned} \tag{2.7}$$

where

$$K(t,\tau) = \frac{\tau^{\rho-1}}{\Gamma(\beta)} \times \begin{cases} \int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds, \\ \qquad 0 < \tau < t \leq 1, \\ \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds, \\ \qquad 0 < t < \tau \leq 1, \end{cases} \tag{2.8}$$

is called the Green function of the boundary value problem (2.4)-(2.6).

Proof. First, by applying the right-sided Katugampola fractional integral ${}^{\rho}I^{\beta}_{1^-}$ defined by (1.11) to both sides of equation (2.4), using theorem 1.4 and equation (2.6), we get

$$\left({}^{\rho}D^{\alpha}_{0^+}+\lambda\right)u(t)=u_1+\left({}^{\rho}I^{\beta}_{1^-}h\right)(t). \tag{2.9}$$

Following the same idea of theorem 1.3, with (2.5), equation (2.9) can be written as

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \\ &+\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha}\right)[u_1+\ {}^{\rho}I^{\beta}_{1^-}h(s)]s^{\rho-1}ds, \end{aligned}$$

so

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \\ &+u_1\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds \\ &+\frac{1}{\Gamma(\beta)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1} \\ &\left[\int_s^1\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}\tau^{\rho-1}h(s)d\tau\right]ds. \end{aligned} \tag{2.10}$$

Applying Fubini's theorem, (2.10) can be re-written as

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right)+u_1\left(\frac{t^\rho}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \\ &+\frac{1}{\Gamma(\beta)}\int_0^t\tau^{\rho-1}h(\tau)\int_0^\tau\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha}\right) \\ &\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}dsd\tau \\ &+\frac{1}{\Gamma(\beta)}\int_t^1\tau^{\rho-1}h(\tau)\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha}\right) \\ &\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}dsd\tau, \end{aligned}$$

which can be simplified to

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right)+u_1\left(\frac{t^\rho}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \\ &+\int_0^1K(t,\tau)\,h(\tau)d\tau. \end{aligned}$$

The proof is finished. ■

Let us define

$$K(t,\tau)=\begin{cases} K_1(t,\tau)\,,\ 0<\tau<t\le 1, \\ K_2(t,\tau)\,,\ 0<t<\tau\le 1. \end{cases}$$

where

$$K_1(t,\tau) = \frac{\tau^{\rho-1}}{\Gamma(\beta)} \int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} ds.$$

$$K_2(t,\tau) = \frac{\tau^{\rho-1}}{\Gamma(\beta)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} ds.$$

In the next lemma, we present some properties of the Green function $K(t,\tau)$, that form the basis of our main work.

Lemma 2.2 *For $\alpha, \beta \in (0,1)$, ρ, $\lambda > 0$, the function $K(t,\tau)$ that mentioned in lemma 2.1 satisfies the following estimates*

1) The function $K(t,\tau)$ is nonnegative.

2)

$$|K_1(t,\tau)| \le \frac{\tau^{\rho-1}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}}{\Gamma(\beta+1)\Gamma(\alpha)}, \; 0 < \tau < t \le 1\,. \tag{2.11}$$

3)

$$|K_2(t,\tau)| \le \frac{\tau^{\rho-1}\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{\beta-1}}{\Gamma(\alpha+1)\Gamma(\beta)}, \; 0 < t < \tau \le 1. \tag{2.12}$$

4)

$$\int_0^1 K(t,\tau)\, d\tau \le \frac{\rho^\alpha + \rho^\beta}{\rho^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} = \Lambda_0. \tag{2.13}$$

Proof. 1) It is obvious that $K(t,\tau) \ge 0$.

2) For $0 < \tau < t \le 1$, by lemma 1.1, we obtain

$$\begin{aligned} & |K_1(t,\tau)| \\ = & \left|\frac{\tau^{\rho-1}}{\Gamma(\beta)} \int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} ds\right| \\ \le & \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)} \left|\int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds\right|, \end{aligned}$$

since

$$\begin{aligned} & \int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds \\ \le & \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \int_0^\tau \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds \\ \le & \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \left[\frac{-1}{\beta\rho^\beta} \int_0^\tau \frac{d}{ds} (\tau^\rho - s^\rho)^\beta\, ds\right] \\ \le & \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \left[\frac{-(\tau^\rho - s^\rho)^\beta}{\beta\rho^\beta}\right]_0^\tau \\ \le & \frac{\left(\frac{\tau^\rho}{\rho}\right)^\beta}{\beta} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}, \end{aligned}$$

then

$$\begin{aligned} |K_1(t,\tau)| &\leq \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)} \frac{\left(\frac{\tau^\rho}{\rho}\right)^\beta}{\beta} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \\ &\leq \frac{\tau^{\rho-1}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}}{\Gamma(\beta+1)\Gamma(\alpha)}. \end{aligned}$$

3) For $0 < t < \tau \leq 1$, similarly with 2), by lemma 1.1, we get

$$\begin{aligned} & |K_2(t,\tau)| \\ = & \left| \frac{\tau^{\rho-1}}{\Gamma(\beta)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} ds \right| \\ \leq & \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)} \left| \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds \right|, \end{aligned}$$

we have

$$\begin{aligned} & \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds \\ \leq & \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} ds \\ \leq & \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1} \left[\frac{-(t^\rho - s^\rho)^\alpha}{\alpha\rho^\alpha}\right]_0^t \\ \leq & \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1} \left[\frac{t^{\rho\alpha}}{\alpha\rho^\alpha}\right] \\ \leq & \frac{\left(\frac{t^\rho}{\rho}\right)^\alpha}{\alpha} \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1}, \end{aligned}$$

then

$$\begin{aligned} |K_2(t,\tau)| &\leq \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)} \frac{\left(\frac{t^\rho}{\rho}\right)^\alpha}{\alpha} \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1} \\ &\leq \frac{\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{\beta-1}\tau^{\rho-1}}{\Gamma(\alpha+1)\Gamma(\beta)}. \end{aligned}$$

4) By (2.11) and (2.12), we observe that

$$\begin{aligned}
\int_0^1 K(t,\tau)\,d\tau &= \int_0^t K_1(t,\tau)\,d\tau + \int_t^1 K_2(t,\tau)\,d\tau \\
&\leq \frac{1}{\Gamma(\beta+1)\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}d\tau \\
&\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)}\int_t^1 \left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{\beta-1}\tau^{\rho-1}d\tau \\
&\leq \frac{t^{\rho\alpha}}{\rho^\alpha\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{(1-t^\rho)^\beta}{\rho^\beta\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\leq \frac{1}{\rho^\alpha\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{1}{\rho^\beta\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\leq \frac{\rho^\alpha+\rho^\beta}{\rho^{\alpha+\beta}\Gamma(\beta+1)\Gamma(\alpha+1)} = \Lambda_0.
\end{aligned}$$

■

2.2 FUNDAMENTAL LEMMAS

We now turn to the question of existence for the boundary value problem (2.1)-(2.3).

Lemma 2.3 *The following expressions hold*

$$\begin{aligned}
&\left|\int_0^\tau \left[\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right) - \left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)\right]\right. \\
&\left.\times\left(\tfrac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right| \leq O((t_2^\rho-t_1^\rho)^\alpha), \qquad \textit{for}\ \ 0<\tau<t_1<t_2\leq 1.
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
&\left|\int_0^{t_1} \left[\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} - \left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right]\left(\tfrac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right| \\
&\leq \left(\tfrac{\tau^\rho-t_2^\rho}{\rho}\right)^{\beta-1}.O((t_2^\rho-t_1^\rho)^\alpha), \qquad \textit{for}\ \ 0<t_1<t_2<\tau\leq 1.
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
&\left|\int_0^{t_1} \left[\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right) - \left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)\right]\right. \\
&\left.\times\left(\tfrac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right| \leq \left[\left(\tfrac{\tau^\rho}{\rho}\right)^{\beta} - \left(\tfrac{\tau^\rho-t_1^\rho}{\rho}\right)^{\beta}\right].O((t_2^\rho-t_1^\rho)^\alpha), \\
&\qquad\qquad \textit{for}\ \ 0<t_1<\tau<t_2\leq 1.
\end{aligned} \tag{2.16}$$

Proof. For $0<\tau<t_1<t_2\leq 1$, it follows from lemma 1.1 and the mean value theorem that

$$\begin{aligned}
&\left|\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right) - \left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)\right| \\
&\leq \left|\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} - \left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right| E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right) \\
&+ \left|E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right) - E_{\alpha,\alpha}\left(-\lambda\left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)\right|\left(\tfrac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} \\
&\leq O\left((t_2^\rho-t_1^\rho)^\alpha\right),
\end{aligned}$$

which yields

$$\begin{aligned}
&\left|\int_0^\tau \left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha}\right) - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha}\right)\right]\right.\\
&\left.\times\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right|\\
&\leq \int_0^\tau \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds.O((t_2^\rho - t_1^\rho)^\alpha)\\
&\leq \frac{1}{\beta}\left(\frac{\tau^\rho}{\rho}\right)^{\beta}.O((t_2^\rho - t_1^\rho)^\alpha)\\
&\leq O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}$$

For $0 < t_1 < t_2 < \tau \leq 1$, we obtain

$$\begin{aligned}
&\left|\int_0^{t_1}\left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right|\\
\leq\ &\left|\int_0^{t_1}\left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right] s^{\rho-1}ds\right|\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1},
\end{aligned}$$

we have

$$\begin{aligned}
&\left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right] s^{\rho-1}\\
=\ &\rho^{1-\alpha}s^{\rho-1}\left[\left(t_1^\rho - s^\rho\right)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1}\right]\\
=\ &\frac{-1}{\alpha\rho^\alpha}\frac{d}{ds}\left[\left(t_1^\rho - s^\rho\right)^{\alpha} - (t_2^\rho - s^\rho)^{\alpha}\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\int_0^{t_1}\left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right] s^{\rho-1}ds\right|\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1}\\
\leq\ &\frac{1}{\alpha\rho^\alpha}\left|\int_0^{t_1}\frac{d}{ds}\left[\left(t_1^\rho - s^\rho\right)^{\alpha} - (t_2^\rho - s^\rho)^{\alpha}\right]ds\right|\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1}\\
\leq\ &\frac{1}{\alpha\rho^\alpha}\left[t_2^{\rho\alpha} - t_1^{\rho\alpha} + (t_2^\rho - t_1^\rho)^\alpha\right]\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1}\\
\leq\ &\left(\frac{\tau^\rho - t_2^\rho}{\rho}\right)^{\beta-1}.O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}$$

In a similar way as for (2.14) and (2.15), for $0 < t_1 < \tau < t_2 \leq 1$, we obtain

$$\begin{aligned} & \left| \int_0^{t_1} \left[\left(\frac{t_1^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(-\lambda \left(\frac{t_1^\rho - s^\rho}{\rho} \right)^{\alpha} \right) - \right. \right. \\ & \left. \left. \left(\frac{t_2^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(-\lambda \left(\frac{t_2^\rho - s^\rho}{\rho} \right)^{\alpha} \right) \right] \left(\frac{\tau^\rho - s^\rho}{\rho} \right)^{\beta-1} s^{\rho-1} ds \right| \\ \leq & \int_0^{t_1} \left(\frac{\tau^\rho - s^\rho}{\rho} \right)^{\beta-1} s^{\rho-1} ds \cdot O((t_2^\rho - t_1^\rho)^\alpha) \\ \leq & \left[\left(\frac{\tau^\rho}{\rho} \right)^{\beta} - \left(\frac{\tau^\rho - t_1^\rho}{\rho} \right)^{\beta} \right] .O((t_2^\rho - t_1^\rho)^\alpha). \end{aligned}$$

■

Lemma 2.4 *We have the following properties*

$$|K_1(t_2,\tau) - K_1(t_1,\tau)| = \tau^{\rho-1} \cdot O((t_2^\rho - t_1^\rho)^\alpha), \textit{ for } \ 0 < \tau < t_1 < t_2 \leq 1. \tag{2.17}$$

$$\begin{aligned} |K_2(t_2,\tau) - K_2(t_1,\tau)| = \tau^{\rho-1} \left(\tfrac{\tau^\rho - t_2^\rho}{\rho} \right)^{\beta-1} .O((t_2^\rho - t_1^\rho)^\alpha), \\ \textit{for } \ 0 < t_1 < t_2 < \tau \leq 1. \end{aligned} \tag{2.18}$$

$$\begin{aligned} |K_1(t_2,\tau) - K_2(t_1,\tau)| = \tau^{\rho-1} \left[\left(\tfrac{\tau^\rho}{\rho} \right)^{\beta} - \left(\tfrac{\tau^\rho - t_1^\rho}{\rho} \right)^{\beta} \right] \cdot O((t_2^\rho - t_1^\rho)^\alpha), \\ \textit{for } \ 0 < t_1 < \tau < \ t_2 \leq 1. \end{aligned} \tag{2.19}$$

Proof. For $0 < \tau < t_1 < t_2 \leq 1$, by (2.14), we get

$$\begin{aligned} & |K_1(t_2,\tau) - K_1(t_1,\tau)| \\ \leq & \frac{\tau^{\rho-1}}{\Gamma(\beta)} \left| \int_0^{\tau} \left[\left(\frac{t_1^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(-\lambda \left(\frac{t_1^\rho - s^\rho}{\rho} \right)^{\alpha} \right) - \right. \right. \\ & \left. \left. \left(\frac{t_2^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(-\lambda \left(\frac{t_2^\rho - s^\rho}{\rho} \right)^{\alpha} \right) \right] \left(\frac{\tau^\rho - s^\rho}{\rho} \right)^{\beta-1} s^{\rho-1} ds \right| \\ \leq & \tau^{\rho-1} \cdot O((t_2^\rho - t_1^\rho)^\alpha). \end{aligned}$$

For $0 < t_1 < t_2 < \tau \leq 1$, by (2.15) and lemma 1.1, we find

$$\begin{aligned}
& \left|K_2(t_2,\tau) - K_2(t_1,\tau)\right| \\
= & \left|\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\int_0^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds\right.\right. \\
& \left.\left. - \int_0^{t_1}\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds\right]\right| \\
\leq & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\int_0^{t_1}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds\right| \\
& + \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds\right| \\
& - \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\int_0^{t_1}\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds\right| \\
\leq & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\left|\int_0^{t_1}\left(\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha}\right) - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right.\right.\right. \\
& \left.\left.\left. \times E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha}\right)\right)\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right|\right] \\
& + \frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha}\right) s^{\rho-1}ds \\
\leq & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left(\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right)\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right|\right] \\
& + \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds \\
\leq & \frac{\tau^{\rho-1}}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{\tau^\rho - t_2^\rho}{\rho}\right)^{\beta-1} \cdot O((t_2^\rho - t_1^\rho)^\alpha) \\
& + \frac{\tau^{\rho-1}\left(\frac{\tau^\rho - t_2^\rho}{\rho}\right)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}ds \\
\leq & \tau^{\rho-1}\left(\frac{\tau^\rho - t_2^\rho}{\rho}\right)^{\beta-1} \cdot O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}$$

In the same way, for $0 < t_1 < \tau < t_2 \leq 1$, by (2.16) and lemma 1.1, we have

$$\begin{aligned}
&\left|K_1(t_2,\tau) - K_2(t_1,\tau)\right| \\
= &\left|\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\int_0^{\tau}\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right.\right. \\
&\left.\left.-\int_0^{t_1}\left(\frac{t_1^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right]\right| \\
\leq &\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\left[\int_0^{t_1}\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right.\right. \\
&\left.\left.-\int_0^{t_1}\left(\frac{t_1^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right]\right| \\
&+\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_{t_1}^{\tau}\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds \\
\leq &\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\left|\int_0^{t_1}\left[\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right)-\left(\frac{t_1^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\right.\right.\right. \\
&\left.\left.\left.\times\ E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right)\right]\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}s^{\rho-1}ds\right|\right] \\
&+\frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\int_{t_1}^{\tau}\left(\frac{t_2^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}s^{\rho-1}ds \\
\leq &\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_0^{t_1}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}s^{\rho-1}ds\cdot O((t_2^{\rho}-t_1^{\rho})^{\alpha}) \\
\leq &\tau^{\rho-1}\left[\left(\frac{\tau^{\rho}}{\rho}\right)^{\beta}-\left(\frac{\tau^{\rho}-t_1^{\rho}}{\rho}\right)^{\beta}\right]\cdot O((t_2^{\rho}-t_1^{\rho})^{\alpha}).
\end{aligned}$$

■

In view of lemma 2.1, we define the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ as follows

$$\begin{aligned}
Fu(t) = &u_0\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)+u_1\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) \\
&+\int_0^1 K(t,\tau)\ f(\tau,u(\tau))d\tau, \qquad (2.20)
\end{aligned}$$

where $K(t,\tau)$ is defined by (2.8).

Lemma 2.5 *F is a completely continuous operator*

Proof. Firstly, according to (H_1), we note that the operator F is well defined. Next, choosing

$$\eta \geq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)}+\frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)}+M\Lambda_0.$$

We define

$$\Omega_\eta = \{u \in C([0,1],\mathbb{R}) : \ \|u\|_\infty \le \eta, \ \eta > 0\}. \tag{2.21}$$

Clearly, Ω_η is a nonempty, bounded, closed and convex subset of $C([0,1],\mathbb{R})$.

We show that $F(\Omega_\eta)$ is uniformly bounded. Let $u \in \Omega_\eta$, in fact for any $t \in [0,1]$, by condition (H_3), from (i) of lemma 1.1 and equation (2.13), we obtain

$$\begin{aligned}
|Fu(t)| &\le \left| u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right| \\
&\quad + \left| u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right| \\
&\quad + \int_0^1 K(t,\tau)\,|f(\tau,u(\tau))|\,d\tau \\
&\le \frac{|u_0|\, t^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|\, t^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + M\int_0^1 K(t,\tau)\,d\tau \\
&\le \frac{|u_0|\, t^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|\, t^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{M(\rho^\alpha \ + \rho^\beta)}{\rho^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\le \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0 \\
&\le \eta.
\end{aligned}$$

Consequently,

$$\|Fu\|_\infty \le \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0 < \infty, \ \text{for all } u \in \Omega_\eta$$

and hence $F(\Omega_\eta)$ is uniformly bounded.

Now, we show that F is equicontinuous. Let $t_1, t_2 \in [0,1]$, with $t_1 < t_2$, $\forall u \in \Omega_\eta$, by the mean value theorem, we obtain

$$\begin{aligned}
&|Fu(t_2) - Fu(t_1)| \\
\le\ & \left| u_0 \left[\left(\frac{t_2^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_1^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho}{\rho}\right)^\alpha\right)\right]\right| \\
& + \left| u_1 \left[\left(\frac{t_2^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t_2^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_1^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t_1^\rho}{\rho}\right)^\alpha\right)\right]\right| \\
& + \int_0^1 |K(t_2,\tau) - K(t_1,\tau)| \ |f(\tau,u(\tau))|\,d\tau \\
\le\ & |u_0|\, O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + |u_1|\, O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) \\
& + M\int_0^1 |K(t_2,\tau) - K(t_1,\tau)|\,d\tau \\
\le\ & O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + M\int_0^1 |K(t_2,\tau) - K(t_1,\tau)|\,d\tau.
\end{aligned}$$

It remains to show that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. In what follows, we divide the proof into three cases

Case1: For $\tau < t_1 < t_2$, by (2.17), we have

$$\begin{aligned} |Fu(t_2) - Fu(t_1)| &\leq O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + M\int_0^1 |k_1(t_2,\tau) - k_1(t_1,\tau)|\, d\tau \\ &\leq O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + M\int_0^1 O((t_2^\rho - t_1^\rho)^\alpha)\tau^{\rho-1} d\tau \\ &= O((t_2^\rho - t_1^\rho)^\alpha). \end{aligned}$$

Case 2: For $t_1 < t_2 < \tau$, by (2.18), we get

$$\begin{aligned} & |Fu(t_2) - Fu(t_1)| \\ \leq\ & O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + M\int_0^1 |k_2(t_2,\tau) - k_2(t_1,\tau)|\, d\tau \\ \leq\ & O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + M\int_0^1 \tau^{\rho-1}\left(\frac{\tau^\rho - t_2^\rho}{\rho}\right)^{\beta-1} .O((t_2^\rho - t_1^\rho)^\alpha)) d\tau \\ =\ & O((t_2^\rho - t_1^\rho)^\alpha). \end{aligned}$$

Case 3: In the same way, for $t_1 < \tau < t_2$, by (2.19), we obtain

$$\begin{aligned} & |Fu(t_2) - Fu(t_1)| \\ \leq\ & O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + M\int_0^1 |k_1(t_2,\tau) - k_2(t_1,\tau)|\, d\tau \\ \leq\ & O\left(\left(t_2^\rho - t_1^\rho\right)^\alpha\right) + \frac{M}{\Gamma(\beta+1)}\int_0^1 \tau^{\rho-1}\left[\left(\frac{\tau^\rho}{\rho}\right)^\beta - \left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^\beta\right] .O((t_2^\rho - t_1^\rho)^\alpha)) d\tau \\ =\ & O((t_2^\rho - t_1^\rho)^\alpha). \end{aligned}$$

Consequently, by this three cases, we have $|Fu(t_2) - Fu(t_1)| \to 0$ as $t_2 \to t_1$. Finally, by the means of the Ascoli-Arzela theorem 1.6, we have $F : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ is completely continuous. ■

Next, we study the existence and uniqueness of solution for the BVP (2.1)-(2.3).

2.3 EXISTENCE RESULTS OF AT LEAST ONE SOLUTION

In this subsection, we discuss the existence of solution for the nonlinear mixed-type FDEs with boundary conditions (2.1)-(2.3) by applying some fixed point theorems (Schauder's fixed point theorem, the nonlinear alternative of Leray-Schauder type and Schaefer's fixed point theorem).

Now, we demonstrate the first following solution's existence result, by using the fixed point theorem of Schauder.

Theorem 2.1 *Assume that* (H_1) *and* (H_3) *are satisfied, then the problem (2.1)-(2.3) has at least one solution on* $[0,1]$.

Proof. Let the operator F defined in (2.20), then we shall show that F satisfies the assumptions of Schauder's fixed point theorem. It means, we will prove that the operator F is continuous.
Let's first show that F is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([0,1],\mathbb{R})$, for each $t \in [0,1]$, by (2.13), we get

$$\begin{aligned}
|Fu_n(t) - Fu(t)| &= \left| \int_0^1 K(t,\tau) f(\tau, u_n(\tau)) d\tau - \int_0^1 K(t,\tau) f(\tau, u(\tau)) d\tau \right| \\
&\leq \int_0^1 K(t,\tau) \left| f(\tau, u_n(\tau)) - f(\tau, u(\tau)) \right| d\tau \\
&\leq \int_0^1 K(t,\tau) \sup \left| f(\tau, u_n(\tau)) - f(\tau, u(\tau)) \right| d\tau \\
&\leq \| f(.,u_n) - f(.,u) \|_\infty \int_0^1 K(t,\tau)\, d\tau \\
&\leq \frac{(\rho^\alpha + \rho^\beta) \| f(.,u_n) - f(.,u) \|_\infty}{\rho^{\alpha+\beta} \Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\leq \Lambda_0 \| f(.,u_n) - f(.,u) \|_\infty ,
\end{aligned}$$

then, $\|Fu_n - Fu\|_\infty \leq \Lambda_0 \| f(.,u_n) - f(.,u) \|_\infty .$

Since f is continuous, then $\| f(.,u_n) - f(.,u) \|_\infty \to 0$ as $n \to \infty$. Consequently, F is continuous. From lemma 2.5, we know that F is a completely continuous operator. As a consequence of Schauder's fixed point theorem 1.8, we deduce that F has a fixed point which is a solution of the problem (2.1)-(2.3) on $[0,1]$. ■

Next, we demonstrate the second following solution's existence result, by using the fixed point theorem of Schaefer.

Theorem 2.2 *Assume that* (H_1) *and* (H_3) *hold, then the problem (2.1)-(2.3) has at least one solution.*

Proof. Consider F as in (2.20). Clearly, F is a continuous and completely continuous operator.

Now, it remains to show that the set $\mathcal{E} = \{u \in C([0,1],\mathbb{R}) : u = \lambda u,\ \lambda \in (0,1)\}$ is bounded. Let $u \in \mathcal{E}$. Then, $u = \lambda F u$ for some $\lambda \in (0,1)$. For each $t \in [0,1]$, by (H_3), lemma 1.1 and equation (2.13), we obtain

$$\begin{aligned} |u(t)| &= \lambda \left| u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) + u_1\left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \right. \\ &\quad \left. + \int_0^1 K(t,\tau)\ f(\tau,u(\tau))d\tau \right| \\ &\leq \left| u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right)\right| + \left| u_1\left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right)\right| \\ &\quad + \int_0^1 K(t,\tau)\ |f(\tau,u(\tau))|\, d\tau \\ &\leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0 < \infty. \end{aligned}$$

This shows that the set $\mathcal{E}$ is bounded. Hence the fixed point theorem of Schaefer guarantees that F has a fixed point, which is a solution of (2.1)-(2.3). ∎

Our third solution's existence result for (2.1)-(2.3) is based on the non-linear alternative of Leray-Schauder type.

Theorem 2.3 *Assume that hypotheses (H_1) and (H_3) hold, and that there exists $\theta > 0$, such that*

$$\frac{1}{\theta}\left(\frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0\right) < 1, \tag{2.22}$$

then the problem (2.1)-(2.3) has at least one solution on $[0,1]$.

Proof. Consider the operator F defined in (2.20), then we shall show that all assumption of Leray-Schauder fixed point theorem 1.9 are satisfied by the operator F. The proof will be given in several claims.

Claim 1: Clearly F is continuous.

Claim 2: F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

Actually, it is enough to show that for any $\theta > 0$, there exists $l > 0$ such that for each $u \in D_\theta = \{u \in C\ ([0,1],\mathbb{R}) : \|u\|_\infty \leq \theta\}$, we have $\|Fu\|_\infty \leq l$.

Let $u \in D_\theta$, for each $t \in [0,1]$, we have

$$\begin{aligned} |Fu(t)| &\leq \left| u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right| \\ &\quad + \left| u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right| \\ &\quad + \int_0^1 K(t,\tau)\, |f(\tau,u(\tau))|\, d\tau \\ &\leq \frac{|u_0|\, t^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|\, t^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{M(\rho^\alpha + \rho^\beta)}{\rho^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0. \end{aligned}$$

Thus

$$\|Fu\|_\infty \leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0 := l < \infty. \tag{2.23}$$

Claim 3: It is clear that F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. From Claim1-Claim3, we conclude that $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is continuous and complectly continuous.

Claim 4: A priori bounds.

Let $u \in \partial D_\theta$, such that $u = \mu F u$, for some $0 < \mu < 1$. From (2.23), we obtain

$$\begin{aligned} \|u\|_\infty &= \mu \|Fu\|_\infty \leq \|Fu\|_\infty, \\ &\leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0, \end{aligned}$$

and thus

$$\theta \leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0,$$

hence,

$$\frac{1}{\theta}\left[\frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0\right] \geq 1,$$

which contradicts (2.22). Consequently, by the nonlinear alternative of Leray-Schauder fixed point theorem 1.9, the problem (2.1)-(2.3) has at least one solution on $[0,1]$. ∎

2.4 UNIQUENESS OF SOLUTION

Finally, we show the last result of the solution's existence and uniqueness for the problem (2.1)-(2.3), which is based on the Banach's contraction principle.

Theorem 2.4 *Assume that $(H_1) - (H_2)$ hold, we give $\alpha, \beta \in (0,1)$, ρ, $\lambda > 0$. Then the problem (2.1)-(2.3) has a unique solution on $[0,1]$, provided that*

$$0 < L\Lambda_0 < 1. \tag{2.24}$$

Proof. Consider the operator $F : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ given by (2.20), we shall show that F is a contraction mapping. Let $u, v \in C([0,1], \mathbb{R})$, for any $t \in [0,1]$, according to (H_2) and by equation (2.13), we obtain

$$\begin{aligned} |Fu(t) - Fv(t)| &= \left| \int_0^1 K(t,\tau)\ f(\tau, u(\tau))d\tau - \int_0^1 K(t,\tau)\ f(\tau, v(\tau))d\tau \right| \\ &\leq \int_0^1 K(t,\tau) \left| f(\tau, u(\tau)) - f(\tau, v(\tau)) \right| d\tau \\ &\leq L \int_0^1 K(t,\tau) \left| u(\tau) - v(\tau) \right| d\tau, \end{aligned}$$

then

$$\|Fu - Fv\|_\infty \leq L\Lambda_0 \|u - v\|_\infty .$$

By the condition (2.24), F is a contraction mapping, using the principle of Banach fixed point theorem 1.7, we deduce that there exists a unique solution of the problem (2.1)-(2.3) on $[0,1]$. ∎

2.5 ULAM-HYERS STABILITY

In this part, we discuss the Ulam-Hyers stability of the problem (2.1)-(2.3).

Let $\widetilde{\varepsilon} > 0$ and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function.

For the mixed fractional BVP (2.1)-(2.3), we emphasize on the following inequality

$$\left| {}^C D_{1^-}^{\beta,\rho} \left({}^\rho D_{0^+}^{\alpha} + \lambda \right) w(t) - f(t, w(t)) \right| \leq \widetilde{\varepsilon}, \quad t \in [0,1]. \tag{2.25}$$

In a similar way as in [61–63], we introduce the following definition and remark.

Definition 2.1 *The mixed fractional boundary value problem (2.1)-(2.3) is Ulam-Hyers stable if there exists a constant $\delta_0 > 0$ such that for each $\widetilde{\varepsilon} > 0$ and for each solution $w \in C([0,1], \mathbb{R})$ of inequality (2.25) there exists a solution $u \in C([0,1], \mathbb{R})$ of (2.1)-(2.3) with*

$$|w(t) - u(t)| \leq \delta_0 \widetilde{\varepsilon}, \quad t \in [0,1].$$

Remark 2.1 *A function $w \in C([0,1], \mathbb{R})$ is a solution of inequality (2.25) if and only if there exists a function $\varphi \in C([0,1], \mathbb{R})$ such that*

(i) $|\varphi(t)| \leq \widetilde{\varepsilon}$, $t \in [0,1]$,

(ii) ${}^C D_{1^-}^{\beta,\rho} \left({}^\rho D_{0^+}^{\alpha} + \lambda \right) w(t) = f(t, w(t)) + \varphi(t)$, $t \in [0,1]$.

Theorem 2.5 *Suppose that* (H_1) *and* (H_2) *hold, then the mixed fractional BVP (2.1)-(2.3) is Ulam-Hyers stable if* $L\Lambda_0 < 1$.

Proof. Let $0 < \alpha, \beta < 1$ and let $w \in C([0,1], \mathbb{R})$ be a solution of inequality (2.25) with $\left({}^{\rho}I_{0^+}^{1-\alpha} w\right)(0) = u_0$, $\left({}^{\rho}D_{0^+}^{\alpha} + \lambda\right) w(1) = u_1$, then by Remark 2.1, we have

$$ {}^{C}D_{1^-}^{\beta,\rho} \left({}^{\rho}D_{0^+}^{\alpha} + \lambda\right) w(t) = f(t, w(t)) + \varphi(t), \ t \in [0,1]. $$

By adopting the same arguments as in the proof of lemma 2.1, we can write

$$ \begin{aligned} w(t) &= u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) + u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \\ &\quad + \int_0^1 K(t,\tau) f(\tau, w(\tau)) d\tau + \int_0^1 K(t,\tau) \ \varphi(\tau) d\tau. \end{aligned} $$

From this equation and by (2.13), it follows that

$$ \begin{aligned} &\left| w(t) - u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \right. \\ &\left. -u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) - \int_0^1 K(t,\tau) f(\tau, w(\tau)) d\tau \right| \\ =\ & \left| \int_0^1 K(t,\tau) \ \varphi(\tau) d\tau \right| \\ \leq\ & \int_0^1 K(t,\tau) \left| \varphi(\tau) \right| d\tau \\ \leq\ & \Lambda_0 \widetilde{\varepsilon}. \end{aligned} \tag{2.26} $$

Now, let $u \in C([0,1], \mathbb{R})$ be a unique solution of (2.1)-(2.3), then for each $t \in [0,1]$, we have

$$ \begin{aligned} |w(t) - u(t)| &= \left| w(t) - u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \right. \\ &\quad \left. -u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) - \int_0^1 K(t,\tau) \ f(\tau, u(\tau)) d\tau \right| \\ &\leq \left| w(t) - u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \right. \\ &\quad -u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) - \int_0^1 K(t,\tau) \ f(\tau, w(\tau)) d\tau \\ &\quad \left. + \int_0^1 K(t,\tau) \ f(\tau, w(\tau)) d\tau - \int_0^1 K(t,\tau) \ f(\tau, u(\tau)) \ d\tau \right| \\ &\leq \left| w(t) - u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) \right. \\ &\quad \left. -u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right) - \int_0^1 K(t,\tau) \ f(\tau, w(\tau)) d\tau \right| \\ &\quad + \int_0^1 K(t,\tau) \ \left| f(\tau, w(\tau)) - f(\tau, u(\tau)) \right| \ d\tau. \end{aligned} $$

From (H_2), (2.26) and (2.13), we obtain

$$\|w-u\| \leq \Lambda_0\widetilde{\varepsilon} + L\Lambda_0 \|w-u\|,$$

which implies

$$\|w-u\| \leq \delta_0\widetilde{\varepsilon},$$

where $\delta_0 = \frac{\Lambda_0}{1-L\Lambda_0} > 0$. Then the mixed fractional boundary value problem (2.1)-(2.3) is Ulam-Hyers stable. ■

2.6 EXAMPLES

In this last subsection, we present two examples to explain the applicability of our main results.

Example 2.1 *Consider the following boundary value problem with two different FDs*

$$\begin{cases} {}^{C}D_{1^-}^{\frac{1}{5},1}\left({}^{1}D_{0^+}^{\frac{1}{3}} + \frac{5}{4}\right)u(t) = \frac{\cos(t)}{1+u^2}, \quad t \in J = [0,1], \\ \left({}^{1}I_{0^+}^{\frac{2}{3}}u\right)(0) = \frac{1}{2}, \ \left({}^{1}D_{0^+}^{\frac{1}{3}} + \frac{5}{4}\right)u(1) = 1. \end{cases} \tag{2.27}$$

Here, $f(t,u(t)) = \frac{\cos(t)}{1+u^2}$, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{5}$, $\alpha+\beta = \frac{8}{15} < 1$, $\lambda = \frac{5}{4}$, $\rho = 1$, $u_0 = \frac{1}{2}$ *and* $u_1 = 1$.
The function f *is continuous for any* $t \in [0,1]$ *and we have* $|f(t,u)| \leq M = 1, \forall (t,u) \in [0,1]\times\mathbb{R}$. *Hence the condition* (H_3) *holds. It follows from theorem 2.1 and theorem 2.2, that the problem (2.27) has at least one solution.*

Example 2.2 *Consider the following mixed fractional boundary value problem*

$$\begin{cases} {}^{C}D_{1^-}^{\frac{1}{4},1}\left({}^{1}D_{0^+}^{\frac{1}{2}} + 2\right)u(t) = t + \frac{u(t)}{5e^t(1+u(t))}, \quad t \in J = [0,1], \\ \left({}^{1}I_{0^+}^{\frac{1}{2}}u\right)(0) = u_0, \ \left({}^{1}D_{0^+}^{\frac{1}{2}} + 2\right)u(1) = u_1. \end{cases} \tag{2.28}$$

Here, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\alpha+\beta = \frac{6}{8} < 1$, $\lambda = 2$, $\rho = 1$ *and* $f(t,u(t)) = t + \frac{u(t)}{5e^t(1+u(t))}$.
The function f *is continuous for any* $t \in [0,1]$, *then we have* $|f(t,u) - f(t,v)| \leq \frac{1}{5}|u-v|$, $L = \frac{1}{5}$, $\Lambda_0 = \frac{2}{\Gamma(\frac{1}{4}+1)\Gamma(\frac{1}{2}+1)} \approx 2.49$ *and* $L\Lambda_0 \approx 0.50 < 1$.
By theorem 2.4, the problem (2.28) has a unique solution u *on* $[0,1]$.

Now, let $w \in C([0,1],\mathbb{R})$ *be a solution of the inequality*

$$\left|{}^{CK}D_{1^-}^{\frac{1}{4},1}\left({}^{1}D_{0^+}^{\frac{1}{2}} + 2\right)w(t) - \left(t + \frac{w(t)}{5e^t\,(1+w(t))}\right)\right| \leq \widetilde{\varepsilon},\ \widetilde{\varepsilon} > o,\ t \in J = [0,1],$$

then, from theorem 2.5, the mixed fractional BVP (2.28) is Ulam-Hyers stable with

$$\delta_0 = \frac{\Lambda_0}{1-L\Lambda_0} = \frac{2.49}{0.5} = 4.98 > 0.$$

3

EXISTENCE RESULTS OF SELF-SIMILAR SOLUTIONS OF THE SPACE-FRACTIONAL DIFFUSION EQUATION INVOLVING THE GENERALIZED RIESZ-CAPUTO FRACTIONAL DERIVATIVE

In this chapter, we have discussed the problem of existence and uniqueness of solutions under the self-similar form to the space-fractional diffusion equation, (see [54]). The space-fractional derivative which will be used is the GRCFD (known as the Riesz-Caputo Katugampola's FD). Based on the similarity variable η, we have introduced the equation satisfied by the self-similar solutions for the aforementioned problem. To study the existence and uniqueness of self-similar solutions for this problem, we have applied some known fixed point theorems (i.e., Banach's contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type).

The problem is given by

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{\alpha,\rho} u(x,t)}{\partial |x|^{\alpha}}, \quad (x,t) \in [0,X] \times [t_0, \infty[\,, \ 1 < \alpha \leq 2 \tag{3.1}$$

under the following self-similar form

$$u(x,t) = t^{\beta} f\left(\frac{x}{t^{\frac{1}{\alpha\rho}}}\right), \text{ with } (x,t) \in [0,X] \times [t_0, \infty[\,, \tag{3.2}$$

where $u(x,t)$ is a scalar function of space variables $x \in [0,X]$, and time $t \in [t_0, \infty)$, with $X, t_0 > 0$, $\frac{\partial^{\alpha,\rho}}{\partial |x|^{\alpha}}$ is the GRCFD of order α with $\rho > 0$ and which is the main motivation of the present research, the "basic profile" f in (3.2) is not known in advance and is to be identified and $\beta \in \mathbb{R}$ is a constant chosen so that the solutions exist.

3.1 STATEMENT OF THE PROBLEM

In this subsection, we consider the following problem of the space-fractional diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^{\alpha,\rho} u(x,t)}{\partial |x|^{\alpha}}, & (x,t) \in [0,X] \times [t_0, \infty[\,, 1 < \alpha \leq 2 \\ u(0,t) + u(X,t) = t^{\beta} M, & \frac{\partial u(0,t)}{\partial x} + \frac{\partial u(X,t)}{\partial x} = 0, \ \beta, M \in \mathbb{R}. \end{cases} \tag{3.3}$$

Under the self-similar form

$$u(x,t) = t^{\beta} f\left(\frac{x}{t^{\frac{1}{\alpha\rho}}}\right), \ \beta \in \mathbb{R}. \tag{3.4}$$

We should first deduce the equation satisfied by the function f in (3.4).

Theorem 3.1 *Let $\alpha, \beta, \rho \in \mathbb{R}$ provided that $1 < \alpha \leq 2$, $\rho > 0$ and $(x,t) \in [0,X] \times [t_0, \infty[$ for some $X, t_0 > 0$. Then the transformation*

$$u(x,t) = t^{\beta} f(\eta), \text{ with } \eta = \frac{x}{t^{\frac{1}{\alpha\rho}}}$$

reduces the fractional partial differential equation (3.1) to the fractional differential equation of the form

$${}_{0}^{RC}D_{\mu}^{\alpha,\rho} f(\eta) = \beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta), \ \eta \in [0,\mu],$$

where $\mu = X t_0^{\frac{-1}{\alpha\rho}}$.

Proof. Let $\eta = \frac{x}{t^{\frac{1}{\alpha\rho}}}$. From (3.4), we obtain

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \beta t^{\beta-1} f(\eta) + t^{\beta}\left[-\frac{1}{\alpha\rho} t^{-\frac{1}{\alpha\rho}-1} x f'(\eta)\right] \\ &= t^{\beta-1}\left[\beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta)\right]. \end{aligned} \tag{3.5}$$

Furtheremore, for $1 < \alpha \leq 2$, $\rho > 0$, by the definition 1.23 of the G-RCFD, equation (3.4) and by putting $\zeta = \frac{s}{t^{\frac{1}{\alpha\rho}}}$, we get

$$\begin{aligned}
\frac{\partial^{\alpha,\rho}u(x,t)}{\partial |x|^{\alpha}} &= \frac{t^{\beta}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{0}^{X} |(x^{\rho}-s^{\rho})|^{n-\alpha-1} s^{\rho-1}\left(s^{1-\rho}\frac{d}{ds}\right)^{n} f\left(\frac{s}{t^{\frac{1}{\alpha\rho}}}\right) ds \\
&= \frac{t^{\beta}}{2}\left({}^{C}D_{0^+}^{\alpha,\rho} + {}^{C}D_{X^-}^{\alpha,\rho}\right) f\left(\frac{x}{t^{\frac{1}{\alpha\rho}}}\right) \\
&= \frac{1}{2}\left[\frac{t^{\beta}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{0}^{x} (x^{\rho}-s^{\rho})^{n-\alpha-1} s^{\rho-1}\left(s^{1-\rho}\frac{d}{ds}\right)^{n} f\left(\frac{s}{t^{\frac{1}{\alpha\rho}}}\right) ds\right. \\
&\quad \left. + \frac{t^{\beta}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{x}^{X} (s^{\rho}-x^{\rho})^{n-\alpha-1} s^{\rho-1}\left(s^{1-\rho}\frac{d}{ds}\right)^{n} f\left(\frac{s}{t^{\frac{1}{\alpha\rho}}}\right) ds\right] \\
&= \frac{1}{2}\left[\frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{0}^{\eta} \left(x^{\rho}-\left(\zeta t^{\frac{1}{\alpha\rho}}\right)^{\rho}\right)^{n-\alpha-1} \left(\zeta t^{\frac{1}{\alpha\rho}}\right)^{\rho-1}\left(\left(\zeta t^{\frac{1}{\alpha\rho}}\right)^{1-\rho}\frac{d}{t^{\frac{1}{\alpha\rho}}d\zeta}\right)^{n} f(\zeta)\, d\zeta\right. \\
&\quad \left. + \frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{\eta}^{\mu} \left(\left(\zeta t^{\frac{1}{\alpha\rho}}\right)^{\rho}-x^{\rho}\right)^{n-\alpha-1} \left(\zeta t^{\frac{1}{\alpha\rho}}\right)^{\rho-1}\left(\left(\zeta t^{\frac{1}{\alpha\rho}}\right)^{1-\rho}\frac{d}{t^{\frac{1}{\alpha\rho}}d\zeta}\right)^{n} f(\zeta)\, d\zeta\right] \\
&= \frac{1}{2}\left[\frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{0}^{\eta} t^{\frac{1}{\alpha}[n-\alpha-1]+\frac{1}{\alpha\rho}[\rho-1+n(1-\rho)-n]} (\eta^{\rho}-\zeta^{\rho})^{n-\alpha-1}\zeta^{\rho-1}\left(\zeta^{1-\rho}\frac{d}{d\zeta}\right)^{n} f(\zeta)\, d\zeta\right. \\
&\quad \left. + \frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{\eta}^{\mu} t^{\frac{1}{\alpha}[n-\alpha-1]+\frac{1}{\alpha\rho}[\rho-1+n(1-\rho)-n]} (\zeta^{\rho}-\eta^{\rho})^{n-\alpha-1}\zeta^{\rho-1}\left(\zeta^{1-\rho}\frac{d}{d\zeta}\right)^{n} f(\zeta)\, d\zeta\right] \\
&= t^{\beta-1}\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{0}^{\mu} |(\eta^{\rho}-\zeta^{\rho})|^{n-\alpha-1}\zeta^{\rho-1}\left(\zeta^{1-\rho}\frac{d}{d\zeta}\right)^{n} f(\zeta)\, d\zeta \\
&= t^{\beta-1}\, {}_{0}^{RC}D_{\mu}^{\alpha,\rho} f(\eta). \qquad (3.6)
\end{aligned}$$

By substituting (3.5) and (3.6) in (3.1) , we get the following equation

$${}_{0}^{RC}D_{\mu}^{\alpha,\rho} f(\eta) = \beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta),\ \eta \in [0,\mu],$$

where $\mu = Xt_0^{\frac{-1}{\alpha\rho}}$. The proof is finished. ■

Now, to study the following problem, we will need the results in subsection 3.1 along with theorem 4.1.

$${}_{0}^{RC}D_{\mu}^{\alpha,\rho} f(\eta) = \beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta),\ 1 < \alpha \leq 2,\ \eta \in [0,\mu], \qquad (3.7)$$

with the conditions

$$f(0) + f(\mu) = M,\ f'(0) + f'(\mu) = 0, \qquad (3.8)$$

where $\beta \in \mathbb{R}$ and $\rho, \mu > 0$.

3.2 DEFINITION OF INTEGRAL SOLUTION OF THE BASIC PROFILE f

In this lemma, we give the formula of solution to the problem $(3.7)-(3.8)$.

Lemma 3.1 *Let $\alpha,\beta,\rho,\mu\in\mathbb{R}$ provided that $1<\alpha\leq 2$ and $\rho,\mu>0$. For a given $f,f',{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f\in C[0,\mu]$. Then the problem $(3.7)-(3.8)$ is equivalent to the following integral equation*

$$\begin{aligned} f(\eta) &= w+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta \\ &\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta, \end{aligned} \tag{3.9}$$

$\forall\eta\in[0,\mu]$, where

$$w=\frac{M}{2}-\frac{\mu^{\rho}}{2\rho}f'(\mu). \tag{3.10}$$

Proof. First, by applying the generalized Riesz fractional integral ${}_{0}^{RG}I_{\mu}^{\alpha,\rho}$ defined in (1.35) to both sides of equation (3.7), we obtain

$${}_{0}^{RG}I_{\mu}^{\alpha,\rho}\,{}_{0}^{RC}D_{\mu}^{\alpha,\rho}={}_{0}^{RG}I_{\mu}^{\alpha,\rho}\left(\beta f(\eta)-\frac{1}{\alpha\rho}\eta f'(\eta)\right). \tag{3.11}$$

From lemma 1.5 and remark 1.3, we get

$$\begin{aligned} {}_{0}^{RG}I_{\mu}^{\alpha,\rho}\,{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f(\eta) &= f(\eta)-\frac{1}{2}[f(0)+f(\mu)] \\ &\quad-\frac{\eta^{\rho}}{2\rho}\left[f'(0)+f'(\mu)\right]+\frac{\mu^{\rho}}{2\rho}f'(\mu). \end{aligned}$$

Then, the fractional integral equation (3.11), can be re-written as follows

$$\begin{aligned} f(\eta) &= {}_{0}^{RG}I_{\mu}^{\alpha,\rho}\left(\beta f(\eta)-\frac{1}{\alpha\rho}\eta f'(\eta)\right)+\frac{1}{2}[f(0)+f(\mu)] \\ &\quad+\frac{\eta^{\rho}}{2\rho}\left[f'(0)+f'(\mu)\right]-\frac{\mu^{\rho}}{2\rho}f'(\mu). \end{aligned} \tag{3.12}$$

Applying (3.8) to (3.12) yields

$$\begin{aligned} f(\eta) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu}|(\zeta^{\rho}-\eta^{\rho})|^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right) \\ &\quad+\frac{M}{2}-\frac{\mu^{\rho}}{2\rho}f'(\mu). \end{aligned} \tag{3.13}$$

Then, according to (3.10), the problem $(3.7)-(3.8)$ is equivalent to

$$\begin{aligned} f(\eta) &= w+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta \\ &\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta. \end{aligned} \tag{3.14}$$

The proof is finished. ■

3.3 FUNDAMENTAL LEMMAS

In what follows, to derive the principal theorems, we will need the following lemmas.

Lemma 3.2 *Let $\mu > 0$, we define*

$$E = \left\{ f \in C\left[0,\mu\right] \mid \; f'\left(0\right) + f'\left(\mu\right) = 0 \right\}. \tag{3.15}$$

Then $\left(E, \|.\|_{\infty}\right)$ is a Banach space.

Proof. Let μ be a positif parameter. It is obvious that the space E with the norm $\|.\|_{\infty}$ is a subspace of the Banach space $C\left[0,\mu\right]$. So, to show that E is a Banach space, it is enought to demonstrate that this later is closed in $C\left[0,\mu\right]$.

Let $\left(f_n\right)_{n\in\mathbb{N}} \in E$ be a real sequence such that $\lim\limits_{n\to\infty} f_n = f$ in $C\left[0,\mu\right]$. Then we demonstrate that $f \in E$. Let $\eta, v \in \left[0,\mu\right] \times \left[0,\mu\right]$. We have

$$\begin{cases} \frac{d}{d\eta}\left[f_n\left(\eta\right) - f\left(\eta\right)\right] = \frac{d}{d\eta} f_n\left(\eta\right) - \frac{d}{d\eta} f\left(\eta\right). \\ \frac{d}{dv}\left[f_n\left(v\right) - f\left(v\right)\right] = \frac{d}{dv} f_n\left(v\right) - \frac{d}{dv} f\left(v\right). \end{cases}$$

Since f_n is continuous, we get

$$\begin{cases} \lim\limits_{n\to\infty} \frac{d}{d\eta} f_n\left(\eta\right) = \frac{d}{d\eta} f\left(\eta\right) \\ \lim\limits_{n\to\infty} \frac{d}{dv} f_n\left(v\right) = \frac{d}{dv} f\left(v\right) \end{cases}, \forall \eta, v \in \left[0,\mu\right] \times \left[0,\mu\right],$$

then

$$\sup_{\eta} \lim_{n\to\infty} \left| \frac{d}{d\eta} f_n\left(\eta\right) - \frac{d}{d\eta} f\left(\eta\right) \right| = \lim_{n\to\infty} \sup_{\eta} \left| \frac{d}{d\eta} f_n\left(\eta\right) - \frac{d}{d\eta} f\left(\eta\right) \right| = 0$$

and

$$\sup_{v} \lim_{n\to\infty} \left| \frac{d}{dv} f_n\left(v\right) - \frac{d}{dv} f\left(v\right) \right| = \lim_{n\to\infty} \sup_{v} \left| \frac{d}{dv} f_n\left(v\right) - \frac{d}{dv} f\left(v\right) \right| = 0.$$

This implies that

$$\begin{cases} \lim\limits_{n\to\infty} \left\| \frac{d}{d\eta} f_n\left(\eta\right) - \frac{d}{d\eta} f\left(\eta\right) \right\|_{\infty} = 0. \\ \lim\limits_{n\to\infty} \left\| \frac{d}{dv} f_n\left(v\right) - \frac{d}{dv} f\left(v\right) \right\|_{\infty} = 0. \end{cases}$$

Thus

$$\begin{aligned} & \lim_{n\to\infty} \left\| \frac{d}{d\eta} f_n\left(\eta\right) - \frac{d}{d\eta} f\left(\eta\right) + \frac{d}{dv} f_n\left(v\right) - \frac{d}{dv} f\left(v\right) \right\|_{\infty} \\ \leq \;& \lim_{n\to\infty} \left\| \frac{d}{d\eta} f_n\left(\eta\right) - \frac{d}{d\eta} f\left(\eta\right) \right\|_{\infty} + \lim_{n\to\infty} \left\| \frac{d}{dv} f_n\left(v\right) - \frac{d}{dv} f\left(v\right) \right\|_{\infty} \\ \leq \;& 0. \end{aligned}$$

Therfore

$$\lim_{n\to\infty}\left\|\frac{d}{d\eta}f_n(\eta)-\frac{d}{d\eta}f(\eta)+\frac{d}{dv}f_n(v)-\frac{d}{dv}f(v)\right\|_\infty=0.$$

Then, for $\eta=0$ and $v=\mu$, we have also

$$\lim_{n\to\infty}\left(\frac{d}{d\eta}f_n\right)(0)+\lim_{n\to\infty}\left(\frac{d}{dv}f_n\right)(v)=f'(0)+f'(\mu)=0,\text{ then } f\in E.$$

Consequently, the subspace E is closed in $C[0,\mu]$. Hence $(E,\|.\|_\infty)$ is a Banach space. ■

Lemma 3.3 *Let T be an integral operator defined by*

$$\begin{aligned}Tf(\eta) &= w+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\eta(\eta^\rho-\zeta^\rho)^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta\\ &\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\eta^\mu(\zeta^\rho-\eta^\rho)^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta.\end{aligned}\tag{3.16}$$

provided that the supremum norm is

$$\|Tf\|_\infty=\sup_{0\le\eta\le\mu}|Tf(\eta)|.$$

Then, T maps E into itself $(T:E\to E)$.

Proof. Let $1<\alpha\le 2$ and $f\in E$ satisfies ${}_0^{RC}D_\mu^{\alpha,\rho}f(\eta)=\beta f(\eta)-\frac{1}{\alpha\rho}\eta f'(\eta)$, where E is the Banach space which defined in (3.15). Then, from (3.16), we have

$$\begin{aligned}\frac{d}{d\eta}Tf(\eta) &= \frac{d}{d\eta}\left[w+{}_0^{RG}I_\mu^{\alpha,\rho}\left(\beta f(\eta)-\frac{1}{\alpha\rho}\eta f'(\eta)\right)\right]\\ &= \frac{d}{d\eta}\left[{}_0^{RG}I_\mu^{\alpha,\rho}\left(\beta f(\eta)-\frac{1}{\alpha\rho}\eta f'(\eta)\right)\right]\\ &= {}_0^{RG}I_\mu^{\alpha-1,\rho}\left(\beta f(\eta)-\frac{1}{\alpha\rho}\eta f'(\eta)\right)\\ &= {}_0^{RG}I_\mu^{\alpha-1,\rho}\,{}_0^{RC}D_\mu^{\alpha,\rho}f(\eta).\end{aligned}$$

It follows from (1.38) and (1.39) in remark 1.4 that

$$\frac{d}{d\eta}Tf(\eta)={}_0^{RG}I_\mu^{\alpha-1,\rho}\,{}_0^{RC}D_\mu^{\alpha,\rho}f(\eta)=f'(\eta).$$

Therefore, $\frac{d}{d\eta}Tf(0)+\frac{d}{d\eta}Tf(\mu)=f'(0)+f'(\mu)=0$. Hence, $T(E)\subset E$. The proof is finished. ■

Next, we will deal with the existence and uniqueness of solution for the problem (3.7) − (3.8).

3.4 EXISTENCE AND UNIQUENESS RESULTS OF THE BASIC PROFILE f

Firstly, using Banach's fixed point theorem, we will derive the conditions of the solutions' existence.

Theorem 3.2 *Let $\alpha, \beta, \rho, \mu \in \mathbb{R}$, provided that $1 < \alpha \leq 2$, $\rho > 0$ and $\mu \in \left(0, \left(\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\right)^{\frac{1}{\rho(\alpha-1)+1}}\right)$. If*

$$\frac{2\mu^{\rho\alpha}\left|\beta\right|}{\rho^{\alpha}\Gamma\left(\alpha+1\right)-2\mu^{\rho(\alpha-1)+1}} < 1. \tag{3.17}$$

Then, the problem (3.7) − (3.8) has a unique solution on $[0, \mu]$.

Proof. first, we will transform the problem (3.7) − (3.8) into a fixed point problem. By lemma 3.1, we define the operator $T : E \to E$ as follows

$$\begin{aligned} Tf\left(\eta\right) &= w + \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{0}^{\eta}\left(\eta^{\rho}-\zeta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta f\left(\zeta\right)-\frac{1}{\alpha\rho}\zeta f'\left(\zeta\right)\right)d\zeta \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{\eta}^{\mu}\left(\zeta^{\rho}-\eta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta f\left(\zeta\right)-\frac{1}{\alpha\rho}\zeta f'\left(\zeta\right)\right)d\zeta. \end{aligned} \tag{3.18}$$

Since the problem (3.7) − (3.8) can be written in the form of the fractional integral equation (3.18), the fixed points of T are to be considered a solutions for the problem (3.7) − (3.8).

Let $f, G \in E$, provided that

$$ {}_{0}^{RC}D_{\mu}^{\alpha,\rho}f\left(\eta\right) = \beta f\left(\eta\right) - \frac{1}{\alpha\rho}\eta f'\left(\eta\right),\ {}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\left(\eta\right) = \beta G\left(\eta\right) - \frac{1}{\alpha\rho}\eta G'\left(\eta\right).$$

Then

$$\begin{aligned} Tf\left(\eta\right)-TG\left(\eta\right) &= \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{0}^{\eta}\left(\eta^{\rho}-\zeta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta\left(f\left(\zeta\right)-G\left(\zeta\right)\right)-\frac{\zeta}{\alpha\rho}\left(f'\left(\zeta\right)-G'\left(\zeta\right)\right)\right)d\zeta \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{\eta}^{\mu}\left(\zeta^{\rho}-\eta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta\left(f\left(\zeta\right)-G\left(\zeta\right)\right)-\frac{\zeta}{\alpha\rho}\left(f'\left(\zeta\right)-G'\left(\zeta\right)\right)\right)d\zeta. \end{aligned}$$

Therefore

$$\begin{aligned} \left|Tf\left(\eta\right)-TG\left(\eta\right)\right| &\leq \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{0}^{\eta}\left(\eta^{\rho}-\zeta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left|{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f\left(\zeta\right)-{}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\left(\zeta\right)\right|d\zeta \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{\eta}^{\mu}\left(\zeta^{\rho}-\eta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left|{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f\left(\zeta\right)-{}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\left(\zeta\right)\right|d\zeta. \end{aligned} \tag{3.19}$$

Moreover, for each $\eta \in [0, \mu]$, we have

$$\begin{aligned} \left|{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f\left(\eta\right)-{}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\left(\eta\right)\right| &= \left|\beta\left(f\left(\eta\right)-G\left(\eta\right)\right)-\frac{\eta}{\alpha\rho}\left(f'\left(\eta\right)-G'\left(\eta\right)\right)\right| \\ &\leq \left|\beta\right|\left|f\left(\eta\right)-G\left(\eta\right)\right|+\frac{\mu}{\alpha\rho}\left|f'\left(\eta\right)-G'\left(\eta\right)\right|. \end{aligned}$$

Using (1.40) in remark 1.4, we find

$$\left\|{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f-{}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\right\|_{\infty}\leq|\beta|\,\|f-G\|_{\infty}+\frac{2\mu^{\rho(\alpha-1)+1}}{\rho^{\alpha}\Gamma(\alpha+1)}\left\|{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f-{}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\right\|_{\infty},$$

which yields

$$\left[1-\frac{2\mu^{\rho(\alpha-1)+1}}{\rho^{\alpha}\Gamma(\alpha+1)}\right]\left\|{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f-{}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\right\|_{\infty}\leq|\beta|\,\|f-G\|_{\infty}.$$

As $\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}>0$, we get

$$\left\|{}_{0}^{RC}D_{\mu}^{\alpha,\rho}f-{}_{0}^{RC}D_{\mu}^{\alpha,\rho}G\right\|_{\infty}\leq\frac{|\beta|\,\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\|f-G\|_{\infty}.$$

Thus, (3.19) can be re-written as

$$\begin{aligned}\|Tf-TG\|_{\infty} &\leq \frac{|\beta|\,\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\|f-G\|_{\infty}\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{0}^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}d\zeta\right.\\ &\quad\left.+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}d\zeta\right]\\ &\leq \left(\frac{2\mu^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right)\left(\frac{|\beta|\,\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\right)\|f-G\|_{\infty}\\ &\leq \frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\|f-G\|_{\infty}.\end{aligned}$$

By the condition (3.17), T is a contraction mapping. Using the principle of Banach's fixed point theorem 1.7, we deduce that T admits a unique fixed point which is a unique solution of the problem $(3.7)-(3.8)$ on $[0,\mu]$. The proof is ended. ∎

Secondly, using the fixed point theorem of Schauder, we will derive the conditions of the solutions' existence.

Theorem 3.3 *Let $\rho,\mu>0$, $\beta\in\mathbb{R}$ and $1<\alpha\leq 2$. If*

$$\frac{\mu^{\rho(\alpha-1)+1}+\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1)}<\frac{1}{2}. \tag{3.20}$$

Then, the problem $(3.7)-(3.8)$ has at least one solution on $[0,\mu]$.

Proof. Let the operator T defined in (3.18). We have already transformed the problem $(3.7)-(3.8)$ into a fixed point problem

$$\begin{aligned}Tf(\eta) &= w+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{0}^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta\\ &\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\right)d\zeta.\end{aligned}$$

We shall show that T satisfies the assumption of Schauder's fixed point theorem 1.8. The proof will be given in three claims.

Claim 1: T is continuous operator.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence provided that $\lim_{n\to\infty} f_n = f$ in E. Then for each $\eta \in [0,\mu]$, we have

$$\begin{aligned} |Tf_n(\eta) - Tf(\eta)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \left| \beta (f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho} (f_n'(\zeta) - f'(\zeta)) \right| d\zeta \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \left| \beta (f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho} (f_n'(\zeta) - f'(\zeta)) \right| d\zeta, \end{aligned} \tag{3.25}$$

where

$${}_0^{RC}D_\mu^{\alpha,\rho} f_n(\eta) = \beta f_n(\eta) - \frac{\eta}{\alpha\rho} f_n'(\eta) \text{ and } {}_0^{RC}D_\mu^{\alpha,\rho} f(\eta) = \beta f(\eta) - \frac{\eta}{\alpha\rho} f'(\eta).$$

We have

$$\begin{aligned} \left| {}_0^{RC}D_\mu^{\alpha,\rho} f_n(\eta) - {}_0^{RC}D_\mu^{\alpha,\rho} f(\eta) \right| &= \left| \beta (f_n(\eta) - f(\eta)) - \frac{\eta}{\alpha\rho} (f_n'(\eta) - f'(\eta)) \right| \\ &\leq |\beta| \, |f_n(\eta) - f(\eta)| + \frac{\mu}{\alpha\rho} |f_n'(\eta) - f'(\eta)|. \end{aligned}$$

Using (1.40) in remark 1.4, we find

$$\left\| {}_0^{RC}D_\mu^{\alpha,\rho} f_n - {}_0^{RC}D_\mu^{\alpha,\rho} f \right\|_\infty \leq |\beta| \, \|f_n - f\|_\infty + \frac{2\mu^{\rho(\alpha-1)+1}}{\rho^\alpha \Gamma(\alpha+1)} \left\| {}_0^{RC}D_\mu^{\alpha,\rho} f_n - {}_0^{RC}D_\mu^{\alpha,\rho} f \right\|_\infty,$$

which yields

$$\left[1 - \frac{2\mu^{\rho(\alpha-1)+1}}{\rho^\alpha \Gamma(\alpha+1)}\right] \left\| {}_0^{RC}D_\mu^{\alpha,\rho} f_n - {}_0^{RC}D_\mu^{\alpha,\rho} f \right\|_\infty \leq |\beta| \, \|f_n - f\|_\infty$$

As $\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1} > 2\mu^{\rho\alpha} |\beta| > 0$, we get

$$\left\| {}_0^{RC}D_\mu^{\alpha,\rho} f_n - {}_0^{RC}D_\mu^{\alpha,\rho} f \right\|_\infty \leq \frac{|\beta| \, \rho^\alpha \Gamma(\alpha+1)}{\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} \|f_n - f\|_\infty.$$

Because $f_n \to f$ as $n \to \infty$, then we get ${}_0^{RC}D_\mu^{\alpha,\rho} f_n \to {}_0^{RC}D_\mu^{\alpha,\rho} f$ as $n \to \infty$ for every $\eta \in [0,\mu]$.

Now let $S_0 > 0$, such that for every $\eta \in [0,\mu]$, we have

$$\left| {}_0^{RC}D_\mu^{\alpha,\rho} f_n \right| \leq S_0 \text{ and } \left| {}_0^{RC}D_\mu^{\alpha,\rho} f \right| \leq S_0.$$

Then, we have

$$\begin{aligned}
|Tf_n(\eta) - Tf(\eta)| &\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \left| \beta(f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho}(f_n'(\zeta) - f'(\zeta)) \right| d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \left| \beta(f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho}(f_n'(\zeta) - f'(\zeta)) \right| d\zeta \\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \left| {}_0^{RC}D_\mu^{\alpha,\rho} f_n(\zeta) - {}_0^{RC}D_\mu^{\alpha,\rho} f(\zeta) \right| d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \left| {}_0^{RC}D_\mu^{\alpha,\rho} f_n(\zeta) - {}_0^{RC}D_\mu^{\alpha,\rho} f(\zeta) \right| d\zeta \\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \left[\left| {}_0^{RC}D_\mu^{\alpha,\rho} f_n(\zeta) \right| + \left| {}_0^{RC}D_\mu^{\alpha,\rho} f(\zeta) \right| \right] d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \left[\left| {}_0^{RC}D_\mu^{\alpha,\rho} f_n(\zeta) \right| + \left| {}_0^{RC}D_\mu^{\alpha,\rho} f(\zeta) \right| \right] d\zeta \\
&\le \frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} d\zeta + \frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} d\zeta.
\end{aligned}$$

Since the functions $\zeta \to \frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)} \left[(\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \right]$ and $\frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)} \left[(\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \right]$ are integrable on $[0,\eta]$ and $[\eta,\mu]$ respectively for each $\eta \in [0,\mu]$, then the Lebesgue dominated convergence theorem and (3.21) implies that

$$|Tf_n(\eta) - Tf(\eta)| \to 0 \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} \|Tf_n - Tf\|_\infty = 0.$$

Consequently, T is continuous.

Claim 2: According to (3.20), by putting

$$R \ge \left(1 + \frac{2\mu^{\rho\alpha}|\beta|}{\rho^\alpha \Gamma(\alpha+1) - 2\left[\mu^{\rho(\alpha-1)+1} + \mu^{\rho\alpha}|\beta|\right]} \right) |w|,$$

and by defining a subset

$$E_R = \{ f \in E : \|f\|_\infty \le R,\ R > 0 \}.$$

So, E_R is a closed, bounded and convex subset of E.

Let $f \in E_R$ and T be the integral operator defined in (3.18). Then, we prove that $T(E_R) \subset E_R$. In fact, by (1.40) in remark 1.4, we have

$$\begin{aligned}
\left| {}_0^{RC}D_\mu^{\alpha,\rho} f(\eta) \right| &= \left| \beta f(\eta) - \frac{\eta}{\alpha\rho} f'(\eta) \right| \\
&\le |\beta| |f(\eta)| + \frac{\mu}{\alpha\rho} |f'(\eta)|.
\end{aligned}$$

This implies that

$$\left\| {}_{0}^{RC}D_{\mu}^{\alpha,\rho} f \right\|_{\infty} \leq \frac{|\beta|\,\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} R. \tag{3.22}$$

Therefore

$$\begin{aligned}
|Tf(\eta)| &\leq |w| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} (\eta^{\rho} - \zeta^{\rho})^{\alpha-1} \zeta^{\rho-1} \left| \beta f(\zeta) - \frac{\zeta}{\alpha\rho} f'(\zeta) \right| d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta}^{\mu} (\zeta^{\rho} - \eta^{\rho})^{\alpha-1} \zeta^{\rho-1} \left| \beta f(\zeta) - \frac{\zeta}{\alpha\rho} f'(\zeta) \right| d\zeta \\
&\leq |w| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} (\eta^{\rho} - \zeta^{\rho})^{\alpha-1} \zeta^{\rho-1} \left| {}_{0}^{RC}D_{\mu}^{\alpha,\rho} f(\zeta) \right| d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta}^{\mu} (\zeta^{\rho} - \eta^{\rho})^{\alpha-1} \zeta^{\rho-1} \left| {}_{0}^{RC}D_{\mu}^{\alpha,\rho} f(\zeta) \right| d\zeta \\
&\leq |w| + \frac{|\beta|\,\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} R \left[\frac{2\mu^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \right] \\
&\leq |w| + \frac{2\mu^{\rho\alpha}\,|\beta|\,R}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} \\
&\leq \frac{|w| \left(1 + \frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1) - 2\left[\mu^{\rho(\alpha-1)+1} + \mu^{\rho\alpha}|\beta|\right]} \right)}{\left(1 + \frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1) - 2\left[\mu^{\rho(\alpha-1)+1} + \mu^{\rho\alpha}|\beta|\right]} \right)} + \frac{2\mu^{\rho\alpha}\,|\beta|\,R}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} \\
&\leq \frac{R \left(\rho^{\alpha}\Gamma(\alpha+1) - 2\left[\mu^{\rho(\alpha-1)+1} + \mu^{\rho\alpha}\,|\beta| \right] \right)}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} + \frac{2\mu^{\rho\alpha}\,|\beta|\,R}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} \\
&\leq R.
\end{aligned}$$

Thus $T(E_R) \subset E_R$, hence $T(E_R)$ is bounded.

Claim 3: $T(E_R)$ is relatively compact.

Let $f \in E_R$, $\eta_1, \eta_2 \in [0, \mu]$ with $\eta_1 < \eta_2$, by (3.22), we get

$$
\begin{aligned}
\left|Tf(\eta_1) - Tf(\eta_2)\right| = & \left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta_1}\left(\eta_1^\rho-\zeta^\rho\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right)d\zeta\right. \\
& +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_1}^{\mu}\left(\zeta^\rho-\eta_1^\rho\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right)d\zeta \\
& -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta_2}\left(\eta_2^\rho-\zeta^\rho\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right)d\zeta \\
& \left.-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_2}^{\mu}\left(\zeta^\rho-\eta_2^\rho\right)^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right)d\zeta\right| \\
\leq & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta_1}\left|\left[\zeta^{\rho-1}\left(\eta_1^\rho-\zeta^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\eta_2^\rho-\zeta^\rho\right)^{\alpha-1}\right]\right|\left|{}_0^{RC}D_\mu^{\alpha,\rho}f(\zeta)\right|d\zeta \\
& +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_2}^{\mu}\left|\left[\zeta^{\rho-1}\left(\zeta^\rho-\eta_1^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\zeta^\rho-\eta_2^\rho\right)^{\alpha-1}\right]\right|\left|{}_0^{RC}D_\mu^{\alpha,\rho}f(\zeta)\right|d\zeta \\
& +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}\left(\eta_2^\rho-\zeta^\rho\right)^{\alpha-1}\left|{}_0^{RC}D_\mu^{\alpha,\rho}f(\zeta)\right|d\zeta \\
& +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}\left(\zeta^\rho-\eta_1^\rho\right)^{\alpha-1}\left|{}_0^{RC}D_\mu^{\alpha,\rho}f(\zeta)\right|d\zeta \\
\leq & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\times\frac{|\beta|\rho^\alpha\Gamma(\alpha+1)}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}R\times \\
& \left\{\int_0^{\eta_1}\left|\left[\zeta^{\rho-1}\left(\eta_1^\rho-\zeta^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\eta_2^\rho-\zeta^\rho\right)^{\alpha-1}\right]\right|d\zeta\right. \\
& +\int_{\eta_2}^{\mu}\left|\left[\zeta^{\rho-1}\left(\zeta^\rho-\eta_1^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\zeta^\rho-\eta_2^\rho\right)^{\alpha-1}\right]\right|d\zeta \\
& \left.+\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}\left(\eta_2^\rho-\zeta^\rho\right)^{\alpha-1}d\zeta+\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}\left(\zeta^\rho-\eta_1^\rho\right)^{\alpha-1}d\zeta\right\}. \qquad (3.23)
\end{aligned}
$$

We have

$$
\zeta^{\rho-1}\left(\eta_1^\rho-\zeta^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\eta_2^\rho-\zeta^\rho\right)^{\alpha-1}=-\frac{1}{\alpha\rho}\frac{d}{d\zeta}\left[\left(\eta_1^\rho-\zeta^\rho\right)^\alpha-\left(\eta_2^\rho-\zeta^\rho\right)^\alpha\right]
$$

and

$$
\zeta^{\rho-1}\left(\zeta^\rho-\eta_1^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\zeta^\rho-\eta_2^\rho\right)^{\alpha-1}=\frac{1}{\alpha\rho}\frac{d}{d\zeta}\left[\left(\zeta^\rho-\eta_1^\rho\right)^\alpha-\left(\zeta^\rho-\eta_2^\rho\right)^\alpha\right]
$$

then

$$
\int_0^{\eta_1}\left|\left[\zeta^{\rho-1}\left(\eta_1^\rho-\zeta^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\eta_2^\rho-\zeta^\rho\right)^{\alpha-1}\right]\right|d\zeta\leq\frac{1}{\alpha\rho}\left[\left(\eta_2^\rho-\eta_1^\rho\right)^\alpha+\left(\eta_2^{\rho\alpha}-\eta_1^{\alpha\rho}\right)\right] \qquad (3.24)
$$

and

$$
\begin{aligned}
\int_{\eta_2}^{\mu}\left|\left[\zeta^{\rho-1}\left(\zeta^\rho-\eta_1^\rho\right)^{\alpha-1}-\zeta^{\rho-1}\left(\zeta^\rho-\eta_2^\rho\right)^{\alpha-1}\right]\right|d\zeta \leq & \frac{1}{\alpha\rho}\times \\
& \left[\left(\mu^\rho-\eta_1^\rho\right)^\alpha-\left(\eta_2^\rho-\eta_1^\rho\right)^\alpha-\left(\mu^\rho-\eta_2^\rho\right)^\alpha\right] \qquad (3.25)
\end{aligned}
$$

we have also

$$\begin{aligned}\int_{\eta_1}^{\eta_2} \zeta^{\rho-1}\left(\eta_2^{\rho}-\zeta^{\rho}\right)^{\alpha-1} d\zeta &= -\frac{1}{\alpha\rho}\left[\left(\eta_2^{\rho}-\zeta^{\rho}\right)^{\alpha}\right]_{\eta_1}^{\eta_2} \\ &= \frac{1}{\alpha\rho}\left(\eta_2^{\rho}-\eta_1^{\rho}\right)^{\alpha} \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\int_{\eta_1}^{\eta_2} \zeta^{\rho-1}\left(\zeta^{\rho}-\eta_1^{\rho}\right)^{\alpha-1} d\zeta &= \frac{1}{\alpha\rho}\left[\left(\zeta^{\rho}-\eta_1^{\rho}\right)^{\alpha-1}\right]_{\eta_1}^{\eta_2} \\ &= \frac{1}{\alpha\rho}\left(\eta_2^{\rho}-\eta_1^{\rho}\right)^{\alpha} \end{aligned} \tag{3.27}$$

By substituting (3.24), (3.25), (3.26) and (3.27) in (3.23), we obtain

$$\begin{aligned}\left|Tf(\eta_1)-Tf(\eta_2)\right| \leq\ & \frac{|\beta|\, R}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}} \times \\ & \left[\left(\mu^{\rho}-\eta_1^{\rho}\right)^{\alpha}+2\left(\eta_2^{\rho}-\eta_1^{\rho}\right)^{\alpha}+\left(\eta_2^{\rho\alpha}-\eta_1^{\alpha\rho}\right)-\left(\mu^{\rho}-\eta_2^{\rho}\right)^{\alpha}\right].\end{aligned}$$

So, the right-hand side of the above inequality tends to zero as $\eta_2 \to \eta_1$. Hence, we obtain that $T(E_R)$ is equicontinuous. Therefore, combining claims 1 to 3 and by the means of the Ascoli-Arzela theorem (1.6), we get that $T : E_R \to E_R$ is continuous and relatively compact. As a consequence, Schauder's fixed point theorem assures the existence of at least one fixed point of operator (3.18) which is the solution of the problem $(3.7)-(3.8)$. ■

Finally, using the fixed point theorem of Leray-Schauder type, we will derive the conditions of the solutions' existence.

Theorem 3.4 *Let $\alpha, \beta, \rho, \mu \in \mathbb{R}$, provided that $1 < \alpha \leq 2$, $\rho > 0$ and $\mu \in \left(0, \left(\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\right)^{\frac{1}{\rho(\alpha-1)+1}}\right)$. Then, the problem $(3.7)-(3.8)$ admits at least one solution on $[0,\mu]$.*

Proof. Consider the operator T defined in (3.18). We shall show that all assumption of Leray-Schauder fixed point theorem 1.9 are satisfied by the operator T. The proof will be devided in four claims.

Claim 1: It is clear that T is countinuous.

Claim 2: T maps bounded sets into bounded sets in E

Actually, it is enough to show that for any $\theta > 0$, there exists $N > 0$ such that for each $f \in D_{\theta} = \{f \in E : \|f\|_{\infty} \leq \theta\}$, we have $\|Tf\|_{\infty} \leq N$. Let $f \in D_{\theta}$, for each $\eta \in [0,\mu]$, we have

$$\begin{aligned}\left|Tf(\eta)\right| \leq\ & |w| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}\left(\eta^{\rho}-\zeta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left|\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right| d\zeta \\ & +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}\left(\zeta^{\rho}-\eta^{\rho}\right)^{\alpha-1}\zeta^{\rho-1}\left|\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right| d\zeta \end{aligned} \tag{3.28}$$

As a similar way as in (3.22) , we have

$$\left|\beta f(\eta)-\frac{\eta}{\alpha\rho}f'(\eta)\right|\leq\frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\theta.$$

Therefore, (3.28) implies that

$$\|Tf\|_{\infty}\leq|w|+\frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\theta=N.$$

Claim 3: It is clear that T maps bounded sets into equicontinuous sets of E. From Claim1-Claim3, we conclude that $T:E\to E$ is continuous and completely continuous.

Claim 4: A priori bounds

Now, we shaw that there exists an open set $H\subset E$ with $f\neq\lambda T(f)$ for some $\lambda\in(0,1)$ and $f\in\partial H$.

Let $f\in E$ and $f=\lambda T(f)$ for $0<\lambda<1$. Then, we have for each $\eta\in[0,\mu]$

$$\begin{aligned}|f(\eta)| &= |\lambda Tf(\eta)|=\left|\lambda w+\frac{\lambda\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu}|(\eta^{\rho}-\zeta^{\rho})|^{\alpha-1}\zeta^{\rho-1}\left(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right)d\zeta\right|\\ &\leq |w|+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left|\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right|d\zeta\\ &\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left|\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right|d\zeta\\ &\leq |w|+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left|{}_0^{RC}D_{\mu}^{\alpha,\rho}f(\zeta)\right|d\zeta\\ &\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}\left|{}_0^{RC}D_{\mu}^{\alpha,\rho}f(\zeta)\right|d\zeta. \end{aligned}\tag{3.29}$$

We have

$$\begin{aligned}\left|{}_0^{RC}D_{\mu}^{\alpha,\rho}f(\eta)\right| &= \left|\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\right|\leq|\beta||f(\eta)|+\frac{\mu}{\alpha\rho}|f'(\eta)|\\ &\leq |\beta||f(\eta)|+\frac{2\mu^{\rho(\alpha-1)+1}}{\rho^{\alpha}\Gamma(\alpha+1)}\sup_{0\leq\eta\leq\mu}\left|{}_0^{RC}D_{\mu}^{\alpha,\rho}f(\eta)\right|,\end{aligned}$$

which implies that

$$\sup_{0\leq\eta\leq\mu}\left|{}_0^{RC}D_{\mu}^{\alpha,\rho}f(\eta)\right|\leq\frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\sup_{0\leq\eta\leq\mu}|f(\eta)|.$$

Thus, (3.29) gives

$$\begin{aligned}\sup_{0\leq\eta\leq\mu}|f(\eta)| &\leq |w|+\frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha)\left[\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}\right]}\times\\ &\quad\left[\rho^{1-\alpha}\int_0^{\eta}\zeta^{\rho-1}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\left\{\sup_{0\leq\eta\leq\mu}|f(\zeta)|\right\}d\zeta\right.\\ &\quad\left.+\rho^{1-\alpha}\int_{\eta}^{\mu}\zeta^{\rho-1}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\left\{\sup_{0\leq\eta\leq\mu}|f(\zeta)|\right\}d\zeta.\right]\end{aligned}\tag{3.30}$$

By using the generalized Gronwall lemma 1.7, (3.30) can be re-written as

$$\begin{aligned}\sup_{0\le\eta\le\mu}|f(\eta)| &\le |w|\,E_{\alpha,1}\left(\rho^{-\alpha}\Gamma(\alpha)(\mu^{\rho}-\eta^{\rho})^{\alpha}\left[\frac{|\beta|\,\rho^{\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha)\left[\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}\right]}\right]\right)\\ &\quad\times E_{\alpha,1}\left(\rho^{-\alpha}\Gamma(\alpha)\,\eta^{\rho\alpha}\left[\frac{|\beta|\,\rho^{\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha)\left[\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}\right]}\right]\right)\end{aligned}$$

which can be simplified to

$$\|f\|_{\infty}\le|w|\,E_{\alpha,1}\left(\frac{(\mu^{\rho}-\eta^{\rho})^{\alpha}|\beta|\,\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\right)E_{\alpha,1}\left(\frac{\eta^{\rho\alpha}|\beta|\,\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\right)=N_1.$$

Let

$$H=\{f\in E:\|f\|_{\infty}<N_1+1\}.$$

By choosing of H, there is no $f\in\partial H$, such that $f=\lambda T(f)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder's fixed point theorem 1.9, the operator T has a fixed point f in H, which is a solution to the problem $(3.7)-(3.8)$ on $[0,\mu]$. The proof is ended.

■

Now, we prove the principle theorems

3.5 EXISTENCE AND UNIQUENESS RESULTS TO THE ORIGINAL PROBLEM

In this subsection, we demonstrate the existence and uniqueness of solutions of the following space-fractional diffusion equation

$$\begin{cases}\frac{\partial u(x,t)}{\partial t}=\frac{\partial^{\alpha,\rho}u(x,t)}{\partial|x|^{\alpha}}, & (x,t)\in[0,X]\times[t_0,\infty[,\ 1<\alpha\le 2\\ u(0,t)+u(X,t)=t^{\beta}M, & \frac{\partial u(0,t)}{\partial x}+\frac{\partial u(X,t)}{\partial x}=0,\ \beta,M\in\mathbb{R}.\end{cases}\tag{3.31}$$

Under the self-similar form, which is

$$u(x,t)=t^{\beta}f\left(\frac{x}{t^{\frac{1}{\alpha\rho}}}\right),\quad (x,t)\in[0,X]\times[t_0,\infty[.\tag{3.32}$$

Theorem 3.5 *Let $\alpha,\beta,\rho,X,t_0\in\mathbb{R}$, provided that $1<\alpha\le 2$, $\rho,t_0>0$ and $X\in\left(0,\left(\frac{t_0^{\frac{\rho(\alpha-1)+1}{\rho\alpha}}\rho^{\alpha}\Gamma(\alpha+1)}{2}\right)^{\frac{1}{\rho(\alpha-1)+1}}\right)$. If*

$$\frac{2X^{\rho\alpha}|\beta|}{t_0\rho^{\alpha}\Gamma(\alpha+1)-2X^{\rho(\alpha-1)+1}t_0^{\frac{\rho-1}{\rho\alpha}}}<1.\tag{3.33}$$

Then, for $f\in E$, the problem (3.31) has a unique solution in the self-similar form (3.32).

Proof. The transformation (3.32) reduces the space-fractional diffusion equation (3.31) to the fractional differential equtiom of the following form

$$ {}_{0}^{RC}D_{\mu}^{\alpha,\rho} f(\eta) = \beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta), \tag{3.34}$$

where

$$\mu = X t_0^{\frac{-1}{\alpha\rho}}, \text{ with } X \in (0, \left(\frac{t_0^{\frac{\rho(\alpha-1)+1}{\rho\alpha}} \rho^{\alpha}\Gamma(\alpha+1)}{2}\right)^{\frac{1}{\rho(\alpha-1)+1}}), t_0, \rho > 0, 1 < \alpha \leq 2.$$

with the conditions

$$f(0) + f(\mu) = M, \ f'(0) + f'(\mu) = 0. \tag{3.35}$$

Let $f \in E$ be a continuous function. By using (3.32), the condition (3.33), is equivalent to (3.17), which is

$$\frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} < 1. \tag{3.36}$$

We already proved in theorem 3.2, the existence and uniqueness of a solution of the problem (3.34) – (3.35) such that the condition (3.36) is satisfied. As a consequence, there exists a unique solution of the problem (3.31)under the self-similar form (3.32) provided that the condition (3.33) holds. The proof is ended. ∎

Remark 3.1 *When $\rho \to 1$, (3.33) reduces to*

$$\frac{2X^{\alpha}|\beta|}{t_0\Gamma(\alpha+1) - 2X^{\alpha}} < 1, \tag{3.37}$$

which represent the standard Riesz-Caputo derivative case.

When $\alpha = 2$, (3.37) reduces to

$$\frac{2X^{2}|\beta|}{t_0\Gamma(3) - 2X^{2}} < 1, \tag{3.38}$$

which gives the integer order derivative case of the space- fractional diffusion equation in (3.1).

Theorem 3.6 *Let $\alpha, \beta, \rho, X, t_0 \in \mathbb{R}$, provided that $1 < \alpha \leq 2$ and $\rho, t_0, X > 0$. If*

$$\frac{X^{\rho(\alpha-1)+1} t_0^{\frac{\rho-1}{\rho\alpha}} + X^{\rho\alpha}|\beta|}{t_0\rho^{\alpha}\Gamma(\alpha+1)} < \frac{1}{2}. \tag{3.39}$$

Then, for $f \in E_R$, the problem (3.31) has at least one solution in the self-similar form (3.32).

Proof. By considering theorem 3.3, and using the same steps followed in the proof of theorem 3.5, we can prove that the problem (3.31) has at least one solution in the self-similar form (3.32), if the condition (3.39) is satisfied. The proof is ended. ∎

Remark 3.2 *When $\rho \to 1$, (3.39) reduces to*

$$\frac{X^{\alpha}(1+|\beta|)}{t_0\Gamma(\alpha+1)} < \frac{1}{2}, \tag{3.40}$$

which represent the standard Riesz-Caputo derivative case.

When $\alpha = 2$, (3.40) reduces to

$$\frac{X^{2}(1+|\beta|)}{t_0\Gamma(3)} < \frac{1}{2}, \tag{3.41}$$

which gives the integer order derivative case of the space- fractional diffusion equation in (3.1).

Theorem 3.7 *Let $\alpha, \beta, \rho, X, t_0 \in \mathbb{R}$, provided that $1 < \alpha \leq 2$, $\rho, t_0 > 0$ and $X \in (0, \left(\frac{t_0^{\frac{\rho(\alpha-1)+1}{\rho\alpha}} \rho^{\alpha}\Gamma(\alpha+1)}{2}\right)^{\frac{1}{\rho(\alpha-1)+1}})$. Then, for $f \in E$, the problem (3.31) has at least one solution in the self-similar form (3.32).*

Proof. Based on theorem 3.4, and using the same steps followed in the proof of theorem 3.5, we can prove the existence of at least one solution of the problem (3.31) in the self-similar form (3.32). The proof is ended. ∎

4

SELF-SIMILAR SOLUTIONS OF TIME-SPACE FRACTIONAL DIFFUSION EQUATION INVOLVING HILFER-KATUGAMPOLA DERIVATIVE

In this chapter, we have determined a symmetry group of scaling transformations for a time and space FPDE of orders α and β respectively. The time and space FDs which will be used is the Hilfer-Katugampola's FD. Based on the similarity variable η, we have introduced the equation satisfied by the self-similar solutions for the aforementioned problem. We have applied the successive approximation method to obtain the self-similar solutions in an explicit form. The obtained self-similar solutions of this equation are given in terms of the generalized Wright function.

We have also provided the self-similar solutions to the following time-space FPDE (known as time-space fractional diffusion equation)

$$^{\rho}D^{\alpha,\gamma}_{0,t}u(x,t) = {}^{\rho}D^{\beta,\gamma}_{0,x}u(x,t)\ ,$$

where $u(x,t)$ is a scalar function of space and time variables $x, t > 0,\ \rho > 0,\ 0 \leq \gamma \leq 1$, D is the HKFD of order $0 < \alpha \leq 1$ and $1 < \beta \leq 2$ respectively.

4.1 DETERMINATION OF A SYMMETRY GROUP OF SCALING TRANSFORMATION

Consider the time-space FPDE

$$^{\rho}D^{\alpha,\gamma}_{0,t}u(x,t) = {}^{\rho}D^{\beta,\gamma}_{0,x}u(x,t)\ ,\ x > 0,\ t > 0, \tag{4.1}$$

with the following conditions

$$^{\rho}I^{2-m}_{0,x}u(0^+,t) = t^{\mu+\frac{\alpha\rho}{\beta}(2-m)}B,\ \delta_x\, {}^{\rho}I^{2-m}_{0,x}u(0^+,t) = t^{\mu+\frac{\alpha\rho}{\beta}(1-m)}A, \tag{4.2}$$

where $0 < \alpha \leq 1,\ 0 \leq \gamma \leq 1,\ 1 < \beta \leq 2,\ \rho > 0,\ m = \beta + \gamma(2-\beta),\ A$ and B are constants and $\delta_x = \left(x^{1-\rho}\frac{\partial}{\partial x}\right)$.

Firstly, we determine a symmetry group of scaling transformation for a time-space FPDE (4.1) by using the similarity method, we introduce new independent and dependent variables.

$$\bar{x} = \lambda x,\ \bar{t} = \lambda^b t,\ \bar{u} = \lambda^c u.$$

Then, by property 1, the time fractional derivative of order $0 < \alpha \leq 1$ becomes ($\gamma_1 = \gamma(1-\alpha)$, $\sigma_1 = \alpha + \gamma - \alpha\gamma$)

$$\begin{aligned}
& {}^{\rho}D_{0,t}^{\alpha,\gamma}\bar{u}(\bar{x},\bar{t}) \\
= \;& {}^{\rho}I_{0,t}^{\gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}I_{0,t}^{(1-\gamma)(1-\alpha)}\bar{u}(\bar{x},\bar{t}) \\
= \;& {}^{\rho}I_{0,t}^{\gamma_1}\,{}^{\rho}D_{0,t}^{\sigma_1}\bar{u}(\bar{x},\bar{t}) \\
= \;& {}^{\rho}I_{0,t}^{\gamma_1}\left(\frac{\rho^{\sigma_1}}{\Gamma(1-\sigma_1)}\left(t^{1-\rho}\frac{\partial}{\partial t}\right)\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{-\sigma_1}\bar{u}(\bar{x},\lambda^b s)ds\right) \\
= \;& {}^{\rho}I_{0,t}^{\gamma_1}\left(\frac{\lambda^c\rho^{\sigma_1}}{\Gamma(1-\sigma_1)}\left(\left(\lambda^{-b}\bar{t}\right)^{1-\rho}\frac{\partial}{\partial(\lambda^{-b}\bar{t})}\right)\int_0^{\lambda^{-b}\bar{t}} s^{\rho-1}\left((\lambda^{-b}\bar{t})^\rho - s^\rho\right)^{-\sigma_1}u(\bar{x},\lambda^b s)ds\right) \\
= \;& {}^{\rho}I_{0,t}^{\gamma_1}\,(\frac{\lambda^{c+b\rho+b\rho\sigma_1}\rho^{\sigma_1}}{\Gamma(1-\sigma_1)}\left(\bar{t}^{1-\rho}\frac{\partial}{\partial\bar{t}}\right)\int_0^{\lambda^{-b}\bar{t}} s^{\rho-1}\left(\bar{t}^\rho - (\lambda^b s)^\rho\right)^{-\sigma_1}u(\bar{x},\lambda^b s)ds),
\end{aligned}$$

put $\tau = \lambda^b s$, we get

$$\begin{aligned}
& {}^{\rho}D_{0,t}^{\alpha,\gamma}\bar{u}(\bar{x},\bar{t}) \\
= \;& {}^{\rho}I_{0,t}^{\gamma_1}\left(\frac{\lambda^{c+b\rho\sigma_1}\rho^{\sigma_1}}{\Gamma(1-\sigma_1)}\left(\bar{t}^{1-\rho}\frac{\partial}{\partial\bar{t}}\right)\int_0^{\bar{t}} \tau^{\rho-1}\left(\bar{t}^\rho - \tau^\rho\right)^{-\sigma_1}u(\bar{x},\tau)d\tau\right) \\
= \;& \frac{\lambda^{c+b\rho\sigma_1}\rho^{1-\gamma_1}}{\Gamma(\gamma_1)}\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\gamma_1-1}\,{}^{\rho}D_{0,\bar{t}}^{\sigma_1}u(\bar{x},\lambda^b s)\,ds \\
= \;& \frac{\lambda^{c+b\rho\sigma_1-b\rho\gamma_1+b\rho}\rho^{1-\gamma_1}}{\Gamma(\gamma_1)}\int_0^{\lambda^{-b}\bar{t}} s^{\rho-1}\left(\bar{t}^\rho - (\lambda^b s)^\rho\right)^{\gamma_1-1}\,{}^{\rho}D_{0,\bar{t}}^{\sigma_1}u(\bar{x},\lambda^b s)\,ds \\
= \;& \frac{\lambda^{c+b\rho\alpha}\rho^{1-\gamma_1}}{\Gamma(\gamma_1)}\int_0^{\bar{t}} \tau^{\rho-1}\left(\bar{t}^\rho - \tau^\rho\right)^{\gamma_1-1}\,{}^{\rho}D_{0,\bar{t}}^{\sigma_1}u(\bar{x},\tau)\,d\tau \\
= \;& \lambda^{c+b\rho\alpha}\;{}^{\rho}I_{0,\bar{t}}^{\gamma_1}\,{}^{\rho}D_{0,\bar{t}}^{\sigma_1}u(\bar{x},\bar{t}) \\
= \;& \lambda^{c+b\rho\alpha}\,{}^{\rho}D_{0,\bar{t}}^{\alpha,\gamma}u(\bar{x},\bar{t})\,.
\end{aligned} \tag{4.3}$$

We can follow the same steps for the space-FD of order $1 < \beta \leq 2$, we obtain

$${}^{\rho}D_{0,x}^{\beta,\gamma}\bar{u}(\bar{x},\bar{t}) = \lambda^{c+\beta\rho}\,{}^{\rho}D_{0,\bar{x}}^{\beta,\gamma}u(\bar{x},\bar{t})\,. \tag{4.4}$$

By substituting (4.3) and (4.4) in (4.1), we get

$${}^{\rho}D_{0,t}^{\alpha,\gamma}\bar{u}(\bar{x},\bar{t}) - {}^{\rho}D_{0,x}^{\beta,\gamma}\bar{u}(\bar{x},\bar{t}) = \lambda^{c+b\rho\alpha}\,{}^{\rho}D_{0,\bar{t}}^{\alpha,\gamma}u(\bar{x},\bar{t}) - \lambda^{c+\beta\rho}\,{}^{\rho}D_{0,\bar{x}}^{\beta,\gamma}u(\bar{x},\bar{t}) = 0,$$

if $b = \frac{\beta}{\alpha}$, c is an arbitrary constant. Using this last relation, then, we choose the following invariant of scaling transformation

$$u(x,t) = t^{\mu} f(\eta),\ \eta = xt^{\frac{-\alpha}{\beta}},\ \mu > 0,$$

to determine the self-similar solutions of the FPDE (4.1).

4.2 STATEMENT OF THE PROBLEM

As a similar way in chapter 3, we first need to find the equivalent approximate to the following problem of the space-time fractional diffusion equation

$$\begin{cases} {}^{\rho}D_{0,t}^{\alpha,\gamma} u(x,t) = {}^{\rho}D_{0,x}^{\beta,\gamma} u(x,t)\ ,\ x,t > 0, 0 < \alpha \leq 1,\ 0 \leq \gamma \leq 1,\ 1 < \beta \leq 2 \\ {}^{\rho}I_{0,x}^{2-m} u(0^{+},t) = t^{\mu + \frac{\alpha\rho}{\beta}(2-m)} B,\ m = \beta + \gamma\left(2 - \beta\right),\ B \in \mathbb{R} \\ \delta_{x}\, {}^{\rho}I_{0,x}^{2-m} u(0^{+},t) \ = t^{\mu + \frac{\alpha\rho}{\beta}(1-m)} A,\ A \in \mathbb{R}. \end{cases}$$

Under the following self similar form

$$u(x,t) = t^{\mu} f\left(\eta\right), \text{ with } \eta = xt^{\frac{-\alpha}{\beta}}. \tag{4.5}$$

We should first deduce the equation satisfied by the function f in the following theorem

Theorem 4.1 *Let $\alpha, \beta, \rho \in \mathbb{R}$ provided that $0 < \alpha \leq 1,\ 1 < \beta \leq 2,\ 0 \leq \gamma \leq 1$ and $\rho > 0$. Then the transformation*

$$u(x,t) = t^{\mu} f\left(\eta\right), \text{ with } \eta = xt^{\frac{-\alpha}{\beta}}$$

reduces the FPDE (4.1) to the FDE of the form

$${}^{\rho}D_{0,\eta}^{\beta,\gamma} f(\eta) = \rho^{\alpha} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} f(\eta), \tag{4.6}$$

with boundary conditions

$${}^{\rho}I_{0^{+}}^{2-m} f(0^{+}) = B,\ \delta_{\eta}\, {}^{\rho}I_{0^{+}}^{2-m} f(0^{+}) \ = A. \tag{4.7}$$

where $\gamma_1 = \gamma(1-\alpha)$, $\sigma_1 = \alpha + \gamma - \alpha\gamma$, $\gamma_3 = 1 - \sigma_1$, and $\delta_{\eta} = \left(\eta^{1-\rho} \frac{d}{d\eta}\right)$.

Proof. Firstly, we calculate the time FD of order $0 < \alpha \leq 1$ in terms of $f(\eta)$ by using the transformation (4.5), and the definition of HKFD in (1.41), for $n = 1$, $(0 < \alpha \leq 1)$, we obtain

$$\begin{aligned} {}^{\rho}D_{0,t}^{\alpha,\gamma} u(x,t) &= {}^{\rho}D_{0,t}^{\alpha,\gamma}\left(t^{\mu} f(xt^{\frac{-\alpha}{\beta}})\right) \\ &= {}^{\rho}I_{0,t}^{\gamma(1-\alpha)}\left(t^{1-\rho}\frac{\partial}{\partial t}\right){}^{\rho}I_{0,t}^{(1-\gamma)(1-\alpha)} t^{\mu} f(xt^{\frac{-\alpha}{\beta}}) \\ &= {}^{\rho}I_{0,t}^{\gamma_1}\left(t^{1-\rho}\frac{\partial}{\partial t}\right){}^{\rho}I_{0,t}^{1-\sigma_1} t^{\mu} f(xt^{\frac{-\alpha}{\beta}}) \\ &= {}^{\rho}I_{0,t}^{\gamma_1}\left(t^{1-\rho}\frac{\partial}{\partial t}\right){}^{\rho}I_{0,t}^{\gamma_3}\, t^{\mu} f(xt^{\frac{-\alpha}{\beta}}), \end{aligned} \tag{4.8}$$

we have

$${}^{\rho}I_{0,t}^{\gamma_3}\, t^{\mu} f(xt^{\frac{-\alpha}{\beta}}) = \frac{\rho^{1-\gamma_3}}{\Gamma(\gamma_3)}\int_0^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\gamma_3-1} s^{\mu} f(xs^{\frac{-\alpha}{\beta}})ds,$$

by substituting $s = t\left(\frac{\eta}{\tau}\right)^{\frac{\beta}{\alpha}}$, we get

$$\begin{aligned} {}^{\rho}I_{0,t}^{\gamma_3}\, t^{\mu} f(xt^{\frac{-\alpha}{\beta}}) &= \frac{\beta\rho^{1-\gamma_3}}{\alpha\Gamma(\gamma_3)}\int_{\eta}^{+\infty}(t\eta^{\frac{\beta}{\alpha}}\tau^{\frac{-\beta}{\alpha}})^{\rho-1+\mu}\left(t^{\rho}-(t\eta^{\frac{\beta}{\alpha}}\tau^{\frac{-\beta}{\alpha}})^{\rho}\right)^{\gamma_3-1} \\ &\quad \times(t\eta^{\frac{\beta}{\alpha}}\tau^{\frac{-\beta}{\alpha}})^{\frac{\alpha}{\beta}+1}\eta^{-1}t^{\frac{-\alpha}{\beta}} f(\tau)d\tau \\ &= \frac{\beta\rho^{1-\gamma_3}t^{\mu+\rho\gamma_3}\eta^{\frac{\beta\rho}{\alpha}\left(\frac{\mu}{\rho}+1\right)}}{\alpha\Gamma(\gamma_3)}\int_{\eta}^{+\infty}\left(\tau^{\frac{\beta\rho}{\alpha}}-\eta^{\frac{\beta\rho}{\alpha}}\right)^{\gamma_3-1}\tau^{\frac{-\beta\rho}{\alpha}\left(\gamma_3+\frac{\mu}{\rho}\right)-1} f(\tau)d\tau \\ &= \rho^{-\gamma_3}t^{\mu+\rho\gamma_3}\left(J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+1,\gamma_3} f\right)(\eta), \end{aligned} \tag{4.9}$$

where $J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+1,\gamma_3}$ is left-sided Erdélyi-Kober FI of order γ_1 defined by (1.20). Then, taking the derivative of (4.9), we arrive to the following

$$\left(t^{1-\rho}\frac{\partial}{\partial t}\right)\left(\rho^{-\gamma_3}t^{\mu+\rho\gamma_3}J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+1,\gamma_3} f(xt^{\frac{-\alpha}{\beta}})\right) = \rho^{-\gamma_3}t^{1-\rho}\frac{\partial}{\partial t}\left(t^{\mu+\rho\gamma_3}J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+1,\gamma_3} f(xt^{\frac{-\alpha}{\beta}})\right),$$

in view of relation (see [44])$\left(\eta = xt^{\frac{-\alpha}{\beta}},\ \Phi \in C^1(0,\infty)\right)$

$$t\frac{\partial}{\partial t}\Phi(\eta) = tx\left(\frac{-\alpha}{\beta}\right)t^{\frac{-\alpha}{\beta}-1}\Phi'(\eta) = \frac{-\alpha}{\beta}\eta\frac{d}{d\eta}\Phi(\eta),$$

we arrive at

$$\begin{aligned} &\rho^{-\gamma_3}t^{1-\rho}\frac{\partial}{\partial t}\left(t^{\mu+\rho\gamma_3}\left(J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+1,\gamma_3} f\right)(xt^{\frac{-\alpha}{\beta}})\right) \\ &= \rho^{-\gamma_3}t^{1-\rho}\left[t^{\mu+\rho\gamma_3-1}\left(\mu+\rho\gamma_3-\frac{\alpha}{\beta}xt^{\frac{-\alpha}{\beta}}\frac{d}{d\eta}\right)\left(J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+1,\gamma_3} f\right)(\eta)\right] \\ &= \rho^{1-\gamma_3}t^{\mu+\rho[\gamma_3-1]}\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta\rho}\eta\frac{d}{d\eta}\right)\left(J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+1,\gamma_3} f\right)(\eta) \\ &= \rho^{1-\gamma_3}t^{\mu+\rho[\gamma_3-1]}\left(P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} f\right)(\eta), \end{aligned}$$

where $P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}$ is the left-sided Erdélyi-Kober FD of order σ_1, which is defined by equation (1.21).

Using the above result and the same substitution as above $s = t\left(\frac{\eta}{\tau}\right)^{\frac{\beta}{\alpha}}$, then the expression (4.8) can be rewritten as

$$\begin{aligned}
& {}^{\rho}I_{0,t}^{\gamma_1}\left(t^{1-\rho}\frac{\partial}{\partial t}\right)\ {}^{\rho}I_{0,t}^{\gamma_3}\ t^{\mu}f(xt^{\frac{-\alpha}{\beta}}) \\
= & \ {}^{\rho}I_{0,t}^{\gamma_1}\left(\rho^{1-\gamma_3}t^{\mu+\rho[\gamma_3-1]}\left(P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f\right)(xt^{\frac{-\alpha}{\beta}})\right) \\
= & \ \frac{\rho^{2-\gamma_1-\gamma_3}}{\Gamma(\gamma_1)}\int_0^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\gamma_1-1}s^{\mu+\rho[\gamma_3-1]}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(xs^{\frac{-\alpha}{\beta}})ds \\
= & \ \frac{\beta\rho^{2-\gamma_1-\gamma_3}t^{\rho\left(\gamma_1+\gamma_3+\frac{\mu}{\rho}-1\right)}\eta^{\frac{\beta\rho}{\alpha}\left(\gamma_3+\frac{\mu}{\rho}\right)}}{\alpha\Gamma(\gamma_1)}\int_{\eta}^{+\infty}\left(\tau^{\frac{\beta\rho}{\alpha}}-\eta^{\frac{\beta\rho}{\alpha}}\right)^{\gamma_1-1} \\
& \times\tau^{\frac{-\beta\rho}{\alpha}\left(\gamma_1+\gamma_3+\frac{\mu}{\rho}-1\right)-1}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(\tau)d\tau \\
= & \ \rho^{1-\gamma_1-\gamma_3}t^{\rho\left(\gamma_1+\gamma_3+\frac{\mu}{\rho}-1\right)}\times\left[\frac{\beta\rho\eta^{\frac{\beta\rho}{\alpha}\left(\gamma_3+\frac{\mu}{\rho}\right)}}{\alpha\Gamma(\gamma_1)}\right. \\
& \left.\int_{\eta}^{+\infty}\left(\tau^{\frac{\beta\rho}{\alpha}}-\eta^{\frac{\beta\rho}{\alpha}}\right)^{\gamma_1-1}\tau^{\frac{-\beta\rho}{\alpha}\left(\gamma_1+\gamma_3+\frac{\mu}{\rho}-1\right)-1}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(\tau)d\tau\right] \\
= & \ \rho^{1-\gamma_1-\gamma_3}t^{\rho\left(\gamma_1+\gamma_3+\frac{\mu}{\rho}-1\right)}J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(\eta),
\end{aligned}$$

the power $\rho\left(\gamma_1+\gamma_3+\frac{\mu}{\rho}-1\right) = \mu-\alpha\rho$, and hence the time FD is given by

$$ {}^{\rho}D_{0,t}^{\alpha,\gamma}u(x,t) = \rho^{1-\gamma_1-\gamma_3}t^{\mu-\alpha\rho}J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(\eta). \tag{4.10}$$

Secondly, we calculate the space-FD of order $1 < \beta \leq 2$ in terms of $f(\eta)$

$$\begin{aligned}
{}^{\rho}D_{0,x}^{\beta,\gamma}u(x,t) &= {}^{\rho}D_{0,x}^{\beta,\gamma}\left(t^{\mu}f(xt^{\frac{-\alpha}{\beta}})\right) \\
&= {}^{\rho}I_{0,x}^{\gamma(2-\beta)}\left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2\ {}^{\rho}I_{0,x}^{(1-\gamma)(2-\beta)}t^{\mu}f(xt^{\frac{-\alpha}{\beta}}) \\
&= t^{\mu}\ {}^{\rho}I_{0,x}^{\gamma_2}\left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2\ {}^{\rho}I_{0,x}^{\gamma_4}\ f(xt^{\frac{-\alpha}{\beta}}),
\end{aligned} \tag{4.11}$$

where $\gamma_2 = \gamma(2-\beta)$ and $\gamma_4 = (1-\gamma)(2-\beta)$.

we have

$$ {}^{\rho}I_{0,x}^{\gamma_4}\ f(xt^{\frac{-\alpha}{\beta}}) = \frac{\rho^{1-\gamma_4}}{\Gamma(\gamma_4)}\int_0^x s^{\rho-1}\left(x^{\rho}-s^{\rho}\right)^{\gamma_4-1}f(st^{\frac{-\alpha}{\beta}})ds, \tag{4.12}$$

using the substitution $\xi = st^{\frac{-\alpha}{\beta}}$, then equation (4.12) can be reduced as follows

$$
\begin{aligned}
{}^{\rho}I_{0,x}^{\gamma_4} f(xt^{\frac{-\alpha}{\beta}}) &= \frac{\rho^{1-\gamma_4}}{\Gamma(\gamma_4)} \int_0^{xt^{\frac{-\alpha}{\beta}}} \left(\xi t^{\frac{\alpha}{\beta}}\right)^{\rho-1} \left(x^{\rho} - \left(\xi t^{\frac{\alpha}{\beta}}\right)^{\rho}\right)^{\gamma_4-1} t^{\frac{\alpha}{\beta}} f(\xi) d\xi \\
&= \frac{\rho^{1-\gamma_4} t^{\frac{\alpha\rho\gamma_4}{\beta}}}{\Gamma(\gamma_4)} \int_0^{\eta} \xi^{\rho-1} \left(\eta^{\rho} - \xi^{\rho}\right)^{\gamma_4-1} f(\xi) d\xi \\
&= t^{\frac{\alpha\rho\gamma_4}{\beta}} {}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta), \qquad (4.13)
\end{aligned}
$$

computing the second derivative of the above integral (4.13), we get

$$
\begin{aligned}
\left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2 {}^{\rho}I_{0,x}^{\gamma_4} f(xt^{\frac{-\alpha}{\beta}}) &= \left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2 \left(t^{\frac{\alpha\rho\gamma_4}{\beta}} {}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right) \\
&= t^{\frac{\alpha\rho\gamma_4}{\beta}} \left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2 \left({}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right),
\end{aligned}
$$

since $\frac{\partial f(\eta)}{\partial x} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$, then

$$
\begin{aligned}
\left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2 \left({}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right) &= \left(x^{1-\rho}\frac{\partial}{\partial x}\right)\left(x^{1-\rho}\frac{\partial}{\partial x}\right)\left({}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right) \\
&= \left(x^{1-\rho}\frac{\partial}{\partial x}\right)\left[\left(\eta t^{\frac{\alpha}{\beta}}\right)^{1-\rho}\left(t^{\frac{-\alpha}{\beta}}\frac{d}{d\eta} {}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right)\right] \\
&= \left(x^{1-\rho}\frac{\partial}{\partial x}\right)\left[t^{\frac{-\alpha\rho}{\beta}}\left(\eta^{1-\rho}\frac{d}{d\eta}\right) {}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right] \\
&= t^{\frac{-\alpha\rho}{\beta}}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left[\left(x^{1-\rho}\frac{\partial}{\partial x}\right)\left({}^{\rho}I_{0,\eta}^{\gamma_4} f(xt^{\frac{-\alpha}{\beta}})\right)\right] \\
&= t^{\frac{-\alpha\rho}{\beta}}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)\left[\left(\eta t^{\frac{\alpha}{\beta}}\right)^{1-\rho}\left(t^{\frac{-\alpha}{\beta}}\frac{d}{d\eta} {}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right)\right] \\
&= t^{\frac{-2\alpha\rho}{\beta}}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)^2\left({}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right), \qquad (4.14)
\end{aligned}
$$

then equation (4.14) can be simplified to

$$
\begin{aligned}
\left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2 {}^{\rho}I_{0,x}^{\gamma_4} f(xt^{\frac{-\alpha}{\beta}}) &= t^{\frac{\alpha\rho\gamma_4}{\beta}}\left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2\left({}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right) \\
&= t^{\frac{\alpha\rho}{\beta}[\gamma_4-2]}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)^2\left({}^{\rho}I_{0,\eta}^{\gamma_4} f(\eta)\right).
\end{aligned}
$$

Now, we do the same for the first integral of (4.12), with $\xi = st^{\frac{-\alpha}{\beta}}$, we obtain

$$\begin{aligned}
& {}^{\rho}I_{0,x}^{\gamma_2}\left(x^{1-\rho}\frac{\partial}{\partial x}\right)^2 {}^{\rho}I_{0,x}^{\gamma_4}\, f(xt^{\frac{-\alpha}{\beta}}) \\
= & \ {}^{\rho}I_{0,x}^{\gamma_2}\left(t^{\frac{\alpha\rho}{\beta}[\gamma_4-2]}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)^2\left({}^{\rho}I_{0,\eta}^{\gamma_4}\, f(\eta)\right)\right) \\
= & \ \frac{\rho^{1-\gamma_2}t^{\frac{\alpha\rho}{\beta}[\gamma_4-2]}}{\Gamma(\gamma_2)}\int_0^x s^{\rho-1}\left(x^\rho - s^\rho\right)^{\gamma_2-1} \\
& \times\left(\left(st^{\frac{-\alpha}{\beta}}\right)^{1-\rho}\frac{d}{d\left(st^{\frac{-\alpha}{\beta}}\right)}\right)^2 {}^{\rho}I_{0,\eta}^{\gamma_4}\, f(st^{\frac{-\alpha}{\beta}})ds \\
= & \ \frac{\rho^{1-\gamma_2}t^{\frac{\alpha\rho}{\beta}[\gamma_4-2+\gamma_2]}}{\Gamma(\gamma_2)}\int_0^\eta \xi^{\rho-1}\left(\eta^\rho - \xi^\rho\right)^{\gamma_2-1}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^2 {}^{\rho}I_{0,\eta}^{\gamma_4}\, f(\xi)d\xi \\
= & \ t^{-\alpha\rho}\, {}^{\rho}I_{0,\eta}^{\gamma_2}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)^2 {}^{\rho}I_{0,\eta}^{\gamma_4}\ f(\eta),
\end{aligned} \tag{4.15}$$

Substituting (4.15) in (4.11), then, the space FD can be rewritten as

$$ {}^{\rho}D_{0,x}^{\beta,\gamma}u(x,t) = t^{\mu-\alpha\rho}\, {}^{\rho}D_{0,\eta}^{\beta,\gamma}\ f(\eta). \tag{4.16}$$

If we replace (4.10) and (4.16) in (4.1), we get

$$\rho^{1-\gamma_1-\gamma_3}t^{\mu-\alpha\rho}J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(\eta) = t^{\mu-\alpha\rho}\, {}^{\rho}D_{0,\eta}^{\beta,\gamma}\ f(\eta),$$

which can be simplified to (4.6)

$${}^{\rho}D_{0,\eta}^{\beta,\gamma}\ f(\eta) = \rho^{\alpha}J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(\eta).$$

■

4.3 DEFINITION OF INTEGRAL SOLUTION OF THE BASIC PROFILE f

In this lemma, we give the formula of solution to the problem (4.6)-(4.7).

Lemma 4.1 *Let $\alpha, \beta, \rho \in \mathbb{R}$ provided that $0 < \alpha \leq 1,\ 1 < \beta \leq 2,\ 0 \leq \gamma \leq 1$ and $\rho > 0$. Then the problem (4.6)-(4.7) is equivalent to the following integral equation*

$$f(\eta) = \frac{A\rho^{1-m}}{\Gamma(m)}\eta^{\rho(m-1)} + \frac{B\rho^{2-m}}{\Gamma(m-1)}\eta^{\rho(m-2)} + \rho^{\alpha}\, {}^{\rho}I_{0,\eta}^{\beta}J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}f(\eta), \tag{4.17}$$

where $\gamma_1 = \gamma(1-\alpha),\ \sigma_1 = \alpha + \gamma - \alpha\gamma,\ \gamma_3 = 1 - \sigma_1$.

Proof. Applying Katugampola's FI of order $1 < \beta \leq 2$ to both sides of equation (4.6), using property 1, equation (1.28), and conditions (4.7), we obtain

$$
{}^{\rho}I_{0,\eta}^{\beta}\ {}^{\rho}D_{0,\eta}^{\beta,\gamma} f(\eta) = {}^{\rho}I_{0,\eta}^{\beta}\ \rho^{\alpha} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} f(\eta), \tag{4.18}
$$

we have $(0 \leq \gamma \leq 1,\ 1 < \beta \leq 2)$

$$
\begin{aligned}
{}^{\rho}I_{0,\eta}^{\beta}\ {}^{\rho}D_{0,\eta}^{\beta,\gamma} f(\eta) &= {}^{\rho}I_{0,\eta}^{\beta} \left[{}^{\rho}I_{0,\eta}^{\gamma(2-\beta)} \left(\eta^{1-\rho} \frac{d}{d\eta} \right)^2 {}^{\rho}I_{0,\eta}^{(1-\gamma)(2-\beta)} f(\eta) \right] \\
&= {}^{\rho}I_{0,\eta}^{\beta+2\gamma-\beta\gamma} \left(\eta^{1-\rho} \frac{d}{d\eta} \right)^2 {}^{\rho}I_{0,\eta}^{2-(\beta-\beta\gamma+2\gamma)} f(\eta) \\
&= {}^{\rho}I_{0,\eta}^{\beta+\gamma(2-\beta)} \left(\eta^{1-\rho} \frac{d}{d\eta} \right)^2 {}^{\rho}I_{0,\eta}^{2-(\beta+\gamma(2-\beta))} f(\eta) \\
&= {}^{\rho}I_{0,\eta}^{\beta+\gamma(2-\beta)}\ {}^{\rho}D_{0,\eta}^{\beta+\gamma(2-\beta)} f(\eta) \\
&= {}^{\rho}I_{0,\eta}^{m}\ {}^{\rho}D_{0,\eta}^{m} f(\eta) \\
&= f(\eta) - \frac{\rho^{1-m}\ {}^{\rho}I_{0,\eta}^{1-m} f(0^+)}{\Gamma(m)} \eta^{\rho(m-1)} - \frac{\rho^{2-m}\ {}^{\rho}I_{0,\eta}^{2-m} f(0^+)}{\Gamma(m-1)} \eta^{\rho(m-2)} \\
&= f(\eta) - \frac{\rho^{1-m} A}{\Gamma(m)} \eta^{\rho(m-1)} - \frac{\rho^{2-m} B}{\Gamma(m-1)} \eta^{\rho(m-2)},
\end{aligned} \tag{4.19}
$$

by substituting (4.19) in (4.18), we get

$$
f(\eta) = \frac{A\rho^{1-m}}{\Gamma(m)} \eta^{\rho(m-1)} + \frac{B\rho^{2-m}}{\Gamma(m-1)} \eta^{\rho(m-2)} + \rho^{\alpha}\ {}^{\rho}I_{0,\eta}^{\beta} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} f(\eta).
$$

■

4.4 DETERMINATION OF THE EXPLICIT SOLUTION f

In this subsection, we determine the exiplicit solution f, by solving the integral equation (4.17), by applying the successive approximation method in the following theorem

Theorem 4.2 *Let $\alpha, \beta, \rho, \mu, \gamma \in \mathbb{R}$ provided that $0 < \alpha \leq 1,\ 1 < \beta \leq 2,\ 0 \leq \gamma \leq 1$ and $\rho, \mu > 0$. Then the explicit solution f of the integral equation (4.17) is given by*

$$
\begin{aligned}
f(\eta) = {} & A\rho^{1-m} \Gamma\left(\frac{\mu}{\rho} + 1 - \frac{\alpha}{\beta}(m-1) \right) \eta^{\rho(m-1)} \\
& \times W_{(\beta,m),\left(-\alpha, \frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)} \left(\rho^{\alpha-\beta} \eta^{\rho\beta} \right) \\
& + B\rho^{2-m} \Gamma\left(\frac{\mu}{\rho} + 1 - \frac{\alpha}{\beta}(m-2) \right) \eta^{\rho(m-2)} \\
& \times W_{(\beta,m-1),\left(-\alpha, \frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)} \left(\rho^{\alpha-\beta} \eta^{\rho\beta} \right).
\end{aligned} \tag{4.20}
$$

Proof. Let $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $0 \leq \gamma \leq 1$ and $\rho, \mu > 0$. Then, we apply the successive approximation method to solve the integral equation (4.17).

According to this method, we set

$$f_0(\eta) = \frac{A\rho^{1-m}}{\Gamma(m)}\eta^{\rho(m-1)} + \frac{B\rho^{2-m}}{\Gamma(m-1)}\eta^{\rho(m-2)}, \tag{4.21}$$

so, the nth term f_n is given by

$$f_n(\eta) = f_0(\eta) + \rho^{\alpha}\, {}^{\rho}I^{\beta}_{0,\eta}\, J^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}_{\frac{\beta\rho}{\alpha}} P^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}_{\frac{\beta\rho}{\alpha}} f_{n-1}(\eta). \tag{4.22}$$

Now, we compute f_1 as follows

$$f_1(\eta) = f_0(\eta) + \rho^{\alpha}\, {}^{\rho}I^{\beta}_{0,\eta}\, J^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}_{\frac{\beta\rho}{\alpha}} P^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}_{\frac{\beta\rho}{\alpha}} f_0(\eta), \tag{4.23}$$

where

$$\begin{aligned} {}^{\rho}I^{\beta}_{0,\eta} J^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}_{\frac{\beta\rho}{\alpha}} P^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}_{\frac{\beta\rho}{\alpha}} f_0(\eta) &= {}^{\rho}I^{\beta}_{0,\eta}\, J^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}_{\frac{\beta\rho}{\alpha}} P^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}_{\frac{\beta\rho}{\alpha}} \left[\frac{A\rho^{1-m}}{\Gamma(m)}\eta^{\rho(m-1)}\right. \\ &\quad \left. + \frac{B\rho^{2-m}}{\Gamma(m-1)}\eta^{\rho(m-2)}\right], \end{aligned} \tag{4.24}$$

using equation (1.22), we obtain

$$\begin{aligned} & P^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}_{\frac{\beta\rho}{\alpha}} \left[\frac{A\rho^{1-m}}{\Gamma(m)}\eta^{\rho(m-1)} + \frac{B\rho^{2-m}}{\Gamma(m-1)}\eta^{\rho(m-2)}\right] \\ &= \frac{A\rho^{1-m}}{\Gamma(m)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1)} \\ &\quad + \frac{B\rho^{2-m}}{\Gamma(m-1)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2)}. \end{aligned}$$

Then, by (1.23), we obtain also

$$\begin{aligned} & J^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}_{\frac{\beta\rho}{\alpha}} P^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}_{\frac{\beta\rho}{\alpha}} \left[\frac{A\rho^{1-m}}{\Gamma(m)}\eta^{\rho(m-1)} + \frac{B\rho^{2-m}}{\Gamma(m-1)}\eta^{\rho(m-2)}\right] \\ &= J^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}_{\frac{\beta\rho}{\alpha}} \left[\frac{A\rho^{1-m}}{\Gamma(m)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1)}\right. \\ &\quad \left. + \frac{B\rho^{2-m}}{\Gamma(m-1)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2)}\right] \\ &= \frac{A\rho^{1-m}}{\Gamma(m)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1)} \\ &\quad + \frac{B\rho^{2-m}}{\Gamma(m-1)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2)}, \end{aligned} \tag{4.25}$$

Substituting equation (4.25) in (4.24) and using (1.26), we get

$$\begin{aligned}
&{}^{\rho}I_{0,\eta}^{\beta} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} f_0(\eta) \\
= &\ {}^{\rho}I_{0,\eta}^{\beta} \left[\frac{A\rho^{1-m}}{\Gamma(m)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-1)\right)} \eta^{\rho(m-1)} \right. \\
&+ \left. \frac{B\rho^{2-m}}{\Gamma(m-1)} \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-2)\right)} \eta^{\rho(m-2)} \right] \\
= &\ \frac{A\rho^{1-m-\beta}}{\Gamma(m+\beta)} \frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)} \eta^{\rho(m-1+\beta)} \\
&+ \frac{B\rho^{2-m-\beta}}{\Gamma(m-1+\beta)} \frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)} \eta^{\rho(m-2+\beta)}. \qquad (4.26)
\end{aligned}$$

Thus, by replacing (4.26) in (4.23), $f_1(\eta)$ is given by

$$\begin{aligned}
f_1(\eta) = &\ A\rho^{1-m} \left[\frac{1}{\Gamma(m)} \eta^{\rho(m-1)} + \frac{\rho^{\alpha-\beta}}{\Gamma(m+\beta)} \frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)} \eta^{\rho(m-1+\beta)} \right] \\
&+ B\rho^{2-m} \left[\frac{1}{\Gamma(m-1)} \eta^{\rho(m-2)} \right. \\
&+ \left. \frac{\rho^{\alpha-\beta}}{\Gamma(m-1+\beta)} \frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)} \eta^{\rho(m-2+\beta)} \right].
\end{aligned}$$

As a similar way, we compute $f_2(\eta)$.

$$\begin{aligned}
f_2(\eta) = &\ f_0(\eta) + \rho^{\alpha}\, {}^{\rho}I_{0,\eta}^{\beta}\, J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} f_1(\eta) \\
= &\ f_0(\eta) + \rho^{\alpha}\, {}^{\rho}I_{0,\eta}^{\beta}\, J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} \left[f_0(\eta) \right. \\
&+ \frac{A\rho^{1-m+\alpha-\beta}}{\Gamma(m+\beta)} \frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)} \eta^{\rho(m-1+\beta)} \\
&+ \left. \frac{B\rho^{2-m+\alpha-\beta}}{\Gamma(m-1+\beta)} \frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)} \eta^{\rho(m-2+\beta)} \right], \qquad (4.27)
\end{aligned}$$

we have

$$
\begin{aligned}
& {}^{\rho}I_{0,\eta}^{\beta} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} \eta^{\rho(m-1+\beta)} \\
= \; & {}^{\rho}I_{0,\eta}^{\beta} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} \left[\frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-1+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-1+\beta)\right)} \eta^{\rho(m-1+\beta)} \right] \\
= \; & {}^{\rho}I_{0,\eta}^{\beta} \left[\frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-1+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-1+\beta)\right)} \right. \\
& \left. \times \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-1+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-1+\beta)\right)} \eta^{\rho(m-1+\beta)} \right] \\
= \; & \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-1+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-1+\beta)\right)} \frac{\rho^{-\beta}\Gamma(m+\beta)}{\Gamma(m+2\beta)} \eta^{\rho(m-1+2\beta)} \\
= \; & \frac{\rho^{-\beta}\Gamma(m+\beta)\,\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma(m+2\beta)\,\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-1)\right)} \eta^{\rho(m-1+2\beta)},
\end{aligned}
$$

and

$$
\begin{aligned}
& {}^{\rho}I_{0,\eta}^{\beta} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} P_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\sigma_1} \eta^{\rho(m-2+\beta)} \\
= \; & {}^{\rho}I_{0,\eta}^{\beta} J_{\frac{\beta\rho}{\alpha}}^{\frac{\mu}{\rho}+\gamma_3,\gamma_1} \left[\frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-2+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-1+\beta)\right)} \eta^{\rho(m-2+\beta)} \right] \\
= \; & {}^{\rho}I_{0,\eta}^{\beta} \left[\frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-2+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-2+\beta)\right)} \right. \\
& \left. \times \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3-\frac{\alpha}{\beta}(m-2+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-2+\beta)\right)} \eta^{\rho(m-2+\beta)} \right] \\
= \; & \frac{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\sigma_1-\frac{\alpha}{\beta}(m-2+\beta)\right)}{\Gamma\left(\frac{\mu}{\rho}+\gamma_3+\gamma_1-\frac{\alpha}{\beta}(m-2+\beta)\right)} \frac{\rho^{-\beta}\Gamma(m+\beta)}{\Gamma(m+2\beta)} \eta^{\rho(m-2+2\beta)} \\
= \; & \frac{\rho^{-\beta}\Gamma(m-1+\beta)\,\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma(m-1+2\beta)\,\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-2)\right)} \eta^{\rho(m-2+2\beta)}.
\end{aligned}
$$

Then, $f_2(\eta)$ can be rewritten as

$$\begin{aligned}
f_2(\eta) &= f_0(\eta)+\rho^{\alpha}\left[\frac{A\rho^{1-m-\beta}}{\Gamma(m+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1+\beta)}\right.\\
&+\frac{B\rho^{2-m-\beta}}{\Gamma(m-1+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2+\beta)}\\
&+\frac{A\rho^{1-m+\alpha-\beta}}{\Gamma(m+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)}\\
&\times\frac{\rho^{-\beta}\Gamma(m+\beta)\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma(m+2\beta)\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1+2\beta)}\\
&+\frac{B\rho^{2-m+\alpha-\beta}}{\Gamma(m-1+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)}\\
&\left.\times\frac{\rho^{-\beta}\Gamma(m-1+\beta)\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma(m-1+2\beta)\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2+2\beta)}\right],
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
f_2(\eta) &= A\rho^{1-m}\left[\frac{1}{\Gamma(m)}\eta^{\rho(m-1)}+\frac{\rho^{\alpha-\beta}}{\Gamma(m+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1+\beta)}\right.\\
&\left.+\frac{\rho^{2(\alpha-\beta)}}{\Gamma(m+2\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1+2\beta)}\right]\\
&+B\rho^{2-m}\left[\frac{1}{\Gamma(m-1)}\eta^{\rho(m-2)}+\frac{\rho^{\alpha-\beta}}{\Gamma(m-1+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)}\right.\\
&\left.\times\eta^{\rho(m-2+\beta)}+\frac{\rho^{2(\alpha-\beta)}}{\Gamma(m-1+2\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2+2\beta)}\right].
\end{aligned}$$

As a similar way, we compute $f_3(\eta)$ as follows

$$f_3(\eta)=f_0(\eta)+\rho^{\alpha}\ {}^{\rho}I^{\beta}_{0,\eta}J^{\frac{\mu}{\rho}+\gamma_3,\gamma_1}_{\frac{\beta\rho}{\alpha}}P^{\frac{\mu}{\rho}+\gamma_3,\sigma_1}_{\frac{\beta\rho}{\alpha}}f_2(\eta)$$

and we get to

$$\begin{aligned}
f_3(\eta) &= A\rho^{1-m}\left[\frac{1}{\Gamma(m)}\eta^{\rho(m-1)} + \frac{\rho^{\alpha-\beta}}{\Gamma(m+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1+\beta)}\right. \\
&+ \frac{\rho^{2(\alpha-\beta)}}{\Gamma(m+2\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1+2\beta)} \\
&+ \left.\frac{\rho^{3(\alpha-\beta)}}{\Gamma(m+3\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-3\alpha-\frac{\alpha}{\beta}(m-1)\right)}\eta^{\rho(m-1+3\beta)}\right] \\
&+ B\rho^{2-m}\left[\frac{1}{\Gamma(m-1)}\eta^{\rho(m-2)} + \frac{\rho^{\alpha-\beta}}{\Gamma(m-1+\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-\alpha-\frac{\alpha}{\beta}(m-2)\right)}\right. \\
&\times \eta^{\rho(m-2+\beta)} + \frac{\rho^{2(\alpha-\beta)}}{\Gamma(m-1+2\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-2\alpha-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2+2\beta)} \\
&+ \left.\frac{\rho^{3(\alpha-\beta)}}{\Gamma(m-1+3\beta)}\frac{\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}{\Gamma\left(\frac{\mu}{\rho}+1-3\alpha-\frac{\alpha}{\beta}(m-2)\right)}\eta^{\rho(m-2+3\beta)}\right].
\end{aligned}$$

This can be generalized to get the nth order term $f_n(\eta)$, $(n \in \mathbb{N})$ given by

$$\begin{aligned}
f_n(\eta) &= A\rho^{1-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)\eta^{\rho(m-1)} \\
&\times \sum_{k=0}^{n}\frac{\rho^{k(\alpha-\beta)}\eta^{\rho\beta k}}{\Gamma(m+\beta k)\,\Gamma\left(\frac{\mu}{\rho}+1-\alpha k-\frac{\alpha}{\beta}(m-1)\right)} \\
&+ B\rho^{2-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)\eta^{\rho(m-2)} \\
&\times \sum_{k=0}^{n}\frac{\rho^{k(\alpha-\beta)}\eta^{\rho\beta k}}{\Gamma(m-1+\beta k)\,\Gamma\left(\frac{\mu}{\rho}+1-\alpha k-\frac{\alpha}{\beta}(m-2)\right)}.
\end{aligned}$$

When $n \to +\infty$, we obtain the following explicit solution $f(\eta)$ to the integral equation (4.17)

$$\begin{aligned}
f(\eta) &= A\rho^{1-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)\eta^{\rho(m-1)} \\
&\times \sum_{k=0}^{\infty}\frac{\rho^{k(\alpha-\beta)}\eta^{\rho\beta k}}{\Gamma(m+\beta k)\,\Gamma\left(\frac{\mu}{\rho}+1-\alpha k-\frac{\alpha}{\beta}(m-1)\right)} \\
&+ B\rho^{2-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)\eta^{\rho(m-2)} \\
&\times \sum_{k=0}^{\infty}\frac{\rho^{k(\alpha-\beta)}\eta^{\rho\beta k}}{\Gamma(m-1+\beta k)\,\Gamma\left(\frac{\mu}{\rho}+1-\alpha k-\frac{\alpha}{\beta}(m-2)\right)}.
\end{aligned} \tag{4.28}$$

Taking into account the relation (1.7), we rewrite the solution (4.28) in terms of the generalized Wright function $W_{(\mu,a),(\nu,b)}(z)$

$$\begin{aligned} f(\eta) &= A\rho^{1-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)\eta^{\rho(m-1)} \\ &\times W_{(\beta,m),\left(-\alpha,\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}\left(\rho^{\alpha-\beta}\eta^{\rho\beta}\right) \\ &+B\rho^{2-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)\eta^{\rho(m-2)} \\ &\times W_{(\beta,m-1),\left(-\alpha,\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}\left(\rho^{\alpha-\beta}\eta^{\rho\beta}\right). \end{aligned}$$

■

4.5 THE EXACT FORM OF THE SELF-SIMILAR SOLUTIONS

In this part, we present the exact form for the self-similar solutions of the time-space fractional diffusion equation (4.1) by substituting (4.20) in the transformation (4.5) in the following theorem

Theorem 4.3 *Let $\alpha, \beta, \rho, \mu, \gamma \in \mathbb{R}$ provided that $0 < \alpha \leq 1,\ 1 < \beta \leq 2,\ 0 \leq \gamma \leq 1$ and $\rho, \mu > 0$. Then the self-similar (according to the transformation (4.5)) solutions of the time-space fractional diffusion equation (4.1) with conditions (4.7) have the following form*

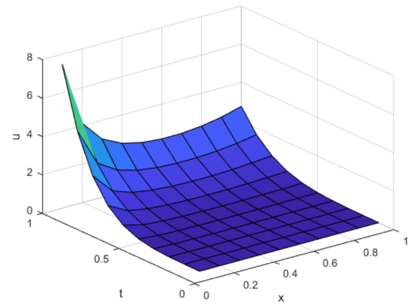
$$\begin{aligned} u(x,t) &= t^{\mu}\left[A\rho^{1-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)\eta^{\rho(m-1)}\right. \\ &\times W_{(\beta,m),\left(-\alpha,\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-1)\right)}\left(\rho^{\alpha-\beta}\eta^{\rho\beta}\right) \\ &+ B\rho^{2-m}\Gamma\left(\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)\eta^{\rho(m-2)} \\ &\left.\times W_{(\beta,m-1),\left(-\alpha,\frac{\mu}{\rho}+1-\frac{\alpha}{\beta}(m-2)\right)}\left(\rho^{\alpha-\beta}\eta^{\rho\beta}\right)\right], \end{aligned} \tag{4.29}$$

where $m = \beta + \gamma(2-\beta)$.

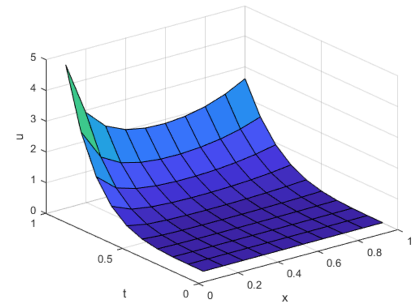
Remark 4.1 *the case of $\rho = 1$, for which the HKFD reduces to the Hilfer FD was considered in [3] by Al-Musalhi and Karimov, when for the case of $\rho = 1$ and $\gamma = 0$, for which the HKFD reduces to the Riemann-Liouville FD was treated in [44] by Luchko and Gorenflo.*

4.6 GRAPHICAL REPRESENTATION OF THE EXACT SOLUTION

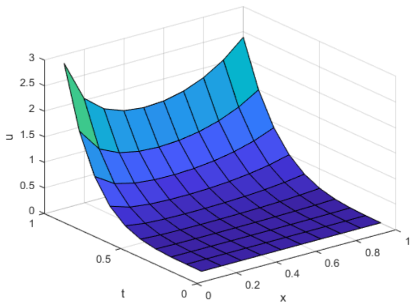
In Fig. 1 and Fig. 2, we plot the solution $u(t,x)$ in (4.29) versus the variables t and x in $3D$ and $2D$ respectively, for different values of the parameters α and β, where $\gamma_1 = 0.5$, $\mu = 5$, $\rho = 2$, A and B are set to 1.



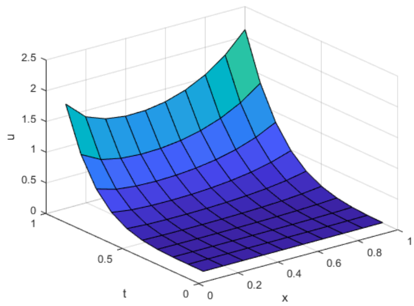
(a) α =0.1, β=1.1



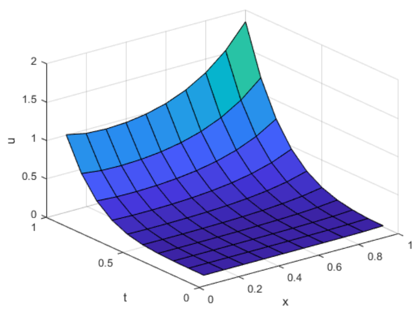
(b) α =0.3, β=1.3



(c) α =0.5, β=1.5

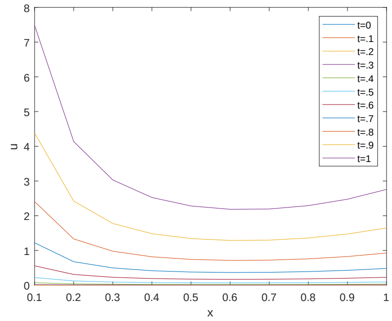


(d) α =0.7, β=1.7

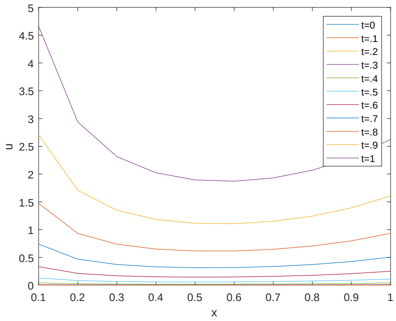


(e) α =0.9, β=1.9

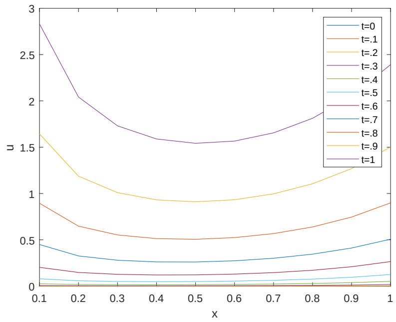
Figure 4.1: The graphical representation of the exact solution in 3D.



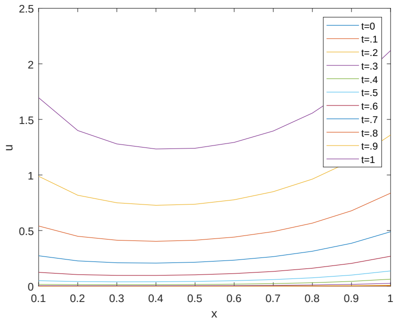
(a) α =0.1, β=1.1



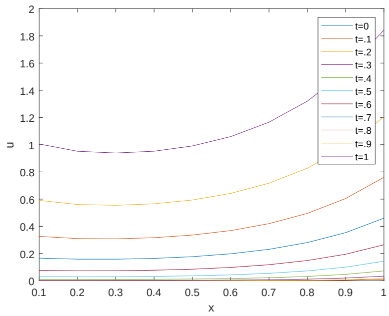
(b) α =0.3, β=1.3



(c) α =0.5, β=1.5

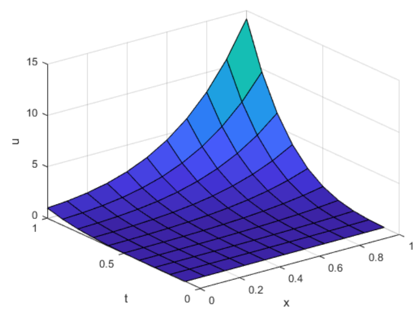


(d) α =0.7, β=1.7

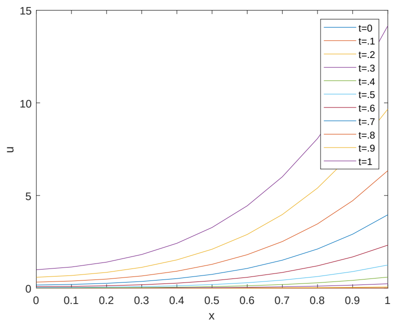


(e) α =0.9, β=1.9

Figure 4.2: The graphical representation of the exact solution in 2D.



(a) α =1, β=2, ρ=1.



(b) α =1, β=2, ρ=1.

Figure 4.3: The graphical representation of the exact solution, for the classical case.

Remark 4.2 *In the classical case for which $\alpha = 1$, $\beta = 2$, and $\rho = 1$, we have the graphical representation of the exact solution in Fig. 3, in 3D and 2D, respectively.*

CONCLUSION AND PERSPECTIVES

This thesis is mainly devoted to study the existence and uniqueness of solutions of some classes of mixed-FDEs and FPDEs. We have also introduced the self-similar solutions in an explicit form of space-time fractional diffusion equation.

In the first chapter, we have gone through a journey on FC. The content of this chapter has helped to work on and understand the various results in the following next chapters.

In chapter 2 and 3, we have investigated the existence and uniqueness of solutions in Banach space, for both non-linear mixed FDEs with two different FDs and for nonlinear space-fractional diffusion equation involving the GRCFD respectively, both with boundary conditions. The main tools we have used in our work are the Banach's contraction principle, Schauder's, Schaefer's fixed point theorems and the nonlinear alternative of Leray-Schauder's type (see [26]).

In the last chapter, we have derived self-similar solutions in an explicit form of space-time fractional diffusion equation involving HKFD by applying the successive approximation method (see [39]).

As this being said, we can claim that our work will pave the way for further developments and investigations in solving more BVPs related to FDEs and FPDES. This will serve in solving some real-world problems.

Despite the contributions made in this thesis and in other works on the studied topic, several problems remain to be solved. Indeed, one may consider the following points:

- Investigate some numerical methods to solve the aforementioned problems.
- Considering different fractional operators in the same or a different framework and context.
- Extend the space of study in a one broader than that of Banach.
- Extend the outcomes of this work to the case of systems of FDEs and FPDEs.

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مُلَخَّص:

في هذه الرسالة ، نقدم بعض نتائج الوجود والوحدانية لحلول المعادلات التفاضلية الكسرية العادية غير الخطية المختلطة (FDEs) مع اثنين من مشتقات كسور مختلفة وللمعادلات التفاضلية الجزئية غير الخطية (FPDEs) التي تتضمن مشتق Riesz – Caputo المعمم (G-RCFD)) ، على التوالي. نعتمد شروطاً حدية في فضاء باناخ.

نستخدم مبدأ التقلص الخاص بـ Banach ، ونظريات النقطة الثابتة لكلّ من Schauder و Schaefer ، وتقنية البديل غير الخطي من نوع Leray-Schauder. نشتق أيضًا الحلول المتشابهة ذاتيًا في شكل صريح من معادلة الانتشار الجزئي للزمان والمكان التي تتضمن مشتقًا جزئيًا من نوع Hilfer-Katugampola من خلال تطبيق طريقة التقريبات المتتالية.

الكلمات المفتاحيّة : مسائل القيمة الحدية (BVPs) ، المعادلات التفاضلية العادية الكسرية (FDEs) ، المعادلات التفاضلية الجزئية الكسرية (FPDEs) ، الحلول المتشابهة ، طريقة التقريبات المتتالية ، نظرية النقطة الثابتة ، فضاء Banach ، الوجود ، الوحدانية ، استقرار Ulam-Hyers ، مشتقات Katugampola's و Katugampola من نوع Caputo ، مشتقات Hilfer-Katugampola الكسرية.

Abstract

In this thesis, we provide some existence and uniqueness results of solutions in Banach space, for nonlinear mixed fractional differential equations (FDEs) with two different fractional derivatives and for nonlinear fractional partial differential equations (FPDEs) involving the generalized Riesz–Caputo fractional derivative (GRCFD) respectively, both with boundary conditions. We use the Banach's contraction principle, Schauder's and Schaefer's fixed point theorems, and the technique of the nonlinear alternative of Leray-Schauder type. We also derive the self-similar solutions in an explicit form of space-time fractional diffusion equation involving Hilfer- Katugampola's fractional derivative (HKFD) by applying the successive approximation method**.**

Key words: Boundary value problems (BVPs), fractional differential equations (FDEs), fractional partial differential equations (FPDEs), self similar solutions, successive approximation method, fixed point theorem, Banach space, existence, uniqueness, Ulam-Hyers stability, Katugampola's and Caputo-type Katugampola's fractional derivatives, Hilfer-Katugampola's fractional derivative.

Résumé

Dans cette thèse, nous nous intéressons à l'existence et l'unicité de solutions dans un espace de Banach, pour des équations différentielles non-linéaires d'ordre fractionnaire avec deux dérivées fractionnaires différentes et des équations aux dérivées partielles non linéaires d'ordre fractionnaire de type Riesz-Caputo généralisée, ces deux équations, avec des conditions aux limites. On utilise le principe de contraction de Banach, la technique alternative non linéaire de type Leray-Schauder et les théorèmes de point fixe de Schauder et Schaefer. On dérive aussi une solution auto-similaire dans sa forme explicite d'équation (de diffusion d'espace-temps) aux dérivées partielles non linéaires d'ordre fractionnaire de type Hilfer-Katugampola en utilisant la méthode des approximations successives.

Mots clés : Problèmes aux limites, équations différentielles fractionnaires, équations aux dérivées partielles fractionnaires, solutions auto-similaires, méthode des approximations successives, théorème de point fixe, espace de Banach, existence, unicité, stabilité de Ulam-Hyers, dérivées fractionnaires de Katugampola et Caputo-Katugampola, dérivée fractionnaire de Hilfer-Katugampola.