PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
UNIVERSITY OF MOHAMED BOUDIAF /M'SILA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCES
DEPARTMENT OF MATHEMATICS

N^o d'ordre:..........................

THESIS

Presented for obtaining the diploma
of doctoral degree 3rd cycle LMD

Speciality
Mathematics

Option
Algebra And Discrete Mathematics

By

Saadi Djazia

Theme

Picture Fuzzy Orderings and Related Structures

Thesis defended publicly on: 11/07/2024, before a jury composed of:

Mihoubi Douadi	Prof.	University of M'sila	**Chair**
Amroune Abdelaziz	Prof.	University of M'sila	**Supervisor**
Boudaoud Abdelmadjid	Prof.	University of M'sila	**Examiner**
Oumhani Ali	MCA	ENS Bousaâda	**Examiner**
Milles Soheyb	MCA	University Center of Barika	**Examiner**

University Year: 2023/2024

Picture Fuzzy Orderings and Related Structures

DJAZIA SAADI

11/07/2024

Acknowledgements

Praise be to Allah, who guided me, directed my steps, and blessed me to complete my doctoral thesis.

My sincere gratitude and appreciation go to my supervisor, Professor Amroune Abdelaziz, who taught me honesty and dedication in the pursuit of knowledge. His valuable guidance, precious advice, and endless support at every step played a significant role in the completion of this work.

I extend my deep thanks to the president of the jury, Prof. Mihoubi Douadi, and the examiners, Prof. Boudaoud Abdelmadjid, Dr. Oumhani Ali, and Dr. Milles Soheyb, for their acceptance to be members of my thesis examination committee and for dedicating their time and effort to read and review my work.

I owe a great deal of love and gratitude to my mother and father, who were my first supporters, my greatest motivators, and my source of strength and support after Allah. Their prayers made hardships easier for me, and their care, attention, and trust in me gave me immense strength to continue and always progress. Words cannot express even a fraction of their favor.

I am deeply grateful to my brothers and sisters and their families for their encouragement and support in all aspects of life. They had a profound impact on overcoming many obstacles and difficulties, and I don't know if I would be where I am today without them in my life. I thank Allah for my family's blessings.

My heartfelt thanks go to my friends, who stood by me throughout my educational journey, shared its steps with me, and eased its burdens. To my companions: I am very grateful to you.

I would like to express my deep appreciation to every teacher who taught me throughout my educational journey, from primary school to university.

My thanks and appreciation to all members of my faculty, Mathematics Department and laboratory of pure and applied mathematics.

I take this opportunity to extend my sincere thanks to everyone who had an impact on shaping my personality: the mosque, Islamic Scouts, the associations, Jil Tarjih Academy, Da'wa and Irched Team, Al-Ibsar Project, Al-Nahda Foundation for the Quran, and all the friends and teachers I met through these institutions.

Finally, I thank everyone who helped me, supported me, encouraged me, and continuously inquired about my progress.

Dedication

I dedicate this work

To our brothers in Gaza and all of Palestine.
To Al-Qassam Brigades in the Islamic Resistance Movement.
To the heroes of the October 7th - Al-Aqsa Flood -.
To all martyrs and prisoners.
To those who have renewed our purposes within us.
To those who taught us that every person has a position to defend. Thus, we have committed ourselves to the position of knowledge to strive through it.

"عِــلْمُـكَ.. مُـقَــاوَمَــةٌ"

أهدي هذا العمل

إلى إخواننا في غزّة وكل فلسطين

إلى كتائب القسّام في حركة المقاومة الإسلامية

إلى أبطال 07 أكتوبر -طوفان الأقصى-

إلى كل الشهداء والأسرى

إلى الذين جددوا فينا الغايات

إلى من علمونا أن لكل انسان ثغر يرابط منه، فلزمنا ثغر العلم نجاهد من خلاله.

Saadi Djazia

Saturday, 27 Ramadan 1445,
April 6th, 2024.

Table of contents

Introduction 1

1 A Brief Overview of L-Fuzzy Sets and Related Structures 5
- 1.1 Fuzzy Sets and L-Fuzzy Sets 5
  - 1.1.1 Fuzzy Binary Relations 7
  - 1.1.2 Fuzzy Lattices 7
- 1.2 Intuitionistic Fuzzy Sets as $\mathbb{L}^*$-Fuzzy Set 10
  - 1.2.1 Intuitionistic Fuzzy Sets 10
  - 1.2.2 Intuitionistic Fuzzy Relations 11
  - 1.2.3 Intuitionistic Fuzzy Lattices 12
  - 1.2.4 Intuitionistic Fuzzy Ordered Lattice 13

2 Background on Picture Fuzzy Sets 16
- 2.1 Picture Fuzzy Sets: Another Point of View 17
  - 2.1.1 Concepts Related to Picture Fuzzy Sets According to Cuong's Definitions 17
  - 2.1.2 Motivations for the Revision of the Definitions of Picture Fuzzy Sets Operations 19
  - 2.1.3 Structure of the Set $\mathbb{D}^*$ 20
  - 2.1.4 Picture Fuzzy Sets Operations in term of $\mathbb{D}^*$ and Some of Their Properties 24
- 2.2 Characteristic Sets of a Picture Fuzzy Set 28
  - 2.2.1 Support and Kernel of a Picture Fuzzy Set 28
  - 2.2.2 α-Cuts and Strong α-Cuts of a Picture Fuzzy Set 29
  - 2.2.3 Picture Fuzzy Line of Degree α of a Picture Fuzzy Set 31
  - 2.2.4 Some Decomposition Theorems of a Picture Fuzzy Set 33
- 2.3 Extension of Some of Atanassov's Modal Operators on Picture Fuzzy Sets . . 35
  - 2.3.1 Necessity and Possibility Operators 35
  - 2.3.2 Operators D_λ and F_α 38

3 $\mathbb{D}^*$-Based Picture Fuzzy Lattices **41**
3.1 Picture Fuzzy Relations 42
3.1.1 Notions of Picture Fuzzy Relations 42
3.1.2 Properties of Picture Fuzzy Relations 46
3.1.3 Picture Fuzzy Order Relation 49
3.2 Picture Fuzzy Lattices 52
3.2.1 Particular Elements of a Picture Fuzzy Ordered Set 52
3.2.2 Concepts Related to Picture Fuzzy Lattices 53
3.2.3 Characterizations of a Lattice in Terms of Their Level Set 55

4 Filters in a Lattices **59**
4.1 Picture Fuzzy Filters in a Lattice 60
4.2 Picture Fuzzy Filters in a Picture Fuzzy Lattice 63
4.3 Picture Fuzzy Prime Filters in a Picture Fuzzy Lattice 67

Conclusion **71**

Bibliography **75**

Introduction

Many problems in daily life contain various levels of uncertainty. Since existing standard mathematical tools may not model such uncertainties, new ones are needed. Fuzzy sets [43] and intuitionistic fuzzy sets [10], were introduced to deal with uncertainty, are some of the well-known mathematical tools for the aforesaid purpose.

Zadeh introduced the ideas of fuzzy sets and fuzzy relations initially, followed by Goguen [26, 43, 44]. Many authors have investigated various approaches to fuzzy lattices, fuzzy filters, and related concepts, see [2–5,18,37]. In [41], Venugopalan presented the idea of a fuzzy ordered set (foset). He also defined the supremum and infimum of a subset A. In 1990, Bo and Wangming proposed the ideas of fuzzy sublattices and fuzzy ideals of a lattice [13]. As a fuzzy algebra, a fuzzy lattice was defined by Ajmal and Thomas [2] and fuzzy sub-lattices were described by the same authors. Using their α-cuts, Chon defined and characterized fuzzy lattices in 2009 and established several fundamental properties of fuzzy lattices [18]. Mezzomo [32] used a fuzzy partial order relation to describe fuzzy lattices. Additionally, he defined filters and α-filters of a fuzzy lattice and he used their α-cuts to describe them.

In 1983, Atanassov first introduced intuitionistic fuzzy sets (IFSs) to address the issue of non-membership [8]. This concept was followed by the introduction in 1984, of a generalized intuitionistic fuzzy set, known as "intuitionistic L-fuzzy set" [9]. It has been shown to be particularly useful for dealing with ambiguity. Based on the intuitionistic fuzzy set concept of Atanassov, Burillo and Bustince [17] introduced the notion of intuitionistic fuzzy relation. Specifically, they presented the intuitionistic fuzzy order relation as a logical extension of the fuzzy order relation introduced by Zadeh [44] before. Many authors have studied the concept of intuitionistic fuzzy order, intuitionistic fuzzy lattice intuitionistic fuzzy filter and intuitionistic fuzzy ideal [2, 6, 15, 33, 39, 46].

Although these sets can model many problems, there are much more problems and uncertainties in real life that they fail to model. For instance, in voting for an election, the decisions of the electorate may be split into three types: yes, no, and abstain. To model this problem and other similar problems, Cuong and Kreinovich put forward in 2013 [20] a new concept called "picture fuzzy sets". The latter is a direct generalization of Zadeh's fuzzy sets and Atanassov's intuitionistic fuzzy sets. The main work of Cuong Bui Cong and Vladik Kreinovich is the incorporation of the concept of the positive, negative and neutral membership degree of an element. In other words, an element of a picture fuzzy set A is characterized by three memberships degrees, positive, negative and neutral membership degree $\mu_A(x), \eta_A(x), \nu_A(x)$, so that the total of the three degrees cannot be greater than one. This gives an unusual but wonderful idea of a mathematician and many valued-logic.

Not only does the resulting notion have a beautiful mathematical structure with connections to various fields of mathematics, but it also has a broad range of applications outside mathematics, for example in decision-making [7,25,42], Medical Diagnosis [24], investment risk [14] and other applications [1,38].

The picture fuzzy set is one of the most reliable techniques to handle the uncertainties in the data throughout the decision-making process, in which an intuitionistic fuzzy set may fail to produce good results. It is an effective mathematical tool for dealing with uncertain real-life issues.

This study aims to:

1. Address the inconsistencies found in Coung's definitions by examining the structure of the set $\mathbb{D}^*$, which serves as a prototype of picture fuzzy sets[1].

2. Extend some of Atanassov's modal operators to the picture fuzzy case with respect to previous modifications[1].

3. Provide some characteristic sets of a picture fuzzy set, such as support, kernel, α-cuts, strong α-cuts and picture fuzzy lines of a degree α of a picture fuzzy set for $\alpha \in \mathbb{D}^*$[1].

4. Study concepts related to relations theory and lattice theory in the picture fuzzy case [2].

5. Study the concepts of picture fuzzy filters in a crisp lattice, crisp filters in a picture fuzzy lattice, picture fuzzy filters in a picture fuzzy lattice, and picture fuzzy prime filters in a picture fuzzy lattice[2].

[1]This part is published in TWMS Journal of Applied and Engineering Mathematics [35].
[2]This part is published in Kragujevac Journal of Mathematics [36].

6. Describe the concepts mentioned in 4 and 5 in terms of α-cuts[2].

This thesis is structured as follows:

The first chapter contains some basic definitions of fuzzy sets, intuitionistic fuzzy sets, fuzzy relations, intuitionistic fuzzy relations, fuzzy lattices, intuitionistic fuzzy lattices, and their filters, which will be extended later on.

The second chapter undertakes a critical examination of picture fuzzy sets which is an important extension of fuzzy sets and intuitionistic fuzzy sets aimed at enhancing the precision of uncertainty representation. It highlights the need for clarity and consistency in its foundational definitions, as some contradictions in specific definitions were found during research. The first section is dedicated to addressing these contradictions. It re-examines and scrutinizes the basic definitions of picture fuzzy sets introduced by Cuong, suggesting corrections and presenting new results. This section not only critiques the current state of definitions but also identifies the motivations for redefining the operations of picture fuzzy sets, supported by a comprehensive analysis of Cuong's definitions, complemented by counterexamples and proofs. This analysis reveals the necessity of theoretical reform to resolve the identified contradictions. A pivotal aspect of this comprehensive reform is the detailed study of the $\mathbb{D}^*$ set structure, which serves as a prototype of picture fuzzy sets. Understanding the structural dynamics of $\mathbb{D}^*$ facilitates the redefinition of operations for picture fuzzy sets, ensuring compatibility with logical coherence and practical applicability. This study introduces new definitions and properties related to the operations of picture fuzzy sets, significantly contributing to the theoretical foundation of this field. The second section explores the extension of Atanassov's modal operators to picture fuzzy sets, focusing on four specific factors: necessity and possibility operators, the operators D_λ and F_α. Furthermore, this section delves into the characteristic sets of a picture fuzzy set, including support, kernel, α-cuts and strong α-cuts, alongside the picture fuzzy line of degree α. Finally, the chapter introduces some decomposition theorems for a picture fuzzy set.

The third chapter focuses on the basic concepts underlying picture fuzzy relations. This includes an extensive discussion of their definitions, the operations that can be performed on them, and an exploration of their properties. This chapter also delves into important aspects of picture fuzzy relations, such as the fuzzy picture order relation and the fuzzy picture chain. It presents a series of results related to these concepts. and examines in detail the structure of the picture fuzzy lattice, where the operations of join and meet in such a lattice are adapted to respect these additional dimensions of membership, enabling more sophisticated aggregation and comparison mechanisms. This section systematically presents concepts of fuzzy picture lattices and fuzzy picture bounded lattices, in addition to the fuzzy picture sublattice, delves into their properties and gives some characterizations using their level sets.

The fourth chapter presents an in-depth exploration of filters structure in the context of lattice theory, adapting and generalizing them through picture fuzzy cases. It delves into various aspects of filters, beginning with the fundamental definitions of picture fuzzy filters in a crisp lattice, crisp filters in a picture fuzzy lattice, picture fuzzy filters in a picture fuzzy lattice, and picture fuzzy prime filters in a picture fuzzy lattice. It then explores their properties and characterizations.

The majority of the results presented in this thesis have been published in international journals. The results included in Chapter 2, Chapter 3 and Chapter 4 have been described in [35] and [36], respectively.

CHAPTER 1

A BRIEF OVERVIEW OF L-FUZZY SETS AND RELATED STRUCTURES

This chapter contains some basic definitions of fuzzy sets, intuitionistic fuzzy sets, fuzzy relations, intuitionistic fuzzy relations, fuzzy lattices, intuitionistic fuzzy lattices, and their filters.

1.1 Fuzzy Sets and L-Fuzzy Sets

A partial order $\leqslant$ is a binary relation on a set X which is reflexive ($x \leqslant x$ for any $x \in X$), antisymmetric ($x \leqslant y$ and $y \leqslant x$ implies $x = y$ for any $x, y \in X$) and transitive (for any $x, y, z \in X, x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$). A partially ordered set, or a poset, denoted by $(X, \leqslant)$, is a set equipped with an order relation. $(X, \leqslant)$ is called a lattice if it has both the least upper bound and the greatest lower bound of a set $\{x, y\}$, for all $x, y \in X$. $x \vee y$ and $x \wedge y$, respectively, indicate the least upper bound and the greatest lower bound of $\{x, y\}$.

If $\vee E$ and $\wedge E$ exist for all $E \subseteq X$, then $(X, \leqslant)$ is called a complete lattice.

Recall that a filter of a lattice $(X, \leqslant)$ is a non-empty subset A of X satisfy the following conditions, for any $x, y \in X$

1. $y \in A$ and $y \leqslant x$ implies $x \in A$,

2. $x, y \in A$ implies $x \wedge y \in A$.

A filter A of $(X, \leqslant)$ is a prime filter if: $x \vee y \in A$ implies $x \in A$ or $y \in A$ for any $x, y \in X$.

Definition 1.1. [43] Let X be a non-empty set. A generalized characteristic function μ_A : $X \longrightarrow [0, 1]$, called a fuzzy subset A on X, and μ_A is said to be the membership function of A.

Definition 1.2. [26] Let X be a non empty set and let L be a complete lattice. An L-fuzzy set μ is any application from $\mu : X \longrightarrow L$.

Notation 1.1. The set of all fuzzy subset of X will be denoted by $F(X)$.

Definition 1.3. [43] Let X be a non empty set, the following expressions are defined by, for all $A, B \in F(X)$ and $x \in X$:

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $(A \cap B)(x) = \mu_A(x) \wedge \mu_B(x)$.
4. $(A \cup B)(x) = \mu_A(x) \vee \mu_B(x)$.
5. $A^c(x) = 1 - \mu_A(x)$.
6. The support of A given by $S(A) = \{x \in X \mid \mu_A(x) > 0\}$.
7. The kernel of A given by $Ker(A) = \{x \in X \mid \mu_A(x) = 1\}$.
8. For $\alpha \in]0, 1]$, the α-cut of A is the crisp subset A_α of X given by

   $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$.
9. For $\alpha \in [0, 1[$, the strong α-cut of A is the crisp subset A_α^+ of X given by

   $A_\alpha^+ = \{x \in X \mid \mu_A(x) > \alpha\}$.
10. For $\alpha \in [0, 1]$, the picture fuzzy line of degree α of A is the crisp subset $L_\alpha(A)$ of X given by $L_\alpha(A) = \{x \in X \mid \mu_A(x) = \alpha\}$.

Knowing that the unite interval $([0, 1], \leq, \min, \max, 0, 1)$ is a complete lattice with $\leq$ is the usual order on the set of real numbers $\mathbb{R}$. It is easy to see that $(F(X), \subseteq, \cap, \cup, \emptyset, X)$ is a complete lattice.

1.1.1 Fuzzy Binary Relations

Definition 1.4. [44] Let X be non-empty set, a fuzzy relation on X, denoted by $\mathcal{R}$, is defined as the fuzzy set

$$\mathcal{R} = \left\{ \left((x,y), \mu_{\mathcal{R}}(x,y) \right) : (x,y) \in X^2 \right\},$$

where the function $\mu_{\mathcal{R}} : X \times X \longrightarrow [0,1]$ is called the membership function and gives the degree of membership of the ordered pair (x,y) in $\mathcal{R}$ associating with each pair (x,y) in X^2 a real number in the interval $[0,1]$.

Definition 1.5. [44] Let X be a non-empty set. A fuzzy binary relation $\mathcal{R}$ on X is called:

1. Reflexive, if $\mathcal{R}(x,x) = 1$, for all $x \in X$.
2. Antisymmetric, if $\mathcal{R}(x,y) > 0$ and $\mathcal{R}(y,x) > 0$ implies $x = y$, for all $x, y \in X$.
3. Transitive, if $\mathcal{R}(x,z) \geq \sup_{y \in X} \min \{\mathcal{R}(x,y), \mathcal{R}(y,z)\}$, for all $x, y, z \in X$.

Definition 1.6. [18] Let X be a non-empty set. A reflexive, antisymmetric and transitive fuzzy relation is called a fuzzy partial order relation. A fuzzy partial order relation $\mathcal{R}$ is a fuzzy total order relation if and only if either $\mathcal{R}(x,y) > 0$ or $\mathcal{R}(y,x) > 0$, for all $x, y \in X$. A set equipped with a fuzzy partial order relation is called a fuzzy partially ordered set (fuzzy poset for short) and we will denote it by $(X, \mathcal{R})$. If $\mathcal{R}$ is a fuzzy total order relation in a set X, then $(X, \mathcal{R})$ is called fuzzy totally ordered set or a fuzzy chain.

1.1.2 Fuzzy Lattices

Definition 1.7. [41] Let $(X, \mathcal{R})$ be a fuzzy poset, and let A be a non-empty subset of X. An element $u \in X$ is said to be an upper bound of a subset A if and only if $\mathcal{R}(a,u) > 0$, for all $a \in A$. An upper bound u_0 of A is the least upper bound of A if and only if $\mathcal{R}(u_0, u) > 0$, for every upper bound u of A. An element $l \in X$ is said to be a lower bound of a subset A if and only if $\mathcal{R}(l,a) > 0$, for all $a \in A$. A lower bound l_0 of A is the greatest lower bound of A if and only if $\mathcal{R}(l, l_0) > 0$, for every lower bound l of A.

A least upper bound of A will be denoted by $\sup A$ and a greatest lower bound by $\inf A$. We denote the least upper bound of the set $\{x, y\}$ by $x \sqcup y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \sqcap y$.

Definition 1.8. [18] A fuzzy poset $(X, \mathcal{R})$ is a fuzzy lattice if and only if $x \sqcup y$ and $x \sqcap y$ exist for all $x, y \in X$.

Example 1.1. Let $X = [0,1]$ and let $\mathcal{R}$ be the fuzzy relation on X defined by

$$\mathcal{R}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ \lambda & (\lambda \in]0,1]) \text{ if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

$(X, \mathcal{R})$ is a fuzzy lattice.

In what follows $(X, \mathcal{R})$ denotes a fuzzy lattice.

Definition 1.9. [29] Let $(X, \mathcal{R})$ be a fuzzy lattice, and let A be a non-empty subset of X. A is a crisp filter of $(X, \mathcal{R})$ if it satisfies the following conditions:

(i) If $y \in X, x \in A$ and $\mathcal{R}(x,y) > 0$, then $y \in A$.

(ii) If $x, y \in A$, then $x \sqcap y \in A$.

Now, we give some types of classical filters in a fuzzy lattice.

Definition 1.10. A filter A of $(X, \mathcal{R})$ such that $A \neq X$, is called proper filter of $(X, \mathcal{R})$.

Definition 1.11. [34] Let A be a proper filter of $(X, \mathcal{R})$. Then, A is a prime filter if for all $x, y \in X, x \sqcup y \in A$ implies $x \in A$ or $y \in A$.

Definition 1.12. [29] Let A be a proper filter of $(X, \mathcal{R})$. We say that the filter A is a maximal filter of $(X, \mathcal{R})$, if $A \subseteq B \subseteq X$, then either $B = A$ or $B = X$, for any filter B.

Definition 1.13. [29] Let $(X, \mathcal{R})$ be a fuzzy lattice and $x \in X$. Then, the set defined by: $\uparrow x = \{y \in X : \mathcal{R}(x,y) > 0\}$ is called principal filter of $(X, \mathcal{R})$ generated by x.

Definition 1.14. [31] Let $(X, \mathcal{R})$ be a fuzzy lattice, $\alpha \in]0,1]$ and let $A \subseteq X$.
A is an α-filter of $(X, \mathcal{R})$, if it satisfies

(i) If $x \in X$, $y \in A$ and $\mathcal{R}(y,x) \geq \alpha$, then $x \in A$;

(ii) $x, y \in A$ implies $x \sqcap y \in A$.

Definition 1.15. Let A be a non-empty subset of X. A filter W in X is said to be generated by A, if $A \subseteq W$ and for any filter Z in X, $A \subseteq Z$ implies $W \subseteq Z$. The filter generated by A will be denoted by $\uparrow A$.

Lemma 1.1. *[5] If A be a non-empty subset of a fuzzy lattice $(X, \mathcal{R})$, then*

$$\uparrow A = \{x \in X \mid \mathcal{R}\left(a_1 \sqcap ... \sqcap a_n, x\right) > 0, \textit{ for Some } a_1, .., a_n \in A\}$$

is the filter generated by A.

Definition 1.16. [5] A fuzzy subset A in $(X, \mathcal{R})$ is called a fuzzy filter of X if it satisfies

(F1) for all $x, y \in X, A(x \sqcap y) \geq \min\{A(x), A(y)\}$;

(F2) A is an order-preserving, that is, for all $x, y \in X, \mathcal{R}(x, y) > 0$ implies $A(x) \leq A(y)$.

Now, we give some types of fuzzy filters in a fuzzy lattice.

Definition 1.17. Let δ be a fuzzy set in $(X, \mathcal{R})$. A fuzzy filter μ in X is said to be generated by δ, if $\delta \subseteq \mu$ and for any fuzzy filter ψ in X, $\delta \subseteq \psi$ imply $\mu \subseteq \psi$. The fuzzy filter generated by δ will be denoted by $\uparrow \delta$.

Theorem 1.1. *[5] If δ is a fuzzy set, then*

$$\uparrow \delta(x) = \max\left\{\min_{i \in \{1,..,n\}} \delta(a_i) \mid \mathcal{R}\left(\sqcap_{i=1}^{n} a_i, x\right) > 0, a_1, a_2, ..., a_n \in X\right\}$$

is the fuzzy filter generated by δ.

Definition 1.18. [5] Let $(X, \mathcal{R})$ be a fuzzy lattice, $\alpha \in]0, 1]$.

A fuzzy set A in $(X, \mathcal{R})$ is called a fuzzy α-filter of X if it satisfies

(F1) for all $x, y \in X$, $A(x \sqcap y) \geq \min\{A(x), A(y)\}$,

(FA) for all $x, y \in X$, $\mathcal{R}(x, y) \geq \alpha$ implies $A(x) \leq A(y)$.

Definition 1.19. [34] Let $(X, \mathcal{R})$ be a fuzzy lattice and A be a fuzzy filter of $(X, \mathcal{R})$.

A is called a fuzzy prime filter if for any $x, y \in X$, $A(x \sqcup y) = \max\{A(x), A(y)\}$.

Example 1.2. Let $(X, \mathcal{R})$ be the fuzzy lattice defined in Example 1.1, and let f be a fuzzy set defined by $f : [0, 1] \to [0, 1], f(x) = x$.

f is a fuzzy prime filter on $(X, \mathcal{R})$.

1.2 Intuitionistic Fuzzy Sets as $\mathbb{L}^*$-Fuzzy Set

In [10,11] Atanassov proposed a generalization of fuzzy sets called intuitionistic fuzzy sets, which contains a degree of membership and a degree of non-membership with a sum of the two values is less than or equal to 1.

In the following, we recall some definitions and properties of the intuitionistic fuzzy sets, intuitionistic fuzzy relations, intuitionistic fuzzy lattices and their filters [10,16].

1.2.1 Intuitionistic Fuzzy Sets

Definition 1.20. [10]Let X be a given non-empty set. An intuitionistic fuzzy set in X is an expression A given by $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ where $\mu_A : X \longrightarrow [0,1]$, $\nu_A : X \longrightarrow [0,1]$ with the condition $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set A.

We will denote by $IFS(X)$ the set of all intuitionistic fuzzy subsets on X. Obviously, when $\nu_A(x) = 1 - \mu_A(x)$ for every x in X, the intuitionistic fuzzy set A is a fuzzy set.

The quantity $\pi_A(x) = 1 - (\mu_A(x) + \upsilon_A(x))$ is called the hesitation degree or the intuitionistic index of x in A.

Since the membership degree and the non-membership degree of an intuitionistic fuzzy set are elements of $[0,1]$ with a sum less than or equal to 1, we study the mathematical model illustrating this phenomenon according to the above notions, i.e., the set L^* given in the following definition.

Definition 1.21. [19] Let $\mathbb{L}^* = \{(a_1, a_2) \in [0,1]^2 \mid a_1 + a_2 \leq 1\}$ and $\leq_{L^*}$ be an order in L^* defined by $\forall (a_1, a_2), (b_1, b_2) \in \mathbb{L}^* : (a_1, a_2) \leq_{\mathbb{L}^*} (b_1, b_2) \Leftrightarrow (a_1 \leq b_1$ and $a_2 \geq b_2)$. $(\mathbb{L}, \leq_{\mathbb{L}^*})$ is a complete lattice with

$$\begin{aligned}(a_1, a_2) \wedge_{\mathbb{L}^*} (b_1, b_2) &= (\min(a_1, b_1), \max(a_2, b_2)),\\ (a_1, a_2) \vee_{\mathbb{L}^*} (b_1, b_2) &= (\max(a_1, b_1), \min(a_2, b_2)),\end{aligned}$$

$0_{\mathbb{L}^*} = (0,1)$ and $1_{\mathbb{L}^*} = (1,0)$ are the units of $\mathbb{L}^*$.

Remark 1.1. Knowing that $(\mathbb{L}^*, \leq_{\mathbb{L}^*}, \wedge_{\mathbb{L}^*}, \vee_{\mathbb{L}^*}, 0_{\mathbb{L}^*}, 1_{\mathbb{L}^*})$ is a complete lattice, it is easy to see that $(IFS_S(X), \subseteq, \cap, \cup, \emptyset, X)$ is a complete lattice.

Throughout the next, we denote by $\mathbb{L}_0^* = \mathbb{L}^* - \{0_{\mathbb{L}^*}\}, \mathbb{L}_1^* = \mathbb{L}^* - \{1_{\mathbb{L}^*}\}$

Definition 1.22. The following expressions are defined in [10,12], for all $A, B \in IFS(X)$

1. $A \subseteq B$ if and only if $A(x) \leq_{\mathbb{L}^*} B(x)$, for all $x \in X$.

   In more detail, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$.

2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

3. $A \cap B = \{\langle x, A(x) \wedge_{\mathbb{L}^*} B(x)\rangle \mid x \in X\} = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$.

4. $A \cup B = \{\langle x, A(x) \vee_{\mathbb{L}^*} B(x)\rangle \mid x \in X\} = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

5. $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$.

6. $S(A) = \{x \in X \mid A(x) \geq_{\mathbb{L}^*} 0_{\mathbb{L}^*}\} = \{x \in X \mid \mu_A(x) > 0$ or $(\mu_A(x) = 0$ and $\nu_A(x) < 1)\}$.

7. $\ker(A) = \{x \in X \mid A(x) = 1_{\mathbb{L}^*}\} = \{x \in X \mid \mu_A(x) = 1$ and $\nu_A(x) = 0\}$.

8. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{L}_0^*$, the α-cut of A given by $A_\alpha = \{x \in X \mid A(x) \geq_{\mathbb{L}^*} \alpha\}$.

   In more detail, $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha_1$ and $\nu_A(x) \leq \alpha_2\}$.

9. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$.

10. $\lozenge A = \{\langle x, 1 - \upsilon_A(x), \upsilon_A(x)\rangle \mid x \in X\}$.

11. $D_\lambda(A) = \{\langle x, \mu_A(x) + \lambda.\pi_A(x), \upsilon_A(x) + (1-\lambda).\pi_A(x)\rangle \mid x \in X\}$, where λ is a fixed number in $[0,1]$.

12. $F_\alpha(A) = \{\langle x, \mu_A(x) + \alpha_1\pi_A(x), \upsilon_A(x) + \alpha_2\pi_A(x)\rangle \mid x \in X\}$, where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{L}^*$.

1.2.2 Intuitionistic Fuzzy Relations

Definition 1.23. [17] Let X, Y be non-empty universes. An intuitionistic fuzzy relation $\mathcal{R}$ from X to Y(for short, IFR) is an intuitionistic fuzzy subset of $X \times Y$ given by the expression $\mathcal{R} = \{\langle (x,y), \mu_{\mathcal{R}}(x,y), \upsilon_{\mathcal{R}}(x,y)\rangle \mid (x,y) \in X \times Y\}$, where $\mu_{\mathcal{R}} : X \times Y \longrightarrow [0,1]$ and $\upsilon_{\mathcal{R}} : X \times Y \longrightarrow [0,1]$ satisfy the condition $\mu_{\mathcal{R}}(x,y) + \upsilon_{\mathcal{R}}(x,y) \leq 1$ for every $(x,y) \in X \times Y$.

In particular, if $\mathcal{R}$ is an intuitionistic fuzzy relation from X to itself, then $\mathcal{R}$ is called an intuitionistic fuzzy binary relation on X, and we will denote the set of all intuitionistic fuzzy relations on X by: $IFR(X)$.

Various operations are defined on intuitionistic fuzzy relations (see, e.g., [16,17]).

Definition 1.24. [17] An intuitionistic fuzzy relation $\mathcal{R}$ on a non-empty set X is said to be

1. reflexive if and only if for all $x \in X,\ \mu_{\mathcal{R}}(x,x) = 1$ and $\upsilon_{\mathcal{R}}(x,y) = 0$;

2. anti-symmetrical intuitionistic, if for every $(x,y) \in X \times X$, $x \neq y$, then
$$\begin{cases} \mu_{\mathcal{R}}(x,y) \neq \mu_{\mathcal{R}}(y,x), \\ \upsilon_{\mathcal{R}}(x,y) \neq \upsilon_{\mathcal{R}}(y,x), \\ \pi_{\mathcal{R}}(x,y) = \pi_{\mathcal{R}}(y,x); \end{cases}$$

3. perfect anti-symmetrical intuitionistic, if for every $(x,y) \in X \times X$ with $x \neq y$ and $\mathcal{R}(x,y) > 0_{\mathbb{L}^*}$, then $\mathcal{R}(x,y) = 0_{\mathbb{L}^*}$. i.e.,
$$\begin{cases} \mu_{\mathcal{R}}(x,y) > 0, \\ \text{or} \\ \mu_{\mathcal{R}}(x,y) = 0 \text{ and } \upsilon_{\mathcal{R}}(x,y) < 1, \end{cases} \quad \text{then} \begin{cases} \mu_{\mathcal{R}}(y,x) = 0, \\ \quad \text{and} \\ \upsilon_{\mathcal{R}}(y,x) = 1; \end{cases}$$

4. transitive if and only if for all $x, y, z \in X$,
$$\begin{cases} \mu_{\mathcal{R}}(x,y) \wedge \mu_{\mathcal{R}}(y,z) \leq \mu_{\mathcal{R}}(x,z), \\ \qquad \text{and} \\ \upsilon_{\mathcal{R}}(x,y) \vee \upsilon_{\mathcal{R}}(y,z) \geq \upsilon_{\mathcal{R}}(x,z). \end{cases}$$

Definition 1.25. [45] A reflexive, perfect antisymmetric and transitive intuitionistic fuzzy relation is called an intuitionistic fuzzy perfect partial ordering relation. An intuitionistic fuzzy perfect partial order relation $\mathcal{R}$ is an intuitionistic fuzzy perfect total order relation if and only if ($\mu_{\mathcal{R}}(x,y) > 0$ or ($\mu_{\mathcal{R}}(x,y) = 0$ and $\upsilon_{\mathcal{R}}(x,y) < 1$)) or ($\mu_{\mathcal{R}}(y,x) > 0$ or ($\mu_{\mathcal{R}}(y,x) = 0$ and $\upsilon_{\mathcal{R}}(y,x) < 1$)) for all $x, y \in X$. A set equipped with an intuitionistic fuzzy perfect partial order relation is called an intuitionistic fuzzy perfect poset.

1.2.3 Intuitionistic Fuzzy Lattices

The idea of an intuitionistic fuzzy lattice was first put forward by Thomas and Nair [39] as an intuitionistic fuzzy set on a crisp lattice stable by the supremum and the infimum of the binary operations $\sqcup$ and $\sqcap$.

Definition 1.26. [39] Let $(X, \leqslant)$ be a crisp lattice, and let L be an intuitionistic fuzzy set on X. We say that L is an intuitionistic fuzzy lattice if the following conditions are satisfied for any $x, y \in X$:

1. $L(x \vee y) \geq_{\mathbb{L}^*} L(x) \wedge_{\mathbb{L}^*} L(y)$,

2. $L(x \wedge y) \geq_{\mathbb{L}^*} L(x) \wedge_{\mathbb{L}^*} L(y)$.

For additional information about intuitionistic fuzzy lattices, see [6,39].

Definition 1.27. [39] Let $(X, \leqslant)$ be a crisp lattice, and let A be an intuitionistic fuzzy set on X. We say that A is an intuitionistic fuzzy filter if the following conditions are satisfied for any $x, y \in X$:

1. $A(x \vee y) \geq_{\mathbb{L}^*} A(x) \vee_{\mathbb{L}^*} A(y)$,

2. $A(x \wedge y) \geq_{\mathbb{L}^*} A(x) \wedge_{\mathbb{L}^*} A(y)$.

Definition 1.28. [33] Let $(X, \leqslant)$ be a crisp lattice, and let A be an intuitionistic fuzzy filter on $(X, \mathcal{R})$. We say that A is a prime intuitionistic fuzzy filter if for any $x, y \in X$,

$$A(x \vee y) \leq_{\mathbb{L}^*} A(x) \vee_{\mathbb{L}^*} A(y).$$

1.2.4 Intuitionistic Fuzzy Ordered Lattice

Here, we review some elementary ideas concerning intuitionistic fuzzy ordered lattices. For more details, see [40].

Definition 1.29. [40] Let $(X, \mathcal{R})$ be an intuitionistic fuzzy ordered set and let $x \in X$. The intuitionistic fuzzy sets $\mathcal{R}_{\geq[x]}$ and $\mathcal{R}_{\leq[x]}$ defined on X by

$$\mathcal{R}_{\geq[x]} = \{\langle y, \mu_{\mathcal{R}}(x, y), \upsilon_{\mathcal{R}}(x, y)\rangle \mid y \in X\},$$

$$\mathcal{R}_{\leq[x]} = \{\langle y, \mu_{\mathcal{R}}(y, x), \upsilon_{\mathcal{R}}(y, x)\rangle \mid y \in X\}.$$

are called, respectively, the dominating class of x and the class dominated by x.

Definition 1.30. [40] Let $(X, \mathcal{R})$ be an intuitionistic fuzzy ordered set and let A be a non-empty subset of X.

1. The set of upper bounds of A is the intuitionistic fuzzy subset of X defined for all $y \in X$ by

$$U(\mathcal{R}, A)(y) = \underset{x \in A}{\cap} \mathcal{R}_{\geq[x]}(y).$$

2. The set of lower bounds of A is the intuitionistic fuzzy subset of X defined for all $y \in X$ by

$$L(\mathcal{R}, A)(y) = \underset{x \in A}{\cap} \mathcal{R}_{\leq [x]}(y).$$

Definition 1.31. [40] Let $(X, \mathcal{R})$ be an intuitionistic fuzzy ordered set and let A be a non-empty subset of X.

An element $x \in X$ is called the least upper bound (or a supremum) of A if:

1. $x \in S(U(\mathcal{R}, A))$,

2. for any $y \in S(U(\mathcal{R}, A))$, $\mu_{\mathcal{R}}(x, y) > 0$ or $(\mu_{\mathcal{R}}(x, y) = 0$ and $\upsilon_{\mathcal{R}}(x, y) < 1)$.

An element $x \in X$ is called the greatest lower bound (or an infimum) of A if:

1. $x \in S(L(\mathcal{R}, A))$,

2. for any $y \in S(L(\mathcal{R}, A))$, $\mu_{\mathcal{R}}(y, x) > 0$ or $(\mu_{\mathcal{R}}(y, x) = 0$ and $\upsilon_{\mathcal{R}}(y, x) < 1)$.

Definition 1.32. [40] An intuitionistic fuzzy poset $(X, \mathcal{R})$ is an intuitionistic fuzzy lattice if and only if a supremum and an infimum of each pair of elements $\{x, y\}$ exist for all $x, y \in X$.

Example 1.3. Let $X = \{a, b, c, d, e\}$ and let $\mathcal{R} : X \times X \to [0, 1]$ be an intuitionistic fuzzy relation defined by

$\mathcal{R}$	a	b	c	d	e
a	$(1.00, 0.00)$	$(0.00, 1.00)$	$(0.00, 0.40)$	$(0.55, 0.45)$	$(0.40, 0.25)$
b	$(0.00, 0.30)$	$(1.00, 0.00)$	$(0.0, 0.20)$	$(0.35, 0.35)$	$(0.45, 0.10)$
c	$(0.00, 1.00)$	$(0.00, 1.00)$	$(1.00, 0.00)$	$(0.00, 0.85)$	$(0.70, 0.15)$
d	$(0.00, 1.00)$	$(0.00, 1.00)$	$(0.00, 1.00)$	$(1.00, 0.00)$	$(0.00, 1.00)$
e	$(0.00, 1.00)$	$(0.00, 1.00)$	$(0.00, 1.00)$	$(0.00, 0.90)$	$(1.00, 0.00)$

It is easy to see that $(X, \mathcal{R})$ is an intuitionistic fuzzy lattice.

Definition 1.33. [40] Let $(X, \mathcal{R})$ be an intuitionistic fuzzy ordered set.

1. An element $\top \in X$ is called the greatest element (the maximum) of X if for all $x \in X$:

$$\begin{cases} \mu_{\mathcal{R}}(x, \top) > 0 \\ \text{or} \\ (\mu_{\mathcal{R}}(x, \top) = 0 \text{ and } \upsilon_{\mathcal{R}}(x, \top) < 1). \end{cases}$$

2. An element $\perp \in X$ is called the smallest element (the minimum) of X if for all $x \in X$:

$$\begin{cases} \mu_{\mathcal{R}}(\perp, x) > 0 \\ \text{or} \\ (\mu_{\mathcal{R}}(\perp, x) = 0 \text{ and } \upsilon_{\mathcal{R}}(\perp, x) < 1) . \end{cases}$$

Definition 1.34. [33] Let $(X, \mathcal{R})$ be an intuitionistic fuzzy ordered lattice and let A be an intuitionistic fuzzy subset on X. Then, we say that A is an intuitionistic fuzzy filter on $(X, \mathcal{R})$ if the following conditions are satisfied for any $x, y \in X$:

1. $A(x \sqcap y) \geq_{\mathbb{L}^*} A(x) \wedge_{\mathbb{L}^*} A(y)$,

2. $A(x) \geq_{\mathbb{L}^*} A(y) \wedge_{\mathbb{L}^*} \mathcal{R}(y, x)$.

Definition 1.35. Let $(X, \mathcal{R})$ be an intuitionistic fuzzy ordered lattice. A non-empty intuitionistic fuzzy set A on X is called a proper intuitionistic fuzzy set, if there exists $x \in X$ such that $\mu_A(x) > 0$ or $\upsilon_A(x) < 1$.

Definition 1.36. [33] Let $(X, \mathcal{R})$ be an intuitionistic fuzzy ordered lattice and let A be a proper intuitionistic fuzzy filter on $(X, \mathcal{R})$. We say that A is a prime intuitionistic fuzzy filter on $(X, \mathcal{R})$ if for any $x, y \in X$,

$$A(x \sqcap y) \leq_{\mathbb{L}^*} A(x) \vee_{\mathbb{L}^*} A(y).$$

CHAPTER 2

BACKGROUND ON PICTURE FUZZY SETS

In the field of fuzzy sets, optimizing conceptual frameworks is central to enhancing theoretical understanding and application. This thesis, specifically this chapter, undertakes a critical examination of picture fuzzy sets [20], an important extension of fuzzy sets and intuitionistic fuzzy sets aimed at enhancing the precision of uncertainty representation. It highlights the need for clarity and consistency in its foundational definitions, as some contradictions in specific definitions were found during research.

The first section is dedicated to addressing these contradictions. It reexamines and scrutinizes the basic definitions of picture fuzzy sets introduced by Cuong, suggesting corrections and presenting new results. This section not only critiques the current state of definitions but also identifies the motivations for redefining the operations of picture fuzzy sets, supported by a comprehensive analysis of Cuong's definitions, complemented by counterexamples and proofs. This analysis reveals the necessity of theoretical reform to resolve the identified contradictions.

A pivotal aspect of this comprehensive reform is the detailed study of the $\mathbb{D}^*$ set structure, which serves as a prototype for picture fuzzy sets. Understanding the structural dynamics of $\mathbb{D}^*$ facilitates the redefinition of operations for picture fuzzy sets, ensuring compatibility with logical coherence and practical applicability. This study introduces new definitions and properties related to the operations of picture fuzzy sets, significantly contributing to the theoretical foundation of this field.

The second section explores the extension of Atanassov's modal operators to picture fuzzy sets, focusing on four specific factors: necessity and possibility operators, the operators D_λ and F_α. Furthermore, this section delves into the characteristic sets of a picture fuzzy set, including support, kernel, α-cuts and strong α-cuts, alongside the picture fuzzy line of degree α. Finally, the chapter introduces some decomposition theorems for a picture fuzzy set.

This chapter aims to illuminate the complexities and potentials of picture fuzzy sets, advocating for a more coherent and robust framework that resolves existing contradictions and lays the groundwork for future advancements. Through a meticulous reconsideration of basic concepts.

2.1 Picture Fuzzy Sets: Another Point of View

Recently, V. Kreinovich and B. C. Cuong in [20] have put forward the concept of picture fuzzy sets as a generalization of fuzzy sets and intuitionistic fuzzy sets to overcome further uncertainties. This section discusses and modifies some basic properties of Cuong's definitions to ensure their consistency.

2.1.1 Concepts Related to Picture Fuzzy Sets According to Cuong's Definitions

Here, we present the concept of picture fuzzy sets and some of their operations, as defined by Cuong that are available in [20].

Definition 2.1. Let X be a non-empty set. A picture fuzzy set (PFS, for short) A on X is an object of the form

$$A = \{\langle x, \mu_A(x), \eta_A(x), \upsilon_A(x)\rangle \mid x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called the degree of positive membership of x in A,
$\eta_A(x) \in [0,1]$ is called the degree of neutral membership of x in A,
$\upsilon_A(x) \in [0,1]$ is called the degree of negative membership of x in A.
μ_A, η_A and υ_A satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \upsilon_A(x) \leq 1, \text{ for any } x \in X.$$

The quantity $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \upsilon_A(x))$ is called the degree of refusal membership of x in A.

Notation 2.1. Let X be a non-empty set. The set of all the picture fuzzy set on X will be denoted by $PFS(X)$.

Example 2.1. Let $X = \{a, b, c\}$ be a universal set. The subsets

$A = \{\langle a, 0.54, 0.21, 0.12\rangle, \langle b, 0.32, 0.25, 0.01\rangle, \langle c, 0.10, 0.31, 0.51\rangle\}$,
$B = \{\langle a, 0.34, 0.29, 0.34\rangle, \langle b, 0.08, 0.21, 0.71\rangle, \langle c, 0.45, 0.21, 0.27\rangle\}$,
and $C = \{\langle a, 0.51, 0.19, 0.22\rangle, \langle b, 0.11, 0.02, 0.66\rangle, \langle c, 0.01, 0.31, 0.53\rangle\}$
are picture fuzzy sets on X.

Example 2.2. Consider diagnosing pneumonia based on the following symptoms:
x_1 : Cough,
x_2 : Fever,
x_3 : Shortness of breath,
x_4 : Severe fatigue.

Let $X = \{x_1, x_2, x_3, x_4\}$ be the universal set of symptoms, and we define the following subsets:

A represents the evaluation results of the symptoms from a doctor, and B represents the evaluation results of the symptoms using laboratory test results, where
$A = \{\langle x_1, 0.85, 0.10, 0.05\rangle, \langle x_2, 0.90, 0.05, 0.05\rangle, \langle x_3, 0.75, 0.15, 0.10\rangle, \langle x_4, 0.70, 0.20, 0.10\rangle\}$,
$B = \{\langle x_1, 0.88, 0.07, 0.05\rangle, \langle x_2, 0.92, 0.03, 0.05\rangle, \langle x_3, 0.78, 0.12, 0.10\rangle, \langle x_4, 0.72, 0.18, 0.10\rangle\}$.
A and B are picture fuzzy sets on X.

Definition 2.2. [20] Let X be a non-empty set and let $A, B \in PFS(X)$.

- $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, $\eta_A(x) \leq \eta_B(x)$ and $\upsilon_A(x) \geq \upsilon_B(x)$, for all $x \in X$.
- $A = B$ if and only if ($A \subseteq B$ and $B \subseteq A$).
- $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \max\{\upsilon_A(x), \upsilon_B(x)\}\rangle \mid x \in X\}$.
- $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \min\{\upsilon_A(x), \upsilon_B(x)\}\rangle \mid x \in X\}$.
- $A^{\mathcal{C}} = \{\langle x, \upsilon_A(x), \eta_A(x), \mu_A(x)\rangle \mid x \in X\}$.

Example 2.3. Consider the picture fuzzy sets given in the previous example, then

- for all $x \in X, \mu_C(x) \leq \mu_A(x), \eta_C(x) \leq \eta_A(x)$ and $\upsilon_C(x) \geq \upsilon_A(x)$.

  Hence $C \subseteq A$, $A \cap C = C$ and $A \cup C = A$.
- $A \cap B = \{\langle x_1, 0.34, 0.21, 0.34\rangle, \langle x_2, 0.08, 0.21, 0.71\rangle, \langle x_3, 0.10, 0.21, 0.51\rangle\}$.
- $A \cup B = \{\langle x_1, 0.54, 0.21, 0.12\rangle, \langle x_2, 0.32, 0.21, 0.01\rangle, \langle x_3, 0.45, 0.21, 0.27\rangle\}$.

- $A^{\mathcal{C}} = \{\langle x_1, 0.12, 0.21, 0.54\rangle, \langle x_2, 0.01, 0.25, 0.32\rangle, \langle x_3, 0.51, 0.31, 0.10\rangle\}$.
- $B^{\mathcal{C}} = \{\langle x_1, 0.34, 0.29, 0.34\rangle, \langle x_2, 0.71, 0.21, 0.08\rangle, \langle x_3, 0.27, 0.21, 0.45\rangle\}$.
- $C^{\mathcal{C}} = \{\langle x_1, 0.22, 0.19, 0.51\rangle, \langle x_2, 0.66, 0.02, 0.11\rangle, \langle x_3, 0.53, 0.31, 0.01\rangle\}$.

2.1.2 Motivations for the Revision of the Definitions of Picture Fuzzy Sets Operations

In what follows, Cuong's definitions of picture fuzzy sets and associated operations are discussed, along with some counter-examples and proofs.

According to Definition 2.2 and respecting the two conditions:

- for all $A \in PFS(X)$, $\emptyset \subseteq A$.
- for all $A \in PFS(X)$, $A \subseteq X$.

The definitions of empty and universal picture fuzzy sets should be, respectively, as follows:

- $\emptyset = \{\langle x, 0, 0, 1\rangle \mid x \in X\}$,
- $X = \{\langle x, 1, 1, 0\rangle \mid x \in X\}$.

There is a contradiction in Definition 2.1 since $1 + 1 + 0 \nleq 1$, i.e., $X \notin PFS(X)$ and $\emptyset^{\mathcal{C}} = \{\langle x, 1, 0, 0\rangle \mid x \in X\} \neq X$.

This also leads to a series of contradictions, such as

- $A \cup \emptyset \neq A$ and $A \cup X \neq X$.
- $A \nsubseteq A \cup B, B \nsubseteq A \cup B$.

Example 2.4. Let $X = \{a, b, c\}$ and let $A, B \in PFS(X)$, where

$A = \{\langle a, 0.01, 0.30, 0.52\rangle, \langle b, 0.02, 0.11, 0.36\rangle, \langle c, 0.13, 0.40, 0.32\rangle\}$,

$B = \{\langle a, 0.01, 0.35, 0.52\rangle, \langle b, 0.28, 0.33, 0.15\rangle, \langle c, 0.21, 0.00, 0.09\rangle\}$.

Then

$A \cup \emptyset = \{\langle a, 0.01, 0.00, 0.52\rangle, \langle b, 0.02, 0.00, 0.36\rangle, \langle c, 0.13, 0.00, 0.32\rangle\} \neq A$,

$A \cup X = \{\langle a, 1.00, 0.30, 0.00\rangle, \langle b, 1.00, 0.11, 0.00\rangle, \langle c, 1.00, 0.40, 0.00\rangle\} \neq X$.

$A \cup B = \{\langle a, 0.01, 0.30, 0.52\rangle, \langle b, 0.28, 0.11, 0.15\rangle, \langle c, 0.21, 0.00, 0.09\rangle\}$.

We can see that

- $A \nsubseteq A \cup B$, since $\eta_A(c) \geq \eta_{A\cup B}(c)$.
- $B \nsubseteq A \cup B$, since $\eta_B(a) \geq \eta_{A\cup B}(a)$ and $\eta_B(b) \geq \eta_{A\cup B}(b)$.

So, picture fuzzy sets operations should be redefined to overcome these contradictions. The best way to avoid these contradictions is the detailed study of the $\mathbb{D}^*$ set structure.

2.1.3 Structure of the Set $\mathbb{D}^*$

The following section is essential to this thesis because it leads to a good understanding of the properties of picture fuzzy sets.

According to [21,27,35], we consider the set $\mathbb{D}^*$ defined by:

$$\mathbb{D}^* = \left\{x = (x_1, x_2, x_3) \in [0,1]^3, x_1 + x_2 + x_3 \leq 1\right\}.$$

This set plays the role of a prototype of a picture fuzzy set, and the study of this set allows us to perform picture fuzzy sets operations using these of $\mathbb{D}^*$.

Obviously, any picture fuzzy set:

$$A = \{\langle x, \mu_A(x), \eta_A(x), \upsilon_A(x)\rangle \mid x \in X\},$$

corresponds to a $\mathbb{D}^*$-fuzzy subset, i.e., a mapping $A : X \longrightarrow \mathbb{D}^*$ given by

$$A(x) = (\mu_A(x), \eta_A(x), \upsilon_A(x)) \in \mathbb{D}^*.$$

We'll suppose that for all $x \in \mathbb{D}^*$, x_1, x_2 and x_3, respectively, refer to the first, second and third components of x, i.e., $x = (x_1, x_2, x_3)$.

Order of $\mathbb{D}^*$

Definition 2.3. According to [22,27,28,35], we define the order relation $\preceq$ on $\mathbb{D}^*$ by for all $x, y \in \mathbb{D}^*$,

$$x \preceq y \text{ if and only if } (x_1, x_3) <_{\mathbb{L}^*} (y_1, y_3) \text{ or } ((x_1, x_3) = (y_1, y_3) \text{ and } x_2 \leq y_2),$$

i.e., $(x_1 < y_1$ and $x_3 \geq y_3)$ or $(x_1 = y_1$ and $x_3 > y_3)$ or $(x_1 = y_1$, $x_3 = y_3$ and $x_2 \leq y_2)$.

Proposition 2.1. *The components of non-comparable elements $x, y \in \mathbb{D}^*$ check $(x_1 > y_1$ and $x_3 > y_3)$ or $(x_1 < y_1$ and $x_3 < y_3)$, and we write $x \parallel y$.*

Proposition 2.2. *[21] $(\mathbb{D}^*, \preceq)$ is a bounded lattice with top element $1_{\mathbb{D}^*} = (1, 0, 0)$ and bottom element $0_{\mathbb{D}^*} = (0, 0, 1)$. And for each $x, y \in \mathbb{D}^*$, $x \curlywedge y$ and $x \curlyvee y$ defined as follows*

$$x \curlywedge y = \begin{cases} x & \text{if } x \preceq y, \\ y & \text{if } y \preceq x, \\ (x_1 \wedge y_1, 1 - (x_1 \wedge y_1) - (x_3 \vee y_3), x_3 \vee y_3) & \text{otherwise.} \end{cases}$$

$$x \curlyvee y = \begin{cases} y & \text{if } x \preceq y, \\ x & \text{if } y \preceq x, \\ (x_1 \vee y_1, 0, x_3 \wedge y_3) & \text{otherwise.} \end{cases}$$

Next, we will give some basic properties for the order of $\mathbb{D}^*$ that will be useful in the sequel.

Remark 2.1. Let $x, y \in \mathbb{D}^*$.

- If $x_1 \neq 0$, then $x_3 \neq 1$.
- $x \succ y$ is equivalent to $(x_1 > y_1$ and $x_3 \leq y_3)$ or $(x_1 = y_1$ and $x_3 < y_3)$ or $(x_1 = y_1$, $x_3 = y_3$ and $x_2 > y_2)$.
- $x \succ 0_{\mathbb{D}^*}$ is equivalent to $x_1 > 0$ or $(x_1 = 0$ and $x_3 < 1)$.

Proposition 2.3. *[35] Let $x, y, z \in \mathbb{D}^*$. Then*

(i) *$x \curlywedge y \preceq x$, $x \curlywedge y \preceq y$.*

(ii) *$x \preceq x \curlyvee y$, $y \preceq x \curlyvee y$.*

(iii) *$x \curlywedge y \preceq x \curlyvee y$.*

(iv) *$x \succ 0_{\mathbb{D}^*}$ and $y \succ 0_{\mathbb{D}^*}$ if and only if $x \curlywedge y \succ 0_{\mathbb{D}^*}$.*

(v) *$x \succeq z$ and $y \succeq z$ if and only if $x \curlywedge y \succeq z$.*

(vi) *$x \succ 0_{\mathbb{D}^*}$ or $y \succ 0_{\mathbb{D}^*}$ if and only if $x \curlyvee y \succ 0_{\mathbb{D}^*}$.*

(vii) *If $x \succeq z$ or $y \succeq z$, then $x \curlyvee y \succeq z$.*

(viii) *$x \preceq z$ and $y \preceq z$ if and only if $x \curlyvee y \preceq z$.*

(ix) *If $y \preceq z$, then $x \curlyvee y \preceq x \curlyvee z$ and $x \curlywedge y \preceq x \curlywedge z$.*

(x) $x \curlywedge (y \curlyvee z) \succeq (x \curlywedge y) \curlyvee (x \curlywedge z)$.

Proof. Let $x = (x_1, x_2, x_3)\,, y = (y_1, y_2, y_3)\,, z = (z_1, z_2, z_3)\,, t = (t_1, t_2, t_3) \in \mathbb{D}^*$.

(i) There are three cases to distinguish,

case 01 $x \preceq y$. It follows that $x \curlywedge y = x$, hence $x \curlywedge y \preceq x$ and $x \curlywedge y \preceq y$.

case 02 $y \preceq x$. It follows that $x \curlywedge y = y$, hence $x \curlywedge y \preceq x$ and $x \curlywedge y \preceq y$.

case 03 $x \parallel y$ means $(x_1 > y_1$ and $x_3 > y_3)$ or $(x_1 < y_1$ and $x_3 < y_3)$.

For the first sub-case, obtain

$x \curlywedge y = (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3) = (y_1, 1 - y_1 - x_3, x_3)\,.$

$$\text{Then} \begin{cases} x_1 \wedge y_1 = y_1 < x_1 \text{ and } x_3 \vee y_3 = x_3, \text{ hence } x \curlywedge y \preceq x, \\ \text{and} \\ x_1 \wedge y_1 = y_1 \text{ and } x_3 \vee y_3 = x_3 > y_3, \text{ hence } x \curlywedge y \preceq y. \end{cases}$$

Similarly, we obtain the same result in the second sub-case.

Therefore, we conclude that $x \curlywedge y \preceq x$ and $x \curlywedge y \preceq y$, for all $x, y \in \mathbb{D}^*$.

The proofs of (**ii**) and **(iii)** are similar to the proof given in (**i**).

(iv) Suppose that $x \succ 0_{\mathbb{D}^*}, y \succ 0_{\mathbb{D}^*}$. Then

$$x \curlywedge y = \begin{cases} x, & \text{if } x \preceq y, \\ y, & \text{if } y \preceq x, \\ (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3)\,, & \text{otherwise.} \end{cases}$$

The result is clear if $x \curlywedge y = x$ or $x \curlywedge y = y$, it remains to prove that the property is true in the case $x \curlywedge y = (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3)\,.$

Since $x \succ 0_{\mathbb{D}^*}$ and $y \succ 0_{\mathbb{D}^*}$, it follow that

$$\begin{cases} x_1 > 0 \text{ and } x_3 < 1, & (a) \\ \text{or} \\ x_1 = 0 \text{ and } x_3 < 1. & (b) \end{cases} \quad \text{and} \quad \begin{cases} y_1 > 0 \text{ and } y_3 < 1, & (a') \\ \text{or} \\ y_1 = 0 \text{ and } y_3 < 1. & (b') \end{cases}$$

Then we distinguish four cases:

case 01 If we have (a) and (a'), i.e., $(x_1 > 0$ and $x_3 < 1)$ and $(y_1 > 0$ and $y_3 < 1)$, then $x_1 \wedge y_1 > 0$ and $x_3 \vee y_3 < 1$. It follows that $x \curlywedge y \succ 0_{\mathbb{D}^*}$.

case 02 If we have (a) and (b'), i.e., $(x_1 > 0$ and $x_3 < 1)$ and $(y_1 = 0$ and $y_3 < 1)$, then $x_1 \wedge y_1 = 0$ and $x_3 \vee y_3 < 1$. It follows that $x \curlywedge y \succ 0_{\mathbb{D}^*}$.

case 03 If we have (b) and (a'), i.e., $(x_1 = 0$ and $x_3 < 1)$ and $(y_1 > 0$ and $y_3 < 1)$, then $x_1 \wedge y_1 = 0$ and $x_3 \vee y_3 < 1$. It follows that $x \curlywedge y \succ 0_{\mathbb{D}^*}$.

case 04 If we have (b) and (b'), i.e., $(x_1 = 0$ and $x_3 < 1)$ and $(y_1 = 0$ and $y_3 < 1)$, then $x_1 \wedge y_1 = 0$ and $x_3 \vee y_3 < 1$. It follows that $x \curlywedge y \succ 0_{\mathbb{D}^*}$.

Conversely, suppose that $x \curlywedge y \succ 0_{\mathbb{D}^*}$ and $x = 0_{\mathbb{D}^*}$. Then $x \curlywedge y = 0_{\mathbb{D}^*} \curlywedge y = 0_{\mathbb{D}^*}$, for each $y \in \mathbb{D}^*$. This is a contradiction. Thus, if $x \curlywedge y \succ 0_{\mathbb{D}^*}$ implies $x \succ 0_{\mathbb{D}^*}$ and $y \succ 0_{\mathbb{D}^*}$.

In the same way, we prove **(v)**.

(vi) Suppose that $x \succ 0_{\mathbb{D}^*}$. Since $x \curlyvee y \succeq x$, then $x \curlyvee y \succ 0_{\mathbb{D}^*}$.

Conversely, suppose that $x \curlyvee y \succ 0_{\mathbb{D}^*}$, $x = 0_{\mathbb{D}^*}$ and $y = 0_{\mathbb{D}^*}$. Then $x \curlyvee y = 0_{\mathbb{D}^*} \curlyvee 0_{\mathbb{D}^*} = 0_{\mathbb{D}^*}$. This is a contradiction. Thus, if $x \curlyvee y \succ 0_{\mathbb{D}^*}$ implies $x \succ 0_{\mathbb{D}^*}$ or $y \succ 0_{\mathbb{D}^*}$.

Similarly, we prove **(vii).**

(viii) Suppose that $x \preceq z$ and $y \preceq z$.

The result is clear if $x \curlyvee y = x$ or $x \curlyvee y = y$.

If $x \curlyvee y = (x_1 \vee y_1, 0, x_3 \wedge y_3)$, then $x_1 \vee y_1 \leq z_1$, $x_3 \wedge y_3 \geq z_3$ and $0 \leq z_2$. Hence $x \curlyvee y \preceq z$.

Conversely, suppose that $x \curlyvee y \preceq z$.

Since $x \preceq x \curlyvee y$ and $y \preceq x \curlyvee y$, then $x \preceq z$ and $y \preceq z$.

(ix) Suppose that $y \preceq z$.

Since $x \preceq x \curlyvee z$ and $y \preceq z \preceq x \curlyvee z$, then from **(viii)** $x \curlyvee y \preceq x \curlyvee z$.

Since $x \curlywedge y \preceq x$ and $x \curlywedge y \preceq y \preceq z$, then from **(v)** $x \curlywedge y \preceq x \curlywedge z$.

(x) Using **(i)**, **(viii)** and **(v)**, we get $x \curlywedge (y \curlyvee z) \succeq (x \curlywedge y) \curlyvee (x \curlywedge z)$.

□

Remark 2.2. Generally, the converse implication in **(vii)** is not true. indeed,

let $x = (0.2, 0.4, 0.3)\,, y = (0.1, 0.3, 0.2)\,, z = (0.2, 0.5, 0.3)\,.$

$x \curlyvee y = (0.2, 0, 0.2) \succeq (0.2, 0.5, 0.3) = z$, but $x \preceq z$ and $y \parallel z$.

Picture Fuzzy Negators on $\mathbb{D}^*$

Picture fuzzy negators on $\mathbb{D}^*$ are an extension of fuzzy negators and intuitionistic fuzzy negators and are defined as follows:

Definition 2.4. [21] A picture fuzzy negator is any non-increasing mapping $N : \mathbb{D}^* \longrightarrow \mathbb{D}^*$ satisfying $N(0_{\mathbb{D}^*}) = 1_{\mathbb{D}^*}$ and $N(1_{\mathbb{D}^*}) = 0_{\mathbb{D}^*}$.

N is called an involutive picture fuzzy negator, if $N(N(x)) = x$, for all $x \in \mathbb{D}^*$.

Proposition 2.4. *Let $x = (x_1, x_2, x_3) \in \mathbb{D}^*$.*

The mappings N_1 and N_S defined respectively by

$N_1(x) = (x_3, x_2, x_1)$ *and* $N_S(x) = (x_3, 1 - x_1 - x_2 - x_3, x_1)$ *, for all* $x \in \mathbb{D}^*$

are involutive picture fuzzy negators and N_S is called the standard picture fuzzy negator.

Example 2.5. .

- $N_1(0.05, 0.60, 0.21) = (0.21, 0.60, 0.05)$ and $N_S(0.05, 0.60, 0.21) = (0.21, 0.14, 0.05)\,,$
- $N_1(0.00, 1.00, 0.00) = (0.00, 1.00, 0.00)$ and $N_S(0.00, 1.00, 0.00) = (0.00, 0.00, 0.00)\,,$
- $N_1(0.00, 0.00, 0.00) = (0.00, 0.00, 0.00)$ and $N_S(0.00, 0.00, 0.00) = (0.00, 1.00, 0.00)\,.$
- $N_1(1.00, 0.00, 0.00) = (0.00, 0.00, 1.00)$ and $N_S(1.00, 0.00, 0.00) = (0.00, 0.00, 1.00)\,,$
- $N_1(0.00, 0.00, 1.00) = (1.00, 0.00, 0.00)$ and $N_S(0.00, 0.00, 1.00) = (0.00, 0.00, 1.00)\,.$

2.1.4 Picture Fuzzy Sets Operations in term of $\mathbb{D}^*$ and Some of Their Properties

Picture Fuzzy Inclusion

Definition 2.5. Let X be a non-empty set and let $A, B \in PFS(X)$. We say that $A \subseteq B$, if $A(x) \preceq B(x)$ for all $x \in X$, where $\preceq$ is the order of $\mathbb{D}^*$. In more detail,

$$A \subseteq B \iff (\mu_A(x), \eta_A(x), \upsilon_A(x)) \preceq (\mu_B(x), \eta_B(x), \upsilon_B(x)),$$
$$\iff \begin{cases} \mu_A(x) < \mu_B(x) \text{ and } \upsilon_A(x) \geq \upsilon_B(x) \\ \text{or} \\ \mu_A(x) = \mu_B(x) \text{ and } \upsilon_A(x) > \upsilon_B(x) \\ \text{or} \\ \mu_A(x) = \mu_B(x), \upsilon_A(x) = \upsilon_B(x) \text{ and } \eta_A(x) \leq \eta_B(x). \end{cases}$$

Definition 2.6. Let X be a non-empty set and let $A, B \in PFS(X)$. We say that $A = B$, if $A(x) = B(x)$ for all $x \in X$.

Picture Fuzzy Intersection and Picture Fuzzy Union

Definition 2.7. [35] Let X be a non-empty set and let $A, B \in PFS(X)$. According to Proposition 2.2, we define the picture fuzzy intersection by $(A \cap B)(x) = A(x) \curlywedge B(x)$. In more detail, $A \cap B = \{\langle x, \mu_{A\cap B}(x), \eta_{A\cap B}(x), \upsilon_{A\cap B}(x)\rangle \mid x \in X\}$, where

$$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x),$$
$$\eta_{A\cap B}(x) = \begin{cases} \eta_A(x) & \text{if } A(x) \preceq B(x), \\ \eta_B(x) & \text{if } B(x) \preceq A(x), \\ 1 - \mu_A(x) \wedge \mu_B(x) - \upsilon_A(x) \vee \upsilon_B(x) & \text{otherwise.} \end{cases}$$
$$\upsilon_{A\cap B}(x) = \upsilon_A(x) \vee \upsilon_B(x).$$

And picture fuzzy union by $(A \cup B)(x) = A(x) \curlyvee B(x)$.

In more detail, $A \cup B = \{\langle x, \mu_{A\cup B}(x), \eta_{A\cup B}(x), \upsilon_{A\cup B}(x)\rangle \mid x \in X\}$, where

$$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x).$$
$$\eta_{A\cup B}(x) = \begin{cases} \eta_B(x) & \text{if } A(x) \preceq B(x), \\ \eta_A(x) & \text{if } B(x) \preceq A(x), \\ 0 & \text{otherwise.} \end{cases}$$
$$\upsilon_{A\cup B}(x) = \upsilon_A(x) \wedge \upsilon_B(x).$$

Example 2.6. Let $X = \{a, b, c\}$ and let $E, F, G \in PFS(X)$, where

$E = \{\langle a, 0.01, 0.30, 0.52\rangle, \langle b, 0.02, 0.11, 0.36\rangle, \langle c, 0.13, 0.40, 0.32\rangle\}$,

$F = \{\langle a, 0.01, 0.35, 0.52\rangle, \langle b, 0.28, 0.33, 0.15\rangle, \langle c, 0.21, 0.00, 0.09\rangle\}$,

$G = \{\langle a, 0.00, 0.44, 0.21\rangle, \langle b, 0.05, 0.51, 0.27\rangle, \langle c, 0.21, 0.07, 0.53\rangle\}$.

Note that for all $x \in X$, $E(x) \preceq F(x)$, then $E \subseteq F$. Moreover, $E \cap F = E$ and $E \cup F = F$.

On the other hand, there exists $a \in X$ such that $E(a) \| G(a)$, then neither $E \subseteq G$ nor $G \subseteq E$. Moreover, $E \cap G = \{\langle a, 0.00, 0.48, 0.52\rangle, \langle b, 0.02, 0.11, 0.36\rangle, \langle c, 0.13, 0.34, 0.53\rangle\}$ and $E \cup G = \{\langle a, 0.01, 0.00, 0.21\rangle, \langle b, 0.05, 0.51, 0.27\rangle, \langle c, 0.21, 0.00, 0.32\rangle\}$.

Note that $E \subseteq E \cup G$ and $G \subseteq E \cup G$. Also, $E \cup \emptyset = E$ and $E \cup X = X$.

Proposition 2.5. *Let X be a non-empty set and let $A, B, C \in PFS(X)$. As in classical set theory, the definitions that we have just given lead us to the following properties*

(i) $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) $A \cap B = B \cap A$, $A \cup B = B \cup A$.

(iii) $A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$.

(iv) $A \cap A = A$, $A \cup A = A$.

(v) $A \cup B \supseteq A \supseteq A \cap B$, $A \cup B \supseteq B \supseteq A \cap B$.

(vi) $A \cup \emptyset = A$, $A \cup X = X$, $\forall A \in PFS(X)$.

(vii) $A \cap \emptyset = \emptyset$ $A \cap X = A$, $\forall A \in PFS(X)$.

Proof. The proofs are direct using the properties in Proposition 2.3. □

Proposition 2.6. $(PFS(X), \subseteq, \cap, \cup, \emptyset, X)$ *is a bounded lattice.*

Picture Fuzzy Complement of a Picture Fuzzy Set

Definition 2.8. Let X be a non empty set and let $A \in PFS(X)$. Using the negators N_S and N_1 in Proposition 2.4, we define $A^{\mathcal{C}}$ and $\overline{A}$ respectively by

$$A^{\mathcal{C}} = N_S(A) = \{\langle x, \upsilon_A(x), 1 - \mu_A(x) - \eta_A(x) - \upsilon_A(x), \mu_A(x)\rangle \mid x \in X\},$$
$$\overline{A} = N_1(A) = \{\langle x, \upsilon_A(x), \eta_A(x), \mu_A(x)\rangle \mid x \in X\}.$$

Example 2.7. The complement of the picture fuzzy sets given in Example 2.6 is

$$E^{\mathcal{C}} = \{\langle a, 0.52, 0.17, 0.01\rangle, \langle b, 0.36, 0.51, 0.02\rangle, \langle c, 0.32, 0.15, 0.13\rangle\},$$
$$F^{\mathcal{C}} = \{\langle a, 0.52, 0.12, 0.01\rangle, \langle b, 0.15, 0.24, 0.28\rangle, \langle c, 0.09, 0.70, 0.21\rangle\},$$
$$G^{\mathcal{C}} = \{\langle a, 0.21, 0.35, 0.00\rangle, \langle b, 0.27, 0.17, 0.05\rangle, \langle c, 0.53, 0.19, 0.21\rangle\}.$$
$$\overline{E} = \{\langle a, 0.52, 0.30, 0.01\rangle, \langle b, 0.36, 0.11, 0.02\rangle, \langle c, 0.32, 0.40, 0.13\rangle\},$$
$$\overline{F} = \{\langle a, 0.52, 0.35, 0.01\rangle, \langle b, 0.15, 0.33, 0.28\rangle, \langle c, 0.09, 0.00, 0.21\rangle\},$$

$\overline{G} = \{\langle a, 0.21, 0.44, 0.00\rangle, \langle b, 0.27, 0.51, 0.05\rangle, \langle c, 0.53, 0.07, 0.21\rangle\}.$
$\emptyset^{\mathcal{C}} = \overline{\emptyset} = X, X^{\mathcal{C}} = \overline{X} = \emptyset.$

Unlike classical subsets, a picture fuzzy set A usually satisfies $A^{\mathcal{C}} \cap A \neq \emptyset$ and $A^{\mathcal{C}} \cup A \neq X$, however some other properties of classical set theory are satisfied, such as:

Proposition 2.7. *Let X be a non-empty set and let $A, B \in PFS(X)$. The complement of picture fuzzy sets $\mathcal{C}$ verify the following properties:*

(i) $(A \cap B)^{\mathcal{C}} = A^{\mathcal{C}} \cup B^{\mathcal{C}}$, $(A \cup B)^{\mathcal{C}} = A^{\mathcal{C}} \cap B^{\mathcal{C}}$.

(ii) $(\emptyset)^{\mathcal{C}} = X$, $(X)^{\mathcal{C}} = \emptyset$.

(iii) $\left(A^{\mathcal{C}}\right)^{\mathcal{C}} = A$.

(iv) $A \subseteq B$ *implies* $B^{\mathcal{C}} \subseteq A^{\mathcal{C}}$.

Proof. **(i)** Let $x \in X$.

$$(A \cap B)(x) = \begin{cases} A(x) & \text{if } A(x) \preceq B(x), \\ B(x) & \text{if } B(x) \preceq A(x), \\ (\mu_{A\cap B}(x), \eta_{A\cap B}(x), \upsilon_{A\cap B}(x)) & \text{otherwise.} \end{cases}$$

Where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x),$
$\eta_{A\cap B}(x) = 1 - \mu_A(x) \wedge \mu_B(x) - \upsilon_A(x) \vee \upsilon_B(x),$
$\upsilon_{A\cap B}(x) = \upsilon_A(x) \vee \upsilon_B(x).$

Hence,

$$(A \cap B)^{\mathcal{C}}(x) = \begin{cases} A^{\mathcal{C}}(x) & \text{if } B^{\mathcal{C}}(x) \preceq A^{\mathcal{C}}(x), \\ B^{\mathcal{C}}(x) & \text{if } A^{\mathcal{C}}(x) \preceq B^{\mathcal{C}}(x), \\ \{\langle x, \upsilon_A(x) \vee \upsilon_B(x), 0, \mu_A(x) \wedge \mu_B(x)\rangle \mid x \in X\} & \text{otherwise.} \end{cases}$$

In other hand,

$$A^{\mathcal{C}} = \{\langle x, \upsilon_A(x), 1 - \mu_A(x) - \eta_A(x) - \upsilon_A(x), \mu_A(x)\rangle \mid x \in X\},$$
$$B^{\mathcal{C}} = \{\langle x, \upsilon_B(x), 1 - \mu_B(x) - \eta_B(x) - \upsilon_B(x), \mu_B(x)\rangle \mid x \in X\}.$$

Hence,

$$\left(A^{\mathcal{C}} \cup B^{\mathcal{C}}\right)(x) = \begin{cases} A^{\mathcal{C}}(x) & \text{if } B^{\mathcal{C}}(x) \preceq A^{\mathcal{C}}(x), \\ B^{\mathcal{C}} & \text{if } A^{\mathcal{C}}(x) \preceq B^{\mathcal{C}}(x), \\ \{\langle x, \upsilon_A(x) \vee \upsilon_B(x), 0, \mu_A(x) \wedge \mu_B(x)\rangle \mid x \in X\} & \text{otherwise.} \end{cases}$$

Therefore, $(A \cap B)^{\mathcal{C}} = A^{\mathcal{C}} \cup B^{\mathcal{C}}$.

In a similar way, it can be shown that $(A \cup B)^{\mathcal{C}} = A^{\mathcal{C}} \cap B^{\mathcal{C}}$.

(ii) For all $x \in X$, $(\emptyset)^{\mathcal{C}}(x) = (\emptyset(x))^{\mathcal{C}} = (0,0,1)^{\mathcal{C}} = (1,0,0) = 1_{\mathbb{D}^*}$.

Hence $(\emptyset)^{\mathcal{C}} = \{\langle x, 1, 0, 0\rangle \mid x \in X\} = X$.

In the same way we obtain, $(X)^{\mathcal{C}} = \emptyset$.

Finally, **(iii)** and **(iv)** are easy to verify. □

Remark 2.3. The complement $\overline{D}$ of a picture fuzzy set D fulfils the previous properties in Proposition 2.7 except for property **(i)**, the latter being true if $A \subseteq B$ or $B \subseteq A$.

2.2 Characteristic Sets of a Picture Fuzzy Set

Among the most important notions in fuzzy set theory are the notions of support, kernel, cuts and fuzzy line of degree α of a fuzzy set, where $\alpha \in \mathbb{D}^*$.

In the following, we generalize these notions to the picture fuzzy setting with respect to the order $\preceq$ in Definition 2.3.

2.2.1 Support and Kernel of a Picture Fuzzy Set

Definition 2.9. Let X be a non empty set and let $A \in PFS(X)$. The support of A is the crisp set on X given by

$$S(A) = \{x \in X \mid A(x) \succ 0_{\mathbb{D}^*}\}.$$

According to Remark 2.1, $S(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \upsilon_A(x) < 1)\}$.

Definition 2.10. Let X be a non empty set and let $A \in PFS(X)$. The kernel of A is the crisp set on X given by

$$\ker(A) = \{x \in X \mid A(x) = 1_{\mathbb{D}^*}\},$$

i.e., $\ker(A) = \{x \in X \mid \mu_A(x) = 1, \eta_A(x) = 0 \text{ and } \upsilon_A(x) = 0\}$.

Example 2.8. Let $X = \{x_1, x_2, x_3, x_4\}$ be a universal set and let $A \in PFS(X)$ where
$A = \{\langle x_1, 1.00, 0.00, 0.00\rangle, \langle x_2, 0.32, 0.25, 0.01\rangle, \langle x_3, 0.00, 0.00, 0.10\rangle, \langle x_4, 0.10, 0.31, 0.51\rangle\}$.

Then $S(A) = \{x_1, x_2, x_4\}$, $\ker(A) = \{x_1\}$.

2.2.2 α-Cuts and Strong α-Cuts of a Picture Fuzzy Set

Definition 2.11. Let X be a non empty set and let $A \in PFS(X)$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_0^*$, the α-cut of A is the crisp subset A_α of X given by

$$A_\alpha = \{x \in X \mid A(x) \succeq \alpha\}.$$

According to Definition 2.3, $A_\alpha = \{x \in X \mid (\mu_A(x) > \alpha_1$ and $\upsilon_A(x) \leq \alpha_3)$
or $(\mu_A(x) = \alpha_1$ and $\upsilon_A(x) < \alpha_3)$ or $(\mu_A(x) = \alpha_1,\ \upsilon_A(x) = \alpha_3$ and $\eta_A(x) \geq \alpha_2)\}$.

Definition 2.12. Let X be a non empty set and let $A \in PFS(X)$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_1^*$, the strong α-cut of A is the crisp subset A_α^+ of X given by

$$A_\alpha^+ = \{x \in X \mid A(x) \succ \alpha\}.$$

According to Remark 2.1, $A_\alpha^+ = \{x \in X \mid (\mu_A(x) > \alpha_1$ and $\upsilon_A(x) \leq \alpha_3)$
or $(\mu_A(x) = \alpha_1$ and $\upsilon_A(x) < \alpha_3)$ or $(\mu_A(x) = \alpha_1,\ \upsilon_A(x) = \alpha_3$ and $\eta_A(x) > \alpha_2)\}$.

Example 2.9. Consider the picture fuzzy set A given in Example 2.8.

$$A_\alpha = \begin{cases} \{x_1, x_2, x_3, x_4\} & \text{if } \alpha \succ 0_{\mathbb{D}^*}, \\ \{x_1, x_2, x_4\} & \text{if } \alpha \succeq (0.10, 0.31, 0.51), \\ \{x_1, x_2\} & \text{if } \alpha \succeq (0.32, 0.25, 0.01), \\ \{x_1\} & \text{if } \alpha = 1_{\mathbb{D}^*}. \end{cases}$$

$$A_\alpha^+ = \begin{cases} \{x_1, x_2, x_4\} & \text{if } \alpha \succeq 0_{\mathbb{D}^*}, \\ \{x_1, x_2\} & \text{if } \alpha \succeq (0.10, 0.31, 0.51), \\ \{x_1\} & \text{if } \alpha \succeq (0.32, 0.25, 0.01). \end{cases}$$

Proposition 2.8. *Let X be a non empty set and let $A, B \in PFS(X)$. Then for all $\alpha, \beta \in \mathbb{D}^*$ the following properties hold:*

(i) $A_\alpha^+ \subseteq A_\alpha$.

(ii) $A \subseteq B$ *if and only if* $A_\alpha \subseteq B_\alpha$, *for all* $\alpha \in \mathbb{D}_0^*$.

(iii) *$A \subseteq B$ if and only if $A^+_\alpha \subseteq B^+_\alpha$, for all $\alpha \in \mathbb{D}^*_1$.*

(iv) *$\alpha \preceq \beta$ implies $A_\alpha \supseteq A_\beta$, for all $\alpha, \beta \in \mathbb{D}^*_0$.*

(v) *$\alpha \preceq \beta$ implies $A^+_\alpha \supseteq A^+_\beta$, for all $\alpha, \beta \in \mathbb{D}^*_1$.*

(vi) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.

(vii) $(A \cup B)_\alpha \supseteq A_\alpha \cup B_\alpha$.

Proof. Let $x \in X$.

(i) Clear.

(ii) Suppose that $A \subseteq B$. Let $x \in A_\alpha$, then $A(x) \succeq \alpha$. Since $B(x) \succeq A(x)$ for all $x \in X$, it follows that $B(x) \succeq \alpha$. Thus $x \in B_\alpha$.

Conversely, suppose that $A_\alpha \subseteq B_\alpha$.

Put $A(x) = \alpha$. It clear that if $\alpha = 0_{\mathbb{D}^*}$, $B(x) \succeq 0_{\mathbb{D}^*}$.

If $\alpha \neq 0_{\mathbb{D}^*}$, then for all $x \in A_\alpha$ implies $x \in B_\alpha$. Thus $B(x) \succeq \alpha = A(x)$, for all $x \in X$. Hence $A \subseteq B$.

(iii) The direct implication is similar to the previous proof.

For the converse implication, suppose for a contradiction that $A^+_\alpha \subseteq B^+_\alpha$, for all $\alpha \in \mathbb{D}^*_1$ but that $A \nsubseteq B$.

Then there exists $x \in X$ such that $A(x) \succ B(x)$ or $A(x) \| B(x)$.

If $A(x) \succ B(x)$, we can take α between $A(x)$ and $B(x)$ i.e., $A(x) \succ \alpha \succ B(x)$. This contradicts the fact that $A^+_\alpha \subseteq B^+_\alpha$.

If $A(x) \| B(x)$, we have two cases: $\mu_A(x) < \mu_B(x)$ and $\nu_A(x) < \nu_B(x)$ or $\mu_B(x) < \mu_A(x)$ and $\nu_B(x) < \nu_A(x)$.

For the first case, take $\lambda = \left(\mu_A(x), \frac{\eta_A(x)+\eta_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\right)$, it is clear that $\lambda < A(x)$ and $\lambda \| B(x)$. This is also a contradiction.

Similarly, we obtain the same result in the second case.

(iv) Assume $\alpha \preceq \beta$ and suppose that $x \in A_\beta$. Then $A(x) \succeq \beta \succeq \alpha$. Thus $x \in A_\alpha$.

(vi)
$$\begin{aligned}(A \cap B)_\alpha &= \{x \in X \mid (A \cap B)(x) \succeq \alpha\}\\ &= \{x \in X \mid A(x) \curlywedge B(x) \succeq \alpha\}\\ &= \{x \in X \mid A(x) \succeq \alpha \text{ and } B(x) \succeq \alpha\}\\ &= \{x \in X \mid A(x) \succeq \alpha\} \cap \{x \in X \mid B(x) \succeq \alpha\}\\ &= A_\alpha \cap B_\alpha.\end{aligned}$$

In the same way, we can prove **(v)** and **(vii).** □

Remark 2.4.
- Generally, the converse of **(iv)** and **(v)** are not true. Indeed,

  Let $X = \{a, b\}$ and let $A, B \in PFS(X)$ given by

  $A = \{\langle a, 1, 0, 0\rangle, \langle b, 0.55, 0.23, 0.11\rangle\}$,

  $B = \{\langle a, 0.75, 0.12, 0.02\rangle, \langle b, 1, 0, 0\rangle\}$.

  And take $\alpha = (0.61, 0.01, 0.21), \beta = (0.62, 0.01, 0.22)$. It is easy to observe that $A_\alpha = \{a\} \subseteq A_\beta = \{a\}$ and $A^+_\alpha = \{a\} \subseteq A^+_\beta = \{a\}$ but $\alpha \| \beta$.

- The converse of **(iv)** and **(v)** are true when α and β are comparable.

- As seen in Remark 2.2, the converse of **(vii)** is not true.

Corollary 2.1. **(i)** $A = B$, if and only if $A_\alpha = B_\alpha$, for all $\alpha \in \mathbb{D}^*_0$.

(ii) $A = B$, if and only if $A^+_\alpha = B^+_\alpha$, for all $\alpha \in \mathbb{D}^*_1$.

(iii) If A is a crisp subset of X, then $A_\alpha = A$, for all $\alpha \in \mathbb{D}^*_0$

(iv) if $\alpha = 0_{\mathbb{D}^*}$, then $A^+_\alpha = S(A)$ and $A_\alpha = X$.

(v) if $\alpha = 1_{\mathbb{D}^*}$, then A_α is the kernel of A.

2.2.3 Picture Fuzzy Line of Degree α of a Picture Fuzzy Set

Definition 2.13. Let X be a non empty set and let $A \in PFS(X)$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}^*$, the picture fuzzy line of degree α of A is the crisp subset $L_\alpha(A)$ of X given by

$$L_\alpha(A) = \{x \in X \mid A(x) = \alpha\}.$$

Example 2.10. Consider the picture fuzzy set A given in Example 2.8.

$$L_\alpha(A)=\begin{cases}\{x_3\} & \text{if } \alpha=0_{\mathbb{D}^*},\\ \{x_1\} & \text{if } \alpha=1_{\mathbb{D}^*},\\ \{x_2\} & \text{if } \alpha=(0.32,0.25,0.01),\\ \{x_4\} & \text{if } \alpha=(0.10,0.31,0.51),\\ \emptyset & \text{otherwise.}\end{cases}$$

Proposition 2.9. *Let X be a non empty set and let $A, B \in PFS(X)$. Then for all $\alpha, \beta \in \mathbb{D}^*$, the following properties hold.*

(i) $L_\alpha(A) \subseteq A_\alpha$, $L_{1_{\mathbb{D}^*}}(A) = Ker(A)$.

(ii) $A = B$ *if and only if* $L_\alpha(A) = L_\alpha(B)$.

(iii) *If* $\alpha \neq \beta$, *then* $L_\alpha(A) \cap L_\beta(A) = \emptyset$.

(iv) $L_\alpha(A) \cap L_\alpha(B) \subseteq L_\alpha(A \cap B)$.

(v) $L_\alpha(A) \cup L_\beta(B) \subseteq L_{\alpha \curlyvee \beta}(A \cup B)$.

(vi) $L_\alpha(A \cap B) \subseteq A_\alpha \cap B_\alpha$.

Proof. Let $x \in X$ and let $\alpha, \beta \in \mathbb{D}^*$.

(i) Clear.

(ii) Suppose that $A = B$. Let $x \in L_\alpha(A)$, then $B(x) = A(x) = \alpha$. Hence $x \in L_\alpha(B)$.

Conversely, suppose that $L_\alpha(A) = L_\alpha(B)$.

Put $A(x) = \alpha$, that is, $x \in L_\alpha(A)$. This equivalent to $x \in L_\alpha(B)$, i.e., $B(x) = \alpha$. Therefore, $A = B$.

(iii) Suppose that $\alpha \neq \beta$ and $x \in L_\alpha(A)$, that is, $A(x) = \alpha \neq \beta$, then $x \notin L_\beta(A)$. Thus $L_\alpha(A) \cap L_\beta(A) = \emptyset$.

(iv) Suppose that $x \in L_\alpha(A) \cap L_\alpha(B)$. Then $x \in L_\alpha(A)$ and $x \in L_\alpha(B)$, imply that $A(x) = \alpha$ and $B(x) = \alpha$. Hence $A(x) \curlywedge B(x) = \alpha$, this gives $(A \cap B)(x) = \alpha$. Consequently $x \in L_\alpha(A \cap B)$.

(v) Similar to **(iv).**

(vi) Suppose that $x \in L_\alpha (A \cap B)$. Then $(A \cap B)(x) = \alpha$, that is, $A(x) \curlywedge B(x) = \alpha$, it follows that $A(x) \succeq \alpha$ and $B(x) \succeq \alpha$. Thus $x \in A_\alpha$ and $x \in B_\alpha$. Hence $x \in A_\alpha \cap B_\alpha$.

□

Remark 2.5. .

- The converse of **(iii)** holds if $L_\alpha (A) \neq \emptyset$ or $L_\beta (A) \neq \emptyset$.
- Generally, the converse inclusion of **(iv)** and **(v)** are not true. Indeed,

  Let $X = \{a, b\}$ and let $A, B \in PFS(X)$ given by

  $A = \{\langle a, 0.10, 0.30, 0.40\rangle , \langle b, 0.30, 0.20, 0.10\rangle\}$,

  $B = \{\langle a, 0.20, 0.20, 0.50\rangle , \langle b, 0.20, 0.40, 0.30\rangle\}$.

  Then, $A \cap B = \{\langle a, 0.10, 0.40, 0.50\rangle , \langle b, 0.20, 0.40, 0.30\rangle\}$.

  Take $\alpha = (0.10, 0.40, 0.50)$.

  Hence $L_\alpha (A \cap B) = \{a\}$, $L_\alpha (A) = \emptyset$ and $L_\alpha (B) = \emptyset$.

  Note that $L_\alpha (A \cap B) \nsubseteq L_\alpha (A) \cap L_\alpha (B)$.

  In the same way, take $\alpha = (0.20, 0.10, 0.50)$ and $\beta = (0.08, 0.31, 0.40)$, It is easy to see that $L_{\alpha \curlyvee \beta} (A \cup B) = \{a\}$, $L_\alpha (A) = \emptyset$ and $L_\beta (B) = \emptyset$. Hence

  $L_{\alpha \curlyvee \beta} (A \cup B) \nsubseteq L_\alpha (A) \cup L_\beta (B)$.

2.2.4 Some Decomposition Theorems of a Picture Fuzzy Set

These theorems permit us to express any picture fuzzy subset of X in terms of its α-cuts, strong α-cuts and picture fuzzy lines of degree α.

Theorem 2.1. *Let X be a non empty set and let $A \in PFS(X)$. Then*

$$A(x) = \curlyvee_{\alpha \in \mathbb{D}^*} \alpha A_\alpha (x) , \text{ for all } x \in X.$$

where

$$A_\alpha (x) = \begin{cases} 1 & \text{if } x \in A_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $x \in X$.

Put $A(x) = \lambda$, where $\lambda \in \mathbb{D}_0^*$, then $A_\lambda(x) = 1$.

We can express $\mathbb{D}_0^*$ as the union of three sets D_1, D_2, D_3, where $D_1 = \{\alpha \in \mathbb{D}_0^* \mid \alpha \preceq \lambda\}$, $D_2 = \{\alpha \in \mathbb{D}_0^* \mid \alpha \succ \lambda\}$ and $D_3 = \{\alpha \in \mathbb{D}_0^* \mid \alpha \shortparallel \lambda\}$.

It holds that

$\underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha A_\alpha(x) = \left(\underset{\alpha \in D_1}{\curlyvee} \{\alpha A_\alpha(x)\} \right) \curlyvee \left(\underset{\alpha \in D_2}{\curlyvee} \{\alpha A_\alpha(x)\} \right) \curlyvee \left(\underset{\alpha \in D_3}{\curlyvee} \{\alpha A_\alpha(x)\} \right).$

If $\alpha \in D_1$, then $A_\alpha(x) = 1$. Hence $\underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha A_\alpha(x) = \underset{\alpha \in D_1}{\curlyvee} \alpha A_\alpha(x) = \underset{\alpha \in D_1}{\curlyvee} \alpha = \lambda$.

Otherwise, $A_\alpha(x) = 0$. Then $\underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha A_\alpha(x) = 0$.

Therefore, $\underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha A_\alpha(x) = \lambda \curlyvee 0 \curlyvee 0 = \lambda = A(x)$. □

Theorem 2.2. *Let X be a non empty set and let $A \in PFS(X)$. Then*

$$A(x) = \underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha A_\alpha^+(x), \textit{ for all } x \in X.$$

Proof. Similar to the previous proof, it suffice to take

$D_1 = \{\alpha \in \mathbb{D}_0^* \mid \alpha \prec \lambda\}$, $D_2 = \{\alpha \in \mathbb{D}_0^* \mid \alpha \succeq \lambda\}$ and $D_3 = \{\alpha \in \mathbb{D}_0^* \mid \alpha \shortparallel \lambda\}$. □

Theorem 2.3. *Let X be a non empty set and let $A \in PFS(X)$. Then*

$$A(x) = \underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha L_\alpha(A)(x), \textit{ for all } x \in X.$$

where

$$L_\alpha(A)(x) = \begin{cases} 1 & \textit{if } x \in L_\alpha(A), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $x \in X$.

Put $A(x) = \lambda$, where $\lambda \in \mathbb{D}_0^*$, then $L_\lambda(A)(x) = 1$.

$\underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha L_\alpha(A)(x) = \left(\underset{\alpha = \lambda}{\curlyvee} \{\alpha L_\alpha(A)(x)\} \right) \curlyvee \left(\underset{\alpha \neq \lambda}{\curlyvee} \{\alpha L_\alpha(A)(x)\} \right) = \lambda \curlyvee 0 = \lambda = A(x)$. □

Proposition 2.10. *Let X be a non empty set and let $A \in PFS(X)$. Then for all $\alpha, \lambda \in \mathbb{D}^*$,*

$$A_\alpha = \underset{\alpha \preceq \lambda}{\cup} A_\lambda.$$

Proof. Direct. □

2.3 Extension of Some of Atanassov's Modal Operators on Picture Fuzzy Sets

This section aims to build on the preliminary work accomplished regarding the redefinition of certain definitions of picture fuzzy sets by extending some operators within this framework.

In what follows, we denote by $\mathbb{D}_0^* = \mathbb{D}^* - \{0_{\mathbb{D}^*}\}, \mathbb{D}_1^* = \mathbb{D}^* - \{1_{\mathbb{D}^*}\}$.

This section extends some of Atanassov's modal operators to the picture fuzzy case.

2.3.1 Necessity and Possibility Operators

Now, we define two operators on the set of picture fuzzy sets that transform every picture fuzzy set into an intuitionistic fuzzy set. These operators extend Atanasov's operators ([10,12]) "necessity" and "possibility" defined in certain modal logics.

Definition 2.14. [35] Let X be a non empty set. For $A \in PFS(X)$, the following associated picture fuzzy sets $\Box A$ (necessity) and $\Diamond A$ (possibility) on X are defined by:

$$\Box A = \{\langle x, \mu_A(x), \eta_A(x), 1 - \eta_A(x) - \mu_A(x)\rangle \mid x \in X\}.$$

$$\Diamond A = \{\langle x, 1 - \eta_A(x) - \upsilon_A(x), \eta_A(x), \upsilon_A(x)\rangle \mid x \in X\}.$$

Example 2.11. Take $X = \mathbb{N}$ and let $A \in PFS(X)$ given by
$A = \left\{\left\langle n, \frac{1}{n}, \frac{3}{2n}, 1 - \frac{3}{n}\right\rangle \mid n \in \mathbb{N}\right\}$.
Then

$$\begin{aligned} \Box A &= \left\{\left\langle n, \tfrac{1}{n}, \tfrac{3}{2n}, 1 - \tfrac{3}{2n} - \tfrac{1}{n}\right\rangle \mid n \in \mathbb{N}\right\} \\ &= \left\{\left\langle n, \tfrac{1}{n}, \tfrac{3}{2n}, 1 - \tfrac{5}{2n}\right\rangle \mid n \in \mathbb{N}\right\}. \end{aligned}$$

$$\begin{aligned} \Diamond A &= \left\{\left\langle n, 1 - \tfrac{3}{2n} - \left(1 - \tfrac{3}{n}\right), \tfrac{3}{2n}, 1 - \tfrac{3}{n}\right\rangle \mid n \in \mathbb{N}\right\} \\ &= \left\{\left\langle n, \tfrac{3}{2n}, \tfrac{3}{2n}, 1 - \tfrac{3}{n}\right\rangle \mid n \in \mathbb{N}\right\}. \end{aligned}$$

With the involutive negator N_1 in Proposition 2.4, the operators $\Box, \Diamond$ verify a similar relation like that seen between modal operators on Łukasiewicz-Moisil algebras.

Recall that $\overline{A} = N_1(A) = \{\langle x, \upsilon_A(x), \eta_A(x), \mu_A(x)\rangle \mid x \in X\}$.

Proposition 2.11. *Let X be a non empty set and let $A \in PFS(X)$. Then*

(i) $\overline{\Box\overline{A}} = \Diamond A.$

(ii) $\overline{\Diamond\overline{A}} = \Box A.$

Proof. Let $x \in X$,

(i)
$$\begin{aligned}\overline{\Box\overline{A}} &= \overline{\Box\overline{\{\langle x, \mu_A(x), \eta_A(x), \upsilon_A(x)\rangle \mid x \in X\}}},\\ &= \overline{\Box\{\langle x, \upsilon_A(x), \eta_A(x), \mu_A(x)\rangle \mid x \in X\}},\\ &= \overline{\{\langle x, \upsilon_A(x), \eta_A(x), 1-\upsilon_A(x)-\eta_A(x)\rangle \mid x \in X\}},\\ &= \{\langle x, 1-\upsilon_A(x)-\eta_A(x), \eta_A(x), \upsilon_A(x)\rangle \mid x \in X\},\\ &= \Diamond A.\end{aligned}$$

(ii) Is obtained dually.

□

Proposition 2.12. *Let X be a non empty set and let $A \in PFS(X)$. The operators $\Box$ and $\Diamond$ verify the following proprieties.*

(i) *$\Diamond$ is extensive and $\Box$ is retractive (i.e., $\Box A \subseteq A \subseteq \Diamond A$).*

(ii) *$\Diamond, \Box$ are idempotent (i.e., $\Diamond\Diamond A = \Diamond A$ and $\Box\Box A = \Box A$).*

(iv) $\Box\Diamond A = \Diamond A.$

(v) $\Diamond\Box A = \Box A.$

Proof. Direct. □

Remark 2.6. Obviously, if A is a crisp set, then

$$\Box A = A = \Diamond A.$$

Proposition 2.13. *[35] Let X be a non empty set and let $A \in PFS(X)$. Then the following properties hold:*

(i) $S(\Box A) \subseteq S(A)$.

(ii) $S(\Diamond A) = S(A)$.

(iii) $Ker(A) = Ker(\Box A)$.

(iv) $Ker\,(A) \subseteq Ker\,(\Diamond A)\,.$

Proof. Let $x \in X$.

(i) Suppose that $x \in S\,(\Box A)$ i.e., $\Box A\,(x) \succ 0_{\mathbb{D}^*}$, which means $\mu_{\Box A}\,(x) = \mu_A\,(x) > 0$ or $\mu_{\Box A}\,(x) = \mu_A\,(x) = 0$ and $\upsilon_{\Box A}\,(x) = 1 - \mu_A\,(x) - \eta_A\,(x) < 1$. Thus $\mu_A\,(x) > 0$ or $\mu_A\,(x) = 0$ and $\eta_A\,(x) > 0$. Since $\mu_A\,(x) + \eta_A\,(x) + \upsilon_A\,(x) \leq 1$, it follows that $\mu_A\,(x) > 0$ or $\mu_A\,(x) = 0$ and $\upsilon_A\,(x) < 1$, that is, $A\,(x) \succ 0$. Hence $x \in S\,(A)\,.$

(ii) The direct inclusion is similar to the previous proof. It remains to show that, $S\,(A) \subseteq S\,(\Diamond A)\,.$

Suppose that $x \in S\,(A)$ i.e., $A\,(x) \succ 0_{\mathbb{D}^*}$, means $\mu_A\,(x) > 0$ or $\mu_A\,(x) = 0$ and $\upsilon_A\,(x) < 1$. We discuss two cases:

If $\mu_A\,(x) > 0$, then $\mu_{\Diamond A}\,(x) = 1 - \eta_A\,(x) - \upsilon_A\,(x) \geq \mu_A\,(x) > 0$. Thus $x \in S\,(\Diamond A)\,.$

If $\mu_A\,(x) = 0$ and $\upsilon_A\,(x) < 1$, then $\mu_{\Diamond A}\,(x) = 1 - \eta_A\,(x) - \upsilon_A\,(x) \geq \mu_A\,(x) = 0$ and $\upsilon_{\Diamond A}\,(x) = \upsilon_A\,(x) < 1$. Thus $x \in S\,(\Diamond A)\,.$

(iii) Suppose that $x \in Ker\,(A)$ i.e., $A\,(x) = 1_{\mathbb{D}^*}$, that is, $\mu_A\,(x) = 1$, $\eta_A\,(x) = 0$ and $\upsilon_A\,(x) = 0$. Which is equivalent to $\mu_{\Box A}\,(x) = \mu_A\,(x) = 1$, $\eta_{\Box A}\,(x) = \eta_A\,(x) = 0$ and $\upsilon_{\Box A}\,(x) = 1 - \mu_A\,(x) - \eta_A\,(x) = 0$. Hence $x \in Ker\,(\Box A)\,.$

(iv) Suppose that $x \in Ker\,(A)$ i.e., $A\,(x) = 1_{\mathbb{D}^*}$, that is, $\mu_A\,(x) = 1$, $\eta_A\,(x) = 0$ and $\upsilon_A\,(x) = 0$. This implies that, $\mu_{\Diamond A}\,(x) = 1 - \eta_A\,(x) - \upsilon_A\,(x) = 1$, $\eta_{\Diamond A}\,(x) = \eta_A\,(x) = 0$ and $\upsilon_{\Diamond A}\,(x) = \upsilon_A\,(x) = 0$. Hence $x \in Ker\,(\Diamond A)\,.$

□

From the operators $\Box, \Diamond$, two new relations are defined as follows:
$A \subseteq_{\Box} B$ iff for all $x \in X$, $\mu_A\,(x) \leq \mu_B\,(x)$ and $\eta_A\,(x) \leq \eta_B\,(x)\,.$
$A \subseteq_{\Diamond} B$ iff for all $x \in X$, $\eta_A\,(x) \geq \eta_B\,(x)$ and $\upsilon_A\,(x) \geq \upsilon_A\,(x)\,.$
This two new relations lead to the following results, which are straightforward.

Proposition 2.14. *Let X be a non empty set and let $A, B \in PFS(X)$.*

(i) *$A \subseteq_{\Box} B$ iff $\Box A \subseteq \Box B$.*

(ii) *$A \subseteq_{\Diamond} B$ iff $\Diamond A \subseteq \Diamond B$.*

(iii) *$A \subseteq_{\Box} B$ and $A \subseteq_{\Diamond} B$ implies $A \subseteq B$.*

2.3.2 Operators D_λ and F_α

In the following, two other Atanassov's modal operators D_λ and F_α (see [12]) will be extended to the picture fuzzy set case. These operators are an extension of the operators $\Box$ and $\Diamond$.

Definition 2.15. [35] Let X be a non empty set and let $A \in PFS(X)$. Define the modal operator D_λ by

$$D_\lambda(A) = \{\langle x, \mu_A(x) + \lambda.\pi_A(x), \eta_A(x), \upsilon_A(x) + (1-\lambda).\pi_A(x)\rangle \mid x \in X\},$$

where λ is a fixed number in $[0,1]$.

Example 2.12. Consider the picture fuzzy set $A = \left\{\left\langle n, \frac{1}{n}, \frac{3}{2n}, 1-\frac{3}{n}\right\rangle \mid n \in \mathbb{N}\right\}$ given in the previous example.

We have $\pi_A(n) = 1 - (\mu_A(n) + \eta_A(n) + \upsilon_A(n))$
$= 1 - \left(\frac{1}{n} + \frac{3}{2n} + \left(1 - \frac{3}{n}\right)\right)$
$= \frac{1}{2n}$.

Then

$D_\lambda(A) = \left\{\left\langle n, \frac{1}{n} + \frac{1}{2n}\lambda, \frac{3}{2n}, 1 - \frac{3}{n} + (1-\lambda)\frac{1}{2n}\right\rangle \mid n \in \mathbb{N}, \lambda \in [0,1]\right\}.$

Proposition 2.15. *Let X be a non empty and let $A \in PFS(X)$. Then for all $\lambda_1, \lambda_2 \in [0,1]$, the following properties hold:*

(i) *If $\lambda_1 \le \lambda_2$, then $D_{\lambda_1}(A) \subseteq D_{\lambda_1}(A)$.*

(ii) $D_0(A) = \Box A$.

(iii) $D_1(A) = \Diamond A$.

Proof. Direct. $\Box$

Definition 2.16. [35] Let X be a non empty set and let $A \in PFS(X)$. For all $\alpha \in \mathbb{D}^*$, define the modal operator F_α by

$$F_\alpha(A) = \{\langle x, \mu_A(x) + \alpha_1\pi_A(x), \eta_A(x) + \alpha_2\pi_A(x), \upsilon_A(x) + \alpha_3\pi_A(x)\rangle \mid x \in X\}.$$

Example 2.13. Consider the picture fuzzy set $A = \left\{\left\langle n, \frac{1}{n}, \frac{3}{2n}, 1-\frac{3}{n}\right\rangle \mid n \in \mathbb{N}\right\}$ given in the previous example. Then

$F_\alpha(A) = \left\{\left\langle n, \frac{1}{n} + \frac{1}{2n}\alpha_1, \frac{3}{2n} + \frac{1}{2n}\alpha_2, 1 - \frac{3}{n} + \frac{1}{2n}\alpha_3\right\rangle \mid n \in \mathbb{N}, \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}^*\right\}.$

Proposition 2.16. *Let $A \in PFS(X)$ and let $\alpha, \beta \in \mathbb{D}^*$. Then the following properties hold:*

(i) *If $\alpha \leq \beta$, then $F_\alpha(A) \subseteq F_\beta(A)$.*

(ii) $F_{0_{\mathbb{D}^*}}(A) = \Box A.$

(iii) $F_{1_{\mathbb{D}^*}}(A) = \Diamond A.$

(iv) $D_\lambda(A) = F_{(\lambda,0,1-\lambda)}(A).$

(v) $\overline{F_\alpha\left(\overline{A}\right)} = F_{\overline{\alpha}}(A).$

Proof. The statements (i), (ii), (iii) and (iv) are easy to check. Using the negator N_1, the proof of (v) is direct. □

Theorem 2.4. *Let $A \in PFS(X)$ and let $\alpha, \beta \in \mathbb{D}^*$. Then the following property holds:*
$F_\alpha(F_\beta(A)) = F_\gamma(A)$, where
$\gamma = (\beta_1 + \alpha_1(1 - \beta_1 - \beta_2 - \beta_3), \beta_2 + \alpha_2(1 - \beta_1 - \beta_2 - \beta_3), \beta_3 + \alpha_3(1 - \beta_1 - \beta_2 - \beta_3)).$

Proof. Let $\alpha, \beta \in \mathbb{D}^*$. For all $x \in X$,

$F_\beta(A) = \left\{\left\langle x, \mu_{F_\beta(A)}(x), \eta_{F_\beta(A)}(x), \upsilon_{F_\beta(A)}(x)\right\rangle \mid x \in X\right\}$ where

$$\begin{cases} \mu_{F_\beta(A)}(x) = \mu_A(x) + \beta_1\pi_A(x), \\ \eta_{F_\beta(A)}(x) = \eta_A(x) + \beta_2\pi_A(x), \\ \upsilon_{F_\beta(A)}(x) = \upsilon_A(x) + \beta_3\pi_A(x). \end{cases} \quad \text{Then}$$

$F_\alpha(F_\beta(A)) = \left\{\left\langle x, \mu_{F_\alpha(F_\beta(A))}(x), \eta_{F_\alpha(F_\beta(A))}(x), \upsilon_{F_\alpha(F_\beta(A))}(x)\right\rangle \mid x \in X\right\}$, where

$$\begin{aligned} \mu_{F_\alpha(F_\beta(A))}(x) &= (\mu_A(x) + \beta_1\pi_A(x)) + \alpha_1\pi_{F_\beta(A)}(x), \\ &= (\mu_A(x) + \beta_1\pi_A(x)) + \alpha_1(1 - \mu_A(x) - \beta_1\pi_A(x) - \\ &\quad \eta_A(x) - \beta_2\pi_A(x) - \upsilon_A(x) - \beta_3\pi_A(x)), \\ &= \mu_A(x) + (\beta_1 + \alpha_1 - \alpha_1\beta_1 - \alpha_1\beta_2 - \alpha_1\beta_3)\pi_A(x). \\ &= \mu_A(x) + (\beta_1 + \alpha_1(1 - \beta_1 - \beta_2 - \beta_3))\pi_A(x). \end{aligned}$$

$$\begin{aligned} \eta_{F_\alpha(F_\beta(A))}(x) &= (\eta_A(x) + \beta_2\pi_A(x)) + \alpha_2\pi_{F_\beta(A)}(x), \\ &= (\eta_A(x) + \beta_2\pi_A(x)) + \alpha_2(1 - \mu_A(x) - \beta_1\pi_A(x) - \\ &\quad \eta_A(x) - \beta_2\pi_A(x) - \upsilon_A(x) - \beta_3\pi_A(x)), \\ &= \eta_A(x) + (\beta_2 + \alpha_2 - \alpha_2\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3)\pi_A(x), \\ &= \eta_A(x) + (\beta_2 + \alpha_2(1 - \beta_1 - \beta_2 - \beta_3))\pi_A(x). \end{aligned}$$

$$\begin{aligned}
\upsilon_{F_\alpha\left(F_\beta(A)\right)}(x) &= \left(\upsilon_A(x) + \beta_3\pi_A(x)\right) + \alpha_3\pi_{F_\beta(A)}(x), \\
&= \ \left(\upsilon_A(x) + \beta_3\pi_A(x)\right) + \alpha_3\left(1 - \mu_A(x) - \beta_1\pi_A(x) - \right. \\
&\quad \left. \eta_A(x) - \beta_2\pi_A(x) - \upsilon_A(x) - \beta_3\pi_A(x)\right), \\
&= \upsilon_A(x) + \left(\beta_3 + \alpha_3 - \alpha_3\beta_1 - \alpha_3\beta_2 - \alpha_3\beta_3\right)\pi_A(x), \\
&= \upsilon_A(x) + \left(\beta_3 + \alpha_3\left(1 - \beta_1 - \beta_2 - \beta_3\right)\right)\pi_A(x).
\end{aligned}$$

□

CHAPTER 3

$\mathbb{D}^*$-BASED PICTURE FUZZY LATTICES

This study lies at the core of the intersection between lattice theory and picture fuzzy sets. Picture fuzzy lattices are not just an expansion in fuzzy set cases; they represent a qualitative leap that incorporates the concepts of positive, neutral, and negative membership into the lattice structure. This approach allows for the evaluation of information in a manner that acknowledges ambiguity and uncertainty, providing a more comprehensive and flexible model for analysis and making picture fuzzy lattices a powerful tool for decision-making, information processing, and modelling in areas where traditional binary or fuzzy methods may not adequately capture the richness or ambiguity of data.

Through its detailed exploration of picture fuzzy relations and lattices, this chapter aims to contribute significantly to the field of picture fuzzy sets. By providing a theoretical examination of lattice structures and the relational properties in picture fuzzy cases, it seeks to enhance understanding of the capabilities of picture fuzzy sets in modelling and analysis, laying the groundwork for further research and application in complex and uncertain environments. The first section focuses on elucidating the basic concepts underlying picture fuzzy relations. This includes an extensive discussion of their definitions, the operations that can be performed on them, and an exploration of their properties. This section also delves into important aspects of picture fuzzy relations, such as the picture fuzzy order relation and the picture fuzzy chain. It presents a series of results related to these concepts.

The second section examines in detail the structure of the picture fuzzy lattice, where the operations of join and meet in such a lattice are adapted to respect these additional dimensions of membership. This section presents concepts of picture fuzzy lattices and picture fuzzy bounded lattices, in addition to the picture fuzzy sub-lattice, delves into their properties and gives some characterizations using their level sets.

3.1 Picture Fuzzy Relations

"Picture fuzzy relations" is a concept that extends the traditional notions of fuzzy relations. In traditional fuzzy sets, introduced by Lotfi Zadeh in the 1960s [43], elements belong to sets to varying degrees, characterized by a membership function taking values between 0 and 1.

Picture fuzzy relations build upon this concept by introducing a more complex structure for representing uncertainty. They incorporate three parameters: positive membership (representing the degree to which an element belongs to a set), neutral membership (representing the degree of neutrality or hesitation), and negative membership (representing the degree to which an element does not belong to a set). This framework allows for a more nuanced representation of uncertain information, especially in situations where neutrality or hesitation is an important factor. And used in various applications where precise information is unavailable or inapplicable.

In this section, we recall the concept of picture fuzzy relations introduced by Cuong [21], and then examine and extend some results related to this concept in the direction we studied in the previous chapter. Some existing results in ([3, 16–18, 23, 34, 44]) have been extended.

3.1.1 Notions of Picture Fuzzy Relations

Here, we recall the definition of a picture fuzzy relation and investigate its main properties.

Definition 3.1. [21] Let X be a non-empty set. A picture fuzzy binary relation $\mathcal{R} : X \times X \longrightarrow \mathbb{D}^*$ is defined by

$$\mathcal{R}(x, y) = (\mu_{\mathcal{R}}(x, y), \eta_{\mathcal{R}}(x, y), \upsilon_{\mathcal{R}}(x, y)), \text{ for all } x, y \in X,$$

where $\mu_{\mathcal{R}} : X \times X \longrightarrow [0, 1]$, $\eta_{\mathcal{R}} : X \times X \longrightarrow [0, 1]$ and $\upsilon_{\mathcal{R}} : X \times X \longrightarrow [0, 1]$ satisfy the condition

$$0 \leq \mu_{\mathcal{R}}(x, y) + \eta_{\mathcal{R}}(x, y) + \upsilon_{\mathcal{R}}(x, y) \leq 1, \text{ for every } (x, y) \in X \times X.$$

Notation 3.1. The set of all the picture fuzzy relations on X will be denoted by $PFR(X)$.

Definition 3.2. [14]Let $\mathcal{R} \in PFR(X)$.

(1) If for all $x, y \in X$, $\mu_{\mathcal{R}}(x, y) = \eta_{\mathcal{R}}(x, y) = 0$ and $\upsilon_{\mathcal{R}}(x, y) = 1$, then $\mathcal{R}$ is called a null PFR.

(2) If for all $x, y \in X$, $\mu_{\mathcal{R}}(x,y) = 1$ and $\eta_{\mathcal{R}}(x,y) = \upsilon_{\mathcal{R}}(x,y) = 0$, then $\mathcal{R}$ is called an absolute PFR.

(3) If for all $x, y \in X$, $\mu_{\mathcal{R}}(x,y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$, $\eta_{\mathcal{R}}(x,y) = 0$ and $\upsilon_{\mathcal{R}}(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$, then R is called an identity PFR.

Definition 3.3. [36] Let X be a non-empty set and let $\mathcal{R}, \mathcal{P} \in PFR(X)$. Using Definition 2.3, we can define

(i) The picture fuzzy inclusion by $\mathcal{R} \subseteq \mathcal{P}$ if and only if $(\mu_{\mathcal{R}}(x,y), \eta_{\mathcal{R}}(x,y), \upsilon_{\mathcal{R}}(x,y)) \preceq (\mu_{\mathcal{P}}(x,y), \eta_{\mathcal{P}}(x,y), \upsilon_{\mathcal{P}}(x,y))$, for all $x, y \in X$. In more detail,

$$\mathcal{R} \subseteq \mathcal{P} \Longleftrightarrow \begin{cases} \mu_{\mathcal{R}}(x,y) < \mu_{\mathcal{P}}(x,y) \text{ and } \upsilon_{\mathcal{R}}(x,y) \geq \upsilon_{\mathcal{P}}(x,y) \\ \text{or} \\ \mu_{\mathcal{R}}(x,y) = \mu_{\mathcal{P}}(x,y) \text{ and } \upsilon_{\mathcal{R}}(x,y) > \upsilon_{\mathcal{P}}(x,y) \\ \text{or} \\ \mu_{\mathcal{R}}(x,y) = \mu_{\mathcal{P}}(x,y) \text{ and } \upsilon_{\mathcal{R}}(x,y) = \upsilon_{\mathcal{P}}(x,y) \text{ and } \eta_{\mathcal{R}}(x,y) \leq \eta_{\mathcal{P}}(x,y). \end{cases}$$

(ii) The picture fuzzy intersection $\mathcal{R} \cap \mathcal{P}$ by $(\mathcal{R} \cap \mathcal{P})(x,y) = \mathcal{R}(x,y) \curlywedge \mathcal{P}(x,y)$, for all $x, y \in X$. In more detail,

$$\mathcal{R} \cap \mathcal{P} = \begin{cases} \mathcal{R} & \text{if } \mathcal{R} \subseteq \mathcal{P}, \\ \mathcal{P} & \text{if } \mathcal{P} \subseteq \mathcal{R}, \\ \{\langle (x,y), \mu_{\mathcal{R}\cap\mathcal{P}}(x,y), \eta_{\mathcal{R}\cap\mathcal{P}}(x,y), \upsilon_{\mathcal{R}\cap\mathcal{P}}(x,y) \rangle \mid x, y \in X\} & \text{otherwise.} \end{cases}$$

Where
$$\mu_{\mathcal{R}\cap\mathcal{P}}(x,y) = \min\{\mu_{\mathcal{R}}(x,y), \mu_{\mathcal{P}}(x,y)\},$$
$$\eta_{\mathcal{R}\cap\mathcal{P}}(x,y) = 1 - \min\{\mu_{\mathcal{R}}(x,y), \mu_{\mathcal{P}}(x,y)\} - \max\{\upsilon_{\mathcal{R}}(x,y), \upsilon_{\mathcal{P}}(x,y)\},$$
$$\upsilon_{\mathcal{R}\cap\mathcal{P}}(x,y) = \max\{\upsilon_{\mathcal{R}}(x,y), \upsilon_{\mathcal{P}}(x,y)\}.$$

(iii) The picture fuzzy union $\mathcal{R} \cup \mathcal{P}$ by $(\mathcal{R} \cup \mathcal{P})(x,y) = \mathcal{R}(x,y) \curlyvee \mathcal{P}(x,y)$, for all $x, y \in X$. In more detail,

$$\mathcal{R} \cup \mathcal{P} = \begin{cases} \mathcal{R} & \text{if } \mathcal{P} \subseteq \mathcal{R}, \\ \mathcal{P} & \text{if } \mathcal{R} \subseteq \mathcal{P}, \\ \{\langle (x,y), \mu_{\mathcal{R}\cup\mathcal{P}}(x,y), \eta_{\mathcal{R}\cup\mathcal{P}}(x,y), \upsilon_{\mathcal{R}\cup\mathcal{P}}(x,y) \rangle \mid x, y \in X\} & \text{otherwise.} \end{cases}$$

Where
$$\mu_{\mathcal{R}\cup\mathcal{P}}(x,y) = \max\{\mu_{\mathcal{R}}(x,y), \mu_{\mathcal{P}}(x,y)\},$$
$$\eta_{\mathcal{R}\cup\mathcal{P}}(x,y) = 0,$$
$$\upsilon_{\mathcal{R}\cup\mathcal{P}}(x,y) = \min\{\upsilon_{\mathcal{R}}(x,y), \upsilon_{\mathcal{P}}(x,y)\}.$$

(iv) The support of a picture fuzzy relation by $S(\mathcal{R}) = \{(x,y) \in X \times X \mid \mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}\}$.

(v) The kernel of a picture fuzzy relation by $\ker(\mathcal{R}) = \{(x, y) \in X \times X \mid \mathcal{R}(x, y) = 1_{\mathbb{D}^*}\}$.

(vi) For all $\alpha \in \mathbb{D}_0^*$, we define the α-cut of a picture fuzzy relation by

$\mathcal{R}_\alpha = \{(x, y) \in X \times X \mid \mathcal{R}(x, y) \succeq \alpha\}$.

In more detail, $\mathcal{R}_\alpha = \{(x, y) \in X \times X \mid (\mu_{\mathcal{R}}(x, y) > \alpha_1$ and $\upsilon_{\mathcal{R}}(x, y) \leq \alpha_3)$ or $(\mu_{\mathcal{R}}(x, y) = \alpha_1$ and $\upsilon_{\mathcal{R}}(x, y) < \alpha_3)$ or $(\mu_{\mathcal{R}}(x, y) = \alpha_1,\ \upsilon_{\mathcal{R}}(x, y) = \alpha_3$ and $\eta_{\mathcal{R}}(x, y) \geq \alpha_2)\}$.

(vii) The inverse of $\mathcal{R}$ by $\mathcal{R}^{-1}(y, x) = \mathcal{R}(x, y)$, i.e., $\mu_{\mathcal{R}^{-1}}(y, x) = \mu_{\mathcal{R}}(x, y), \eta_{\mathcal{R}^{-1}}(y, x) = \eta_{\mathcal{R}}(x, y)$ and $\upsilon_{\mathcal{R}^{-1}}(y, x) = \upsilon_{\mathcal{R}}(x, y)$.

(viii) The complement of $\mathcal{R}$ by $\mu_{\mathcal{R}^C}(x, y) = \upsilon_{\mathcal{R}}(x, y), \eta_{\mathcal{R}^C}(x, y) = 1 - \mu_{\mathcal{R}}(x, y) - \eta_{\mathcal{R}}(x, y) - \upsilon_{\mathcal{R}}(x, y)$ and $\upsilon_{\mathcal{R}^C}(x, y) = \mu_{\mathcal{R}}(x, y)$.

Example 3.1. Let $X = \{x_1, x_2, x_3\}$ and let $\mathcal{R}, \mathcal{P} \in PFR(X)$ given by

$$\mathcal{R} = \begin{pmatrix} (0.30, 0.00, 0.50) & (0.70, 0.10, 0.10) & (0.00, 0.20, 0.40) \\ (0.10, 0.30, 0.20) & (1.00, 0.00, 0.00) & (0.00, 0.00, 0.26) \\ (0.00, 0.00, 0.00) & (0.80, 0.10, 0.00) & (0.00, 0.60, 0.30) \end{pmatrix},$$

$$\mathcal{P} = \begin{pmatrix} (0.40, 0.00, 0.40) & (0.07, 0.10, 0.20) & (0.20, 0.10, 0.50) \\ (0.20, 0.40, 0.30) & (0.00, 0.00, 1.00) & (0.00, 0.20, 0.26) \\ (0.00, 0.00, 0.00) & (0.71, 0.22, 0.01) & (0.10, 0.50, 0.20) \end{pmatrix}.$$

Then

- The picture fuzzy intersection and the picture fuzzy union of $\mathcal{R}$ and $\mathcal{P}$ are given as follows:

$$\mathcal{R} \cap \mathcal{P} = \begin{pmatrix} (0.30, 0.00, 0.50) & (0.07, 0.10, 0.20) & (0.00, 0.50, 0.50) \\ (0.10, 0.60, 0.30) & (0.00, 0.00, 1.00) & (0.00, 0.00, 0.26) \\ (0.00, 0.00, 0.00) & (0.71, 0.22, 0.01) & (0.00, 0.60, 0.30) \end{pmatrix},$$

$$\mathcal{R} \cup \mathcal{P} = \begin{pmatrix} (0.40, 0.00, 0.40) & (0.70, 0.10, 0.10) & (0.20, 0.00, 0.40) \\ (0.20, 0.00, 0.20) & (1.00, 0.00, 0.00) & (0.00, 0.20, 0.26) \\ (0.00, 0.00, 0.00) & (0.80, 0.10, 0.00) & (0.10, 0.50, 0.20) \end{pmatrix}$$

- The support of $\mathcal{P}$ and the kernel of $\mathcal{R}$ are given as follows:

$S(\mathcal{P}) = \{(x_1, x_1), (x_1, x_2), (x_1, x_3), (x_2, x_1), (x_2, x_3), (x_3, x_1), (x_3, x_2), (x_3, x_3)\}$.

$\ker(\mathcal{R}) = \{(x_2, x_2)\}$.

- For $\alpha = (0.10, 0.20, 0.30) \in \mathbb{D}_0^*$, the α-cut of $\mathcal{R}$ are given as follows:

  $\mathcal{R}_{\alpha} = \{(x_1, x_2), (x_2, x_1), (x_2, x_2), (x_3, x_2)\}$.

- The inverse and the complement of $\mathcal{R}$ are given as follows:

$$\mathcal{R}^{-1} = \begin{pmatrix} (0.3, 0, 0.5) & (0.1, 0.3, 0.2) & (0, 0, 0) \\ (0.7, 0.1, 0.1) & (1, 0, 0) & (0.8, 0.1, 0) \\ (0, 0.2, 0.4) & (0, 0, 0) & (0, 0.6, 0.3) \end{pmatrix},$$

$$\mathcal{R}^{C} = \begin{pmatrix} (0.5, 0.2, 0.3) & (0.1, 0.1, 0.7) & (0.4, 0.4, 0) \\ (0.2, 0.4, 0.1) & (0, 0, 1) & (0, 0, 0) \\ (0, 0, 0) & (0, 0.1, 0.8) & (0.3, 0.1, 0) \end{pmatrix}.$$

Remark 3.1. In general, for each $\mathcal{R} \in PFR(X)$, $\mathcal{R} \cap \mathcal{R}^C = \emptyset$ and $\mathcal{R} \cup \mathcal{R}^C = X$ don't hold. Indeed.

Example 3.2. Consider the picture fuzzy relation $\mathcal{R}$ in the previous example. Then

$$\mathcal{R} \cap \mathcal{R}^{C} = \begin{pmatrix} (0.3, 0.2, 0.5) & (0.1, 0.2, 0.7) & (0, 0.6, 0.4) \\ (0.1, 0.7, 0.2) & (0, 0, 1) & (0, 0, 0) \\ (0, 0, 0) & (0, 0.2, 0.8) & (0, 0.7, 0.3) \end{pmatrix} \neq \emptyset,$$

$$\mathcal{R} \cup \mathcal{R}^{C} = \begin{pmatrix} (0.5, 0, 0.3) & (0.7, 0, 0.1) & (0.4, 0, 0) \\ (0.2, 0, 0.1) & (1, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (0.8, 0, 0) & (0.3, 0, 0) \end{pmatrix} \neq X.$$

Proposition 3.1. *Let $\mathcal{R}, \mathcal{P}, \mathcal{Q} \in PFR(X)$. Then*

1. $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$.
2. $\mathcal{R} \subseteq \mathcal{P} \Longrightarrow \mathcal{R}^{-1} \subseteq \mathcal{P}^{-1}$.
3. $(\mathcal{R} \cup \mathcal{P})^{-1} = \mathcal{R}^{-1} \cup \mathcal{P}^{-1}, (\mathcal{R} \cap \mathcal{P})^{-1} = \mathcal{R}^{-1} \cap \mathcal{P}^{-1}$.
4. $(\mathcal{R} \cup \mathcal{P})^{C} = \mathcal{R}^{C} \cap \mathcal{P}^{C}, (\mathcal{R} \cap \mathcal{P})^{C} = \mathcal{R}^{C} \cup \mathcal{P}^{C}$.
5. $\mathcal{R} \subseteq \mathcal{R} \cup \mathcal{P}, \mathcal{P} \subseteq \mathcal{R} \cup \mathcal{P}, \mathcal{R} \cap \mathcal{P} \subseteq \mathcal{R}, \mathcal{R} \cap \mathcal{P} \subseteq \mathcal{P}$.
6. $(\mathcal{R} \supseteq \mathcal{P})$ & $(\mathcal{R} \supseteq \mathcal{Q})$ *if and only if* $\mathcal{R} \supseteq \mathcal{P} \cup \mathcal{Q}$.
7. $(\mathcal{R} \subseteq \mathcal{P})$ & $(\mathcal{R} \subseteq \mathcal{Q})$ *if and only if* $\mathcal{R} \subseteq \mathcal{P} \cap \mathcal{Q}$.

Proof. Let $\mathcal{R}, \mathcal{P}, \mathcal{Q} \in PFR(X)$.

1. $\left(\mathcal{R}^{-1}\right)^{-1}(x,y) = \mathcal{R}^{-1}(y,x) = \mathcal{R}(x,y)$.

2. $$\begin{aligned} \mathcal{R}^{-1}(x,y) &= \mathcal{R}(y,x) \\ &\preceq \mathcal{P}(y,x) \\ &= \mathcal{P}^{-1}(x,y) \end{aligned}$$

3. $$\begin{aligned} (\mathcal{R}\cup\mathcal{P})^{-1}(x,y) &= (\mathcal{R}\cup\mathcal{P})(y,x) \\ &= \mathcal{R}(y,x) \curlyvee \mathcal{P}(y,x) \\ &= \mathcal{R}^{-1}(x,y) \curlyvee \mathcal{P}^{-1}(x,y) \\ &= \left(\mathcal{R}^{-1}\cup\mathcal{P}^{-1}\right)(x,y). \end{aligned}$$

   Similarly, we prove that $(\mathcal{R}\cap\mathcal{P})^{-1} = \mathcal{R}^{-1}\cap\mathcal{P}^{-1}$.

The proof of 4,5,6,7 is the same in Proposition 2.3 and Proposition 2.7. □

Proposition 3.2. *Let $\mathcal{R}, \mathcal{P}, \mathcal{Q} \in PFR(X)$. Then*

1. *(Idempotent laws): $\mathcal{R}\cup\mathcal{R} = \mathcal{R}, \mathcal{R}\cap\mathcal{R} = \mathcal{R}$.*

2. *(Commutative laws): $\mathcal{R}\cup\mathcal{P} = \mathcal{P}\cup\mathcal{R}, \mathcal{R}\cap\mathcal{P} = \mathcal{P}\cap\mathcal{R}$.*

3. *(Associative laws): $\mathcal{R}\cup(\mathcal{P}\cup\mathcal{Q}) = (\mathcal{R}\cup\mathcal{P})\cup\mathcal{Q}, \mathcal{R}\cap(\mathcal{P}\cap\mathcal{Q}) = (\mathcal{R}\cap\mathcal{P})\cap\mathcal{Q}$.*

4. *(Absorption laws): $\mathcal{R}\cup(\mathcal{P}\cap\mathcal{R}) = \mathcal{R}, \mathcal{R}\cap(\mathcal{P}\cup\mathcal{R}) = \mathcal{R}$.*

3.1.2 Properties of Picture Fuzzy Relations

Definition 3.4. [36] Let X be a non-empty set and let $\mathcal{R} \in PFR(X)$. We say that $\mathcal{R}$ is

(i) Reflexive if and only if $\mathcal{R}(x,x) = (1,0,0)$ for all $x \in X$.

(ii) Perfect antisymmetric, if for every $x, y \in X$ with $x \neq y$ and $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(y,x) = 0_{\mathbb{D}^*}$. Or rather,

$$\begin{cases} \mu_{\mathcal{R}}(x,y) > 0, \\ \text{or} \\ \eta_{\mathcal{R}}(x,y) > 0, \\ \text{or} \\ \mu_{\mathcal{R}}(x,y) = 0 \text{ and } \upsilon_{\mathcal{R}}(x,y) < 1, \end{cases} \quad \text{then} \begin{cases} \mu_{\mathcal{R}}(y,x) = 0, \\ \text{and} \\ \eta_{\mathcal{R}}(y,x) = 0, \\ \text{and} \\ \upsilon_{\mathcal{R}}(y,x) = 1. \end{cases}$$

(iii) Transitive if and only if for all $x, y, z \in X$, $\mathcal{R}(x,z) \succeq \mathcal{R}(x,y) \curlywedge \mathcal{R}(y,z)$. In more detail,

If $\mathcal{R}(x,y) \preceq \mathcal{R}(y,z)$, then $\mathcal{R}(x,y) \curlywedge \mathcal{R}(y,z) = \mathcal{R}(x,y)$. Hence $\mathcal{R}$ is transitive if $\mathcal{R}(x,y) \preceq \mathcal{R}(x,z)$.

If $\mathcal{R}(x,y) \succeq \mathcal{R}(y,z)$, then $\mathcal{R}(x,y) \curlywedge \mathcal{R}(y,z) = \mathcal{R}(y,z)$. Hence $\mathcal{R}$ is transitive if $\mathcal{R}(y,z) \preceq \mathcal{R}(x,z)$.

If $\mathcal{R}(x,y) \parallel \mathcal{R}(y,z)$, then $\mathcal{R}(x,y) \curlywedge \mathcal{R}(y,z) = (\mu_{\mathcal{R}}(x,y) \wedge \mu_{\mathcal{R}}(y,z), 1-\mu_{\mathcal{R}}(x,y) \wedge \mu_{\mathcal{R}}(y,z) - \upsilon_{\mathcal{R}}(x,y) \vee \upsilon_{\mathcal{R}}(y,z), \upsilon_{\mathcal{R}}(x,y) \vee \upsilon_{\mathcal{R}}(y,z))$. Hence $\mathcal{R}$ is transitive if

$$(\mu_{\mathcal{R}}(x,y) \wedge \mu_{\mathcal{R}}(y,z), 1-\mu_{\mathcal{R}}(x,y) \wedge \mu_{\mathcal{R}}(y,z) - \upsilon_{\mathcal{R}}(x,y) \vee \upsilon_{\mathcal{R}}(y,z), \upsilon_{\mathcal{R}}(x,y) \vee \upsilon_{\mathcal{R}}(y,z)) \preceq \mathcal{R}(x,z).$$

Remark 3.2. The definition of perfect antisymmetry is equivalent to the following one: for all $x, y \in X$, if $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$ then $x = y$.

Proposition 3.3. *Let $\mathcal{R}, \mathcal{P} \in PFR(X)$.*

1. *$\mathcal{R}$ is reflexive if and only if $\mathcal{R}^{-1}$ is reflexive.*
2. *If $\mathcal{R}$ is reflexive, then $\mathcal{R} \cup \mathcal{P}$ is reflexive, for all $\mathcal{P} \in PFR(X)$.*
3. *$\mathcal{R} \cap \mathcal{P}$ is reflexive if and only if $\mathcal{R}$ and $\mathcal{P}$ are reflexive.*
4. *$\mathcal{R}$ is perfect antisymmetric if and only if $\mathcal{R}^{-1}$ is perfect antisymmetric.*
5. *If $\mathcal{R}$ is perfect antisymmetric, then $\mathcal{R} \cap \mathcal{P}$ is perfect antisymmetric, for all $\mathcal{P} \in PFR(X)$.*
6. *$\mathcal{R}$ and $\mathcal{P}$ are perfect antisymmetric, then $\mathcal{R} \cup \mathcal{P}$ is perfect antisymmetric.*
7. *$\mathcal{R}$ is transitive if and only if $\mathcal{R}^{-1}$ is transitive.*
8. *If $\mathcal{R}$ and $\mathcal{P}$ are transitive, then $\mathcal{R} \cap \mathcal{P}$ is transitive.*

Proof. Let $\mathcal{R}, \mathcal{P} \in PFR(X)$.

1. $\mathcal{R}$ is reflexive if and only if $\mathcal{R}^{-1}$ is reflexive.

$$\begin{aligned} \mathcal{R} \text{ is reflexive} &\iff \mathcal{R}(x,x) = 1_{\mathbb{D}^*} \\ &\iff \mathcal{R}^{-1}(x,x) = 1_{\mathbb{D}^*} \\ &\iff \mathcal{R}^{-1} \text{ is reflexive.} \end{aligned}$$

2. If $\mathcal{R}$ is reflexive, then $\mathcal{R} \cup \mathcal{P}$ is reflexive, for all $\mathcal{P} \in PFR(X)$.

$$\begin{aligned} (\mathcal{R} \cup \mathcal{P})(x,x) &= \mathcal{R}(x,x) \curlyvee \mathcal{P}(x,x) \\ &= 1_{\mathbb{D}^*} \curlyvee \mathcal{P}(x,x) \\ &= 1_{\mathbb{D}^*}. \end{aligned}$$

3. $\mathcal{R} \cap \mathcal{P}$ is reflexive if and only if $\mathcal{R}$ and $\mathcal{P}$ are reflexive.

$$\begin{aligned} \mathcal{R} \cap \mathcal{P} \text{ is reflexive} &\iff (\mathcal{R} \cap \mathcal{P})(x,x) = 1_{\mathbb{D}^*} \\ &\iff \mathcal{R}(x,x) \curlywedge \mathcal{P}(x,x) = 1_{\mathbb{D}^*} \\ &\iff \mathcal{R}(x,x) = 1_{\mathbb{D}^*} \curlywedge \mathcal{P}(x,x) = 1_{\mathbb{D}^*} \\ &\iff \mathcal{R} \text{ and } \mathcal{P} \text{ are reflexive.} \end{aligned}$$

4. $\mathcal{R}$ is perfect antisymmetric if and only if $\mathcal{R}^{-1}$ is perfect antisymmetric.

$\mathcal{R}^{-1}(x,y) = \mathcal{R}(y,x) \succ 0_{\mathbb{D}^*}$ implies $\mathcal{R}(x,y) = \mathcal{R}^{-1}(y,x) = 0_{\mathbb{D}^*}$.

5. If $\mathcal{R}$ is perfect antisymmetric, then $\mathcal{R} \cap \mathcal{P}$ is perfect antisymmetric, for all $\mathcal{P} \in PFR(X)$.

$(\mathcal{R} \cap \mathcal{P})(x,y) \succ 0_{\mathbb{D}^*}$ implies that $\mathcal{R}(x,y) \curlywedge \mathcal{P}(x,y) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(x,y) \curlywedge \mathcal{P}(x,y) \succ 0_{\mathbb{D}^*}$. Since $\mathcal{R}$ is perfect antisymmetric, we have $(\mathcal{R} \cap \mathcal{P})(y,x) = \mathcal{R}(y,x) \curlywedge \mathcal{P}(y,x) = 0_{\mathbb{D}^*} \curlywedge \mathcal{P}(y,x) = 0_{\mathbb{D}^*}$.

6. $\mathcal{R}$ and $\mathcal{P}$ are perfect antisymmetric, then $\mathcal{R} \cup \mathcal{P}$ is perfect antisymmetric.

$(\mathcal{R} \cup \mathcal{P})(x,y) \succ 0_{\mathbb{D}^*}$ implies that $\mathcal{R}(x,y) \curlyvee \mathcal{P}(x,y) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ or $\mathcal{R}(x,y) \curlywedge \mathcal{P}(x,y) \succ 0_{\mathbb{D}^*}$. Since $\mathcal{R}$ and $\mathcal{P}$ are perfect antisymmetric, we have $(\mathcal{R} \cup \mathcal{P})(y,x) = \mathcal{R}(y,x) \curlywedge \mathcal{P}(y,x) = 0_{\mathbb{D}^*} \curlyvee 0_{\mathbb{D}^*} = 0_{\mathbb{D}^*}$.

7. $\mathcal{R}$ is transitive if and only if $\mathcal{R}^{-1}$ is transitive.

Suppose that $\mathcal{R}$ is transitive.

$$\begin{aligned} \mathcal{R}^{-1}(x,z) &= \mathcal{R}(z,x) \\ &\succeq \mathcal{R}(z,y) \curlywedge \mathcal{R}(y,x) \\ &= \mathcal{R}^{-1}(y,z) \curlywedge \mathcal{R}^{-1}(x,y). \end{aligned}$$

conversely, suppose that $\mathcal{R}^{-1}$ is transitive.

$$\begin{aligned} \mathcal{R}(x,z) &= \mathcal{R}^{-1}(z,x) \\ &\succeq \mathcal{R}^{-1}(z,y) \curlywedge \mathcal{R}^{-1}(y,x) \\ &= \mathcal{R}(y,z) \curlywedge \mathcal{R}(x,y). \end{aligned}$$

8. If $\mathcal{R}$ and $\mathcal{P}$ are transitive, then $\mathcal{R} \cap \mathcal{P}$ is transitive.

$$\begin{aligned}(\mathcal{R} \cap \mathcal{P})(x,z) &= \mathcal{R}(x,z) \curlywedge \mathcal{P}(x,z) \\ &\succeq \mathcal{R}(x,y) \curlywedge \mathcal{R}(y,z) \curlywedge \mathcal{P}(x,y) \curlywedge \mathcal{P}(y,z) \\ &= (\mathcal{R} \cap \mathcal{P})(x,y) \curlywedge (\mathcal{R} \cap \mathcal{P})(y,z).\end{aligned}$$

□

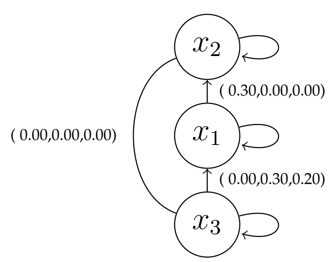
3.1.3 Picture Fuzzy Order Relation

Definition 3.5. Let X be a non-empty set and let $\mathcal{R} \in PFR(X)$. Then $\mathcal{R}$ is called a picture fuzzy ordering, or a picture fuzzy partial ordering, if it is reflexive, transitive and perfect antisymmetric.

A set equipped with a picture fuzzy partial order relation is called a picture fuzzy poset (PF-poset, for short).

Example 3.3. Let $X = \{x_1, x_2, x_3\}$ and let $\mathcal{R}$ be a picture fuzzy relation on X given by the following table and represented by the given Hasse diagram.

$\mathcal{R}$	x_1	x_2	x_3
x_1	$(1.00, 0.00, 0.00)$	$(0.30, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
x_2	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
x_3	$(0.00, 0.30, 0.20)$	$(0.00, 0.00, 0.00)$	$(1.00, 0.00, 0.00)$



Hasse diagram of a PF-ordering $(X, \mathcal{R})$, where $X = \{x_1, x_2, x_3\}$.

It is easy to see that $\mathcal{R}$ is a picture fuzzy ordering.

Corollary 3.1. Let $\mathcal{R}, \mathcal{P} \in PFR(X)$.

1. $\mathcal{R}$ is a picture fuzzy ordering if and only if $\mathcal{R}^{-1}$ is a picture fuzzy ordering.

2. If $\mathcal{R}$ and $\mathcal{P}$ are picture fuzzy ordering, then $\mathcal{R} \cap \mathcal{P}$ is a picture fuzzy ordering.

Definition 3.6. Let $(X, \mathcal{R})$ be a PF-poset and let x, y be two elements of X. We say that x is comparable with y in $(X, \mathcal{R})$ if $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ or $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$.

Definition 3.7. A picture fuzzy ordering $\mathcal{R}$ is linear (or total) on X if for every $x, y \in X$, $\mathcal{R}$ $(x, y) \succ 0_{\mathbb{D}^*}$ or $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$.

A picture fuzzy ordered set $(X, \mathcal{R})$ in which $\mathcal{R}$ is linear is called a linearly picture fuzzy ordered set or a picture fuzzy chain.

Example 3.4. Consider the set of all natural numbers $\mathbb{N}$ and let $\mathcal{R}$ be a picture fuzzy relation defined on $\mathbb{N}$ by

$$\mu_{\mathcal{R}}(m, n) = \begin{cases} 1 & if \quad m = n \\ 1 - \frac{m}{n} & if \quad m < n \\ 0 & if \quad n > m \end{cases}$$

$$\eta_{\mathcal{R}}(m, n) = \begin{cases} 0 & if \quad m = n \\ \frac{m}{4n} & if \quad m < n \\ 0 & if \quad n > m \end{cases}$$

$$\upsilon_{\mathcal{R}}(m, n) = \begin{cases} 0 & if \quad m = n \\ \frac{m}{2n} & if \quad m < n \\ 1 & if \quad n > m \end{cases}$$

It's simple to check that $\mathcal{R}$ is a picture fuzzy chain.

Lemma 3.1. *Let X be a non-empty set and let $\mathcal{R} \in PFR(X)$. If $\mathcal{R}$ is a picture fuzzy ordering relation on X, then $S(\mathcal{R})$ and $\ker(\mathcal{R})$ are order relations on X.*

Proof. Assume that $(X, \mathcal{R})$ is a picture fuzzy poset.

The reflexivity of $S(\mathcal{R})$ is direct. Since $\mathcal{R}(x, x) = (1, 0, 0) \succ 0_{\mathbb{D}^*}$ for all $x \in X$, then $(x, x) \in S(\mathcal{R})$.

For the antisymmetry. Suppose that $(x, y), (y, x) \in S(\mathcal{R})$, i.e., $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$. Then, from the perfect antisymmetric of $\mathcal{R}$, we obtain $x = y$.

Concerning the transitivity, suppose that $(x, y) \in S(\mathcal{R})$ and $(y, z) \in S(\mathcal{R})$, that is, $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$. Since $\mathcal{R}(x, z) \succeq \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$. Using Proposition 2.3(4), we obtain $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ for all $x, z \in X$, thus $(x, z) \in S(\mathcal{R})$.

Therefore $S(\mathcal{R})$ is a partial order relation on X.

Similarly, we obtain the same result for $\ker(\mathcal{R})$. □

Remark 3.3. The fact that $S(\mathcal{R})$ is a partial order relation on X does not imply that $\mathcal{R}$ is a picture fuzzy ordering relation on X. Indeed.

Example 3.5. Let $X = \{a, b\}$. Consider the relation $\mathcal{R}$ defined on X by

$\mathcal{R}$	a	b
a	$(0.10, 0.30, 0.00)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.50, 0.03, 0.20)$

its support is given by

$S(\mathcal{R})$	a	b
a	1	0
b	0	1

It is not difficult to see that $S(\mathcal{R})$ is a partial order relation on X but $\mathcal{R}$ is not a picture fuzzy ordering relation on X.

Proposition 3.4. *Let X be a non-empty set and let $\mathcal{R} \in PFR(X)$. $\mathcal{R}$ is a picture fuzzy ordering relation if and only if all cuts $\mathcal{R}_\alpha$ are order relations on X, for any $\alpha \in \mathbb{D}_0^*$.*

Proof. Let $\alpha \in \mathbb{D}_0^*$. Assume $(X, \mathcal{R})$ is a picture fuzzy poset and let $x \in X$.

Since $\mathcal{R}(x, x) = (1, 0, 0)$, then $\mathcal{R}(x, x) \succeq \alpha$, for all $\alpha \in \mathbb{D}_0^*$, so $(x, x) \in \mathcal{R}_\alpha$. Thus $\mathcal{R}_\alpha$ is reflexive.

Suppose that $(x, y) \in \mathcal{R}_\alpha$ and $(y, x) \in \mathcal{R}_\alpha$, then $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, x) \succeq \alpha$. This implies that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$. From the perfect antisymmetric of $\mathcal{R}$, we obtain $x = y$. Thus $\mathcal{R}_\alpha$ is antisymmetric.

Suppose that $(x, y) \in \mathcal{R}_\alpha$ and $(y, z) \in \mathcal{R}_\alpha$. Then $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, z) \succeq \alpha$. From the transitivity of $\mathcal{R}$ and Proposition 2.3, we obtain $\mathcal{R}(x, z) \succeq \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z) \succeq \alpha$ for all $x, y, z \in X$, this implies that $(x, z) \in \mathcal{R}_\alpha$. Thus $\mathcal{R}_\alpha$ is transitive.

Hence, if $\mathcal{R}$ is a picture fuzzy ordering relation, then all cuts $\mathcal{R}_\alpha$ are order relations on X.

Conversely, assume that for all $\alpha \in \mathbb{D}_0^*$, $\mathcal{R}_\alpha$ is a partial ordering relation on X.

If $\alpha = 1_{\mathbb{D}^*} \in \mathbb{D}_0^*$, then $(x, x) \in \mathcal{R}_{1_{\mathbb{D}^*}}$ for all $x \in X$, i.e., $\mathcal{R}(x, x) = 1_{\mathbb{D}^*}$, Thus $\mathcal{R}$ is reflexive.

Suppose that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$. Then there exists $\alpha, \beta \in \mathbb{D}_0^*$ such that $\mathcal{R}(x, y) = \alpha$ and $\mathcal{R}(y, x) = \beta$. Put $\gamma = \alpha \curlywedge \beta$.

It's clear that $\mathcal{R}(x, y) \succeq \gamma$ and $\mathcal{R}(x, y) \succeq \gamma$, that is, $(x, y) \in \mathcal{R}_\gamma$ and $(y, x) \in \mathcal{R}_\gamma$. From the antisymmetry of $\mathcal{R}_\gamma$ we obtain $x = y$. Thus $\mathcal{R}$ is perfect antisymmetric.

Let $x, y, z \in X$, and put $\alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$.

If $\alpha = 0_{\mathbb{D}^*}$, it is easy to see that $\alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z) \preceq \mathcal{R}(x, z)$.

If $\alpha \succ 0_{\mathbb{D}^*}$, then we have $\mathcal{R}(x, y) \succeq \alpha$ and $\mathcal{R}(y, z) \succeq \alpha$, that is, $(x, y) \in \mathcal{R}_\alpha$ and $(y, z) \in \mathcal{R}_\alpha$. Using the transitivity of $\mathcal{R}_\alpha$ we obtain $(x, z) \in \mathcal{R}_\alpha$, i.e., $\mathcal{R}(x, z) \succeq \alpha = \mathcal{R}(x, y) \curlywedge \mathcal{R}(y, z)$.

Thus, $\mathcal{R}$ is transitive. Hence $\mathcal{R}$ is a picture fuzzy ordering. □

3.2 Picture Fuzzy Lattices

Picture fuzzy sets are a generalization of fuzzy sets and intuitionistic fuzzy sets, introduced to better handle data uncertainty by allowing for the representation of positive, neutral and negative membership degrees for elements in relation to a set. This approach provides a richer framework for expressing and managing uncertainty, as it distinguishes between degrees of approval, disapproval, and hesitation or neutrality towards information or criteria. In a fuzzy picture lattice, the elements of the lattice are evaluated based on these three degrees of membership. This means that the structure not only accounts for how much an element belongs to a set (as in traditional fuzzy lattices), but also how much it does not belong and to what extent its belonging is undecided. The operations of join and meet in such a lattice are adapted to respect these additional dimensions of membership, enabling more sophisticated aggregation and comparison mechanisms. This complexity allows for a more nuanced handling of uncertainty and vagueness in various applications, such as decision-making processes, information filtering, and artificial intelligence systems.

In this section, we first extend the concept of fuzzy lattices and intuitionistic fuzzy lattices studied in [2, 6, 18, 34, 37, 39] to picture fuzzy cases. Hence, we extend some results in this direction.

3.2.1 Particular Elements of a Picture Fuzzy Ordered Set

Many important properties of a picture fuzzy ordered set X can be described in terms of the existence of certain upper or lower bounds of picture fuzzy sets of X. Here we present the basic theory of these ordered sets, and consider picture fuzzy lattices as picture ordered sets.

Definition 3.8. Let $(X, \mathcal{R})$ be a picture fuzzy poset and let E be a non-empty subset of X.

An element u in X is considered an upper bound of E if and only if, for all $x \in E, \mathcal{R}(x, u) \succ 0_{\mathbb{D}^*}$.

A particular upper bound, denoted as u_0, is identified as the least upper bound of E when, for every upper bound u of E, $\mathcal{R}(u_0, u) \succ 0_{\mathbb{D}^*}$.

Conversely, an element l in X is considered a lower bound of E if, for every x in E, $\mathcal{R}(l, x) \succ 0_{\mathbb{D}^*}$.

The greatest lower bound of E, represented by l_0, is defined when $\mathcal{R}(l, l_0) \succ 0_{\mathbb{D}^*}$ for every lower bound l of E.

If the greatest lower bound and the least upper bound of the set $\{x, y\}$ exist, they are denoted, respectively, by $x \sqcap y$ and $x \sqcup y$.

Remark 3.4. Let $(X, \mathcal{R})$ be a picture fuzzy poset and let E be a non-empty subset of X. If the least upper bound and the greatest lower bound of E exist, then by the perfect antisymmetry of $\mathcal{R}$ they are unique.

Definition 3.9. Let $(X, \mathcal{R})$ be a picture fuzzy poset.

- We say X has a bottom element if there exists $\perp \in X$ such that $\mathcal{R}(\perp, x) \succ 0_{\mathbb{D}^*}$, for all $x \in X$.
- Dually, X has a top element if there exists $\top \in X$ such that $\mathcal{R}(x, \top) \succ 0_{\mathbb{D}^*}$, for all $x \in X$.

Remark 3.5. Note that $\perp$ and $\top$ are unique when they exist. (The uniqueness comes from the perfect antisymmetry of $\mathcal{R}$).

3.2.2 Concepts Related to Picture Fuzzy Lattices

Here, we will define picture fuzzy lattice and picture fuzzy sub-lattice in a crisp lattice. We accomplish this by generalizing some existing notions and results in the Zadeh's fuzzy sets and Antanssov's intuitionistic fuzzy sets (see [2–5,37]) to the picture fuzzy case.

In the following, the symbol $\preceq$ indicates the picture fuzzy ordering defined on the set $\mathbb{D}^*$ as seen in Definition 2.3 and $\succeq$ is its dual.

Definition 3.10. A picture fuzzy poset $(X, \mathcal{R})$ is a picture fuzzy lattice if and only if $x \sqcup y$ and $x \sqcap y$ are exist for all $x, y \in X$.

Example 3.6. In Example 3.3, $(X, \mathcal{R})$ is a picture fuzzy lattice. Indeed, the following tables describe the supremum and the infimum of each subset of two elements $\{x, y\}$ of X.

$\sqcap$	x_1	x_2	x_3
x_1	x_1	x_1	x_3
x_2	x_1	x_2	x_3
x_3	x_3	x_3	x_3

$\sqcup$	x_1	x_2	x_3
x_1	x_1	x_2	x_1
x_2	x_2	x_2	x_2
x_3	x_1	x_2	x_3

Definition 3.11. A picture fuzzy lattice $(X, \mathcal{R})$ which has a top and a bottom elements is called a bounded picture fuzzy lattice.

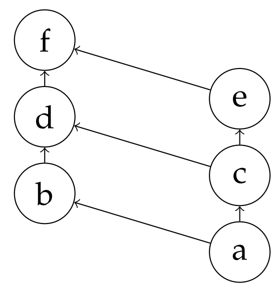
Definition 3.12. Let $(X, \leqslant)$ be a crisp lattice and let E be a picture fuzzy subset on X. E is called a picture fuzzy sublattice of $(X, \leqslant)$ if the following conditions are satisfied for all $x, y \in X$:

(i) $E(x \wedge y) \succeq E(x) \curlywedge E(y)$;

(ii) $E(x \vee y) \succeq E(x) \curlywedge E(y)$.

Example 3.7. Let $X = \{a, b, c, d, e, f\}$ and let $(X, \leqslant)$ be the lattice given by the following table and represented by the given Hasse diagram.

$(X, \leqslant)$	a	b	c	d	e	f
a	1	1	1	1	1	1
b	0	1	0	1	0	1
c	0	0	1	1	1	1
d	0	0	0	1	0	1
e	0	0	0	0	1	1
f	0	0	0	0	0	1



Hasse diagram of the crisp lattice $(X, \leqslant)$, where $X = \{a, b, c, d, e, f\}$.

The picture fuzzy set E defined on X by

X	$E(x)$
a	$(0.10, 0.50, 0.40)$
b	$(0.20, 0.10, 0.40)$
c	$(0.10, 0.20, 0.30)$
d	$(0.20, 0.00, 0.40)$
e	$(0.10, 0.20, 0.30)$
f	$(0.40, 0.40, 0.10)$

is a picture fuzzy sublattice of $(X, \leqslant)$.

Proposition 3.5. *Let $(X, \leqslant)$ be a crisp lattice. If E, F are picture fuzzy sublattices of $(X, \leqslant)$, then $E \cap F$ is also a picture fuzzy sublattice of $(X, \leqslant)$.*

Proof. Let E and F be two picture fuzzy sublattices of $(X, \leqslant)$.

We have $(E \cap F)(x) = E(x) \curlywedge F(x)$, then

$$\begin{aligned} (E \cap F)(x \wedge y) &= E(x \wedge y) \curlywedge F(x \wedge y) \\ &\succeq (E(x) \curlywedge E(y)) \curlywedge (F(x) \curlywedge F(y)) \\ &= (E(x) \curlywedge F(x)) \curlywedge (E(y) \curlywedge F(y)) \\ &= (E \cap F)(x) \curlywedge (E \cap F)(y) . \end{aligned}$$

$$\begin{aligned}(E\cap F)(x\vee y) &= E(x\vee y)\curlywedge F(x\vee y)\\ &\succeq (E(x)\curlywedge E(y))\curlywedge (F(x)\curlywedge F(y))\\ &= (E(x)\curlywedge F(x))\curlywedge (E(y)\curlywedge F(y))\\ &= (E\cap F)(x)\curlywedge (E\cap F)(y).\end{aligned}$$

Hence $E\cap F$ is a picture fuzzy sublattice of $(X,\leqslant)$. □

Remark 3.6. The union of two picture fuzzy sublattices of $(X,\leqslant)$ need not be a picture fuzzy sublattice of X.

Proposition 3.6. *Let $(X,\leqslant)$ be a crisp lattice. If E is a picture fuzzy sublattice of $(X,\leqslant)$, then $S(E)$ and $\ker(E)$ are sublattices on $(X,\leqslant)$.*

Proof. Suppose that E is a picture fuzzy sublattice of $(X,\leqslant)$ and let $x,y\in E$.

- If $x,y\in S(E)$, then $E(x)\succ 0_{\mathbb{D}^*}$ and $E(y)\succ 0_{\mathbb{D}^*}$.

  We have $E(x\wedge y)\succeq E(x)\curlywedge E(y)\succ 0_{\mathbb{D}^*}$, then $E(x\wedge y)\succ 0_{\mathbb{D}^*}$. Hence $x\wedge y\in S(E)$.

  and $E(x\vee y)\succeq E(x)\curlywedge E(y)\succ 0_{\mathbb{D}^*}$, then $E(x\vee y)\succ 0_{\mathbb{D}^*}$. Hence $x\vee y\in S(E)$.

- If $x,y\in\ker(E)$, then $E(x)=1_{\mathbb{D}^*}$ and $E(y)=1_{\mathbb{D}^*}$.

  We have $E(x\wedge y)\succeq E(x)\curlywedge E(y)=1_{\mathbb{D}^*}$, then $E(x\wedge y)=1_{\mathbb{D}^*}$. Hence $x\wedge y\in\ker(E)$.

  and $E(x\vee y)\succeq E(x)\curlywedge E(y)=1_{\mathbb{D}^*}$, then $E(x\vee y)=1_{\mathbb{D}^*}$. Hence $x\vee y\in\ker(E)$.

Therefore, $S(E)$ and $\ker(E)$ are sublattices on $(X,\leqslant)$. □

3.2.3 Characterizations of a Lattice in Terms of Their Level Set

We now turn to a characterization of the relationship between a picture fuzzy lattice and its level sets.

Proposition 3.7. *Let $(X,\mathcal{R})$ be a picture fuzzy poset. If $(X,\mathcal{R})$ is a picture fuzzy lattice, then $(X,S(\mathcal{R}))$ is a lattice.*

Proof. Suppose that $(X,\mathcal{R})$ is a picture fuzzy lattice. Let $x,y\in X$.

It is clear that the least upper bound is exists if x and y are comparable elements.

If x and y are non-comparable elements, i.e., $(x,y),(y,x)\notin S(\mathcal{R})$, then $\mathcal{R}(x,y)=\mathcal{R}(y,x)=0_{\mathbb{D}^*}$. Since $(X,\mathcal{R})$ is a picture fuzzy lattice, then there exists $l\in X$, such that $\mathcal{R}(x,l)\succ 0_{\mathbb{D}^*},\mathcal{R}(y,l)\succ 0_{\mathbb{D}^*}$, and $\mathcal{R}(l,u)\succ 0_{\mathbb{D}^*}$, for any upper bound u of $\{x,y\}$. Thus, there exists

l such that $(x,l) \in S(\mathcal{R})$, $(y,l) \in S(\mathcal{R})$ and $(l,u) \in S(\mathcal{R})$ for all upper bound u of $\{x,y\}$. So, there exists a least upper bound l of $\{x,y\}$ on $(X, S(\mathcal{R}))$.

Similarly, there exists a greatest lower bound g of $\{x,y\}$ on $(X, S(\mathcal{R}))$.

□

Proposition 3.8. *Let $(X,\mathcal{R})$ be a picture fuzzy poset. If $(X,\mathcal{R}_\alpha)$ are lattices for all $\alpha \in \mathbb{D}_0^*$, then $(X,\mathcal{R})$ is a picture fuzzy lattice.*

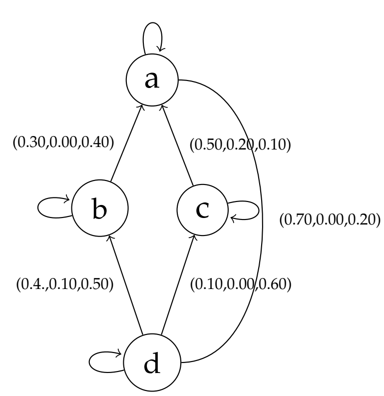
Proof. Suppose that for all $\alpha \in \mathbb{D}_0^*$, $(X,\mathcal{R}_\alpha)$ are crisp lattices. Let $x, y \in X$, then there exists $r \in X$, such that $(x,r) \in \mathcal{R}_\alpha$, $(y,r) \in \mathcal{R}_\alpha$, and $(r,u) \in \mathcal{R}_\alpha$, for every upper bound u of $\{x,y\}$. That is, there exists r such that $\mathcal{R}(x,r) \succ 0_{\mathbb{D}^*}$, $\mathcal{R}(y,r) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(r,u) \succ 0_{\mathbb{D}^*}$ for all upper bound u of $\{x,y\}$. Hence there exists a least upper bound r of $\{x,y\}$ on $(X,\mathcal{R})$. Thus $(X,\mathcal{R})$ is a picture fuzzy lattice.

Similarly, there exists a greatest lower bound l of $\{x,y\}$ on $(X,\mathcal{R})$. □

Remark 3.7. If $(X,\mathcal{R})$ is a picture fuzzy lattice, then $(X,\mathcal{R}_\alpha)$ may not be a crisp lattice. Indeed.

Example 3.8. Let $(X,\mathcal{R})$ be a picture fuzzy lattice, where $X = \{a,b,c,d\}$ and $\mathcal{R}$ defined by the following table and represented by the given Hasse diagram.

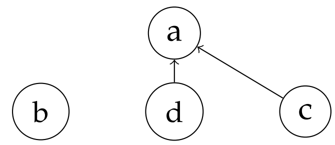
$\mathcal{R}$	a	b	c	d
a	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
b	$(0.30, 0.00, 0.40)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
c	$(0.50, 0.20, 0.10)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$
d	$(0.70, 0.00, 0.20)$	$(0.4., 0.10, 0.50)$	$(0.10, 0.00, 0.60)$	$(1.00, 0.00, 0.00)$



Hasse diagram of a PFL $(X,\mathcal{R})$, where $X = \{a,b,c,d\}$.

Consider the relation $\mathcal{R}_{(0.5,0.2,0.3)}$

$\mathcal{R}_{(0.5,0.2,0.3)}$	a	b	c	d
a	1.00	0.00	0.00	0.00
b	0.00	1.00	0.00	0.00
c	1.00	0.00	1.00	0.00
d	1.00	0.00	0.00	1.00



Hasse diagram of $(X, \mathcal{R}_{(0.5,0.2,0.3)})$.

It is not difficult to see that $\{b, c\}$ has neither the least upper bound nor the greatest lower bound. So, $(X, \mathcal{R}_{0.5,0.2,0.3})$ is a poset but not a crisp lattice.

Proposition 3.9. *Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let $x, y, z \in X$. Then*

1. $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}, \mathcal{R}(y, x \sqcup y) \succ 0_{\mathbb{D}^*}, \mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}, \mathcal{R}(x \sqcap y, y) \succ 0_{\mathbb{D}^*}$.

2. $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$ *implies* $\mathcal{R}(x \sqcup y, z) \succ 0_{\mathbb{D}^*}$.

3. $\mathcal{R}(z, x) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(z, y) \succ 0_{\mathbb{D}^*}$ *implies* $\mathcal{R}(z, x \sqcap y) \succ 0_{\mathbb{D}^*}$.

4. $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ *if and only if* $x \sqcup y = y$.

5. $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ *if and only if* $x \sqcap y = x$.

6. *If* $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$, *then* $\mathcal{R}(x \sqcap y, x \sqcap z) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(x \sqcup y, x \sqcup z) \succ 0_{\mathbb{D}^*}$.

7. *If* $\mathcal{R}(x \sqcup y, z) \succ 0_{\mathbb{D}^*}$, *then* $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$.

8. *If* $\mathcal{R}(x, y \sqcap z) \succ 0_{\mathbb{D}^*}$, *then* $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ *and* $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$.

Proof. Assume that $(X, \mathcal{R})$ is a picture fuzzy lattice and let $x, y, z \in X$.

1. Since $x \sqcup y$ is the least upper bound of a set $\{x, y\}$, then $x \sqcup y$ is an upper bound of x and y. Similarly, $x \sqcap y$ is the greatest lower bound of $\{x, y\}$, then $x \sqcap y$ is a lower bound of x and y.

2. Suppose that $\mathcal{R}(x, z) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(y, z) \succ 0_{\mathbb{D}^*}$, it follows that z is an upper bound of $\{x, y\}$. Since $x \sqcup y$ is the least upper bound, then $\mathcal{R}(x \sqcup y, z) \succ 0_{\mathbb{D}^*}$.

3. Suppose that $\mathcal{R}(z, x) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(z, y) \succ 0_{\mathbb{D}^*}$, it follows that z is a lower bound of $\{x, y\}$. Since $x \sqcap y$ is the greatest lower bound, then $\mathcal{R}(x \sqcap y, z) \succ 0_{\mathbb{D}^*}$.

4. Suppose that $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$. Since $\mu_{\mathcal{R}}(y,y) = 1 > 0$, then from **(2)** we obtain $\mathcal{R}(x \sqcup y, y) \succ 0_{\mathbb{D}^*}$.

   And since $\mathcal{R}\ (y, x \sqcup y) \succ 0_{\mathbb{D}^*}$, then by the perfect antisymmetric of $\mathcal{R}$, $x \sqcup y = y$.

   Conversely, suppose that $x \sqcup y = y$. Then $\mathcal{R}(x,y) = \mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$

5. Suppose that $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$. Since $\mu_{\mathcal{R}}(x,x) = 1 > 0$, then from **(3)** we obtain $\mathcal{R}(x, x \sqcap y) \succ 0_{\mathbb{D}^*}$.

   And since $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$, then by the perfect antisymmetric of $\mathcal{R}$, $x \sqcap y = x$.

   Conversely, suppose that $x \sqcap y = x$. Then $\mathcal{R}(x,y) = \mathcal{R}(x \sqcap y, y) \succ 0_{\mathbb{D}^*}$.

6. Suppose that $\mathcal{R}(y,z) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(x \sqcap y, z) \succ 0_{\mathbb{D}^*}$.

   Since $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$, then $x \sqcap y$ is a lower bound of $\{x, z\}$.

   And since $x \sqcap z$ is the greatest lower bound of $\{x, z\}$, thus $\mathcal{R}(x \sqcap y, x \sqcap z) \succ 0_{\mathbb{D}^*}$.

   Similarly, we prove that $\mathcal{R}(x \sqcup y, x \sqcup z) \succ 0_{\mathbb{D}^*}$.

7. Suppose that $\mathcal{R}(x \sqcup y, z) \succ 0_{\mathbb{D}^*}$.

   Since $\mathcal{R}(y, x \sqcup y) \succ 0_{\mathbb{D}^*}$, then by the transitivity of $\mathcal{R}$, $\mathcal{R}(y,z) \succ 0_{\mathbb{D}^*}$.

   Since $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$, then by the transitivity of $\mathcal{R}$, $\mathcal{R}(x,z) \succ 0_{\mathbb{D}^*}$.

8. Suppose that $\mathcal{R}(x, y \sqcap z) \succ 0_{\mathbb{D}^*}$.

   Since $\mathcal{R}(y \sqcap z, y) \succ 0_{\mathbb{D}^*}$, then by the transitivity of $\mathcal{R}$, $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$.

   Since $\mathcal{R}(y \sqcap z, z) \succ 0_{\mathbb{D}^*}$, then by the transitivity of $\mathcal{R}$, $\mathcal{R}(x,z) \succ 0_{\mathbb{D}^*}$.

□

Proposition 3.10. *Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let $x, y, z \in X$. Then*

1. $x \sqcup x = x$, $x \sqcap x = x$.

2. $x \sqcup y = y \sqcup x$, $x \sqcap y = y \sqcap x$.

3. $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$, $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$.

4. $(x \sqcup y) \sqcap x = x$, $(x \sqcap y) \sqcup x = x$.

CHAPTER 4

FILTERS IN A LATTICES

In lattice theory, the concept of filters is fundamental. A lattice is a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and an infimum (also called a greatest lower bound or meet). A filter in a lattice is a specific, non-empty subset that relates to the lattice's structure and ordering. A filter is closed under the meet operation; it is typically used to capture the notion of "largeness" in the sense that it contains elements that are greater or equal to a certain threshold. This concept is pivotal in lattice theory and has applications in various areas of mathematics and computer science, particularly the study of order, algebraic structures, and logic.

When extending the concept of filters to the picture fuzzy case in lattice theory, we introduce a nuanced approach to dealing with uncertainties and partial membership characteristics. This extension naturally leads to the concept of picture fuzzy filters in lattices, which adapt the traditional definitions to accommodate the additional complexity brought by picture fuzzy sets. This concepts enhance the ability of lattice theory to model and analyze complex systems where uncertainty and partial information play critical roles, thus expanding the applicability of lattice theory into new and emerging areas of research and application.

This chapter presents an in-depth exploration of filters structure in the context of lattice theory, adapting and generalizing them through picture fuzzy case. It delves into various aspects of filters, starting from their fundamental definitions in crisp lattice and picture fuzzy lattice to exploring some of their properties and characterizations.

The first section delineates some basic concepts of picture fuzzy filters in a crisp lattice. By generalizing concepts and results existing in Zadeh's fuzzy sets and Atanassov's intuitionistic fuzzy sets into the context of picture fuzzy, this work paves the way for a deeper examination of filters in this new setting, also shedding light on the subtle differences and extensions introduced by picture fuzzy.

The second section broadens the discourse further by focusing on extending crisp filters and picture fuzzy filters from fuzzy lattice to picture fuzzy lattice. It provides a detailed examination of the properties and characterizations of picture fuzzy filters.

The final section focuses on a specific type of picture fuzzy filter, which is a picture fuzzy prime filter in a picture fuzzy lattice. It presents a set of properties and results that are associated with this specific structure.

Through a detailed exploration of filters in both crisp lattice and picture fuzzy lattice, this chapter aims to significantly contribute to the fields of picture fuzzy sets and lattice theory. By providing a meticulous examination of these concepts and their interrelations, it seeks to enhance theoretical understanding of lattice structures in picture fuzzy settings, paving the way for future research and applications in complex and uncertain environments.

4.1 Picture Fuzzy Filters in a Lattice

In this section, we will define the picture fuzzy filter in a crisp lattice. We accomplish this by generalizing some existing notions and results in the Zadeh's fuzzy sets and Antanssov's intuitionistic fuzzy sets (see [2–5, 37]) to the picture fuzzy case.

Definition 4.1. Let $(X, \leqslant)$ be a crisp lattice. A picture fuzzy set E of X is called a picture fuzzy filter of X, if it satisfies the following conditions, for any $x, y \in X$:

(i) $E(x \wedge y) \succeq E(x) \curlywedge E(y)$;

(ii) $E(x \vee y) \succeq E(x) \curlyvee E(y)$.

Example 4.1. Consider the lattice $(X, \leqslant)$ given in Example 3.7.

The picture fuzzy subset E' on X defined by

X	$E'(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$
c	$(0.10, 0.30, 0.50)$
d	$(0.10, 0.20, 0.40)$
e	$(0.10, 0.30, 0.50)$
f	$(0.40, 0.10, 0.30)$

is a picture fuzzy filter on $(X, \leqslant)$.

Remark 4.1. Every picture fuzzy filter is a picture fuzzy sublattice. But the converse is not true.

Example 4.2. The picture fuzzy filter $E^{'}$ given in Example 4.1 is a picture fuzzy sublattice.

But the picture fuzzy sublattice E given in Example 3.7 is not a picture fuzzy filter, since $E\left(c\vee b\right)=\left(0.20,0.00,0.40\right)$, $E\left(c\right)\curlyvee E\left(b\right)=\left(0.20,0.00,0.30\right)$ and $\left(0.20,0.00,0.40\right)\not\succeq\left(0.20,0.00,0.30\right)$.

Proposition 4.1. *Let $(X,\leqslant)$ be a crisp lattice. If E,F are picture fuzzy filters of $(X,\leqslant)$, then $E\cap F$ is also a picture fuzzy filter of $(X,\leqslant)$.*

Proof. Suppose that E and F are two picture fuzzy filters of $(X,\leqslant)$.

We have $\left(E\cap F\right)\left(x\right)=E\left(x\right)\curlywedge F\left(x\right)$, then

$$\begin{aligned}
\left(E\cap F\right)\left(x\wedge y\right) &= E\left(x\wedge y\right)\curlywedge F\left(x\wedge y\right)\\
&\succeq \left(E\left(x\right)\curlywedge E\left(y\right)\right)\curlywedge\left(F\left(x\right)\curlywedge F\left(y\right)\right)\\
&= \left(E\left(x\right)\curlywedge F\left(x\right)\right)\curlywedge\left(E\left(y\right)\curlywedge F\left(y\right)\right)\\
&= \left(E\cap F\right)\left(x\right)\curlywedge\left(E\cap F\right)\left(y\right).\\
\left(E\cap F\right)\left(x\vee y\right) &= E\left(x\vee y\right)\curlywedge F\left(x\vee y\right)\\
&\succeq \left(E\left(x\right)\curlyvee E\left(y\right)\right)\curlywedge\left(F\left(x\right)\curlyvee F\left(y\right)\right)\\
&\succeq \left(E\left(x\right)\curlywedge F\left(x\right)\right)\curlyvee\left(E\left(x\right)\curlywedge F\left(y\right)\right)\curlyvee\left(E\left(y\right)\curlywedge F\left(x\right)\right)\curlyvee\\
&\quad \left(E\left(y\right)\curlywedge F\left(y\right)\right)\\
&\succeq \left(E\left(x\right)\curlywedge F\left(x\right)\right)\curlyvee\left(E\left(y\right)\curlywedge F\left(y\right)\right)\\
&= \left(E\cap F\right)\left(x\right)\curlyvee\left(E\cap F\right)\left(y\right).
\end{aligned}$$

Hence $E\cap F$ is a picture fuzzy filters of $(X,\leqslant)$. □

Remark 4.2. The union of two picture fuzzy filters of $(X,\leqslant)$ need not be a picture fuzzy filter of $(X,\leqslant)$.

Example 4.3. Consider the lattice $(X,\leqslant)$ given in Example 3.7. Let $E^{'}$ be the picture fuzzy filter given in Example 4.1 and define the picture fuzzy filter E^{*} on X by

X	$E^{*}\left(x\right)$
a	$(0.00,0.00,1.00)$
b	$(0.10,0.40,0.50)$
c	$(0.00,0.00,1.00)$
d	$(0.30,0.10,0.40)$
e	$(0.00,0.00,1.00)$
f	$(0.30,0.20,0.40)$

Then,

X	$\left(E^{'} \cup E^{*}\right)(x)$
a	$(0.00, 0.00, 1.00)$
b	$(0.10, 0.40, 0.50)$
c	$(0.10, 0.30, 0.50)$
d	$(0.30, 0.10, 0.40)$
e	$(0.10, 0.30, 0.50)$
f	$(0.40, 0.10, 0.30)$

Since $\left(E^{'} \cup E^{*}\right)(c \wedge b) = 0_{\mathbb{D}^{*}}$, $\left(E^{'} \cup E^{*}\right)(c) \curlywedge \left(E^{'} \cup E^{*}\right)(b) = (0.1, 0.3, 0.5)$ and $(0,0,1) \not\succeq (0.1, 0.3, 0.5)$,

these imply that $E^{'} \cup E^{*}$ is not a picture fuzzy filter.

Proposition 4.2. *Let $(X, \leqslant)$ be a crisp lattice. If E is a picture fuzzy filter of $(X, \leqslant)$, then $S(E)$ and $\ker(E)$ are filters on $(X, \leqslant)$.*

Proof. Suppose that E is a picture fuzzy filter of $(X, \leqslant)$.

1. Let $x \in S(E)$ and $y \in X$ such that $x \leqslant y$ it follows that $E(x) \succ 0_{\mathbb{D}^{*}}$ and $x \vee y = y$.

   Since $E(y) = E(x \vee y) \succeq E(x) \curlyvee E(y) \succ 0_{\mathbb{D}^{*}}$, then $y \in S(E)$.

2. Let $x, y \in S(E)$. We prove that $x \wedge y \in S(E)$.

   $x, y \in S(E)$ implies $E(x) \succ 0_{\mathbb{D}^{*}}$ and $E(y) \succ 0_{\mathbb{D}^{*}}$. Since $E(x \wedge y) \succeq E(x) \curlywedge E(y)$, then according to Proposition 2.3(4), $E(x \wedge y) \succ 0_{\mathbb{D}^{*}}$. Hence $x \wedge y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Theorem 4.1. *Let $(X, \leqslant)$ be a crisp lattice. A picture fuzzy subset E of X is a picture fuzzy filter if and only if their α-cuts are filters of $(X, \leqslant)$ for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_0^{*}$.*

Proof. Suppose that E is a picture fuzzy filter on $(X, \leqslant)$ and prove that E_α are filters of $(X, \leqslant)$, for all $\alpha \in \mathbb{D}_0^{*}$.

1. Let $x \in E_\alpha$ and $y \in X$ such that $x \leqslant y$. Then $E(x) \succeq \alpha$ and $x \vee y = y$.

   It follows that $E(y) = E(x \vee y) \succeq E(x) \curlyvee E(y) \succeq \alpha$. Hence $y \in E_\alpha$.

2. Let $x, y \in E_\alpha$. Then it holds that $E(x) \succeq \alpha$ and $E(y) \succeq \alpha$.

   Since $E(x \wedge y) \succeq E(x) \curlywedge E(y) \succeq \alpha$, thus $x \wedge y \in E_\alpha$.

Conversely, suppose that E_α are filters of $(X, \leqslant)$, for all $\alpha \in \mathbb{D}_0^*$, and show that E is a picture fuzzy filter on $(X, \leqslant)$.

1. Let $x, y \in X$ and let $\alpha \in \mathbb{D}^*$. Put $E(x) \curlywedge E(y) = \alpha$.

   The case $\alpha = 0_{\mathbb{D}^*}$ is direct.

   If $\alpha \succ 0_{\mathbb{D}^*}$, we have $E(x) \succeq E(x) \curlywedge E(y) = \alpha$ and $E(y) \succeq E(x) \curlywedge E(y) = \alpha$, that is, $x, y \in E_\alpha$. Since E_α is a filter, then $x \wedge y \in E_\alpha$. This implies that $E(x \wedge y) \succeq \alpha = E(x) \curlywedge E(y)$.

2. Let $x, y \in X$ and let $\alpha, \beta \in \mathbb{D}_0^*$ such that $E(x) = \alpha$ and $E(y) = \beta$. Then $x \in E_\alpha$ and $y \in E_\beta$. Since E_α and E_β are filters, it follow that $x \vee y \in E_\alpha$ and $x \vee y \in E_\beta$, i.e., $E(x \vee y) \succeq \alpha$ and $E(x \vee y) \succeq \beta$.

   Hence $E(x \vee y) \succeq \alpha \curlyvee \beta = E(x) \curlyvee E(y)$.

□

4.2 Picture Fuzzy Filters in a Picture Fuzzy Lattice

This section extends the notion of a crisp filter and a fuzzy filter in a fuzzy lattice [30,34] to a crisp filter and a picture fuzzy filter in a picture fuzzy lattice, as well as giving more of their characterizations and proving some properties.

Definition 4.2. Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let E be a non-empty subset of X. E is a crisp filter on $(X, \mathcal{R})$ if it satisfies the following conditions, for any $x, y \in X$:

(i) If $x \in X, y \in E$ and $\mathcal{R}(y, x) \succ 0_{\mathbb{D}^*}$, then $x \in E$.

(ii) If $x, y \in E$, then $x \sqcap y \in E$.

Example 4.4. Let $(X, \mathcal{R})$ be the picture fuzzy lattice defined in Example 3.8, and let $E_1 = \{a, b\}$ be a non-empty subset of X. It easy to see that E_1 is a crisp filter of $(X, \mathcal{R})$.

Definition 4.3. Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let E be a picture fuzzy subset of X. E is a picture fuzzy filter on $(X, \mathcal{R})$ if it meets the conditions below:

(C$_1$) $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$, for all $x, y \in X$.

(C$_2$) $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$ implies $E(x) \preceq E(y)$, for all $x, y \in X$.

Proposition 4.3. *Let $(X,\mathcal{R})$ be a picture fuzzy lattice. If E and F are two picture fuzzy filters of $(X,\mathcal{R})$, then $E \cap F$ is a picture fuzzy filter of $(X,\mathcal{R})$.*

Proof. Suppose that E and F are two picture fuzzy filters of $(X,\mathcal{R})$. Then, for all $x, y \in X$,

(C$_1$)
$$\begin{aligned}(E\cap F)(x\sqcap y) &= E(x\sqcap y) \curlywedge F(x\sqcap y)\\ &\succeq (E(x)\curlywedge E(y)) \curlywedge (F(x)\curlywedge F(y))\\ &= (E(x)\curlywedge F(x)) \curlywedge (E(y)\curlywedge F(y))\\ &= (E\cap F)(x) \curlywedge (E\cap F)(y).\end{aligned}$$

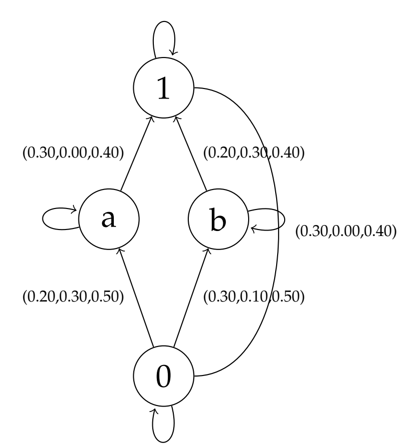
(C$_2$) suppose that $\mathcal{R}(x,y) \succ 0_{\mathbb{D}^*}$. Then $E(x) \preceq E(y)$ and $F(x) \preceq F(y)$, this implies that $E(x) \curlywedge F(x) \preceq E(y) \curlywedge F(y)$. That is, $(E\cap F)(x) \preceq (E\cap F)(y)$.

□

Remark 4.3. The union of two picture fuzzy filters does not necessarily be a picture fuzzy filter, as seen in the following example:

Example 4.5. Let $X = \{0, a, b, 1\}$ and let $(X,\mathcal{R})$ be the picture fuzzy lattice given by the following table and represented by the given Hasse diagram.

$\mathcal{R}$	0	a	b	1
0	$(1.00, 0.00, 0.00)$	$(0.20, 0.30, 0.50)$	$(0.30, 0.10, 0.50)$	$(0.30, 0.00, 0.40)$
a	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.00, 0.00, 1.00)$	$(0.30, 0.00, 0.40)$
b	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$	$(0.20, 0.30, 0.40)$
1	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$	$(1.00, 0.00, 0.00)$



Hasse diagram of $(X,\mathcal{R})$, where $X = \{0, a, b, 1\}$.

We define the two picture fuzzy filters E_1 and E_2 on $(X, \mathcal{R})$ by

X	$E_1(x)$	$E_2(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.20, 0.40, 0.30)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.30, 0.20, 0.20)$	$(0.30, 0.20, 0.10)$

Then,

X	$(E_1 \cap E_2)(x)$	$(E_1 \cup E_2)(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.00, 0.00, 1.00)$	$(0.20, 0.40, 0.30)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.30, 0.20, 0.20)$	$(0.30, 0.20, 0.10)$

It is easy to see that $E_1 \cap E_2$ is a picture fuzzy filter of $(X, \mathcal{R})$, but $E_1 \cup E_2$ is not a picture fuzzy filter. Indeed, $(E_1 \cup E_2)(a \sqcap b) = 0_{\mathbb{D}^*}$, $(E_1 \cup E_2)(a) \curlywedge (E_1 \cup E_2)(b) = (0.1, 0.6, 0.3)$ and $(0, 0, 1) \nsucceq (0.1, 0.6, 0.3)$.

Proposition 4.4. *Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let E be a picture fuzzy subset on X. If E is a picture fuzzy filter of $(X, \mathcal{R})$, then $S(E)$ and $\ker(E)$ are crisp filters of $(X, \mathcal{R})$.*

Proof. Suppose that E is a picture fuzzy filter of $(X, \mathcal{R})$ and let $x, y \in X$.

1. If $x \in S(E)$ and $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ implies $E(y) \succeq E(x) \succ 0_{\mathbb{D}^*}$. Thus $y \in S(E)$.

2. If $x, y \in S(E)$, then $E(x) \succ 0_{\mathbb{D}^*}$ and $E(y) \succ 0_{\mathbb{D}^*}$. Since $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$, according to Proposition 2.3(4), $E(x \sqcap y) \succ 0_{\mathbb{D}^*}$, i.e., $x \sqcap y \in S(E)$.

Similarly, we obtain the same result for $\ker(E)$. □

Proposition 4.5. *Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let E be a picture fuzzy subset on X. E is a picture fuzzy filter of $(X, \mathcal{R})$ if and only if its α-cuts are crisp filters of $(X, \mathcal{R})$.*

Proof. Let $x, y \in X$, $\alpha \in \mathbb{D}_0^*$. Assume that E is a picture fuzzy filter of $(X, \mathcal{R})$.

1. If $x \in E_\alpha$ and $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $E(x) \succeq \alpha$ and $E(y) \succeq E(x) \succeq \alpha$. Hence $y \in E_\alpha$.

2. If $x, y \in E_\alpha$, then $E(x) \succeq \alpha$ and $E(y) \succeq \alpha$. According to Proposition 2.3, we obtain $E(x \sqcap y) \succeq E(x) \curlywedge E(y) \succeq \alpha$, That is, $x \sqcap y \in E_\alpha$.

Conversely, suppose that E_α are crisp filters of $(X, \mathcal{R})$, for all $\alpha \in \mathbb{D}_0^*$.

(C$_1$) Put $E(x) \curlywedge E(y) = \beta$. It is easy to see that if $\beta = 0_{\mathbb{D}^*}$, then $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$.

If $\beta \succ 0_{\mathbb{D}^*}$. We have $E(x) \succeq \beta$ and $E(y) \succeq \beta$ it follows that $x, y \in E_\beta$. Then $x \sqcap y \in E_\beta$. Hence $E(x \sqcap y) \succeq \beta = E(x) \curlywedge E(y)$.

(C$_2$) Suppose that $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$ and put $E(x) = \gamma$. The case $\gamma = 0_{\mathbb{D}^*}$ is trivially.

If $\gamma \succ 0_{\mathbb{D}^*}$. Then $x \in E_\gamma$ implies $y \in E_\gamma$. Hence $E(y) \succeq \gamma = E(x)$.

□

Now, we give some characterizations of picture fuzzy filters of a picture fuzzy lattice.

Theorem 4.2. *E is a picture fuzzy filter of a picture fuzzy lattice $(X, \mathcal{R})$ if and only if it satisfies* **(C$_1$)** *and*

(C$_3$) *for all $x, y \in X, E(x) \succeq E(x \sqcap y) \curlywedge E(y)$.*

Proof. Suppose that E is a picture fuzzy filter of $(X, \mathcal{R})$. It suffices to prove **(C$_3$).**

Since $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$, then according to **(C$_1$)** and **(C$_2$),** $E(x) \succeq E(x \sqcap y) \succeq E(x \sqcap y) \curlywedge E(y)$.

Conversely, suppose that **(C$_1$)** and **(C$_3$)** are satisfied.

If $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $x \sqcap y = x$ implies $E(y) \succeq E(x \sqcap y) \curlywedge E(x) = E(x)$. □

Theorem 4.3. *E is a picture fuzzy filter of a picture fuzzy lattice $(X, \mathcal{R})$ if and only if it satisfies*

(C$_4$) *for any $x, y, z \in X$, $\mathcal{R}(x \sqcap y, z) \succ 0_{\mathbb{D}^*}$ implies $E(z) \succeq E(x) \curlywedge E(y)$.*

Proof. Let $x, y, z \in X$. Suppose that E is a picture fuzzy filter of $(X, \mathcal{R})$.

If $\mathcal{R}(x \sqcap y, z) \succ 0_{\mathbb{D}^*}$, then according to **(C$_1$)** and **(C$_2$),** $E(z) \succeq E(x \sqcap y) \succeq E(x) \curlywedge E(y)$.

Conversely, suppose that **(C$_4$)** is satisfied. Then,

1. Since $\mathcal{R}(x \sqcap y, x \sqcap y) \succ 0_{\mathbb{D}^*}$, then $E(x \sqcap y) \succeq E(x) \curlywedge E(y)$.

2. If $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $\mathcal{R}(x \sqcap x, y) \succ 0_{\mathbb{D}^*}$. It follows that $E(y) \succeq E(x) \curlywedge E(x)$, that is, $E(y) \succeq E(x)$.

□

Theorem 4.4. *E is a picture fuzzy filter of a picture fuzzy lattice $(X, \mathcal{R})$ if and only if it satisfies* **(C_1)** *and*

(C_5) *for all $x, y \in X, E(x \sqcup y) \succeq E(x)$.*

Proof. Let $x, y \in X$.

Suppose that E is a picture fuzzy filter of $(X, \mathcal{R})$. It suffices to prove **(C_5).**

Since $\mathcal{R}(x, x \sqcup y) \succ 0_{\mathbb{D}^*}$, then $E(x \sqcup y) \succeq E(x)$.

Conversely, suppose that **(C_1)** and **(C_5)** are satisfied.

If $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $x \sqcup y = y$. Hence $E(x) \preceq E(x \sqcup y) = E(y)$. □

Theorem 4.5. *E is a picture fuzzy filter of a picture fuzzy lattice $(X, \mathcal{R})$ if and only if the following condition is satisfied:*

(C_6) *for all $x, y \in X, E(x \sqcap y) = E(x) \curlywedge E(y)$.*

Proof. Suppose that E is a picture fuzzy filter of $(X, \mathcal{R})$.

In view of the definition of picture fuzzy filter, it suffices to show that $E(x \sqcap y) \preceq E(x) \curlywedge E(y)$.

Let $x, y \in X$. Since $\mathcal{R}(x \sqcap y, x) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}(x \sqcap y, y) \succ 0_{\mathbb{D}^*}$, then $E(x \sqcap y) \preceq E(x)$ and $E(x \sqcap y) \preceq E(y)$. Hence $E(x \sqcap y) \preceq E(x) \curlywedge E(y)$.

Conversely, suppose that $E(x \sqcap y) = E(x) \curlywedge E(y)$, for all $x, y \in X$.

1. $E(x \sqcap y) = E(x) \curlywedge E(y)$ implies **(C_1).**

2. If $\mathcal{R}(x, y) \succ 0_{\mathbb{D}^*}$, then $x \sqcap y = x$. Thus $E(x) = E(x \sqcap y) = E(x) \curlywedge E(y)$, that is, $E(x) \preceq E(y)$.

□

4.3 Picture Fuzzy Prime Filters in a Picture Fuzzy Lattice

In what follows, we extend the notion of fuzzy prime filters in a fuzzy lattice and intuitionistic fuzzy prime filters in an intuitionistic fuzzy lattice [6, 31, 32, 34] to picture fuzzy prime filters in a picture fuzzy lattice and prove some of its results.

Definition 4.4. Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let E be a crisp filter of $(X, \mathcal{R})$. E is called crisp prime filter if for all $x, y \in X$, $x \sqcup y \in E$ implies that $x \in E$ or $y \in E$.

Definition 4.5. Let $(X, \mathcal{R})$ be a picture fuzzy lattice and let E be a picture fuzzy filter of $(X, \mathcal{R})$. E is called a picture fuzzy prime filter if for all $x, y \in X$, $E(x \sqcup y) = E(x) \curlyvee E(y)$.

Remark 4.4. The intersection of two picture fuzzy prime filters of $(X, \mathcal{R})$ does not be necessarily a picture fuzzy prime filter of $(X, \mathcal{R})$.

Example 4.6. Consider the lattice $(X, \mathcal{R})$ given in Example 4.5.

Let E_1, E_2 be two picture fuzzy prime filters on $(X, \mathcal{R})$ defined by

X	$E_1(x)$	$E_2(x)$
0	$(0.00, 0.00, 1.00)$	$(0.00, 0.00, 1.00)$
a	$(0.20, 0.40, 0.30)$	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$	$(0.10, 0.50, 0.20)$
1	$(0.20, 0.40, 0.30)$	$(0.10, 0.50, 0.20)$

Then,

X	$(E_1 \cap E_2)(x)$
0	$(0.00, 0.00, 1.00)$
a	$(0.00, 0.00, 1.00)$
b	$(0.00, 0.00, 1.00)$
1	$(0.10, 0.60, 0.30)$

It is easy to checked that $E_1 \cap E_2$ is a picture fuzzy filter.

But $(E_1 \cap E_2)(a \sqcup b) = (E_1 \cap E_2)(1) = (0.10, 0.60, 0.30)$, in other hand, $(E_1 \cap E_2)(a) \curlyvee (E_1 \cap E_2)(b) = (0, 0, 1) \curlyvee (0, 0, 1) = (0, 0, 1) \neq (0.10, 0.60, 0.30)$. Hence the picture fuzzy filter $E_1 \cap E_2$ is not prime.

Proposition 4.6. *Let $(X, \mathcal{R})$ be a picture fuzzy lattice. If E is a picture fuzzy prime filter on $(X, \mathcal{R})$, then $S(E)$ and $\ker(E)$ are crisp prime filters on $(X, \mathcal{R})$.*

Proof. Suppose that E is a picture fuzzy prime filter on $(X, \mathcal{R})$. From Proposition 4.4, it holds that $S(E)$ is a filter on $(X, \mathcal{R})$. Next, we prove that $S(E)$ is prime.

Let $x, y \in X$. If $x \vee y \in S(E)$, then $E(x \vee y) \succ 0_{\mathbb{D}^*}$. Since $E(x \vee y) = E(x) \curlyvee E(y)$, this implies from Proposition 2.3(6) that $E(x) \succ 0_{\mathbb{D}^*}$ or $E(y) \succ 0_{\mathbb{D}^*}$. Hence, either $x \in S(E)$ or $y \in S(E)$. □

Theorem 4.6. *Let $(X, \mathcal{R})$ be a picture fuzzy lattice. if for all $\alpha \in \mathbb{D}_0^*$, E_α are crisp prime filters on $(X, \mathcal{R})$, then E is a picture fuzzy prime filter on $(X, \mathcal{R})$.*

Proof. Suppose that E_α are crisp prime filters on $(X, \mathcal{R})$, for all $\alpha \in \mathbb{D}_0^*$. From Proposition 4.5, E is a picture fuzzy filter on $(X, \mathcal{R})$. It remains to show the primality of E, i.e., for all $x, y \in X$, $E(x \sqcup y) = E(x) \curlyvee E(y)$.

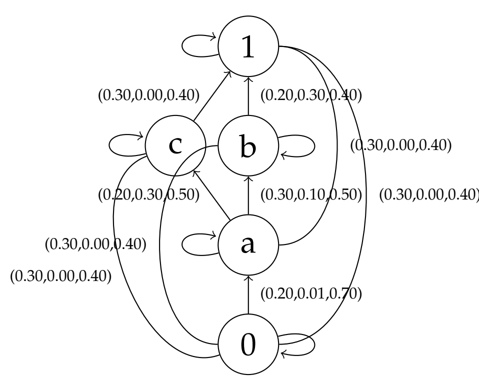
Put $E(x \sqcup y) = \alpha$. Then $x \sqcup y \in E_\alpha$. Since E_α is a prime filter, this implies that $x \in E_\alpha$ or $y \in E_\alpha$. Hence $E(x) \curlyvee E(y) \succeq \alpha = E(x \sqcup y)$.

On other hand, since $\mathcal{R}\ (x, x \sqcup y) \succ 0_{\mathbb{D}^*}$ and $\mathcal{R}\ (y, x \sqcup y) \succ 0_{\mathbb{D}^*}$ these imply that $E(x) \preceq E(x \sqcup y)$ and $E(y) \preceq E(x \sqcup y)$. Hence $E(x) \curlyvee E(y) \preceq E(x \sqcup y)$. □

Remark 4.5. Unlike the fuzzy case, the converse implication in Proposition 4.6 is not true. Indeed.

Example 4.7. Let $(X, \mathcal{R})$ be the lattice given as follows

$\mathcal{R}$	0	a	b	c	1
0	$1_{\mathbb{D}^*}$	$(0.20, 0.01, 0.70)$	$(0.40, 0.20, 0.30)$	$(0.50, 0.10, 0.30)$	$1_{\mathbb{D}^*}$
a	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$(0.40, 0.10, 0.50)$	$(0.10, 0.00, 0.60)$	$(0.70, 0.00, 0.20)$
b	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$(0.30, 0.00, 0.40)$
c	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$	$(0.50, 0.20, 0.10)$
1	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	$1_{\mathbb{D}^*}$



Hasse diagram of $(X, \mathcal{R})$, where $X = \{0, a, b, c, 1\}$.

The following table represented two picture fuzzy subsets: E_1 and its support, and E_2 and its kernel

X	$E_1(x)$	$E_2(x)$	$S(E_1)(x)$	$\ker(E_2)(x)$
0	$0_{\mathbb{D}^*}$	$0_{\mathbb{D}^*}$	0	0
a	$(0.10, 0.60, 0.30)$	$0_{\mathbb{D}^*}$	1	0
b	$(0.10, 0.40, 0.10)$	$(0.10, 0.30, 0.20)$	1	0
c	$(0.20, 0.10, 0.30)$	$1_{\mathbb{D}^*}$	1	1
1	$(0.20, 0.60, 0.10)$	$1_{\mathbb{D}^*}$	1	1

It is easy to see that $S(E_1)$ and $\ker(E_2)$ are crisp prime filters on $(X, \mathcal{R})$, but E_1 and E_2 are not PFPFs on $(X, \mathcal{R})$.

Conclusion

This study has shed light on the complexities and challenges of understanding uncertainty and ambiguity in real phenomena through a strict focus on theoretical aspects, affirming the vital role that picture fuzzy sets play in developing mathematical theories capable of addressing these challenges.

Through the careful theoretical discussion of the definitions presented by Coung, theoretical modifications are suggested that may provide a new horizon for research in this field. By applying these modifications to other theories in mathematics, this work has expanded the horizons of picture fuzzy sets, demonstrating its utility in modeling complex structures in picture fuzzy sets, picture fuzzy relations, picture fuzzy lattices, and the precise concepts of filters in this context. Despite its purely theoretical nature, the insights provided by this study greatly enhance our ability to visualize and analyse complex concepts related to picture fuzzy sets and their future applications.

The results of this thesis indicate great potential for future research, especially in the area of developing more advanced mathematical theories based on the concepts explored here. Future work can build on the theoretical foundations established by this study to explore new applications in mathematics, artificial intelligence, computer science, and beyond, thus expanding the scope of picture fuzzy sets and their contribution to contemporary challenges.

Bibliography

[1] S. Abdullah and S. Ashraf, *Cubic picture fuzzy sets and their application to a petroleum circulation center evaluation problem*, Sigma, **10**(2) (2019), 99-133.

[2] N. Ajmal and K. V. Thomas, *Fuzzy lattices*, Information sciences, **79**(3-4) (1994), 271-291. https://doi.org/10.1016/0020-0255(94)90124-4

[3] A. Amroune and B. Davvaz, *Fuzzy ordered sets and duality for finite fuzzy distributive lattices*, Iranian Journal of Fuzzy Systems, **8**(5) (2011), 1-12. https://doi.org/10.5899/2016/jfsva-00355

[4] A. Amroune and A. Oumhani, *A representation theorem for infinite fuzzy distributive lattices*, Journal of Intelligent and Fuzzy Systems, **32**(1) (2017), 35-42. https://doi.org/10.3233/jifs-151029

[5] A. Amroune, A. Oumhani and B. Davvaz, *Kinds of t-fuzzy Filters of Fuzzy Lattices*, Fuzzy Information and Engineering, **9**(3) (2017), 325-343. https://doi.org/10.1016/j.fiae.2017.09.005

[6] A. Amroune and B. Ziane, *More on Intuitionistic Fuzzy sublattices and their ideals*, Facta Universitatis, Series: Mathematics and Informatics, **34**(5) (2019), 871-888. https://doi.org/10.22190/FUMI1905871A

[7] S. Ashraf, T. Mahmood, S. Abdullah, and Q. Khan, *Different approaches to multi-criteria group decision making problems for picture fuzzy environment*, Bulletin of the Brazilian Mathematical Society, New Series, **50**(2) (2019), 373-397. https://doi.org/10.1007/s00574-018-0103-y

[8] K. T. Atanassov, *Intuitionistic fuzzy sets VII ITKR's Session*, Sofia, **1** (1983), 983.

[9] K. T. Atanassov and S. Stoeva, *Intuitionistic L-fuzzy sets*, in R. Trappl (Ed.), Cybernetics and Systems Research 2 (Elsevier Sci. Publ., Amsterdam), 1984, 539-540.

[10] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1) (1986), 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3

[11] K. T. Atanassov, *More on intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **33**(1) (1989) 37-45. https://doi.org/10.1016/0165-0114(89)90215-7

[12] K. T. Atanassov, *Studies in Fuzziness and Soft Computing*, Springer, (2012). https://doi.org/ 10.1007/978-3-642-29127-2

[13] Y. Bo and W. Wangming, *Fuzzy ideals on a distributive lattice*, Fuzzy sets and systems, **35**(2) (1990), 231-240. https://doi.org/10.1016/0165-0114(90)90196-d

[14] C. Bo and X. Zhang, *New Operations of Picture Fuzzy Relations and Fuzzy Comprehensive Evaluation*, Symmetry, **9**(11) (2017), 268. https://doi.org/10.3390/sym9110268

[15] S. Boudaoud, L. Zedam and S. Milles, *Principal intuitionistic fuzzy ideals and filters on a lattice*, Discussiones Mathematicae-General Algebra and Applications, **40**(1) (2020), 75-88. doi:10.7151/dmgaa.1325

[16] H. Bustince and P. Burillo, *Structure on intuitionistic fuzzy relations*, Fuzzy Sets and Systems, **78** (1996), 293-303. https://doi.org/10.1016/0165-0114(96)84610-0

[17] P. J. Burillo and H. Bustince, *Intuitionistic fuzzy relations (Part I)*, Mathware and soft computing, **2**(1) (1995), 5-38.

[18] I. Chon, *Fuzzy partial order relations and fuzzy lattices*, Korean Journal of Mathematics, **17**(4) (2009), 361-374.

[19] C. Cornelis and G. Deschrijver, *The compositional rule of inference in an intuitionistic fuzzy logic setting*, K. Striegnitz (Ed.), Proceedings of Student Session, Kluwer Academic Publishers, (2001) 83–94.

[20] B. C. Cuong and V. Kreinovich, *Picture Fuzzy Sets-a new concept for computational intelligence problems*, 2013 third world congress on information and communication technologies (WICT 2013). IEEE, (2013). https://doi.org/ 10.1109/WICT.2013.7113099

[21] B. C. Cuong and P.V. Hai, *Some fuzzy logic operators for picture fuzzy sets*, In Proceedings of the IEEE Seventh International Conference on Knowledge and Systems Engineering, Ho Chi Minh, Vietnam, 8–10 October (2015) 132–137. https://doi.org/ 10.1109/KSE.2015.20

[22] C. B. Cuong, R. T. Ngan and B. D. Hai, *An involutive picture fuzzy negator on picture fuzzy sets and some De Morgan triples*, 2015 Seventh International Conference on Knowledge and Systems Engineering (KSE), IEEE. (2015). https://doi.org/ 10.1109/KSE.2015.21

[23] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, Cambridge university press, (2002).

[24] P. Dutta, *Medical diagnosis via distance measures on picture fuzzy sets*, Advances in Modelling and Analysis A, **54**(2) (2017) 657-672.

[25] H. Garg, *Some picture fuzzy aggregation operators and their applications to multicriteria decision-making*, Arabian Journal for Science and Engineering, **42**(12) (2017), 5275-5290. https://doi.org/ 10.1007/s13369-017-2625-9

[26] J. A. Goguen, *L-fuzzy sets*, Journal of mathematical analysis and applications, **18**(1) (1967), 145-174. https://doi.org/10.1016/0022-247X(67)90189-8

[27] E. P. Klement, R. Mesiar and A. Stupňanova, *Picture fuzzy sets and 3-fuzzy sets*, 2018 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), IEEE. (2018). https://doi.org/ 10.1109/FUZZ-IEEE.2018.8491520

[28] E. P. Klement and R. Mesiar, *Intervals and More: Aggregation Functions for Picture Fuzzy Sets*, Beyond Traditional Probabilistic Data Processing Techniques: Interval, Fuzzy etc. Methods and Their Applications 835 (2020), 179-194. https://doi.org/10.1007/978-3-030-31041-7_10

[29] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, *Kinds of Ideals of Fuzzy Lattice*, Second Brasilian Congress on Fuzzy Systems, (2012), 657-671.

[30] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, *On fuzzy ideals of fuzzy lattice*, IEEE International Conference on Fuzzy Systems, (2012), 1-5. https://doi.org/10.1109/fuzz-ieee.2012.6251307

[31] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, *α-ideals of fuzzy lattice*, In IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), (2013), 157-162.

[32] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, *Types of fuzzy ideals in fuzzy lattices*, Journal of Intelligent and Fuzzy Systems, **28** (2015), 929-945. https://doi.org/10.3233/ifs-141374

[33] S. Milles, L. Zedam and E. Rak, *Characterizations of intuitionistic fuzzy ideals and filters based on lattice operations*, Journal of Fuzzy Set Valued Analysis, **2** (2017), 143-159. doi:10.5899/2017/jfsva-00399

[34] A. Oumhani and A. Amroune, *More on fuzzy lattices and their filters*, Annals of Oradea University-Mathematics Fascicola, **25**(2) (2018), 143-155.

[35] D. Saadi and A. Amroune, *More on picture fuzzy sets and their properties*, TWMS Journal of Applied and Engineering Mathematics, **14**(3) (2024), 1136-1151.

[36] D. Saadi and A. Amroune, *Picture fuzzy ordering and $\mathbb{D}^*$-based lattices*, Kragujevac Journal of Mathematics, **50**(5) (2026), 827-845.

[37] B. Šešelja and A. Tepavčević, *Fuzzifying closure systems and fuzzy lattices*, International Workshop on Rough Sets, Fuzzy Sets, Data Mining, and Granular-Soft Computing. Springer, Berlin, Heidelberg, (2007), 111-118. https://doi.org/10.1007/978-3-540-72530-5_13

[38] L. H. Son, *Generalized picture distance measure and applications to picture fuzzy clustering*, Applied Soft Computing, **46**(C) (2016), 284-295. http://dx.doi.org/10.1016/j.asoc.2016.05.009

[39] K. V. Thomas and L. S. Nair, *Intuitionistic fuzzy sublattices and ideals*, Fuzzy information and Engineering, **3**(3) (2011), 321-331. https://doi.org/10.1007/s12543-011-0086-5

[40] B. K. Tripathy, M. K. Satapathy and P. K. Choudhury, *Intuitionistic fuzzy lattices and intuitionistic fuzzy Boolean algebras*, International Journal of Engineering and Technology, **5** (2013), 2352-2361.

[41] P. Venugopalan, *Fuzzy ordered sets*, Fuzzy sets and systems **46**(2) (1992), 221-226. https://doi.org/10.1016/0165-0114(92)90134-p

[42] C. Wang, X. Zhou, H. Tu, and S. Tao, *Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making*, italian journal of pure and applied mathematics, **37** (2017), 477-492.

[43] L. A. Zadeh, *Fuzzy Sets*, Information and Control, **8** (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[44] L. A. Zadeh, *Similarity relation and fuzzy orderings*, Information Sciences, **3** (1971), 177-200. https://doi.org/10.1016/S0020-0255(71)80005-1

[45] L. Zedam, A.Amroune and B.Davvaz, *Szpilrajn theorem for intuitionistic fuzzy orderings*, Annals of Fuzzy Mathematics and Informatics, **9**(5) (2015), 703-718.

[46] B. Ziane and A. Amroune, *Representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorders and fuzzy weak $\mathcal{T}$-orders*, Discussiones Mathematicae: General Algebra & Applications, **41**(1) (2021). https://doi.org/10.7151/dmgaa.1345

ملخّص: في هذه الأطروحة، قمنا بمناقشة تعريفات كيونغ وكراينوفتش ثم تعديلها من خلال دراسة بنية المجموعة *D، التي تعمل كنموذج أولي لمجموعة الضبابية للصورة. من خلال هذه المفاهيم المعدلة، قدمنا بعض المجموعات المميزة مثل الحامل، والنواة، والقطع α، والقطع α الدقيق، والخط الضبابي من الدرجة α لمجموعة الضبابية للصورة، حيث α في *D. لقد أثبتنا بعض خصائص هذه المجموعات المميزة وقدمنا أيضًا بعض نظريات التفكيك لمجموعة ضبابية للصورة. كما قمنا بتوسيع بعض عوامل تحويل أتاناسوف المشروطة لتشمل الحالة الضبابية للصورة. بالإضافة إلى ذلك، قمنا باستكشاف بعض الجوانب الأساسية المتعلقة بعلاقات الضبابية للصورة، وشبكات الضبابية للصورة، ومرشحاتها، وتمييزها باستخدام الحامل، والنواة، والقطع، وما إلى ذلك. كما أثبتنا أيضًا بعض النتائج الموجودة المتعلقة بالمرشحات الأولية.
الكلمات المفتاحية: مجموعة ضبابية للصورة، المجموعة *D، علاقة ضبابية للصورة، شبكة ضبابية للصورة، مرشحات ضبابي للصورة.

Abstract: In this thesis, we have discussed and subsequently modified Coung's definitions through the study of the structure of D* (the set of membership degrees), which serves as a prototype of the picture fuzzy set. Through these modified concepts, we introduced some characteristic sets such as support, kernel, α-cuts, strong α-cuts, and the picture fuzzy lines of degree α of a picture fuzzy set, where α is in D*. We demonstrated some of their properties. Also, we presented some decomposition theorems for a picture fuzzy set and extended some of Atanassov's modal operators to the picture fuzzy case. Additionally, we explored some of the fundamental aspects related to picture fuzzy relations, picture fuzzy lattices, and their filters, characterizing them using support, kernel, a-cuts, etc. We also proved some existing results related to picture fuzzy prime filters.
Keywords: Picture fuzzy set, the set D*, picture fuzzy relation, picture fuzzy lattice, picture fuzzy filters.

Résumé : Dans cette thèse, nous avons étudié et corrigé les définitions de Coung et Kreinovich à travers l'étude de la structure de l'ensemble D*, qui sert de prototype de l'ensemble flou d'image. À travers ces concepts modifiés, nous avons introduit des ensembles spéciaux tels que le support, le noyau, les α-coupe, les α-coupe strict et les lignes de floues de degré α d'un ensemble flou d'image, où α est dans D*. Nous avons prouvé certaines de leurs propriétés et présenté également quelques théorèmes de décomposition pour un ensemble flou d'images. Nous avons étendu certains des opérateurs d'Atanassov au cas de flou d'image. De plus, nous avons exploré certains des aspects fondamentaux liés aux relations floues d'images, aux treillis flous d’image, à leurs filtres, les caractérisation à l'aide du support, du noyau, des α-coupes…, etc. Nous avons également prouvé certains résultats existants relatifs au prime filtre.
Mots-clés : Ensemble flou d'image, l'ensemble D*, relation floue d'image, treillis flou d’image, filtre flou d’image.