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Some properties of Herz-type Sobolev spaces on domains

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Dedicate

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Contents

1 Herz spaces	3
1.1 Definition and basic properties	3
1.2 Technical results	4
2 Herz-type Sobolev spaces	13
2.1 Definition and basic properties	13
2.2 Sobolev embeddings	15
3 Further embeddings	22
3.1 Embeddings of type $\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_p^{\alpha_1,r}(\Omega)$ and $\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_\infty^{\alpha_1,r}(\Omega)$	22
3.2 Embeddings of type $\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_q^{\alpha_1,r}(\Omega), q < p$	27

Notation

- $\mathbb{R}^n$ is the n -dimensional real Euclidean space.
- $\mathbb{N}$ is the collection of all natural numbers.
- $\mathbb{Z}$ is the set of all integer numbers.
- For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.
- The Euclidean scalar product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is given by $x \cdot y = x_1 y_1 + \dots + x_n y_n$.
- The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$ for non-negative functions f and g .
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r .
- For $k \in \mathbb{Z}$, we denote by $R_k = \{x \in \mathbb{R}^n : 2^{k-1} \leq |x| < 2^k\}$
- χ_k the characteristic function of the set R_k

Introduction

Function spaces have been widely used in various areas of analysis, such as harmonic analysis and partial differential equations. Among these spaces, Sobolev spaces are among the most important examples. Interest in these spaces arises not only from theoretical considerations but also from their numerous applications in mathematical analysis. We refer the reader to the monographs [1], [2], [4], and [18] for further details, historical remarks, and references concerning Sobolev spaces.

It is well known that Herz spaces play an important role in harmonic analysis. Since their introduction in [19], the theory of these spaces has undergone remarkable development, partly due to their usefulness in various applications. For instance, Herz spaces appear in the characterization of multipliers on Hardy spaces [3], in the summability of Fourier transforms [10], and in the regularity theory for elliptic equations in divergence form [22]. Moreover, the authors in [23] studied the Cauchy problem for Navier-Stokes equations in Herz and weak Herz spaces. More recently, Herz spaces have appeared in the study of semilinear parabolic equations [7] and the summability of Fourier transforms on mixed-norm Lebesgue spaces [11]. For important recent developments and further results on Herz spaces, we refer the reader to [17], and the monograph [24].

Motivated by Sobolev and Herz spaces, we introduce a class of function spaces called Herz-type Sobolev spaces, which generalize the classical Sobolev spaces. These spaces over R^n were introduced by Lu and Yang in [16], where several applications to partial differential equations were established.

In this thesis, we consider Herz-type Sobolev spaces defined over domains. More precisely, the domains under consideration are assumed to satisfy the cone condition.

The thesis is organized as follows. In Chapter 1, we present some preliminary material, establish notation, and recall basic facts concerning Herz spaces, including approximation by smooth functions.

In Chapter 2, we introduce fundamental properties of Herz-type Sobolev spaces in analogy with classical Sobolev spaces and prove a Sobolev embedding theorem for these spaces. Additional Sobolev embedding results are presented in Chapter 3.

HERZ SPACES

In this chapter, we presented some fundamental proprieties to our work. We also give some key technical lemmas needed in the proofs of the main statements of this thesis.

1.1 Definition and basic properties

In this section we recall and present some properties of Lebesgue spaces.

Definition 1.1.1. Let $0 < p \leq \infty$. The Lebesgue space L^p is the set of all measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty,$$

with $0 < p < \infty$ and

$$\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

Remark 1.1.2. Let $1 \leq p \leq \infty$. The L^p spaces are Banach spaces. For $0 < p < 1$, they are only quasi-Banach spaces, that is, they are complete but the triangle inequality is now in the following form

$$\|f + g\|_p \leq 2^{\frac{1-p}{p}} \left(\|f\|_p + \|g\|_p \right), \quad f, g \in L^p.$$

Now, we recall the Hölder inequality in Lebesgue spaces.

Theorem 1.1.3. Let $0 < p, q, r \leq \infty$, $f \in L^p$ and $g \in L^q$. Then $fg \in L^r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition, we have

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Definition 1.1.4. Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_p^{\alpha, q}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{\text{loc}}^p(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_p^{\alpha, q}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f \chi_k\|_p^q \right)^{1/q} < \infty$$

with the usual modifications when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ are quasi-Banach spaces. If $\alpha = 0$ and $0 < p = q \leq \infty$ then $\dot{K}_p^{0,p}(\mathbb{R}^n)$ coincides with the Lebesgue spaces $L^p(\mathbb{R}^n)$. In addition

$$\dot{K}_p^{\alpha,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n, |\cdot|^{\alpha p}), \quad (\text{Lebesgue space equipped with power weight}),$$

where

$$\|f\|_{L^p(\mathbb{R}^n, |\cdot|^{\alpha p})} = \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^{\alpha p} dx \right)^{1/p}.$$

Let $0 < p, q \leq \infty$ and $\alpha > 0$. Then

$$\dot{K}_p^{\alpha,q}(\mathbb{R}^n) \cap L^p = K_p^{\alpha,q}(\mathbb{R}^n),$$

in the sense of equivalent quasi-norms, where $K_p^{\alpha,q}(\mathbb{R}^n)$ is the nonhomogeneous Herz space;

$$\|f\|_{K_p^{\alpha,q}(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{k\alpha q} \|f \chi_{\hat{R}_k}\|_p^q \right)^{1/q} < \infty$$

where $\hat{R}_k = R_k, k \in \mathbb{N}$ and $\hat{R}_0 = B_0$. If $0 < q_1 \leq q_2 \leq \infty$, then we have the embeddings

$$\dot{K}_p^{\alpha,q_1}(\mathbb{R}^n) \hookrightarrow \dot{K}_p^{\alpha,q_2}(\mathbb{R}^n). \quad (1.1)$$

If $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \rightarrow \mathbb{R}$ a measurable function, then we write $f \in \dot{K}_p^{\alpha,q}(\Omega)$ if $f \chi_\Omega \in \dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and we put

$$\|f\|_{\dot{K}_p^{\alpha,q}(\Omega)} = \|f \chi_\Omega\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)}.$$

1.2 Technical results

Various important results have been proved in the space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ under some assumptions on α, p and q . The conditions $-\frac{n}{p} < \alpha < n(1 - \frac{1}{p}), 1 < p < \infty$ and $1 \leq q \leq \infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ spaces. This fact was first realized by Li and Yang [12] with the proof of the boundedness of the maximal function. As usual, we put

$$\mathcal{M}(f)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| dy, \quad f \in L_{\text{loc}}^1(\mathbb{R}^n), x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes with sides parallel to the axis and $x \in Q$. Also we set

$$\mathcal{M}_t(f) = (\mathcal{M}(|f|^t))^{\frac{1}{t}}, \quad 0 < t < \infty.$$

Lemma 1.2.1. *Let $1 < p < \infty$ and $0 < q \leq \infty$. If f is a locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{p} < \alpha < n(1 - \frac{1}{p})$, then*

$$\|\mathcal{M}(f)\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \leq c \|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)}.$$

A detailed discussion of the properties of these spaces may be found in the recent monograph [24], the papers [15], [17], [20], and references therein.

The next lemma is a Hardy-type inequality which is basically a consequence of Young's inequality in the sequence Lebesgue space ℓ^q .

Lemma 1.2.2. *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=k}^{\infty} a^{j-k} \varepsilon_j$, $k \in \mathbb{Z}$. Then there exists a constant $c > 0$ depending only on a and q such that*

$$\|\{\delta_k\}_{k \in \mathbb{Z}}\|_{\ell^q} \leq c \|\{\varepsilon_k\}_{k \in \mathbb{Z}}\|_{\ell^q}.$$

Let $V_{\alpha,p,q}$ be the set of $(\alpha, p, q) \in \mathbb{R} \times [1, \infty]^2$ such that:

- $\alpha < n - \frac{n}{p}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$,
- $\alpha = n - \frac{n}{p}$, $1 \leq p \leq \infty$ and $q = 1$,

The next lemma gives a necessary and sufficient condition on the parameters α , p and q , in order to make sure that

$$\langle T_f, \varphi \rangle = \int_{\Omega} f(x) \varphi(x) dx, \quad \varphi \in \mathcal{D}(\Omega), f \in \dot{K}_p^{\alpha,q}(\Omega)$$

generates a regular distribution $T_f \in \mathcal{D}'(\Omega)$.

Lemma 1.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set with $0 \in \Omega$, and let $1 \leq p, q \leq \infty$. Then*

$$\dot{K}_p^{\alpha,q}(\Omega) \hookrightarrow L_{\text{loc}}^1(\Omega)$$

if and only if $(\alpha, p, q) \in V_{\alpha,p,q}$.

Proof. We split the proof into two steps.

Step 1. Assume that $(\alpha, p, q) \in V_{\alpha,p,q}$, $f \in \dot{K}_p^{\alpha,q}(\Omega)$, and that $B(0, 2^N) \subset \Omega$ for some $N \in \mathbb{Z}$.

By scaling, it suffices to consider this case.

Using Hölder's inequality and a dyadic decomposition, we obtain

$$\begin{aligned}
\|f\|_{L^1(B(0,2^N))} &= \sum_{i=-\infty}^N \|f\chi_{R_i \cap \Omega}\|_1 \\
&\lesssim \sum_{i=-\infty}^N 2^{i(n-\frac{n}{p})} \|f\chi_{R_i \cap \Omega}\|_p \\
&= c 2^{N(n-\frac{n}{p}-\alpha)} \sum_{i=-\infty}^N 2^{(i-N)(n-\frac{n}{p}-\alpha)} 2^{i\alpha} \|f\chi_{R_i \cap \Omega}\|_p \\
&\lesssim \|f\|_{\dot{K}_p^{\alpha,q}(\Omega)}.
\end{aligned}$$

Step 2. Assume that $(\alpha, p, q) \notin V_{\alpha,p,q}$. We distinguish two cases.

Case 1. $\alpha > n - \frac{n}{p}$. Let $r > 0$ be such that $B(0, r) \subset \Omega$ and define

$$f(x) = |x|^{-n} \chi_{\{0 < |x| < r\}}(x).$$

We show that $f \in \dot{K}_p^{\alpha,q}(\Omega)$ for any $1 \leq p, q \leq \infty$, while $f \notin L_{\text{loc}}^1(\Omega)$.

Indeed, we compute

$$\begin{aligned}
\|f\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q &= \sum_{k \in \mathbb{Z}: 2^k < 2r} 2^{k\alpha q} \|f\chi_{R_k \cap \Omega}\|_p^q \\
&\lesssim \sum_{k \in \mathbb{Z}: 2^k < 2r} 2^{k\alpha q} \left\| |x|^{-n} \chi_{R_k \cap \{|x| < r\}} \right\|_p^q \\
&\lesssim \sum_{k \in \mathbb{Z}: 2^k < 2r} 2^{k\alpha q} (2^{-kn} |R_k|^{1/p})^q \\
&\lesssim \sum_{k \in \mathbb{Z}: 2^k < 2r} 2^{k(\alpha - n + \frac{n}{p})q}.
\end{aligned}$$

Since $\alpha - n + \frac{n}{p} > 0$, the geometric series converges, and hence $\|f\|_{\dot{K}_p^{\alpha,q}(\Omega)} < \infty$ (with the usual modification when $p = \infty$ and/or $q = \infty$).

On the other hand, $f \notin L_{\text{loc}}^1(\Omega)$ since

$$\int_{B(0,r)} |f(x)| dx = \int_{B(0,r)} |x|^{-n} dx = \infty.$$

Case 2. Let $\alpha = n - \frac{n}{p}$ with $1 \leq p \leq \infty$ and $1 < q \leq \infty$. By scaling, we may assume that $B(0, \frac{1}{2}) \subset \Omega$. We define the function f by

$$f(x) = |x|^{-n} (|\log |x||)^{-1} \chi_{\{0 < |x| < \frac{1}{2}\}}(x).$$

A direct computation shows that

$$\|f\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q \lesssim \sum_{k=1}^{\infty} k^{-q} < \infty,$$

and hence $f \in \dot{K}_p^{n-\frac{n}{p},q}(\Omega)$, with the usual modification when $q = \infty$.

On the other hand, it is clear that $f \notin L_{\text{loc}}^1(\Omega)$. □

Remark 1.2.4. We easily see that in general if $0 \notin \Omega$ then the set $V_{\alpha,p,q}$ is not optimal. From this lemma it thus makes sense to talk about weak derivatives of functions in $\dot{K}_p^{\alpha,q}(\Omega)$, in addition the assumption $(\alpha, p, q) \in V_{\alpha,p,q}$ is optimal.

Theorem 1.2.5. *Let $\Omega \subset \mathbb{R}^n$ be open and $0 \in \Omega$, $1 < p < \infty$, $1 \leq q < \infty$ and $\alpha > -\frac{n}{p}$. Then $C_c(\Omega)$ is dense in $\dot{K}_p^{\alpha,q}(\Omega)$.*

Proof. First, we observe that $C_c(\Omega) \subset \dot{K}_p^{\alpha,q}(\Omega)$ if and only if $\alpha > -\frac{n}{p}$.

Indeed, let $\varphi \in C_c(\Omega)$ be such that

$$\varphi(x) = 1 \quad \text{for } x \in B(0, 2^N) \subset \Omega, \quad N \in \mathbb{Z}.$$

We estimate the norm as follows:

$$\begin{aligned} \|\varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|\varphi \chi_{\Omega \cap R_k}\|_p^q \\ &\geq \sum_{k=-\infty}^N 2^{k\alpha q} \|\chi_{B(0,2^N) \cap R_k}\|_p^q \\ &= \sum_{k=-\infty}^N 2^{k\alpha q} \|\chi_{R_k}\|_p^q. \end{aligned}$$

Since $|R_k| \asymp 2^{kn}$, we have

$$\|\chi_{R_k}\|_p = |R_k|^{1/p} \asymp 2^{kn/p},$$

and therefore

$$\|\varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q \gtrsim \sum_{k=-\infty}^N 2^{k(\alpha+\frac{n}{p})q}.$$

The series on the right-hand side diverges if $\alpha + \frac{n}{p} \leq 0$, i.e., $\alpha \leq -\frac{n}{p}$. Hence, $\varphi \notin \dot{K}_p^{\alpha,q}(\Omega)$ in this case, which shows the necessity of $\alpha > -\frac{n}{p}$. It is clear that $C_c(\Omega) \subset \dot{K}_p^{\alpha,q}(\Omega)$ whenever $\alpha > -\frac{n}{p}$. Let $\dot{K}_{p,c}^{\alpha,q}(\Omega)$ be the set of all $g \in \dot{K}_p^{\alpha,q}(\Omega)$ such that $g = 0$ outside a compact. As in [26, Proposition 3.1] we obtain that $\dot{K}_{p,c}^{\alpha,q}(\Omega)$ is dense in $\dot{K}_p^{\alpha,q}(\Omega)$. Therefore we prove the density of $C_c(\Omega)$ in $\dot{K}_p^{\alpha,q}(\Omega)$. Let $f \in \dot{K}_{p,c}^{\alpha,q}(\Omega)$ with $f(x) = 0$ if $x \notin A \subset \Omega$ compact. As in [2, Theorem 2.19], the proof can be restricted to the case f is real-valued and nonnegative. Since f

is measurable, there exists a monotonically increasing sequence $\{u_i\}_{i \in \mathbb{N}_0}$ of nonnegative simple functions converging pointwise to f on Ω and

$$0 \leq u_i \leq f, \quad i \in \mathbb{N}_0.$$

Since

$$0 \leq f - u_i \leq f, \quad i \in \mathbb{N}_0,$$

by dominated convergence theorem $\{u_i\}_{i \in \mathbb{N}_0}$ converge to f in $\dot{K}_p^{\alpha,q}(f)$. Therefore we find an $u \in \{u_i\}_{i \in \mathbb{N}_0}$ such that

$$\|f - u\|_{\dot{K}_p^{\alpha,q}(\Omega)} < \frac{\varepsilon}{2}.$$

Since $0 \leq u \leq f$, $\text{supp } u \subset A$. Let $\theta > 0$ be such that

$$\max\left(0, \frac{-\alpha p}{n}\right) < \theta < 1.$$

Assume that $A \subset V \subset \bar{V} \subset \Omega$, where $\bar{V}$ is compact. We define

$$E = \sum_{k \in \mathbb{Z}: R_k \cap \bar{V} \neq \emptyset} 2^{k\alpha q} |R_k|^{\frac{\theta q}{p}}.$$

By Lusin's theorem, there exists a function $\varphi \in C_c(\Omega)$ such that

$$|\varphi(x)| \leq \|u\|_\infty \quad \text{for all } x \in \Omega, \quad \text{supp}(\varphi) \subset \bar{V},$$

and such that

$$|H| \leq \left(\frac{\varepsilon}{4\|u\|_\infty E^{1/q}} \right)^{\frac{p}{1-\theta}},$$

where $H = \{x \in \Omega : \varphi(x) \neq u(x)\}$. Set $B = H \cap \bar{V} = \{x \in \bar{V} : \varphi(x) \neq u(x)\}$.

We now estimate the quasi-norm:

$$\begin{aligned} \|u - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q &= \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|(u - \varphi)\chi_{R_k \cap \Omega}\|_p^q \\ &= \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|(u - \varphi)\chi_{R_k \cap B}\|_p^q \\ &= \sum_{k \in \mathbb{Z}: R_k \cap B \neq \emptyset} 2^{k\alpha q} \|(u - \varphi)\chi_{R_k \cap B}\|_p^q. \end{aligned}$$

Let $k \in \mathbb{Z}$ be such that $R_k \cap B \neq \emptyset$. Since $|u| \leq \|u\|_\infty$ and $|\varphi| \leq \|u\|_\infty$, we obtain

$$\begin{aligned} \|(u - \varphi)\chi_{R_k \cap B}\|_p &\leq 2\|u\|_\infty \|\chi_{R_k \cap B}\|_p \\ &= 2\|u\|_\infty |R_k \cap B|^{1/p} \\ &\leq 2\|u\|_\infty |R_k|^{\theta/p} |B|^{(1-\theta)/p}. \end{aligned}$$

Consequently,

$$\begin{aligned}
\|u - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q &\leq 2^q \|u\|_\infty^q |B|^{\frac{(1-\theta)q}{p}} \sum_{k: R_k \cap \tilde{V} \neq \emptyset} 2^{k\alpha q} |R_k|^{\frac{\theta q}{p}} \\
&= 2^q \|u\|_\infty^q E |B|^{\frac{(1-\theta)q}{p}} \\
&< \left(\frac{\varepsilon}{2}\right)^q.
\end{aligned}$$

This completes the proof. $\square$

Let $J \in \mathcal{D}(\mathbb{R}^n)$ be a real-valued function such that

$$J(x) \geq 0, \quad \text{if } x \in \mathbb{R}^n, \quad J(x) = 0 \quad \text{if } |x| \geq 1 \quad \text{and} \quad \int_{\mathbb{R}^n} J(x) dx = 1.$$

We put $J_\varepsilon(x) = \varepsilon^{-n} J(\frac{x}{\varepsilon})$, $x \in \mathbb{R}^n$.

Theorem 1.2.6. *Let $\Omega \subset \mathbb{R}^n$ be open, $0 \in \Omega$, $1 \leq p < \infty$, $1 \leq q < \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. Let $f \in \dot{K}_p^{\alpha,q}(\Omega)$ be a function defined on $\mathbb{R}^n$ and vanishes identically outside Ω . Then*

$$\lim_{\varepsilon \rightarrow 0_+} \|J_\varepsilon * f - f\|_{\dot{K}_p^{\alpha,q}(\Omega)} = 0. \quad (1.2)$$

Proof. We divide the proof into two steps.

Step 1. We prove that

$$\lim_{\varepsilon \rightarrow 0_+} \|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)} = 0, \quad (1.3)$$

for every $\varphi \in C_c(\Omega)$ and $\alpha \in \mathbb{R}$.

Substep 1.1. $\alpha > -\frac{n}{p}$.

Assume that

$$\text{supp } \varphi \subset B(0, 2^N) \subset \Omega, \quad N \in \mathbb{N}.$$

Since $|x - y| > 2^N \geq \varepsilon$ for every $x \in \mathbb{R}^n \setminus B(0, 2^{N+1})$ and $y \in B(0, 2^N)$, it follows that

$$J_\varepsilon * \varphi(x) = 0, \quad x \in \mathbb{R}^n \setminus B(0, 2^{N+1}), \quad \varepsilon \leq 2^N.$$

Consequently,

$$\|(J_\varepsilon * \varphi - \varphi)\chi_k\|_{L^p(\Omega)}^p = \int_{\Omega \cap R_k \cap B(0, 2^{N+1})} |J_\varepsilon * \varphi(x) - \varphi(x)|^p dx, \quad k \in \mathbb{Z}.$$

Furthermore,

$$J_\varepsilon * \varphi(x) - \varphi(x) = \int_{B(0,1)} J(z) (\varphi(x - \varepsilon z) - \varphi(x)) dz.$$

Hence,

$$|J_\varepsilon * \varphi(x) - \varphi(x)| \leq \sup_{z \in B(0,1)} |\varphi(x - \varepsilon z) - \varphi(x)|.$$

Since φ is uniformly continuous on its compact support,

$$\sup_{|x| \leq 2^{N+2}} \sup_{z \in B(0,1)} |\varphi(x - \varepsilon z) - \varphi(x)| \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+.$$

Therefore,

$$\begin{aligned} \|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)}^q &= \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|(J_\varepsilon * \varphi - \varphi)\chi_k\|_{L^p(\Omega)}^q \\ &\leq \sum_{k=-\infty}^{N+2} 2^{k\alpha q} \|(J_\varepsilon * \varphi - \varphi)\chi_k\|_{L^p(\Omega)}^q \\ &\leq \left(\sup_{|x| \leq 2^{N+2}} \sup_{z \in B(0,1)} |\varphi(x - \varepsilon z) - \varphi(x)| \right)^q \sum_{k=-\infty}^{N+2} 2^{k(\alpha + \frac{n}{p})q} \\ &\lesssim \left(\sup_{|x| \leq 2^{N+2}} \sup_{z \in B(0,1)} |\varphi(x - \varepsilon z) - \varphi(x)| \right)^q, \end{aligned}$$

where the last inequality follows from the assumption $\alpha > -\frac{n}{p}$.

Passing to the limit as $\varepsilon \rightarrow 0_+$, we obtain (1.3) for every $\varphi \in C_c(\Omega)$ and $\alpha > -\frac{n}{p}$.

Substep 1.2. Let $\alpha < -\frac{n}{p}$. By duality,

$$\|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)} = \sup \left| \int_{\Omega} (J_\varepsilon * \varphi(x) - \varphi(x))g(x) dx \right|,$$

where the supremum is taken over all functions $g \in C_c(\Omega)$ satisfying

$$\|g\|_{\dot{K}_{p'}^{-\alpha,q'}(\Omega)} = 1.$$

Since

$$\int_{\Omega} (J_\varepsilon * \varphi(x) - \varphi(x))g(x) dx = \int_{\Omega} (\tilde{J}_\varepsilon * g(x) - g(x))\varphi(x) dx,$$

where

$$\tilde{J}_\varepsilon(x) = J_\varepsilon(-x), \quad x \in \mathbb{R}^n,$$

it follows from Hölder's inequality that

$$\left| \int_{\Omega} (\tilde{J}_\varepsilon * g(x) - g(x))\varphi(x) dx \right| \leq \|\tilde{J}_\varepsilon * g - g\|_{\dot{K}_{p'}^{-\alpha,q'}(\Omega)} \|\varphi\|_{\dot{K}_p^{\alpha,q}(\Omega)}.$$

Observe that

$$-\alpha > -\frac{n}{p'}$$

since $\alpha < -\frac{n}{p}$. Therefore, by Substep 1.1, for every $\eta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\|\tilde{J}_\varepsilon * g - g\|_{\dot{K}_{p'}^{-\alpha, q'}(\Omega)} < \frac{\eta}{\|\varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)}}$$

whenever $0 < \varepsilon < \varepsilon_0$. Consequently,

$$\|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} \leq \eta$$

for all sufficiently small ε . Since $\eta > 0$ is arbitrary, we conclude that

$$\lim_{\varepsilon \rightarrow 0_+} \|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} = 0.$$

Substep 1.3. $\alpha = -\frac{n}{p}$. Let $\alpha_0 > -\frac{n}{p}$ and $\alpha_1 < -\frac{n}{p}$ be such that $\alpha = \theta\alpha_0 + (1 - \theta)\alpha_1$, $0 < \theta < 1$.

Hölder's inequality yields

$$\begin{aligned} \|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} &\leq \|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha_0, q}(\Omega)}^\theta \|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha_1, q}(\Omega)}^{1-\theta} \\ &\leq \eta \end{aligned}$$

for any $\eta > 0$ and any ε small enough.

Step 2. We prove (1.2). By Theorem 1.2.5, there exists $\varphi \in C_c(\Omega)$ such that

$$\|f - \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} \leq \frac{\eta}{3}$$

for any given $\eta > 0$. Hence, for any sufficiently small $\eta_1 > 0$,

$$\|J_\varepsilon * f - J_\varepsilon * \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} \leq c\|\mathcal{M}(f - \varphi)\|_{\dot{K}_p^{\alpha, q}(\mathbb{R}^n)} \leq c\eta_1,$$

by Lemma 1.2.1, since

$$-\frac{n}{p} < \alpha < n - \frac{n}{p}.$$

Choosing η_1 so that $c\eta_1 < \eta/3$, we obtain

$$\|J_\varepsilon * f - J_\varepsilon * \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} < \frac{\eta}{3}.$$

On the other hand, by Step 1,

$$\|J_\varepsilon * \varphi - \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} \leq \frac{\eta}{3}$$

provided that $\varepsilon > 0$ is sufficiently small. Combining these estimates with

$$\|f - \varphi\|_{\dot{K}_p^{\alpha, q}(\Omega)} \leq \frac{\eta}{3},$$

and using the triangle inequality, we obtain

$$\|J_\varepsilon * f - f\|_{\dot{K}_p^{\alpha,q}(\Omega)} \leq \eta.$$

This proves (1.2) for $p > 1$.

It remains to consider the case $p = 1$. Let $s > 1$. By Hölder's inequality and the assumption $-n < \alpha < 0$, we have

$$\|J_\varepsilon * f - f\|_{\dot{K}_1^{\alpha,q}(\Omega)} \leq \|J_\varepsilon * f - f\|_{\dot{K}_s^{\alpha+n-\frac{n}{s},q}(\Omega)}.$$

Since

$$-\frac{n}{s} < \alpha + n - \frac{n}{s} < n - \frac{n}{s},$$

the result already proved for $p = s$ implies that the right-hand side tends to zero as $\varepsilon \rightarrow 0_+$.

Consequently,

$$\lim_{\varepsilon \rightarrow 0_+} \|J_\varepsilon * f - f\|_{\dot{K}_1^{\alpha,q}(\Omega)} = 0.$$

This completes the proof. □

HERZ-TYPE SOBOLEV SPACES

In this chapter, we present the definition of modulation spaces and prove their basic properties. Also, we prove some elementary embeddings.

2.1 Definition and basic properties

In this chapter, we prove the basic properties of Herz-type Sobolev spaces in analogy to the classical Sobolev spaces.

Definition 2.1.1. Let $\Omega \subset \mathbb{R}^n$ be open, $(\alpha, p, q) \in V_{\alpha,p,q}$ and $m \in \mathbb{N}_0$. We define the Herz-type Sobolev space $\dot{K}_{p,m}^{\alpha,q}(\Omega)$ as the set of functions $f \in \dot{K}_p^{\alpha,q}(\Omega)$ with weak derivatives $D^\beta f \in \dot{K}_p^{\alpha,q}(\Omega)$ for $|\beta| \leq m$. We define the norm of $\dot{K}_{p,m}^{\alpha,q}(\Omega)$ by

$$\|f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \left(\sum_{|\beta| \leq m} \|(D^\beta f)\chi_{R_k \cap \Omega}\|_p^p \right)^{\frac{q}{p}} \right)^{1/q}$$

if $1 \leq p, q < \infty$ and

$$\|f\|_{\dot{K}_{p,m}^{\alpha,\infty}(\Omega)} = \sup_{k \in \mathbb{Z}} 2^{k\alpha} \left(\sum_{|\beta| \leq m} \|(D^\beta f)\chi_{R_k \cap \Omega}\|_p^p \right)^{\frac{1}{p}}.$$

One recognizes immediately that if $p = q$ and $\alpha = 0$, then we have $\dot{K}_{p,m}^{0,p}(\Omega) = W_p^m(\Omega)$.

As in classical Sobolev spaces, see [2, Theorem 3.3], we have the following statements:

Theorem 2.1.2. Let $\Omega \subset \mathbb{R}^n$ be open and $(\alpha, p, q) \in V_{\alpha,p,q}$. For each $m \in \mathbb{N}_0$, the Herz-type Sobolev space $\dot{K}_{p,m}^{\alpha,q}(\Omega)$ is a Banach space.

Exactly in the same way as in the classical Sobolev spaces, see [2], but we use Theorem 1.2.6 we immediately arrive at the following result.

Lemma 2.1.3. Let $\Omega \subset \mathbb{R}^n$ be open, $m \in \mathbb{N}_0$ and $(\alpha, p, q) \in V_{\alpha,p,q}$ with $1 < p < \infty, 1 \leq q < \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. Let Ω' be an open subset of Ω such that $\overline{\Omega'}$ is a compact subset of Ω . Let J_ε be as above and $f \in \dot{K}_{p,m}^{\alpha,q}(\Omega)$. Then

$$\lim_{\varepsilon \rightarrow 0_+} \|J_\varepsilon * f - f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega')} = 0. \quad (2.1)$$

Proof. Let $\varepsilon < \text{dist}(\Omega', \partial\Omega)$ and

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

which is the zero extension of f outside Ω . Let $\varphi \in \mathcal{D}(\Omega')$. Then

$$\int_{\Omega'} J_\varepsilon * f(x) \partial^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\Omega'} J_\varepsilon * \partial^\alpha f(x) \varphi(x) dx.$$

Hence $\partial^\alpha(J_\varepsilon * f) = J_\varepsilon * \partial^\alpha f$ in the distributional sense in Ω' . Observe that Since $\partial^\alpha f \in \dot{K}_{p,m}^{\alpha,q}(\Omega')$, $0 \leq |\alpha| \leq m$ we have by Theorem 1.2.6

$$\lim_{\varepsilon \rightarrow 0} \|\partial^\alpha(J_\varepsilon * f) - \partial^\alpha f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega')} = \lim_{\varepsilon \rightarrow 0} \|J_\varepsilon * \partial^\alpha f - \partial^\alpha f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega')} = 0.$$

Hence we obtain (2.1). □

Similarly as in [2] Theorems 3.6 and 3.17], we have the following statements:

Lemma 2.1.4. *Let $\Omega \subset \mathbb{R}^n$ be open $\alpha \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $m \in \mathbb{N}_0$. Then for any $\varphi \in \mathcal{D}(\Omega)$ there exists $c > 0$ such that for all $f \in \dot{K}_{p,m}^{\alpha,q}(\Omega)$*

$$\|\varphi f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega)} \leq c \|f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega)}.$$

As in classical Sobolev spaces, see [2] and [4], we have the following statements:

Theorem 2.1.5. *Let $\Omega \subset \mathbb{R}^n$ be open, $m \in \mathbb{N}_0$ and $(\alpha, p, q) \in V_{\alpha,p,q}$ with $1 < p < \infty$, $1 \leq q < \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. Then $C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha,q}(\Omega)$ is dense in $\dot{K}_{p,m}^{\alpha,q}(\Omega)$.*

Proof. We set

$$\Omega_k = \{x \in \Omega : x \in B(0, k) \text{ and } \text{dis}(x, \partial\Omega) > k\}. \quad k \in \mathbb{N}$$

and $\Omega_{-1} = \Omega_0 = \emptyset$. Observing that (the complement is respect to the set Ω)

$$\Omega = \bigcup_{k=1}^{\infty} U_k, \quad U_k = \Omega_{k+1} \cap (\overline{\Omega_{k-1}})^c, \quad k \in \mathbb{N}.$$

Let Ψ be a C^∞ partition of unity for Ω subordinate to $\{U_k\}_{k \in \mathbb{N}}$. Let $\psi_k, k \in \mathbb{N}$ denote the sum of the finitely many functions $\psi \in \Psi$ such that $\text{supp} \psi \subset U_k$. Then

$$\psi_k \in \mathcal{D}(U_k), \quad \sum_{k=1}^{\infty} \psi_k(x) = 1, \quad x \in \Omega.$$

Let $f \in \dot{K}_{p,m}^{\alpha,q}(\Omega)$. We see that $\psi_k f \in \Omega$ and

$$\text{supp } J_\varepsilon * (\psi_k f) \subset V_k = \Omega_{k+2} \cap (\overline{\Omega_{k-2}})^c \subset \Omega$$

for any $0 < \varepsilon < \frac{1}{(k+1)(k+2)}$. We may choose ε_k , satisfying $0 < \varepsilon_k < \frac{1}{(k+1)(k+2)}$, such that

$$\|J_{\varepsilon_k} * (\psi_k f) - \psi_k f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega)} = \|J_{\varepsilon_k} * (\psi_k f) - \psi_k f\|_{\dot{K}_{p,m}^{\alpha,q}(V_k)} \leq \frac{\varepsilon}{2^k}.$$

We set

$$\phi = \sum_{k=1}^{\infty} J_{\varepsilon_k} * (\psi_k f).$$

On any $\Omega' \subset \Omega$ such that $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ compact only finitely many terms in the sum can be nonzero. Therefore $\phi \in C^\infty(\Omega)$. Let $x \in \Omega_k$. We have

$$f = \sum_{j=1}^{k+1} \psi_j f$$

and

$$\phi = \sum_{j=1}^{k+1} J_{\varepsilon_j} * (\psi_j f).$$

Consequently

$$\|f - \phi\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega_k)} \leq \sum_{j=1}^{k+1} \|J_{\varepsilon_j} * (\psi_j f) - \psi_j f\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega)} \leq \varepsilon.$$

By the monotone convergence theorem,

$$\|f - \phi\|_{\dot{K}_{p,m}^{\alpha,q}(\Omega)} \leq \varepsilon.$$

The proof is complete. □

2.2 Sobolev embeddings

In this subsection we present some embeddings of the spaces introduced above.

Definition 2.2.1. Let $v \in \mathbb{R}^n \setminus \{0\}$ and for each $x \neq 0$ let $\angle(x, v)$ be the angle between the position vector x and v . Let κ satisfying $0 < \kappa < \pi$. The set

$$C = \{x \in \mathbb{R}^n : x = 0 \text{ or } 0 < |x| \leq \varrho, \angle(x, v) \leq \kappa/2\}$$

is called a finite cone of height ϱ , axis direction v and aperture angle κ with vertex at the origin.

Definition 2.2.2. Let C be a finite cone with vertex at the origin. Note that $x+C = \{x+y : y \in C\}$ is a finite cone with vertex at x but the same dimensions and axis direction as C and is obtained by parallel translation of C .

We are now in a position to state the definition of domain satisfying the cone condition

Definition 2.2.3. Let $\Omega \subset \mathbb{R}^n$ be open. Ω satisfies the cone condition if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C .

Definition 2.2.4. In Definition 2.2.3 the cone C_x is not obtained from C by parallel translation, but simply by rigid motion.

The following statement can be found in [2] Lemma 4.15], that plays an essential for us.

Lemma 2.2.5. Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone condition. Then we can find a positive constant K depending on m, n , and the dimensions ϱ and κ of the cone C specified the cone condition for Ω such that for every $f \in C^\infty(\Omega)$, every $x \in \Omega$, and every r satisfying $0 < r \leq \varrho$, we have

$$|f(x)| \leq K \left(\sum_{|\beta| \leq m-1} r^{|\beta|-n} \int_{C_{x,r}} |D^\beta f(y)| dy + \sum_{|\beta|=m-1} \int_{C_{x,r}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \right),$$

where $C_{x,r} = \{y \in C_x : y \in B(x, r)\}$.

Let $0 < \lambda < n$. The Riesz potential operator $\mathcal{I}_\lambda$ is defined by

$$\mathcal{I}_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} dy.$$

Let p^* be the Sobolev exponent defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{\lambda}{n}$. The following statement plays a crucial role in our embeddings results, see [12].

Theorem 2.2.6. Let $0 < \lambda < n$, $0 < q_0 \leq q_1 \leq \infty$ and $1 < p < \frac{n}{\lambda}$. If

$$\lambda - \frac{n}{p} < \alpha < n - \frac{n}{p},$$

then $\mathcal{I}_\lambda$ is bounded from $\dot{K}_p^{\alpha, q_0}(\mathbb{R}^n)$ into $\dot{K}_{p^*}^{\alpha, q_1}(\mathbb{R}^n)$. Also $\mathcal{I}_\lambda$ is bounded from $K_p^{\alpha, q_0}(\mathbb{R}^n)$ into $K_{p^*}^{\alpha, q_1}(\mathbb{R}^n)$.

Now we state the first embeddings

Theorem 2.2.7. Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone condition, $0 \in \Omega$ and $m \in \mathbb{N}_0$. Let $1 < p < \infty$, $1 \leq r < \infty$, $\alpha_2 \geq \alpha_1$, $m - \frac{n}{p} < \alpha_2 < n - \frac{n}{p}$,

$$m - \alpha_2 + \alpha_1 > 0 \quad \text{and} \quad \frac{n}{q} = \frac{n}{p} - m + \alpha_2 - \alpha_1 > 0. \quad (2.2)$$

Then

$$\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_q^{\alpha_1,r}(\Omega)$$

holds.

Proof. We apply Theorem 2.1.5 and divide the proof into two steps. Let

$$f \in C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha_2,r}(\Omega).$$

Step 1. The case $\alpha_1 = \alpha_2$. By Lemma 2.2.5,

$$|f(x)| \lesssim \sum_{|\beta| \leq m} \mathcal{I}_m((D^\beta f)\chi_\Omega)(x), \quad x \in \Omega.$$

Therefore, applying Theorem 2.2.6, we obtain

$$\|f\|_{\dot{K}_q^{\alpha_1,r}(\Omega)} \lesssim \sum_{|\beta| \leq m} \|(D^\beta f)\chi_\Omega\|_{\dot{K}_p^{\alpha_1,r}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_1,r}(\Omega)}.$$

Step 2. $\alpha_2 > \alpha_1$. We write

$$\begin{aligned} \|f\|_{\dot{K}_q^{\alpha_1,r}(\Omega)}^r &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r \\ &= \sum_{k=-\infty}^{-1} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r + \sum_{k=0}^{\infty} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r \\ &= I_1 + I_2. \end{aligned}$$

Estimate of I_1 . Let ϱ be as in Lemma 2.2.5. We decompose

$$I_1 = I_3 + I_4,$$

where

$$I_3 = \sum_{\substack{k \leq -1 \\ \varrho \leq 2^{k-2}}} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r, \quad I_4 = \sum_{\substack{k \leq -1 \\ \varrho > 2^{k-2}}} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_q^r.$$

We first estimate I_3 . Let $x \in R_k \cap \Omega$. Since $\varrho \leq 2^{k-2}$ and $x \in R_k$, we have

$$C_{x,\varrho} \subset \tilde{R}_k := \{z \in \mathbb{R}^n : 2^{k-2} \leq |z| \leq 2^{k+1}\}.$$

Hence, by Lemma 2.2.5,

$$|f(x)| \lesssim \sum_{|\beta| \leq m-1} \varrho^{|\beta|-n} \int_{C_{x,\varrho}} |D^\beta f(y)| \chi_{\tilde{R}_k}(y) dy + \sum_{|\beta|=m} \int_{C_{x,\varrho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{\tilde{R}_k}(y) dy.$$

Since $\alpha_2 > \alpha_1$, $m < n$, and $|x - y| \leq \varrho$ on $C_{x,\varrho}$, we obtain

$$|f(x)| \lesssim 2^{k(\alpha_2 - \alpha_1)} \sum_{|\beta| \leq m} \mathcal{I}_{m - \alpha_2 + \alpha_1}((D^\beta f)\chi_{\Omega \cap \tilde{R}_k})(x),$$

where the implicit constant is independent of k .

Consequently,

$$\begin{aligned} I_3 &\lesssim \sum_{|\beta| \leq m} \sum_{k \leq -1} 2^{k\alpha_2 r} \left\| \mathcal{I}_{m - \alpha_2 + \alpha_1}((D^\beta f)\chi_{\Omega \cap \tilde{R}_k})\chi_{R_k} \right\|_q^r \\ &\leq \sum_{|\beta| \leq m} \left\| \mathcal{I}_{m - \alpha_2 + \alpha_1}((D^\beta f)\chi_\Omega) \right\|_{\dot{K}_q^{\alpha_2, r}(\mathbb{R}^n)}^r. \end{aligned}$$

Since

$$\frac{1}{q} = \frac{1}{p} - \frac{m - \alpha_2 + \alpha_1}{n},$$

Theorem 2.2.6 yields

$$I_3 \lesssim \sum_{|\beta| \leq m} \|D^\beta f\|_{\dot{K}_p^{\alpha_2, r}(\Omega)}^r \lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2, r}(\Omega)}^r.$$

It remains to estimate I_4 . We set

$$J_{1,k}(x) = \sum_{|\beta| \leq m-1} \varrho^{|\beta| - n} \int_{C_{x,\varrho}} |D^\beta f(y)| dy$$

and

$$J_{2,k}(x) = \sum_{|\beta|=m} \int_{C_{x,\varrho}} \frac{|D^\beta f(y)|}{|x - y|^{n-m}} dy.$$

To estimate the first term, we use the assumptions $m < n$ and $\varrho > 2^{k-2}$. Hence,

$$\begin{aligned} J_{1,k}(x) &\lesssim 2^{(m-n)k} \sum_{|\beta| \leq m-1} \int_{C_{x,2^{k-2}}} |D^\beta f(y)| dy \\ &\quad + \sum_{|\beta| \leq m-1} \varrho^{|\beta| - n} \int_{2^{k-2} \leq |x-y| \leq \varrho} |D^\beta f(y)| \chi_\Omega(y) dy \\ &=: c(J_{1,k}^1(x) + J_{1,k}^2(x)). \end{aligned}$$

We now estimate the terms $J_{1,k}^1$ and $J_{1,k}^2$ separately. By assumption (2.2) and Hölder's inequality, we have

$$\begin{aligned} J_{1,k}^1(x) &\lesssim \sum_{|\beta| \leq m-1} 2^{(m-n)k} \int_{C_{x,2^{k-2}}} |D^\beta f(y)| \chi_{\tilde{R}_k}(y) \chi_\Omega(y) dy \\ &\lesssim 2^{k(\alpha_2 - \alpha_1 - \frac{n}{q})} \sum_{|\beta| \leq m-1} \left\| (D^\beta f)\chi_{\tilde{R}_k \cap \Omega} \right\|_p. \end{aligned}$$

Hence,

$$\|(J_{1,k}^1)\chi_{R_k \cap \Omega}\|_q \lesssim 2^{k(\alpha_2 - \alpha_1)} \sum_{|\beta| \leq m-1} \|(D^\beta f)\chi_{\tilde{R}_k \cap \Omega}\|_p,$$

for every $k \leq -1$ satisfying $\varrho > 2^{k-2}$.

Next, we decompose $J_{1,k}^2$ as

$$J_{1,k}^2 = J_{1,k,1}^2 + J_{1,k,2}^2,$$

where

$$J_{1,k,1}^2(x) = \sum_{|\beta| \leq m-1} \varrho^{|\beta|-n} \int_{2^{k-2} \leq |x-y| \leq 2^{k+2}} |D^\beta f(y)| \chi_\Omega(y) dy,$$

and

$$J_{1,k,2}^2(x) = \sum_{|\beta| \leq m-1} \varrho^{|\beta|-n} \int_{2^{k+2} \leq |x-y| \leq \varrho} |D^\beta f(y)| \chi_\Omega(y) dy.$$

$J_{1,k,1}^2(x)$ can be estimated from above by

$$c 2^{(\alpha_2 - \alpha_1)k} \sum_{|\beta| \leq m-1} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_\Omega)(x)$$

for any $k \leq -1$ such that $\varrho > 2^{k-2}$. Now we consider the second term. We have

$$J_{1,k,2}^2(x) \lesssim \sum_{|\beta| \leq m-1} \int_{2^{k+2} \leq |x-y| \leq \varrho} \frac{|D^\beta f(y)| \chi_\Omega(y)}{|x-y|^{n-m}} dy,$$

which can be estimated by

$$\begin{aligned} & c \sum_{|\beta| \leq m-1} \sum_{i=k+2}^j \int_{2^i \leq |x-y| \leq 2^{i+1}} \frac{|D^\beta f(y)| \chi_\Omega(y)}{|x-y|^{n-m}} \chi_{\tilde{R}_i}(y) dy \\ & \lesssim \sum_{|\beta| \leq m-1} \sum_{i=k+2}^j 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{i\alpha_2} \|(D^\beta f)\chi_{\tilde{R}_i \cap \Omega}\|_p, \end{aligned}$$

where $2^{j-1} \leq \varrho < 2^j$, $j \in \mathbb{Z}$ and we used Hölder's inequality. By assumption (2.2) we obtain

$$2^{k\alpha_1} \|(J_{1,k,2}^2)\chi_{R_k \cap \Omega}\|_q \lesssim 2^{(\frac{n}{p}-m+\alpha_2)k} \sum_{|\beta| \leq m-1} \sum_{i=k+2}^j 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{i\alpha_2} \|(D^\beta f)\chi_{\tilde{R}_i \cap \Omega}\|_p$$

for any $k \leq -1$ such that $\varrho > 2^{k-2}$.

We estimate $J_{2,k}$. We write

$$J_{2,k}(x) = \sum_{|\beta|=m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy + \sum_{|\beta|=m} \int_{B_{x,\varrho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy, \quad (2.3)$$

where

$$B_{x,\varrho,k} = C_x \cap \{y \in \mathbb{R}^n : 2^{k-2} \leq |x-y| < \varrho\}.$$

Estimate of the first term. Using the relation between the parameters α_1 and α_2 , we obtain

$$\begin{aligned} \sum_{|\beta|=m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy &\leq c 2^{k(\alpha_2-\alpha_1)} \sum_{|\beta|\leq m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)| \chi_\Omega(y)}{|x-y|^{n-m-\alpha_1+\alpha_2}} dy \\ &\lesssim 2^{k(\alpha_2-\alpha_1)} \sum_{|\beta|\leq m} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_\Omega)(x), \end{aligned}$$

where $\mathcal{I}_\gamma$ denotes the Riesz potential of order γ .

Decomposition of the second term. We split the remaining integral as

$$J_{2,k}(x) = J_{2,k,1}(x) + J_{2,k,2}(x),$$

where

$$J_{2,k,1}(x) = \sum_{|\beta|=m} \int_{2^{k-2} \leq |x-y| \leq 2^{k+2}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy,$$

and

$$J_{2,k,2}(x) = \sum_{|\beta|=m} \int_{2^{k+2} \leq |x-y| < \varrho} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy.$$

Estimate of $J_{2,k,1}$. Arguing as in the estimate of $J_{1,k,2}^2$, we obtain

$$J_{2,k,1}(x) \lesssim 2^{k(\alpha_2-\alpha_1)} \sum_{|\beta|=m} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_\Omega)(x).$$

Estimate of $J_{2,k,2}$. We proceed as in the corresponding estimate for $J_{1,k,2}^2$. For $x \in R_k \cap \Omega$, we obtain

$$2^{k\alpha_1} \|(J_{2,k,2})\chi_{R_k \cap \Omega}\|_q \lesssim 2^{(\frac{n}{p}-m+\alpha_2)k} \sum_{|\beta|\leq m-1} \sum_{i=k+2}^j 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{i\alpha_2} \|(D^\beta f)\chi_{\tilde{R}_i \cap \Omega}\|_p.$$

Since $\alpha_2 > m - \frac{n}{p}$, the geometric sum in i is convergent. Applying Lemma [1.2.2](#), we deduce that

$$I_4 \lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r.$$

This completes the estimate of $J_{2,k}$.

Estimate of I_2 . Since $\alpha_2 > \alpha_1$, we obtain that

$$I_2 \leq \sup_{k \in \mathbb{N}_0} 2^{kr\alpha_2} \|f\chi_\Omega\|_q^r \lesssim \|f\|_{\dot{K}_q^{\alpha_2,\infty}(\Omega)}^r.$$

Again, by Lemma 2.2.5, we have

$$|f(x)| \lesssim \sum_{|\beta| \leq m} \mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_\Omega)(x), \quad x \in \Omega.$$

Applying Theorem 2.2.6 yields

$$\begin{aligned} I_2 &\lesssim \sum_{|\beta| \leq m} \|\mathcal{I}_{m-\alpha_2+\alpha_1}((D^\beta f)\chi_\Omega)\|_{\dot{K}_q^{\alpha_2, r}(\mathbb{R}^n)}^r \\ &\lesssim \sum_{|\beta| \leq m} \|(D^\beta f)\chi_\Omega\|_{\dot{K}_p^{\alpha_2, r}(\mathbb{R}^n)}^r \\ &\lesssim \|f\|_{\dot{K}_{p, m}^{\alpha_2, r}(\Omega)}^r, \end{aligned}$$

since $m - \frac{n}{p} < \alpha_2 < n - \frac{n}{p}$. The proof is complete. $\square$

Remark 2.2.8. We mention that Theorem 2.2.7 covers the Sobolev inequality. In addition

$$W_p^m(\Omega, |\cdot|^{\alpha_2 p}) \hookrightarrow \dot{K}_q^{\alpha_1, p}(\Omega) \hookrightarrow L^q(\Omega, |\cdot|^{\alpha_1 q}),$$

under the same assumptions of Theorem 2.2.7 with $r = p$. In particular

$$W_p^m(\Omega) \hookrightarrow \dot{K}_q^{0, p}(\Omega) \hookrightarrow L^q(\Omega),$$

holds if $1 < p < \infty, 0 < m < \frac{n}{p}$ and

$$\frac{n}{q} = \frac{n}{p} - m.$$

FURTHER EMBEDDINGS

In this chapter, we develop new embedding results in the framework of Herz-type Sobolev spaces, extending the known theory.

3.1 Embeddings of type $\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_p^{\alpha_1,r}(\Omega)$ and $\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_\infty^{\alpha_1,r}(\Omega)$

The following theorem describes an embedding from $\dot{K}_{p,m}^{\alpha_2,r}(\Omega)$ into $\dot{K}_p^{\alpha_1,r}(\Omega)$.

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone condition, with $0 \in \Omega$, and let $m \in \mathbb{N}$. Let $1 < p < \infty$, $1 \leq r < \infty$, and $\alpha_2 \geq \alpha_1$ with $\alpha_1 + \frac{n}{p} > 0$. Assume that*

$$\max\left(\frac{n}{p} + \alpha_2, \frac{n}{p} + \alpha_2 - \alpha_1\right) < m < n.$$

Then the embedding

$$\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_p^{\alpha_1,r}(\Omega)$$

holds.

Proof. We make use of Theorem [2.1.5](#). Let $f \in C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha_2,r}(\Omega)$ and ϱ be as in Lemma [2.2.5](#).

We write

$$\begin{aligned} \|f\|_{\dot{K}_p^{\alpha_1,r}(\Omega)}^r &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_p^r \\ &= \sum_{2^{k+2} > \varrho} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_p^r + \sum_{2^{k+2} \leq \varrho} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_p^r \\ &=: I_1 + I_2. \end{aligned}$$

We first estimate I_1 . Let $t > 0$ be such that

$$\frac{n}{m} < t < \min\left(p, \frac{n}{\max(0, \alpha_2 + \frac{n}{p})}\right).$$

By Hölder's inequality, we obtain

$$\begin{aligned} |f(x)| &\lesssim \sum_{|\beta| \leq m-1} \varrho^{|\beta|-n} \int_{C_{x,\varrho}} |D^\beta f(y)| dy + \sum_{|\beta|=m} \int_{C_{x,\varrho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \\ &\lesssim \sum_{|\beta| \leq m} \mathcal{M}_t((D^\beta f)\chi_\Omega)(x), \end{aligned}$$

for any $x \in R_k \cap \Omega$. Consequently,

$$\begin{aligned} I_1 &\lesssim \sum_{|\beta| \leq m} \sum_{2^{k+2} > \varrho} 2^{k\alpha_1 r} \|\mathcal{M}_t((D^\beta f)\chi_\Omega)\chi_{R_k}\|_p^r \\ &\lesssim \sum_{|\beta| \leq m} \|\mathcal{M}_t((D^\beta f)\chi_\Omega)\|_{\dot{K}_p^{\alpha_2, r}(\mathbb{R}^n)}^r \\ &\lesssim \|f\|_{\dot{K}_{p, \tilde{m}}^{\alpha_2, r}(\Omega)}^r, \end{aligned}$$

by Lemma [1.2.1](#).

We now proceed to estimate I_2 . We employ the same notation as in Theorem [2.2.7](#). We have

$$J_{1,k}(x) \lesssim \sum_{|\beta| \leq m-1} \varrho^{|\beta|-n} \int_{C_{x,\varrho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy, \quad x \in R_k \cap \Omega.$$

Hence, it suffices to estimate $J_{2,k}$. We decompose it as

$$J_{2,k}(x) = \sum_{|\beta|=m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy + \sum_{|\beta|=m} \int_{B_{x,\varrho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy, \quad (3.1)$$

where $B_{x,\varrho,k} = C_x \cap \{y : 2^{k-2} \leq |x-y| < \varrho\}$.

Let $t > 0$ be such that $m - \frac{n}{t} + \alpha_1 - \alpha_2 > 0$ and $t < p$. By Hölders inequality, the first integral in [\(3.1\)](#) is bounded by

$$\begin{aligned} &c 2^{k(\alpha_2 - \alpha_1)} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)| \chi_{\tilde{R}_k \cap \Omega}(y)}{|x-y|^{n-m-\alpha_1+\alpha_2}} dy \\ &\lesssim 2^{k(\alpha_2 - \alpha_1)} 2^{(m - \frac{n}{t} + \alpha_1 - \alpha_2)k} \mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k \cap \Omega})(x), \quad |\beta| = m. \end{aligned}$$

The boundedness of the HardyLittlewood maximal operator on $L^{\frac{p}{t}}(\mathbb{R}^n)$ implies

$$\|\mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k \cap \Omega})\|_p \lesssim \|(D^\beta f)\chi_{\tilde{R}_k \cap \Omega}\|_p, \quad |\beta| = m.$$

Next, we estimate

$$\begin{aligned} \int_{B_{x,2^{k+2},k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy &\lesssim 2^{(m-n)k} \int_{B_{x,2^{k+2},k}} |D^\beta f(y)| \chi_\Omega(y) dy \\ &\lesssim 2^{mk} \mathcal{M}((D^\beta f)\chi_\Omega)(x) \\ &\lesssim 2^{(m+\alpha_1-\alpha_2)k} 2^{(\alpha_2-\alpha_1)k} \mathcal{M}((D^\beta f)\chi_\Omega)(x), \quad |\beta| = m. \end{aligned}$$

Let $j \in \mathbb{Z}$ be such that $2^{j-1} \leq \varrho < 2^j$. As in Theorem [2.2.7](#), we obtain

$$\begin{aligned} \int_{B_{x,\varrho,k+4}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy &\lesssim \sum_{i=k+2}^j 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{\alpha_2 i} \|(D^\beta f)\chi_{R_i \cap \Omega}\|_p \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}, \quad |\beta| = m. \end{aligned}$$

The desired estimate follows from Lemma [1.2.1](#) together with the assumption $\alpha_1 + \frac{n}{p} > 0$. The proof is complete. $\square$

Now we present an embedding of $\dot{K}_{p,m}^{\alpha_2,r}(\Omega)$ into $\dot{K}_\infty^{\alpha_1,r}(\Omega)$.

Theorem 3.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone condition, with $0 \in \Omega$, and let $m \in \mathbb{N}_0$. Let $1 < p < \infty$, $1 \leq r < \infty$, and assume that $\frac{n}{p} + \alpha_2 < m < n$. Suppose further that $\alpha_2 \geq \alpha_1 > 0$. Then the embedding*

$$\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_\infty^{\alpha_1,r}(\Omega)$$

holds.

Proof. Let $f \in C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha_2,r}(\Omega)$ and ϱ be as in Lemma [2.2.5](#). We write

$$\begin{aligned} \|f\|_{\dot{K}_\infty^{\alpha_1,r}(\Omega)}^r &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_\infty^r \\ &= \sum_{2^{k-2} > \varrho} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_\infty^r + \sum_{2^{k-2} \leq \varrho} 2^{k\alpha_1 r} \|f\chi_{R_k \cap \Omega}\|_\infty^r \\ &= S_1 + S_2. \end{aligned}$$

Estimate of S_1 . By Lemma [2.2.5](#) and Hölders inequality, and since $m > \frac{n}{p}$, we obtain

$$|f(x)| \lesssim \sum_{|\beta| \leq m} \|(D^\beta f)\chi_{\tilde{R}_k \cap \Omega}\|_p$$

for every $x \in R_k \cap \Omega$, noting that $C_{x,\varrho} \subset \tilde{R}_k$.

Therefore,

$$\begin{aligned} S_1 &\lesssim \sum_{|\beta| \leq m} \sum_{2^{k-2} > \varrho} 2^{k(\alpha_1 - \alpha_2)r} 2^{k\alpha_2 r} \|(D^\beta f)\chi_{\tilde{R}_k \cap \Omega}\|_p^r \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r, \end{aligned}$$

because of $\alpha_2 \geq \alpha_1$.

Estimate of S_2 . We begin with the pointwise estimate

$$\begin{aligned} |f(x)| &\lesssim \sum_{|\beta| \leq m} \int_{C_{x,\varrho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \\ &= \sum_{|\beta| \leq m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy + \sum_{|\beta| \leq m} \int_{B_{x,\varrho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \\ &= P_{1,k}(x) + P_{2,k}(x), \end{aligned}$$

see; Lemma 2.2.5, where $B_{x,\varrho,k} = C_x \cap \{y : 2^{k-2} \leq |x-y| < \varrho\}$. Using again Hölder's inequality we obtain

$$P_{1,k}(x) \leq \sum_{|\beta| \leq m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{\tilde{R}_k \cap \Omega}(y) dy \lesssim 2^{k(m-\frac{n}{p})} \sum_{|\beta| \leq m} \|(D^\beta f) \chi_{\tilde{R}_k \cap \Omega}\|_p,$$

because of $m > \frac{n}{p}$. Therefore

$$\begin{aligned} \sum_{2^{k-2} \leq \varrho} 2^{k\alpha_1 r} \sup_{x \in R_k \cap \Omega} (P_{1,k}(x))^r &\lesssim \sum_{|\beta| \leq m} \sum_{2^{k-2} \leq \varrho} 2^{k(m-\frac{n}{p}+\alpha_1-\alpha_2)} 2^{k\alpha_2 r} \|(D^\beta f) \chi_{\tilde{R}_k \cap \Omega}\|_p^r \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r, \end{aligned}$$

since $m - \frac{n}{p} + \alpha_1 - \alpha_2 > 0$. Now we estimate $P_{2,k}$. We decompose $P_{2,k} = T_{1,k} + T_{2,k} + T_{3,k}$, where

$$\begin{aligned} T_{1,k}(x) &= \sum_{|\beta| \leq m} \int_{B_{x,\varrho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{\{|y| \leq \frac{|x|}{2}\}}(y) dy, \\ T_{2,k}(x) &= \sum_{|\beta| \leq m} \int_{B_{x,\varrho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}}(y) dy, \end{aligned}$$

and

$$T_{3,k}(x) = \sum_{|\beta| \leq m} \int_{B_{x,\varrho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} \chi_{\{|y| > 2|x|\}}(y) dy.$$

Let us consider the first term. Using the fact that $|x-y| \geq |y|$ if $|y| \leq \frac{|x|}{2}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} T_{1,k}(x) &\lesssim \sum_{|\beta| \leq m} \int_{|y| \leq 2^k} \frac{|D^\beta f(y)| \chi_\Omega(y)}{|y|^{n-m}} dy \\ &= c \sum_{|\beta| \leq m} \sum_{i=-\infty}^k 2^{i(m-\frac{n}{p}-\alpha_2)} 2^{i\alpha_2} \|(D^\beta f) \chi_{R_i \cap \Omega}\|_p \\ &= c 2^{k(m-\frac{n}{p}-\alpha_2)} \sum_{|\beta| \leq m} \sum_{i=-\infty}^k 2^{(i-k)(m-\frac{n}{p}-\alpha_2)} 2^{i\alpha_2} \|(D^\beta f) \chi_{R_i \cap \Omega}\|_p \\ &\lesssim 2^{k(m-\frac{n}{p}-\alpha_2)} \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}, \end{aligned}$$

since $m - \frac{n}{p} - \alpha_2 > 0$.

This leads to

$$\begin{aligned} \sum_{2^{k-2} \leq \varrho} 2^{k\alpha_1 r} \sup_{x \in R_k \cap \Omega} (T_{1,k}(x))^r &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r \sum_{2^{k-2} \leq \varrho} 2^{k(m - \frac{n}{p} - \alpha_2 + \alpha_1)r} \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r. \end{aligned}$$

Now we estimate $T_{2,k}$. We have

$$\begin{aligned} T_{2,k}(x) &\lesssim \sum_{|\beta| \leq m} 2^{k(m-n)} \int_{\tilde{R}_k \cap \Omega} |D^\beta f(y)| dy \\ &\lesssim 2^{k(m - \frac{n}{p})} \sum_{|\beta| \leq m} \|(D^\beta f) \chi_{\tilde{R}_k \cap \Omega}\|_p, \end{aligned}$$

by Hölder's inequality. Therefore,

$$\begin{aligned} \sum_{2^{k-2} \leq \varrho} 2^{k\alpha_1 r} \sup_{x \in R_k \cap \Omega} (T_{2,k}(x))^r &\lesssim \sum_{|\beta| \leq m} \sum_{2^{k-2} \leq \varrho} 2^{k(m - \frac{n}{p} - \alpha_2 + \alpha_1)r} 2^{k\alpha_2 r} \|(D^\beta f) \chi_{\tilde{R}_k \cap \Omega}\|_p^r \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r. \end{aligned}$$

Finally, we estimate $T_{3,k}$. We use that $|x - y| \geq \frac{|y|}{2}$ whenever $|y| > 2|x|$. Hence,

$$\begin{aligned} T_{3,k}(x) &\lesssim \sum_{|\beta| \leq m} \int_{|y| > 2^k} \frac{|D^\beta f(y)|}{|y|^{n-m}} \chi_\Omega(y) dy \\ &\lesssim \sum_{|\beta| \leq m} \sum_{i=k}^{j+1} 2^{i(m - \frac{n}{p} - \alpha_2)} 2^{i\alpha_2} \|(D^\beta f) \chi_{R_i \cap \Omega}\|_p \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r, \end{aligned}$$

where $2^{j-1} \leq \varrho < 2^j$, $j \in \mathbb{Z}$.

Using that $\alpha_1 > 0$, we obtain

$$\sum_{2^{k-2} \leq \varrho} 2^{k\alpha_1 r} \sup_{x \in R_k \cap \Omega} (T_{3,k}(x))^r \lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r.$$

The proof is complete. □

Collecting the results obtained in Theorems [3.1.1](#) and [3.1.2](#) we have the following statement.

Theorem 3.1.3. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition, $0 \in \Omega$ and $m \in \mathbb{N}_0$. Let $1 < p < q < \infty$, $1 \leq r < \infty$, $\alpha_2 \geq \alpha_1 > 0$ and*

$$\max\left(\frac{n}{p} + \alpha_2, \frac{n}{p} + \alpha_2 - \alpha_1\right) < m < n.$$

Then

$$\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_q^{\alpha_1,r}(\Omega)$$

holds. *thm*

Proof. Let $f \in \dot{K}_{p,m}^{\alpha_2,r}(\Omega)$ and $\theta = \frac{p}{q}$. We have

$$\|f\|_{\dot{K}_q^{\alpha_1,r}(\Omega)} \leq \|f\|_{\dot{K}_p^{\alpha_1,r}(\Omega)}^\theta \|f\|_{\dot{K}_\infty^{\alpha_1,r}(\Omega)}^{1-\theta} \lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)},$$

by Theorems [3.1.1](#) and [3.1.2](#). The proof is complete. $\square$

3.2 Embeddings of type $\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_q^{\alpha_1,r}(\Omega)$, $q < p$.

In the previous results we have not treated the case $q < p$. The next theorem gives a positive answer.

Theorem 3.2.1. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition, $0 \in \Omega$ and $m \in \mathbb{N}_0$. Let $1 < q < p < \infty$, $1 \leq r < \infty$, $\alpha_2 + \frac{n}{p} \geq \alpha_1 + \frac{n}{q} > 0$ and*

$$\max\left(\frac{n}{p}, \frac{n}{p} + \alpha_2\right) < m < n.$$

Then

$$\dot{K}_{p,m}^{\alpha_2,r}(\Omega) \hookrightarrow \dot{K}_q^{\alpha_1,r}(\Omega)$$

holds.

Proof. We use Theorem [2.1.5](#). Let $f \in C^\infty(\Omega) \cap \dot{K}_{p,m}^{\alpha_2,r}(\Omega)$. We adopt the same notation as in Theorem [3.1.1](#). We begin by estimating I_1 .

Let $t > 0$ be such that

$$1 < \frac{n}{m} < t < \min\left(p, \frac{n}{\max(0, \alpha_2 + \frac{n}{p})}\right).$$

Then, for any $x \in R_k \cap \Omega$, we have

$$\begin{aligned} |f(x)| &\lesssim \sum_{|\beta| \leq m-1} \varrho^{|\beta|-n} \int_{C_{x,\varrho}} |D^\beta f(y)| dy + \sum_{|\beta|=m} \int_{C_{x,\varrho}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \\ &\lesssim \sum_{|\beta| \leq m} \mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k \cap \Omega})(x). \end{aligned}$$

Applying Hölders inequality together with the boundedness of the maximal operator on $L^{p/t}(\mathbb{R}^n)$, we obtain

$$\begin{aligned} I_1 &\lesssim \sum_{|\beta| \leq m} \sum_{2^{k+2} > \varrho} 2^{(\alpha_1 + \frac{n}{q} - \frac{n}{p} - \alpha_2)kr} 2^{k\alpha_2 r} \left\| \mathcal{M}_t((D^\beta f)\chi_{\tilde{R}_k \cap \Omega}) \right\|_p^r \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r, \end{aligned}$$

where we used that $\alpha_2 + \frac{n}{p} \geq \alpha_1 + \frac{n}{q}$.

We now turn to I_2 . It suffices to estimate $J_{2,k}$, where

$$J_{2,k}(x) = \sum_{|\beta|=m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy + \sum_{|\beta|=m} \int_{B_{x,\varrho,k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy,$$

and $B_{x,\varrho,k} = C_x \cap \{y : 2^{k-2} \leq |x-y| < \varrho\}$.

By Hölders inequality, the first term satisfies

$$\sum_{|\beta|=m} \int_{C_{x,2^{k-2}}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy \lesssim 2^{k(m-\frac{n}{p})} \|(D^\beta f)\chi_{\tilde{R}_k \cap \Omega}\|_p.$$

Moreover, we have

$$\begin{aligned} \int_{B_{x,2^{k+2},k}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy &\lesssim 2^{(m-n)k} \int_{B_{x,2^{k+2},k}} |D^\beta f(y)| \chi_\Omega(y) dy \\ &\lesssim 2^{mk} \mathcal{M}((D^\beta f)\chi_\Omega)(x), \quad |\beta| = m. \end{aligned}$$

Let $j \in \mathbb{Z}$ be such that $2^{j-1} \leq \varrho < 2^j$. As in Theorem [2.2.7](#), we obtain

$$\begin{aligned} \int_{B_{x,\varrho,k+4}} \frac{|D^\beta f(y)|}{|x-y|^{n-m}} dy &\lesssim \sum_{i=k+2}^j 2^{(m-\frac{n}{p}-\alpha_2)i} 2^{\alpha_2 i} \|(D^\beta f)\chi_{R_i \cap \Omega}\|_p \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}, \quad |\beta| = m. \end{aligned}$$

Combining the above estimates, using Hölders inequality and Lemma [1.2.1](#), we deduce that I_2 is bounded by

$$\begin{aligned} I_2 &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r \sum_{2^{k+2} \leq \varrho} 2^{k(\alpha_1 + \frac{n}{q})r} \\ &\quad + \sum_{|\beta| \leq m} \sum_{2^{k+2} \leq \varrho} 2^{k(m-\frac{n}{p} + \alpha_1 - \alpha_2 + \frac{n}{q})r} 2^{k\alpha_2 r} \\ &\quad \times \left(\|(D^\beta f)\chi_{\tilde{R}_k \cap \Omega}\|_p^r + \|\mathcal{M}((D^\beta f)\chi_\Omega)\chi_{R_k}\|_p^r \right) \\ &\lesssim \|f\|_{\dot{K}_{p,m}^{\alpha_2,r}(\Omega)}^r, \end{aligned}$$

since $\alpha_1 + \frac{n}{q} > 0$ and

$$m - \frac{n}{p} + \alpha_1 - \alpha_2 + \frac{n}{q} > 0.$$

This completes the proof. □

Theorem 3.2.2. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition and $m \in \mathbb{N}$. Let $1 < p < \infty$ and $1 \leq r < \infty$. Assume that $0 \leq \alpha < n - \frac{n}{p}$,*

$$m > \frac{n}{p} + \alpha.$$

Then

$$\dot{K}_{p,m}^{\alpha,r}(\Omega) \hookrightarrow L^\infty(\Omega).$$

holds.

For the proof; see [9].

For an arbitrary nonempty set $\Omega \subset \mathbb{R}^n$ we shall denote by $C_b(\Omega)$ the Banach space of functions f continuous and bounded on Ω with the norm

$$\|f\|_{C_b(\Omega)} = \sup_{x \in \Omega} |f(x)|.$$

Theorem 3.2.3. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition and $m \in \mathbb{N}$. Let $1 < p < \infty$ and $1 \leq r < \infty$. Assume that $0 \leq \alpha \leq n - \frac{n}{p}$ and $m > \frac{n}{p} + \alpha$. Then*

$$\dot{K}_{p,m}^{\alpha,v}(\Omega) \hookrightarrow C_b(\Omega)$$

where

$$v = \begin{cases} 1, & \text{if } \alpha = n - \frac{n}{p}, \\ r, & \text{otherwise.} \end{cases}$$

For the proof; see [9].

Conclusion

In this thesis, we investigated several properties of Herz-type Sobolev spaces defined on domains satisfying the cone condition. These spaces represent a natural extension of classical Sobolev spaces by incorporating the weighted and localized structure of Herz spaces, which makes them suitable for studying more general analytical problems.

We began by presenting the fundamental notions and preliminary results related to Herz spaces, including their basic properties, approximation results, and technical tools required throughout the work. These preliminaries provided the mathematical framework necessary for the development of the subsequent chapters.

Next, we introduced Herz-type Sobolev spaces and established their principal structural properties. In particular, we discussed approximation by smooth functions and showed that these spaces preserve several important characteristics analogous to those of classical Sobolev spaces. We also proved Sobolev-type embedding theorems that describe the relationship between Herz-type Sobolev spaces and Herz spaces under suitable assumptions on the parameters.

Finally, we extended these embedding results by studying additional classes of embeddings involving different parameter configurations. These results highlight the flexibility and richness of Herz-type Sobolev spaces and demonstrate their ability to generalize classical embedding phenomena.

Overall, the results obtained in this thesis contribute to a deeper understanding of Herz-type Sobolev spaces on domains and provide useful analytical tools for future investigations. Possible directions for further research include the study of compact embeddings, applications to partial differential equations, and extensions to variable exponent or generalized function spaces.

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المُلخَص

الهدف من هذا العمل هو دراسة العديد من خصائص فضاءات سوبوليف من نوع هيرتز، مع التركيز بشكل خاص على خواص انغماس سوبوليف.

الكلمات المفتاحية: فضاءات هيرتز، فضاءات سوبوليف من نوع هيرتز، انغماسات سوبوليف.

Abstract

The objective of this work is to investigate several properties of Herz-type Sobolev spaces, with particular emphasis on their Sobolev embedding properties.

Keywords: Herz spaces, Herz-type Sobolev spaces, Embeddings.

Résumé

L'objectif de cette thèse est d'étudier certaines propriétés des espaces de Sobolev de type Herz, avec une attention particulière portée à leurs propriétés d'inclusion de Sobolev.

Mots clés : Espaces de Herz, Espaces de Sobolev de type Herz, Inclusion.